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ON A REVERSE HARDY-LITTLEWOOD-PÓLYA'S INEQUALITY*

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Abstract By the use of the weight coefficients, the idea of introduced parameters and Euler-Maclaurin summation formula, a reverse Hardy-Littlewood-Pólya's inequality with parameters as well as the equivalent forms are provided. The equivalent statements of the best possible constant factor related to a few parameters and some particular cases are given.

Keywords Weight coefficient, Hardy-Littlewood-Pólya's inequality, Euler-Maclaurin summation formula, equivalent statement, parameter.

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1. Introduction

If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, a_m, b_n \geq 0, 0 < \sum_{m=1}^{\infty} a_m^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then we have Hardy-Hilbert's inequality with the best possible constant factor $\frac{\pi}{\sin(\pi/p)}$ as follows:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}(\sum_{m=1}^{\infty} a_m^p)^{\frac{1}{p}}(\sum_{n=1}^{\infty} b_n^q)^{\frac{1}{q}}, \tag{1.1}$$

and the following Hardy-Littlewood-Pólya's inequality:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{\max\{m,n\}} < pq(\sum_{m=1}^{\infty} a_m^p)^{\frac{1}{p}}(\sum_{n=1}^{\infty} b_n^q)^{\frac{1}{q}}, \tag{1.2}$$

where, the constant factor pq is the best possible (cf [4], Theorem 315 and Theorem 341).

In 2006, by introducing a few parameters $\lambda_i \in (0, 2] (i = 1, 2), \lambda_1 + \lambda_2 = \lambda \in (0, 4]$, an extension of (1.1) was provided by [12] as follows:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}} < B(\lambda_1, \lambda_2)[\sum_{m=1}^{\infty} m^{p(1-\lambda_1)-1} a_m^p]^{\frac{1}{p}}[\sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} b_n^q]^{\frac{1}{q}}, \tag{1.3}$$

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where, the constant factor $B(\lambda_1, \lambda_2)$ is the best possible ($B(u, v) = \int_0^\infty \frac{t^{u-1}}{(1+t)^{u+v}} dt$ $(u, v > 0)$ is the beta function). For $\lambda = 1, \lambda_1 = \frac{1}{q}, \lambda_2 = \frac{1}{p}$, inequality (1.2) reduces to (1.1); for $p = q = 2, \lambda_1 = \lambda_2 = \frac{\lambda}{2}$, (1.3) reduces to Yang's inequality in [23]. Recently, applying (1.3), Adiyasuren et al. [1] provided a new Hilbert-type inequality with the kernel $\frac{1}{(m+n)^\lambda}$ involving partial sums.

If $f(x), g(y) \geq 0, 0 < \int_0^\infty f^p(x)dx < \infty$ and $0 < \int_0^\infty g^q(y)dy < \infty$, then we have the following Hardy-Hilbert's integral inequality (cf. [4], Theorem 316):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin(\pi/p)} (\int_0^\infty f^p(x)dx)^{\frac{1}{p}} (\int_0^\infty g^q(y)dy)^{\frac{1}{q}}, \tag{1.4}$$

where, the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. Inequalities (1.1), (1.2) and (1.4) with their extensions and reverses play an important role in analysis and its applications (cf. [2,3,5,6,13,15,19–21,24,28]).

In 1934, a half-discrete Hilbert-type inequality was given as follows (cf. [4], Theorem 351): If $K(t) (t > 0)$ is decreasing, $p > 1, \frac{1}{p} + \frac{1}{q} = 1, 0 < \phi(s) = \int_0^\infty K(t)t^{s-1}dt < \infty$, $a_n \geq 0, 0 < \sum_{n=1}^\infty a_n^p < \infty$, then we have

$$\int_0^\infty x^{p-2} (\sum_{n=1}^\infty K(nx)a_n)^p dx < \phi^p(\frac{1}{q}) \sum_{n=1}^\infty a_n^p. \tag{1.5}$$

In the last ten years, some extensions of (1.5) with their applications and the reverses were provided by [16–18,25,26].

In 2016, by means of the technique of real analysis, Hong et al. [7] considered some equivalent statements of the extensions of (1.1) with the best possible constant factor related to a few parameters. The other similar works about (1.2), (1.4) and (1.5) were given by [8–11,22,27].

In this paper, following the methods of [12] and [7], by the use of the weight coefficients, the idea of introducing parameters and Euler-Maclaurin summation formula, a reverse Hardy-Littlewood-Pólya's inequality with parameters as well as the equivalent forms are provided in Lemma 2.2 and Theorem 3.1. The equivalent statements of the best possible constant factor related to a few parameters and some particular cases are considered in Theorem 3.2 and Remark 3.1.

