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OPTIMAL ITERATIVE PERTURBATION TECHNIQUE FOR SOLVING JEFFERY–HAMEL FLOW WITH HIGH MAGNETIC FIELD AND NANOPARTICLE

Necdet Bildik[1,†] and Sinan Deniz[1]

Abstract In this research paper, a different semi-analytical analysis of modified magnetohydrodynamic Jeffery–Hamel flow is conducted via the newly developed technique. We use the optimal iterative perturbation method with multiple parameters to see the effects of the magnetic field and nanoparticle on the Jeffery–Hamel flow. Comparing our new approximate solutions with some earlier works proved the excellent accuracy of the newly proposed technique. Convergence analysis of the proposed method is also discussed and error estimation is given to anticipate the accuracy of higher-order approximate solutions.

Keywords Optimal iterative perturbation technique, Jeffery–Hamel flows, nanoparticle.

MSC(2010) 70E20, 70K60, 76A02, 76W05.

1. Introduction

In 1915, George Barker Jeffery published a paper about the two-dimensional steady motion of a viscous fluid [28]. After two years, another study about the spiral movements of viscous liquids was carried out German scientist Georg Hamel [25]. The equations resulting from these studies were called as Jeffery – Hamel flows. These flows can be counted as an exact similarity solution of the NavierCStokes equations in the specific case of 2D flow through a channel with inclined plane walls intersecting at a vertex with a point of supply or sink at the vertex [23]. There are many researchers have struggled to obtain approximate solutions to Jeffery – Hamel flow problem. Ganji et al. have used decomposition method to get analytical solution to classical Jeffery – Hamel problem [24]. Adomian decomposition method has been also used for analytical investigation of Jeffery – Hamel flow with high magnetic field and nanoparticle by Rokni et al. [41]. Marinca and Herisanu have implemented the optimal homotopy asymptotic method to deal with nonlinear flow problem [35].

Due to the nonlinearity of most of the mathematical models such as Jeffery – Hamel flows and other fluid mechanic problems, many different analytical and numerical techniques are required to handle these types of equations. For instance, the homotopy analysis method (HAM) is one of the most encountered techniques for

[†]The corresponding author. Email:necdet.bildik@cbu.edu.tr(N. Bildik)
[1]Department of Mathematics, Manisa Celal Bayar University, Manisa, Turkey

solving nonlinear problems. Nonlinear fractional differential equations have been safely solved by the HAM [38–40]. Deniz and Sezer have applied rational Chebyshev collocation method to solve nonlinear heat transfer models. [20]. New approximate solutions to electrostatic differential equations have been obtained by using optimal homotopy asymptotic method [11]. Bildik and Deniz have revisited a model of the polluted lakes system via new numerical scheme [8]. Many recent works of fractional calculus have been considered via various numerical methods [2,3,31,37]. Fractional complex transform and (G'/G)-expansion method have been applied for solving time-fractional differential equations [5]. Gner has found exact travelling wave solutions to the space-time fractional Calogero-Degasperis equation using different analytical methods [26]. Perturbation iteration technique has been recently constructed and used to solve many linear and nonlinear problems [9, 12, 13, 21]. Exact travelling wave solutions of reaction diffusion models of fractional order have been obtained by Q-function method [14]. Fourth-order time-fractional partial differential equations with variable coefficients have been numerically solved by Javidi and Ahmad [29]. Yuan and Alam have implemented the optimal homotopy analysis method based on particle swarm optimization to solve fractional-order differential equation [42]. Recently, a new semi-analytical technique, namely the optimal iterative perturbation technique, has been established to deal with many types of nonlinear differential equations [7, 10, 15, 17, 22].

In the present research, the optimal iterative perturbation method (OIPM) with multiple parameters have been applied to solve Jeffery – Hamel flow with high magnetic field and nanoparticle. In accordance with this aim, we put forward a new idea of convergence-control parameters in the perturbation iteration technique. In order to optimally determine the convergence-control parameters, we make use of the squared residual error. By solving the modified Jeffery – Hamel flow problem, we see that obtained results are more accurate and impressive than those of many other techniques in the literature.