2. Some lemmas

In what follows, we suppose that $0 < p < 1 (q < 0), \frac{1}{p} + \frac{1}{q} = 1, \lambda \in (0, 3], \lambda_i \in (0, \frac{11}{8}] \cap (0, \lambda)$ $(i = 1, 2)$. We also assume that $a_m, b_n \geq 0$ $(m, n \in \mathbf{N} = \{1, 2, \cdots\})$, such that

$$0 < \sum_{m=1}^\infty m^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1} a_m^p < \infty \text{ and } 0 < \sum_{n=1}^\infty n^{q[1-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1} b_n^q < \infty.$$

Lemma 2.1. *Define the following weight coefficient:*

$$\varpi_\lambda(\lambda_2, m) := m^{\lambda-\lambda_2} \sum_{n=1}^\infty \frac{n^{\lambda_2-1}}{(\max\{m, n\})^\lambda} \ (m \in \mathbf{N}). \tag{2.1}$$

We have the following inequality:

$$0 < k_\lambda(\lambda_2)(1 - \frac{\lambda - \lambda_2}{\lambda m^{\lambda_2}})$$
$$< \varpi_\lambda(\lambda_2, m) < k_\lambda(\lambda_2) := \frac{\lambda}{\lambda_2(\lambda - \lambda_2)} \ (m \in \mathbf{N}). \qquad (2.2)$$

Proof. For fixed $m \in \mathbf{N}$, we set function $g_m(t) := \frac{t^{\lambda_2 - 1}}{(\max\{m,t\})^\lambda}$ $(t > 0)$. We find

$$g_m(t) = \begin{cases} \frac{t^{\lambda_2-1}}{m^\lambda}, 0 < t < m, \\ t^{\lambda_2 - \lambda - 1}, t \geq m, \end{cases}$$

$$g'_m(t) = \begin{cases} \frac{(\lambda_2 - 1)t^{\lambda_2-2}}{m^\lambda}, 0 < t < m, \\ (\lambda_2 - \lambda - 1)t^{\lambda_2 - \lambda - 2}, t > m. \end{cases}$$

In the following, we divide two cases of λ_2 to obtain (2.2).

(i) For $\lambda_2 \in (0, 1] \cap (0, \lambda)$, by the decreasingness property of series, we obtain

$$\varpi_\lambda(\lambda_2, m) < m^{\lambda - \lambda_2} \int_0^\infty g_m(t)dt$$
$$= m^{\lambda - \lambda_2}[\int_0^m \frac{t^{\lambda_2 - 1}}{m^\lambda} dt + \int_m^\infty t^{\lambda_2 - \lambda - 1} dt] = k_\lambda(\lambda_2),$$

$$\varpi_\lambda(\lambda_2, m) > m^{\lambda - \lambda_2} \int_1^\infty g_m(t)dt$$
$$= m^{\lambda - \lambda_2}[\int_0^\infty \frac{t^{\lambda_2 - 1}dt}{(\max\{m,t\})^\lambda} - \int_0^1 \frac{t^{\lambda_2 - 1}dt}{(\max\{m,t\})^\lambda}]$$
$$= k_\lambda(\lambda_2) - m^{\lambda - \lambda_2} \int_0^1 \frac{t^{\lambda_2 - 1}}{m^\lambda} dt$$
$$= k_\lambda(\lambda_2)(1 - \frac{\lambda - \lambda_2}{\lambda m^{\lambda_2}}) > 0.$$

In this case, (2.2) follow.

(ii) For $\lambda_2 \in (1, \frac{11}{8}] \cap (0, \lambda)$, by using Euler-Maclaurin summation formula (cf. [26]), for $\rho(t) = t - [t] - \frac{1}{2}$, we find

$$\sum_{n=2}^m g_m(n) = \int_1^m g_m(t)dt + \frac{1}{2} g_m(t)|_1^m + \int_1^m \rho(t) g'_m(t)dt$$
$$= \int_1^m g_m(t)dt + \frac{1}{2} g_m(t)|_1^m + \frac{\lambda_2 - 1}{m^\lambda} \int_1^m \rho(t) t^{\lambda_2 - 2} dt$$
$$= \int_1^m g_m(t)dt + \frac{1}{2} g_m(t)|_1^m + \frac{\lambda_2 - 1}{m^\lambda} \frac{\varepsilon}{12} t^{\lambda_2 - 2}|_1^m$$
$$\leq \int_1^m g_m(t)dt + \frac{1}{2} g_m(t)|_1^m \ (1 < \lambda_2 < 2, 0 < \varepsilon < 1),$$

$$\sum_{n=m+1}^\infty g_m(n) = \int_m^\infty g_m(t)dt + \frac{1}{2} g_m(t)|_m^\infty + \int_m^\infty \rho(t) g'_m(t)dt$$

$$= \int_m^\infty g_m(t)dt + \frac{1}{2}g_m(t)|_m^\infty + \frac{\lambda_2 - \lambda - 1}{12}\varepsilon t^{\lambda_2-\lambda-2}|_m^\infty$$
$$< \int_m^\infty g_m(t)dt + \frac{1}{2}g_m(t)|_m^\infty + \frac{\lambda_2 - \lambda - 1}{12m^{\lambda-\lambda_2+2}}$$
$$(\lambda_2 < \lambda, 0 < \varepsilon < 1),$$

and then it follows that

$$\sum_{n=1}^\infty g_m(n) < \int_1^\infty g_m(t)dt + \frac{1}{2}g_m(1) + \frac{\lambda_2 - \lambda - 1}{12m^{\lambda-\lambda_2+2}}$$
$$= \int_0^\infty g_m(t)dt - h_m(\lambda, \lambda_2),$$

where, for $\lambda \leq 3, \lambda_2 < 2, h(\lambda_2) := 12 - 10\lambda_2 + \lambda_2^2$,

$$h_m(\lambda, \lambda_2) \ : \ = \int_0^1 g_m(t)dt - \frac{1}{2}g_m(1) - \frac{\lambda_2 - \lambda - 1}{12m^{\lambda-\lambda_2+2}}$$
$$= \frac{1}{\lambda_2 m^\lambda} - \frac{1}{2m^\lambda} - \frac{\lambda_2 - \lambda - 1}{12m^{\lambda-\lambda_2+2}}$$
$$> (\frac{1}{\lambda_2} - \frac{1}{2} - \frac{\lambda_2 - \lambda - 1}{12})\frac{1}{m^\lambda} = \frac{h(\lambda_2)}{12\lambda_2 m^\lambda}.$$

Since $h'(\lambda_2) := -10 + 2\lambda_2 < 0$ ($\lambda_2 \in (1, \frac{11}{8}]$), we find

$$h_m(\lambda, \lambda_2) > \frac{h(\lambda_2)}{12\lambda_2 m^\lambda} \geq . \frac{h(11/8)}{12\lambda_2 m^\lambda} = \frac{1}{256\lambda_2 m^\lambda} > 0,$$

and then we obtain

$$\varpi_\lambda(\lambda_2, m) = m^{\lambda-\lambda_2}\sum_{n=1}^\infty g_m(n) < m^{\lambda-\lambda_2}\int_0^\infty g_m(t)dt$$
$$= k_\lambda(\lambda_2) = \frac{\lambda}{\lambda_2(\lambda - \lambda_2)}.$$

On the other hand, we find

$$\sum_{n=2}^m g_m(n) = \int_1^m g_m(t)dt + \frac{1}{2}g_m(t)|_1^m + \frac{\lambda_2 - 1}{m^\lambda}\frac{\varepsilon}{12}t^{\lambda_2-2}|_1^m$$
$$\geq \int_1^m g_m(t)dt + \frac{1}{2}g_m(t)|_1^m - \frac{\lambda_2 - 1}{12m^\lambda}(1 - m^{\lambda_2-2}),$$

$$\sum_{n=m+1}^\infty g_m(n) = \int_m^\infty g_m(t)dt + \frac{1}{2}g_m(t)|_m^\infty + \frac{\lambda_2 - \lambda - 1}{12}\varepsilon t^{\lambda_2-\lambda-2}|_m^\infty$$
$$> \int_m^\infty g_m(t)dt + \frac{1}{2}g_m(t)|_m^\infty,$$

and then for $\frac{1}{2m^\lambda} - \frac{\lambda_2-1}{12m^\lambda} > \frac{1}{2m^\lambda} - \frac{1}{12m^\lambda}$ ($\lambda_2 < 2$), we find

$$\sum_{n=1}^\infty g_m(n) > \int_1^\infty g_m(t)dt + \frac{1}{2}g_m(1) + \frac{\lambda_2 - 1}{12m^\lambda}(1 - m^{\lambda_2-2})$$

$$> \int_1^\infty g_m(t)dt + (\frac{1}{2m^\lambda} - \frac{\lambda_2 - 1}{12m^\lambda}) > \int_1^\infty g_m(t)dt > 0.$$

Hence, in view of the result of the case (i), for $\lambda_2 \in (1, \frac{11}{8}] \cap (0, \lambda)$, we still have (2.2).

The lemma is proved. □

Lemma 2.2. *We have the following reverse Hardy-littlewood-Polya's inequality with parameters:*

$$\begin{aligned} I &:= \sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{(\max\{m,n\})^\lambda} \\ &> k_\lambda^{\frac{1}{p}}(\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1) \{\sum_{m=1}^\infty (1 - \frac{\lambda - \lambda_2}{\lambda m^{\lambda_2}}) m^{p[1-(\frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q})]-1} a_m^p\}^{\frac{1}{p}} \\ &\times \{\sum_{n=1}^\infty n^{q[1-(\frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p})]-1} b_n^q\}^{\frac{1}{q}}. \end{aligned} \tag{2.3}$$

Proof. In the same way of obtaining (2.2), for $n \in \mathbf{N}$, we have the following inequality of the weight coefficient:

$$\begin{aligned} 0 &< k_\lambda(\lambda_1)(1 - \frac{\lambda - \lambda_1}{\lambda n^{\lambda_1}}) \\ &< \omega_\lambda(\lambda_1, n) := n^{\lambda - \lambda_1} \sum_{m=1}^\infty \frac{m^{\lambda_1 - 1}}{(\max\{m,n\})^\lambda} \\ &< k_\lambda(\lambda_1) = \frac{\lambda}{\lambda_1(\lambda - \lambda_1)} \ (n \in \mathbf{N}). \end{aligned} \tag{2.4}$$