The rest of the paper is organized as follows: Derivation of the considered problem is given in the next section. The new optimal iterative perturbation algorithm (OIPA) is formed in section 3. Convergence analysis and error estimation of the algorithms is given in Section 4. Section 5 is devoted to analyzing a comprehensive illustration via new algorithms. Eventually, a general evaluation will be given in the conclusion part.

2. Analysis of governing problem

In this section, the classical mathematical formulation of the Jeffery–Hamel equation is revisited. These derivations have been reviewed by many researchers for reducing the model into the classical nonlinear differential equations [30, 36]. Configuration of the Jeffery–Hamel flow can be pictured as in Fig. 1. In this model, the fluid pressure, the electromagnetic induction and the conductivity of the fluid will be denoted as P, B_0, σ respectively. In order to accomplish our purpose, we assume that there is no change in the flow parameter and no magnetic field along the z-direction of the cylindrical polar coordinates (r, θ, z). Therefore, our equations will depend only on r and θ and can be showed in polar coordinates as:

$$\frac{\rho_{nf}}{r}\frac{\partial}{\partial r}(ru(r,\theta)) = 0, \tag{2.1}$$

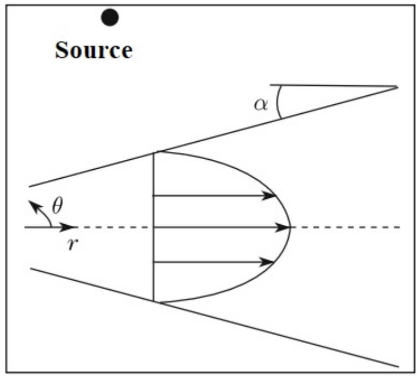

Figure 1. Configuration of the Jeffery–Hamel flow: The rigid walls are considered to be divergent if $\alpha > 0$ and convergent if $\alpha < 0$.

$$\begin{aligned} u(r,\theta)\frac{\partial u(r,\theta)}{\partial r} &= -\frac{1}{\rho_{nf}}\frac{\partial P}{\partial r} + \nu_{nf}\left[\nabla^2 u(r,\theta) - \frac{u(r,\theta)}{r^2}\right] - \frac{\sigma B_0^2}{\rho_{nf} r^2}u(r,\theta) \\ &= -\frac{1}{\rho_{nf}}\frac{\partial P}{\partial r} + \nu_{nf}\left[\frac{\partial^2 u(r,\theta)}{\partial r^2} + \frac{1}{r}\frac{\partial u(r,\theta)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u(r,\theta)}{\partial \theta^2} - \frac{u(r,\theta)}{r^2}\right] \\ &\quad - \frac{\sigma B_0^2}{\rho_{nf} r^2}u(r,\theta), \end{aligned} \tag{2.2}$$

$$0 = -\frac{1}{\rho_{nf} r}\frac{\partial P}{\partial \theta} + \frac{2\nu_{nf}}{r^2}\frac{\partial u(r,\theta)}{\partial \theta}, \tag{2.3}$$

where ρ_{nf} and ν_{nf} represent the fluid density and the coefficient of kinematic viscosity, respectively. In 2009, Aminossadati and Ghasemi have given the effective dynamic viscosity μ_{nf}, the kinematic viscosity ν_{nf} and the effective density ρ_{nf} of the nanofluid for natural convection cooling of a localized heat source at the bottom of a nanofluid-filled enclosure [4]. Taking ϕ as the solid volume fraction, these parameters can be given as:

$$\begin{cases} \mu_{nf} = \dfrac{\mu_f}{(1-\phi)^{2.5}}, \\ \nu_{nf} = \dfrac{\nu_f}{\rho_{nf}}, \\ \rho_{nf} = \rho_f(1-\phi) + \rho_s\phi. \end{cases} \tag{2.4}$$