By the reverse Hölder's inequality (cf. [14]), we obtain

$$\begin{aligned} I &= \sum_{m=1}^\infty \sum_{n=1}^\infty \frac{1}{(\max\{m,n\})^\lambda} [\frac{n^{(\lambda_2-1)/p}}{m^{(\lambda_1-1)/q}} a_m][\frac{m^{(\lambda_1-1)/q}}{n^{(\lambda_2-1)/p}} b_n] \\ &\geq \{\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{1}{(\max\{m,n\})^\lambda} \frac{n^{\lambda_2-1}}{m^{(\lambda_1-1)(p-1)}} a_m^p\}^{\frac{1}{p}} \\ &\times \{\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{1}{(\max\{m,n\})^\lambda} \frac{m^{\lambda_1-1}}{n^{(\lambda_2-1)(q-1)}} b_n^q\}^{\frac{1}{q}} \\ &= \{\sum_{m=1}^\infty \varpi_\lambda(\lambda_2, m) m^{p[1-(\frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q})]-1} a_m^p\}^{\frac{1}{p}} \\ &\times \{\sum_{n=1}^\infty \omega_\lambda(\lambda_1, n) n^{q[1-(\frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p})]-1} b_n^q\}^{\frac{1}{q}}. \end{aligned}$$

Then by (2.2) and (2.4), for $0 < p < 1, q < 0$, we have (2.3).
The lemma is proved. □

Remark 2.1. By (2.3), for $\lambda_1 + \lambda_2 = \lambda \in (0, \frac{11}{4}] (\subset (0,3])$, $\lambda_i \in (0, \frac{11}{8}] \cap (0, \lambda) (i = 1, 2)$,we find

$$0 < \sum_{m=1}^\infty m^{p(1-\lambda_1)-1} a_m^p < \infty \text{ and } 0 < \sum_{n=1}^\infty n^{q(1-\lambda_2)-1} b_n^q < \infty,$$

and the following inequality:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(\max\{m,n\})^{\lambda}}$$
$$> \frac{\lambda}{\lambda_1\lambda_2}[\sum_{m=1}^{\infty}(1-\frac{\lambda_1}{\lambda m^{\lambda_2}})m^{p(1-\lambda_1)-1}a_m^p]^{\frac{1}{p}}[\sum_{n=1}^{\infty}n^{q(1-\lambda_2)-1}b_n^q]^{\frac{1}{q}}. \quad (2.5)$$

Lemma 2.3. *The constant factor* $\frac{\lambda}{\lambda_1\lambda_2}$ *in (2.5) is the best possible.*

Proof. For any $0<\varepsilon<p\lambda_1$, we set

$$\widetilde{a}_m := m^{\lambda_1-\frac{\varepsilon}{p}-1}, \widetilde{b}_n := n^{\lambda_2-\frac{\varepsilon}{q}-1}(m,n\in\mathbf{N}).$$

If there exists a constant $M\geq\frac{\lambda}{\lambda_1\lambda_2}$, such that (2.5) is valid when replacing $\frac{\lambda}{\lambda_1\lambda_2}$ by M, then in particular, substitution of $\widetilde{a}_m = a_m$ and $\widetilde{b}_n = b_n$ in (2.5), we have

$$\widetilde{I} := \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\widetilde{a}_m\widetilde{b}_n}{(\max\{m,n\})^{\lambda}}$$
$$> M[\sum_{m=1}^{\infty}(1-\frac{\lambda_1}{\lambda m^{\lambda_2}})m^{p(1-\lambda_1)-1}\widetilde{a}_m^p]^{\frac{1}{p}}[\sum_{n=1}^{\infty}n^{q(1-\lambda_2)-1}\widetilde{b}_n^q]^{\frac{1}{q}}. \quad (2.6)$$

By (2.5) and the decreasingness property of series, we obtain

$$\widetilde{I} > M(\sum_{m=1}^{\infty}m^{-\varepsilon-1}-\frac{\lambda_1}{\lambda}\sum_{m=1}^{\infty}m^{-\varepsilon-\lambda_2-1})^{\frac{1}{p}}(1+\sum_{n=2}^{\infty}n^{-\varepsilon-1})^{\frac{1}{q}}$$
$$> M(\int_1^{\infty}x^{-\varepsilon-1}dx - O(1))^{\frac{1}{p}}(1+\int_1^{\infty}y^{-\varepsilon-1}dy)^{\frac{1}{q}}$$
$$= \frac{M}{\varepsilon}(1-\varepsilon O(1))^{\frac{1}{p}}(\varepsilon+1)^{\frac{1}{q}}.$$

By (2.4), setting

$$\widehat{\lambda}_1 := \lambda_1-\frac{\varepsilon}{p}\in(0,\frac{11}{8})\cap(0,\lambda),$$
$$0 < \widehat{\lambda}_2 := \lambda_2+\frac{\varepsilon}{p} = \lambda-\widehat{\lambda}_1 < \lambda,$$

we find

$$\widetilde{I} = \sum_{n=1}^{\infty}[n^{\lambda_2+\frac{\varepsilon}{p}}\sum_{m=1}^{\infty}\frac{m^{(\lambda_1-\frac{\varepsilon}{p})-1}}{(\max\{m,n\})^{\lambda}}]n^{-\varepsilon-1} = \sum_{n=1}^{\infty}\omega_{\lambda}(\widehat{\lambda}_1,n)n^{-\varepsilon-1}$$
$$< \frac{\lambda}{\widehat{\lambda}_1\widehat{\lambda}_2}(1+\sum_{n=2}^{\infty}n^{-\varepsilon-1}) < \frac{\lambda}{\widehat{\lambda}_1\widehat{\lambda}_2}(1+\int_1^{\infty}y^{-\varepsilon-1}dy)$$
$$= \frac{\lambda}{\varepsilon(\lambda_1-\frac{\varepsilon}{p})(\lambda_2+\frac{\varepsilon}{p})}(\varepsilon+1).$$

Then we have

$$\frac{\lambda}{(\lambda_1-\frac{\varepsilon}{p})(\lambda_2+\frac{\varepsilon}{p})}(\varepsilon+1) > \varepsilon\widetilde{I} > M(1-\varepsilon O(1))^{\frac{1}{p}}(\varepsilon+1)^{\frac{1}{q}}.$$

For $\varepsilon \to 0^+$, we find $\frac{\lambda}{\lambda_1\lambda_2} \geq M$. Hence, $M = \frac{\lambda}{\lambda_1\lambda_2}$ is the best possible constant factor in (2.5).

The lemma is proved. □

Setting $\widetilde{\lambda}_1 := \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q}$, $\widetilde{\lambda}_2 := \frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p}$, we find

$$\widetilde{\lambda}_1 + \widetilde{\lambda}_2 = \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q} + \frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p} = \frac{\lambda}{p} + \frac{\lambda}{q} = \lambda,$$

and we can rewrite (2.3) as follows:

$$\begin{aligned} I &:= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(\max\{m,n\})^{\lambda}} \\ &> k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1)[\sum_{m=1}^{\infty}(1-\frac{\lambda-\lambda_2}{\lambda m^{\lambda_2}})m^{p(1-\widetilde{\lambda}_1)-1}a_m^p]^{\frac{1}{p}}[\sum_{n=1}^{\infty}n^{q(1-\widetilde{\lambda}_2)-1}b_n^q]^{\frac{1}{q}}. \quad (2.7) \end{aligned}$$

Lemma 2.4. *If inequality (2.7) is valid with the best possible constant factor* $k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1)$, *then for* $\lambda-\lambda_1-\lambda_2 \in (-p\lambda_1, p(\lambda-\lambda_1))$, *we have* $\lambda-\lambda_1-\lambda_2 = 0$, *namely,* $\lambda = \lambda_1+\lambda_2$.

Proof. For $\lambda-\lambda_1-\lambda_2 \in (-p\lambda_1, p(\lambda-\lambda_1))$, we obtain

$$0 < \widetilde{\lambda}_1 = \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q} < \lambda, 0 < \widetilde{\lambda}_2 = \lambda - \widetilde{\lambda}_1 < \lambda.$$

Hence, we have $k_{\lambda}(\widetilde{\lambda}_1) = \frac{\lambda}{\widetilde{\lambda}_1(\lambda-\widetilde{\lambda}_1)} = \frac{\lambda}{\widetilde{\lambda}_1\widetilde{\lambda}_2} \in \mathbf{R}_+ = (0,\infty)$.

If the constant factor $k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1)$ in (2.7) is the best possible, then in view of (10), we have the following inequality:

$$k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1) \geq k_{\lambda}(\widetilde{\lambda}_1).$$

By the reverse Hölder's inequality, we obtain

$$\begin{aligned} k_{\lambda}(\widetilde{\lambda}_1) &= k_{\lambda}(\frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q}) = \int_0^{\infty}\frac{u^{\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}-1}}{(\max\{1,u\})^{\lambda}}du \\ &= \int_0^{\infty}\frac{1}{(\max\{1,u\})^{\lambda}}(u^{\frac{\lambda-\lambda_2-1}{p}})(u^{\frac{\lambda_1-1}{q}})du \\ &\geq [\int_0^{\infty}\frac{u^{\lambda-\lambda_2-1}}{(\max\{1,u\})^{\lambda}}du]^{\frac{1}{p}}[\int_0^{\infty}\frac{u^{\lambda_1-1}}{(\max\{1,u\})^{\lambda}}du]^{\frac{1}{q}} \\ &= k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1). \quad (2.8) \end{aligned}$$

Hence, $k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1) = k_{\lambda}(\widetilde{\lambda}_1)$, namely, (2.8) keeps the form of equality.