Velocity parameter can be described as $f(\theta) = ru(r,\theta)$ by knowing $u_\theta = 0$ for the purely radial flow. Using dimensionless parameters,

$$S(x) = \frac{f(\theta)}{f_{max}} \quad \text{where} \quad x = \frac{\theta}{\alpha} \tag{2.5}$$

and eliminating P between Eqs. (2.2) and (2.3) gives the following nonlinear second orde ordinary differential equation:

$$S'''(x) + 2\alpha ReY^*(1-\phi)^{2.5}S(x)S'(x) + (4 - (1-\phi)^{2.5}Ha)\alpha^2 S'(x) = 0 \tag{2.6}$$

where

$$Re = \frac{\alpha f_{max}}{\nu_f} = \frac{r\alpha U_{max}}{\nu_f}, \tag{2.7}$$

$$Ha = \sqrt{\sigma \frac{B_0^2}{\rho_f \nu_f}}, \tag{2.8}$$

$$Y^* = \frac{\rho_s}{\rho_f}\phi + (1-\phi). \tag{2.9}$$

The equations (2.7) and (2.8) are called as the Reynolds number and the Hartmann number, respectively. The boundary conditions can also be simplified as

$$S(0) = 1, \quad S'(0) = 0, \quad S(1) = 0. \tag{2.10}$$

3. Optimal iterative perturbation technique

Considering the Eq. (2.6), one can rewrite the main problem in a closed form as:

$$F\left(S''', S', S, \varepsilon\right) = 0 \tag{3.1}$$

where $S = S(x)$ and ε is the perturbation parameter. In order to get optimal iterative perturbation algorithms (OIPAs), we take the approximate solution with one correction term in the perturbation straightforward expansion as

$$S_{n+1} = S_n + \varepsilon (S_c)_n \tag{3.2}$$

where $n \in \mathrm{N} \cup \{0\}$ and $(S_c)_n$ is the nth correction term of the iteration algorithm. Upon substitution of (3.2) into (3.1) then expanding it in a Taylor series with nth derivatives yields the OIPA-n s. Taking only first derivatives, we have OIPA-1 as

$$F + F_S(S_c)_n\varepsilon + F_{S'}(S'_c)_n\varepsilon + F_{S'''}(S'''_c)_n\varepsilon + F_\varepsilon\varepsilon = 0 \tag{3.3}$$

where subscripts of F denotes partial differentiation and all derivatives and functions are computed at $\varepsilon = 0$. We can reformulate the above algorithm as follows:

$$\left(S_c^{'''}\right)_n + \frac{F_{S'}}{F_{S'''}}\left(S_c^{'}\right)_n + \frac{F_S}{F_{S'''}}\left(S_c\right)_n = -\frac{\frac{F}{\varepsilon} + F_\varepsilon}{F_{S'''}}. \tag{3.4}$$

An initial function S_0 satisfying the prescribed condition(s) must be selected to obtain the first correction term from the following algorithm:

$$\left(S_c^{'''}\right)_0 + \frac{F_{S'}}{F_{S'''}}\left(S_c^{'}\right)_0 + \frac{F_S}{F_{S'''}}\left(S_c\right)_0 = -\frac{\frac{F}{\varepsilon} + F_\varepsilon}{F_{S'''}}. \tag{3.5}$$

One can use the above equation to get the approximate results in the desired limits. To start the iteration procedure, a first trial function u_0 is selected appropriately according to the prescribed conditions. The first correction term $(u_c)_0$ can be computed from the algorithms (3.4) by using u_0 and given condition(s). Then the first approximate solution u_1 is obtained by using $(u_c)_0$ and so on. To get better and more effective approximations, we propose a new approach to these algorithms.

Based on the idea of HAM [38–40], we insert a convergence-control parameters $P_0, P_1, P_2, ...$ into Eq. (3.2) and then construct new components, defined by

$$\begin{aligned} S_1(x;C_0) &= S_0 + C_0(S_c)_0, \\ S_2(x;C_1) &= S_1 + C_1\left(S_c\right)_1, \\ &\vdots \\ S_m(x;C_{m-1}) &= S_{m-1} + C_{m-1}(S_c)_{m-1}. \end{aligned} \tag{3.6}$$

In order to obtain the optimum values of these paramters, we make use of the similar strategy mentioned by Marinca et al [33,34]. Substituting the approximate solution S_m into the Eq.(3.1), we will get the following residual:

$$R(x, C_0, \ldots, C_{m-1}) = F\left((S_m)''', (S_m)', S_m\right). \tag{3.7}$$

It is clear that, when $R(x, C_0, \ldots, C_{m-1}) = 0$ then the approximation S_m is the exact solution of the problems. Generally such case will not arise for nonlinear equations, but one can minimize the functional

$$J(C_0, \ldots, C_{m-1}) = \int_a^b R^2(x, C_0, \ldots, C_{m-1})dx \tag{3.8}$$

where a and b are elected from the domain of the problem. Optimum values of $C_0, C_1, \ldots$ can be optimally defined from the conditions $J_{C_0} = J_{C_1} = ... = J_{C_{m-1}}$.