We observe that (2.8) keeps the form of equality if and only if there exist constants A and B, such that they are not all zero and (cf. [14])

$$Au^{\lambda-\lambda_2-1} = Bu^{\lambda_1-1} \text{ a.e. in } \mathbf{R}_+.$$

Assuming that $A \neq 0$, we have $u^{\lambda-\lambda_1-\lambda_2} = \frac{B}{A}$ a.e. in $\mathbf{R}_+$, and then $\lambda-\lambda_1-\lambda_2 = 0$, namely, $\lambda = \lambda_1+\lambda_2$.

The lemma is proved. □

3. Main results and some particular inequalities

Theorem 3.1. *Inequality (2.3) is equivalent to the following inequalities:*

$$J_1 := \{\sum_{n=1}^{\infty} n^{p(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})-1}[\sum_{m=1}^{\infty} \frac{a_m}{(\max\{m,n\})^{\lambda}}]^p\}^{\frac{1}{p}}$$
$$> k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1)\{\sum_{m=1}^{\infty}(1-\frac{\lambda-\lambda_2}{\lambda m^{\lambda_2}})m^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}a_m^p\}^{\frac{1}{p}}, \tag{3.1}$$

$$J_2 := \{\sum_{m=1}^{\infty} \frac{m^{q(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})-1}}{(1-\frac{\lambda-\lambda_2}{\lambda m^{\lambda_2}})^{q-1}}[\sum_{n=1}^{\infty} \frac{b_n}{(\max\{m,n\})^{\lambda}}]^q\}^{\frac{1}{q}}$$
$$> k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1)\{\sum_{n=1}^{\infty} n^{q[1-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1}b_n^q\}^{\frac{1}{q}}. \tag{3.2}$$

If the constant factor in (2.3) is the best possible, then so is the constant factor in (3.1) and (3.2).

Proof. Suppose that (3.1) is valid. By the reverse Hölder's inequality, we have

$$I = \sum_{n=1}^{\infty}[n^{\frac{-1}{p}+(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})}\sum_{m=1}^{\infty} \frac{a_m}{(\max\{m,n\})^{\lambda}}][n^{\frac{1}{p}-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})}b_n]$$
$$\geq J_1\{\sum_{n=1}^{\infty} n^{q[1-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1}b_n^q\}^{\frac{1}{q}}. \tag{3.3}$$

Then by (3.1), we obtain (2.3). On the other hand, assuming that (2.3) is valid, we set

$$b_n := n^{p(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})-1}[\sum_{m=1}^{\infty} \frac{a_m}{(\max\{m,n\})^{\lambda}}]^{p-1}, n \in \mathbf{N}.$$

If $J_1 = \infty$, then (3.1) is naturally valid; if $J_1 = 0$, then it is impossible to make (3.1) valid, namely, $J_1 > 0$. Suppose that $0 < J_1 < \infty$. By (2.3), we have

$$\infty > \sum_{n=1}^{\infty} n^{q[1-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1}b_n^q = J_1^p = I$$
$$> k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1)\{\sum_{m=1}^{\infty}(1-\frac{\lambda-\lambda_2}{\lambda m^{\lambda_2}})m^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}a_m^p\}^{\frac{1}{p}}J_1^{p-1},$$
$$J_1 = \{\sum_{n=1}^{\infty} n^{q[1-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1}b_n^q\}^{1/p}$$
$$> k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1)\{\sum_{m=1}^{\infty}(1-\frac{\lambda-\lambda_2}{\lambda m^{\lambda_2}})m^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}a_m^p\}^{\frac{1}{p}},$$

namely, (3.1) follows, which is equivalent to (2.3).

Suppose that (3.2) is valid. By the reverse Hölder's inequality, we have

$$I = \sum_{m=1}^{\infty}[(1-\frac{\lambda-\lambda_2}{\lambda m^{\lambda_2}})^{\frac{1}{p}}m^{\frac{1}{q}-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})}a_m]$$

$$\times[\frac{m^{\frac{-1}{q}+(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})}}{(1-\frac{\lambda-\lambda_2}{\lambda m^{\lambda_2}})^{\frac{1}{p}}}\sum_{n=1}^{\infty}\frac{b_n}{(\max\{m,n\})^{\lambda}}]$$

$$\geq\{\sum_{m=1}^{\infty}(1-\frac{\lambda-\lambda_2}{\lambda m^{\lambda_2}})m^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}a_m^p\}^{\frac{1}{p}}J_2. \tag{3.4}$$

Then by (3.2), we obtain (2.3). On the other hand, assuming that (2.3) is valid, we set

$$a_m := \frac{m^{q(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})-1}}{(1-\frac{\lambda-\lambda_2}{\lambda m^{\lambda_2}})^{q-1}}[\sum_{n=1}^{\infty}\frac{b_n}{(\max\{m,n\})^{\lambda}}]^{q-1}, m\in\mathbf{N}.$$

If $J_2=\infty$, then (3.2) is naturally valid; if $J_2=0$, then it is impossible to make (3.2) valid, namely, $J_2>0$. Suppose that $0<J_2<\infty$. By (2.3), we have

$$\infty>\sum_{m=1}^{\infty}(1-\frac{\lambda-\lambda_2}{\lambda m^{\lambda_2}})m^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}a_m^p=J_2^q=I$$

$$>k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1)J_2^{q-1}\{\sum_{n=1}^{\infty}n^{q[1-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1}b_n^q\}^{1/q},$$

$$J_2=\{\sum_{m=1}^{\infty}(1-\frac{\lambda-\lambda_2}{\lambda m^{\lambda_2}})m^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}a_m^p\}^{1/q}$$

$$>k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1)\{\sum_{n=1}^{\infty}n^{q[1-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1}b_n^q\}^{1/q},$$

namely, (3.2) follows, which is equivalent to (2.3). Hence, inequalities (2.3), (3.1) and (3.2) are equivalent.

If the constant factor in (2.3) is the best possible, then so is the constant factor in (3.1) and (3.2). Otherwise, by (3.3) (or (3.4)), we would reach a contradiction that the constant factor in (2.3) is not the best possible.

The theorem is proved. □

Theorem 3.2. *The following statements (i), (ii), (iii) and (iv) are equivalent:*

(i) Both $k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1)$ and $k_{\lambda}(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})$ are finite and independent of p,q;

(ii) $k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1)$ is expressible as a single integral:

$$k_{\lambda}(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})=\int_0^{\infty}\frac{u^{\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}-1}}{(\max\{1,u\})^{\lambda}}du;$$

(iii) $k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1)$ in (2.3) is the best possible constant factor;

(iv) If $\lambda-\lambda_1-\lambda_2\in(-p\lambda_1,p(\lambda-\lambda_1))$, then $\lambda=\lambda_1+\lambda_2$.

If the statement (iv) follows, namely, $\lambda=\lambda_1+\lambda_2$, then we have (2.5) and the following equivalent inequalities with the best possible constant factor $\frac{\lambda}{\lambda_1\lambda_2}$:

$$\{\sum_{n=1}^{\infty}n^{p\lambda_2-1}[\sum_{m=1}^{\infty}\frac{a_m}{(\max\{m,n\})^{\lambda}}]^p\}^{\frac{1}{p}}$$

$$>\frac{\lambda}{\lambda_1\lambda_2}[\sum_{m=1}^{\infty}(1-\frac{\lambda_1}{\lambda m^{\lambda_2}})m^{p(1-\lambda_1)-1}a_m^p]^{\frac{1}{p}}, \tag{3.5}$$

$$\{\sum_{m=1}^{\infty}\frac{m^{q\lambda_1-1}}{(1-\frac{\lambda_1}{\lambda m^{\lambda_2}})^{q-1}}[\sum_{n=1}^{\infty}\frac{b_n}{(\max\{m,n\})^{\lambda}}]^q\}^{\frac{1}{q}}$$
$$>\frac{\lambda}{\lambda_1\lambda_2}[\sum_{n=1}^{\infty}n^{q(1-\lambda_2)-1}b_n^q]^{\frac{1}{q}}. \tag{3.6}$$

Proof. $(i) => (ii)$. By (i), we have

$$k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1)=\lim_{p\to 1^-}\lim_{q\to-\infty}k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1)=k_{\lambda}(\lambda_2),$$
$$k_{\lambda}(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})=\lim_{p\to 1^-}\lim_{q\to-\infty}k_{\lambda}(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})$$
$$=k_{\lambda}(\lambda-\lambda_2)=k_{\lambda}(\lambda_2),$$

namely, $k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1)$ is expressible as a single integral $k_{\lambda}(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})$.

$(ii) => (iv)$. If $k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1)=k_{\lambda}(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})$, then (2.8) keeps the form of equality. In view of the proof of Lemma 4, it follows that $\lambda=\lambda_1+\lambda_2$.

$(iv) => (i)$. If $\lambda=\lambda_1+\lambda_2$, then

$$k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1)=k_{\lambda}(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})=k_{\lambda}(\lambda_1),$$

which are finite and independent of p,q. Hence, it follows that $(i) <=> (ii) <=> (iv)$.

$(iii) => (iv)$. By Lemma 2.4, we have $\lambda=\lambda_1+\lambda_2$.

$(iv) => (iii)$. By Lemma 2.3, for $\lambda=\lambda_1+\lambda_2$, $k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1)$ is the best possible constant factor in (2.3). Therefore, we have $(iii) <=> (iv)$.

Hence, the statements (i), (ii), (iii) and (iv) are equivalent.