4. Convergence analysis and error estimate

We now investigate the convergence of the proposed optimal iterative perturbation technique with the aid of some theorems. New approximate solution obtained by OIPM are considered in a different way as follows:

$$D_0 = S_0, \quad D_{n+1} = C_n\left(S_c\right)_n. \tag{4.1}$$

Correspondingly, other OIPM solutions can be determined as:

$$\begin{aligned} &S_0 = D_0, \\ &S_1 = S_0 + C_0\left(S_c\right)_0 = D_0 + D_1, \\ &S_2 = S_1 + C_1\left(S_c\right)_1 = D_0 + D_1 + D_2, \\ &S_3 = S_2 + C_2\left(S_c\right)_2 = D_0 + D_1 + D_2 + D_3, \\ &\quad\vdots \\ &S_{n+1} = S_n + C_n\left(S_c\right)_n = D_0 + D_1 + D_2 + \cdots + D_{n+1} = \textstyle\sum_{i=0}^{n+1} D_i. \end{aligned} \tag{4.2}$$

Therefore, one can represent the approximate solution of the problem as:

$$S(x) = \lim_{n\to\infty} S_{n+1}(x) = \sum_{i=0}^{\infty} D_i. \tag{4.3}$$

Theorem 4.1. *Let us assume that B denotes a Banach space and*

$$A : B \to B \tag{4.4}$$

is a kind of nonlinear mapping and also we suppose that

$$\|A[y] - A[\bar{y}]\| \le \beta \|y - \bar{y}\|, y, \bar{y} \in B, \tag{4.5}$$

for $\beta < 1$, where β is some constant. Then, the mapping A has a unique fixed point. Additionally, the following sequence

$$S_{n+1} = A[S_n], \tag{4.6}$$

with an arbitray selection of $S_0 \in B$, converges to the fixed point of the mapping A and

$$\|S_r - S_s\| \le \|S_1 - S_0\| \sum_{j=s-1}^{r-2} \beta^j. \tag{4.7}$$

Banach fixed point theorem may be used to derive the following theorem

Theorem 4.2. *Let B represents a Banach space designated with an appropriate norm $\|.\|$ over which the series $\sum_{i=0}^{\infty} D_i$ is defined and let us assume that the initial mapping $S_0 = D_0$ falls inside the ball of the exact solution $S(x)$. So, the solution $\sum_{i=0}^{\infty} D_i$ converges if there is a β such that*

$$\|D_{n+1}\| \le \beta \|D_n\|. \tag{4.8}$$

Proof. To prove the above theorem, let us first define a sequence as:

$$\begin{aligned} A_0 &= D_0, \\ A_1 &= D_0 + D_1, \\ A_2 &= D_0 + D_1 + D_2, \\ &\vdots \\ A_n &= D_0 + D_1 + D_2 + \cdots + D_n. \end{aligned} \tag{4.9}$$

We must now to show that $\{A_n\}_{n=0}^{\infty}$ is a Cauchy sequence in B. In order to achieve that, we consider

$$\|A_{n+1} - A_n\| = \|S_{n+1}\| \le \beta \|S_n\| \le \beta^2 \|S_{n-1}\| \le \cdots \le \beta^{n+1} \|D_0\|. \tag{4.10}$$

For every $n, k \in \mathrm{N}, n \ge k$, we have

$$\begin{aligned} \|A_n - A_k\| &= \|(A_n - A_{n-1}) + (A_{n-1} - A_{n-2}) + \cdots + (A_{k+1} - A_k)\| \\ &\le \|A_n - A_{n-1}\| + \|A_{n-1} - A_{n-2}\| + \cdots + \|A_{k+1} - A_k\| \\ &\le \beta^n \|D_0\| + \beta^{n-1} \|D_0\| + \cdots + \beta^{k+1} \|D_0\| \\ &= \frac{1 - \beta^{n-k}}{1 - \beta} \beta^{k+1} \|D_0\|. \end{aligned} \tag{4.11}$$