The theorem is proved. □

Remark 3.1. For $\lambda_1=\frac{\lambda}{r}\leq\frac{11}{8},\lambda_2=\frac{\lambda}{s}\leq\frac{11}{8}(r>1,\frac{1}{r}+\frac{1}{s}=1,0<\lambda\leq\frac{11}{8}\min\{r,s\})$ in (2.5), (3.5) and (3.6), we have the following equivalent inequalities with the best possible constant factor $\frac{rs}{\lambda}$:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_mb_n}{(\max\{m,n\})^{\lambda}}$$
$$>\frac{rs}{\lambda}[\sum_{m=1}^{\infty}(1-\frac{1}{rm^{\frac{\lambda}{s}}})m^{p(1-\frac{\lambda}{r})-1}a_m^p]^{\frac{1}{p}}[\sum_{n=1}^{\infty}n^{q(1-\frac{\lambda}{s})-1}b_n^q]^{\frac{1}{q}}, \tag{3.7}$$

$$\{\sum_{n=1}^{\infty}n^{\frac{p\lambda}{s}-1}[\sum_{m=1}^{\infty}\frac{a_m}{(\max\{m,n\})^{\lambda}}]^p\}^{\frac{1}{p}}>\frac{rs}{\lambda}[\sum_{m=1}^{\infty}(1-\frac{1}{rm^{\frac{\lambda}{s}}})m^{p(1-\frac{\lambda}{r})-1}a_m^p]^{\frac{1}{p}}, \tag{3.8}$$

$$\{\sum_{m=1}^{\infty}\frac{m^{\frac{q\lambda}{r}-1}}{(1-\frac{1}{rm^{\frac{\lambda}{s}}})^{q-1}}[\sum_{n=1}^{\infty}\frac{b_n}{(\max\{m,n\})^{\lambda}}]^q\}^{\frac{1}{q}}>\frac{rs}{\lambda}[\sum_{n=1}^{\infty}n^{q(1-\frac{\lambda}{s})-1}b_n^q]^{\frac{1}{q}}. \tag{3.9}$$

In particular, (i) for $\lambda=1$, we have the following equivalent inequalities:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_mb_n}{\max\{m,n\}}>rs[\sum_{m=1}^{\infty}(1-\frac{1}{rm^{\frac{1}{s}}})m^{\frac{p}{s}-1}a_m^p]^{\frac{1}{p}}(\sum_{n=1}^{\infty}n^{\frac{q}{r}-1}b_n^q)^{\frac{1}{q}}, \tag{3.10}$$

$$\{\sum_{n=1}^{\infty} n^{\frac{p}{s}-1}[\sum_{m=1}^{\infty} \frac{a_m}{\max\{m,n\}}]^p\}^{\frac{1}{p}} > rs[\sum_{m=1}^{\infty}(1-\frac{1}{rm^{\frac{1}{s}}})m^{\frac{p}{s}-1}a_m^p]^{\frac{1}{p}}, \tag{3.11}$$

$$\{\sum_{m=1}^{\infty} \frac{m^{\frac{q}{r}-1}}{(1-\frac{1}{rm^{\frac{1}{s}}})^{q-1}}[\sum_{n=1}^{\infty} \frac{b_n}{\max\{m,n\}}]^q\}^{\frac{1}{q}} > rs[\sum_{n=1}^{\infty} n^{q(1-\frac{\lambda}{s})-1}b_n^q]^{\frac{1}{q}}; \tag{3.12}$$

(ii) for $\lambda = \frac{11}{4}$, $r = s = 2$, we have the following equivalent inequalities:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m b_n}{(\max\{m,n\})^{\frac{11}{4}}} > \frac{16}{11}[\sum_{m=1}^{\infty}(1-\frac{1}{2m^{\frac{11}{8}}})m^{\frac{-3p}{8}-1}a_m^p]^{\frac{1}{p}}(\sum_{n=1}^{\infty} n^{\frac{-3q}{8}-1}b_n^q)^{\frac{1}{q}}, \tag{3.13}$$

$$\{\sum_{n=1}^{\infty} n^{\frac{11p}{8}-1}[\sum_{m=1}^{\infty} \frac{a_m}{(\max\{m,n\})^{\frac{11}{4}}}]^p\}^{\frac{1}{p}} > \frac{16}{11}[\sum_{m=1}^{\infty}(1-\frac{1}{2m^{\frac{11}{8}}})m^{\frac{-3p}{8}-1}a_m^p]^{\frac{1}{p}}, \tag{3.14}$$

$$\{\sum_{m=1}^{\infty} \frac{m^{\frac{11q}{8}-1}}{(1-\frac{1}{2m^{\frac{11}{8}}})^{q-1}}[\sum_{n=1}^{\infty} \frac{b_n}{(\max\{m,n\})^{\frac{11}{4}}}]^q\}^{\frac{1}{q}} > \frac{16}{11}(\sum_{n=1}^{\infty} n^{\frac{-3q}{8}-1}b_n^q)^{\frac{1}{q}}. \tag{3.15}$$

4. Conclusions

In this paper, by the use of the weight coefficients, the idea of introduced parameters and Euler-Maclaurin summation formula, a reverse Hardy-Littlewood-Pólya's inequality with parameters and the equivalent forms are given in Lemma 2.2 and Theorem 3.1. The equivalent statements of the best possible constant factor related to a few parameters, and some particular cases are considered in Theorem 3.2 and Remark 3.1. The lemmas and theorems provide an extensive account of this type of inequalities.

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