Since it is known that $0 < \beta < 1$, one can easily get from (4.11)

$$\lim_{n,k \to \infty} \|A_n - A_k\| = 0. \tag{4.12}$$

Finally, $\{A_n\}_{n=0}^{\infty}$ is a Cauchy sequence in B and this implies that the series solution (4.2) is convergent. This completes the proof. □

Theorem 4.3. *If* $S_0 = D_0$ *falls inside the ball of the solution* $S(x)$ *then* $A_n = \sum_{i=0}^{n} D_i$ *remains inside that ball, too.*

Proof. Assume that

$$D_0 \in B_r(S) \tag{4.13}$$

where

$$B_r(S) = \{\, D \in A |\, \|S - D\| < r\} \tag{4.14}$$

is the ball of $D(x)$. From the hypothesis $S = \lim_{n\to\infty} A_n = \sum_{i=0}^{\infty} D_i$ and using Theorem 4.2, we get

$$\|S - A_n\| \le \beta^{n+1} \|D_0\| < \|D_0\| < r \tag{4.15}$$

where $\beta \in (0,1)$ and $n \in \mathrm{N}$. □

Theorem 4.4. *Let us now suppose that* $\sum_{i=0}^{\infty} D_i$ *, i.e. the approximate OIPM solution, is convergent to the desired solution* $S(x)$*. If the truncated series* $\sum_{i=0}^{k} D_i$ *is utilized as an approximation to the* (3.1)*, then the maximum error can be obtained as,*

$$E_k(x) \le \frac{\beta^{k+1}}{1-\beta} \|D_0\| \,. \tag{4.16}$$

Proof. By using the Eq.(4.11), one can get

$$\|A_n - A_k\| \le \frac{1-\beta^{n-k}}{1-\beta} \beta^{k+1} \|D_0\| \tag{4.17}$$

for $n \ge k$. By knowing

$$S(x) = \lim_{n\to\infty} A_n(x) = \sum_{i=0}^{\infty} D_i \tag{4.18}$$

one can write

$$\left\| S(x) - \sum_{i=0}^{k} D_i \right\| \le \frac{1-\beta^{n-k}}{1-\beta} \beta^{k+1} \|D_0\| \tag{4.19}$$

and also it can be rewritten as

$$E_k(x) = \left\| S(x) - \sum_{i=0}^{k} D_i \right\| \le \frac{\beta^{k+1}}{1-\beta} \|D_0\| \tag{4.20}$$

since $1 - \beta^{n-k} < 1$. Here β is chosen as $\beta = \max\{\beta_i, i = 0, 1, \ldots, n\}$ where

$$\beta_i = \frac{\|D_{n+1}\|}{\|D_n\|}. \tag{4.21}$$

□

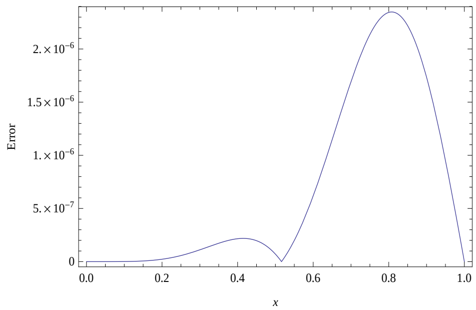

Figure 2. Errors for fifth order OIPM solutions

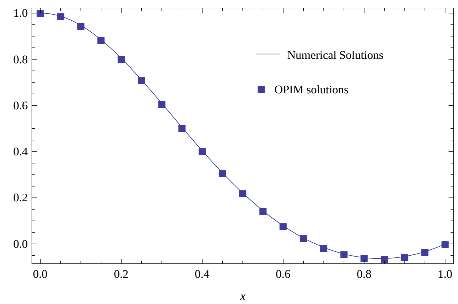

Figure 3. Numerical results and the second order approximation for $\alpha = \frac{\pi}{20}$, $Ha = 0$, $\phi = 0.01$, $Re = 100$.

5. Applications

In this section, we try to find new approximate solutions to modified Jeffery–Hamel flow equation by using perturbation algorithms. Firstly, the Eq. (2.6) with perturbation parameter can be written as :

$$\begin{aligned} F\left(S''', S', S, \varepsilon\right) =& S_n'''(x) + 2\varepsilon\alpha Re Y^*(1-\phi)^{2.5} S_n(x) S_n'(x) \\ & + \varepsilon(4-(1-\phi)^{2.5}Ha)\alpha^2 S_n'(x) = 0. \end{aligned} \tag{5.1}$$

With the aid of the Eqs. (3.2) and (3.4) and setting $\varepsilon = 1$ one can get the following algorithm:

$$(S_c)_n''' = -\left(S_n''' + 2\alpha Re Y^*(1-\phi)^{2.5} S_n(x) S_n'(x) + (4-(1-\phi)^{2.5}Ha)\alpha^2 S_n'(x)\right). \tag{5.2}$$

One can start with the following trial function

$$S_0 = 1 - x^2 \tag{5.3}$$

which satisfies the boundary conditions (2.10). Substituting S_0 into Eq. (5.2) gives a first-order problem:

$$(S_c)_0''' = 2x\alpha^2\left(4 - \mathrm{Ha}(1-\phi)^{2.5}\right) + 4\mathrm{Re}x\left(1-x^2\right)Y^*\alpha(1-\phi)^{2.5}. \tag{5.4}$$

which has solution as:

$$(S_c)_0 = -0.0333333\begin{pmatrix} 10.x^2\alpha^2 - 10.x^4\alpha^2 - 5.\mathrm{Re}x^4Y^*\alpha(1.-1.\phi)^{2.5} + \\ 1.\mathrm{Re}x^6Y^*\alpha(1.-1.\phi)^{2.5} + 2.5\mathrm{Ha}x^4\alpha^2(1.-1.\phi)^{2.5} + \\ 4.\mathrm{Re}x^2Y^*\alpha(1.-1.\phi)^{5/2} - 2.5\mathrm{Ha}x^2\alpha^2(1.-1.\phi)^{5/2} \end{pmatrix}. \tag{5.5}$$

Therefore, first order approximate solution will be in the following form:

$$S_1 = 1 - x^2 - 0.0333333C_0\begin{pmatrix} 10.x^2\alpha^2 - 10.x^4\alpha^2 - 5.\mathrm{Re}x^4Y^*\alpha(1.-1.\phi)^{2.5} + \\ 1.\mathrm{Re}x^6Y^*\alpha(1.-1.\phi)^{2.5} + 2.5\mathrm{Ha}x^4\alpha^2(1.-1.\phi)^{2.5} + \\ 4.\mathrm{Re}x^2Y^*\alpha(1.-1.\phi)^{5/2} - 2.5\mathrm{Ha}x^2\alpha^2(1.-1.\phi)^{5/2} \end{pmatrix}. \tag{5.6}$$

With the solution (5.6) and proceeding as in the Section 3, second order approximate solutions can be obtained as:

$$
\begin{aligned}
S_2 = S_1 - 6.105006\times 10^{-6} C_1 \times \\
\left(\begin{array}{l}
54600.x^2\alpha^2 - 54600.x^4\alpha^2 + 21840.\mathrm{Re}x^2Y^*\alpha\sqrt{1.-1.\phi} - \\
13650.\mathrm{Ha}x^2\alpha^2\sqrt{1.-1.\phi} - 27300.\mathrm{Re}x^4Y^*\alpha(1.-1.\phi)^{2.5} \\
+5460.\mathrm{Re}x^6Y^*\alpha(1.-1.\phi)^{2.5} + 13650.\mathrm{Ha}x^4\alpha^2(1.-1.\phi)^{2.5} - \\
43680.\mathrm{Re}x^2Y^*\alpha\sqrt{1.-1.\phi}\phi - 54600.x^2\alpha^2C_0 + 6.89394\mathrm{HaRe}^2x^{12}Y^{*2}\alpha^4(1.-1.\phi)^{7.5}C_0^2 \\
+27300.\mathrm{Ha}x^2\alpha^2\sqrt{1.-1.\phi}\phi + 21840.\mathrm{Re}x^2Y^*\alpha\sqrt{1.-1.\phi}\phi^2 - \\
13650.\mathrm{Ha}x^2\alpha^2\sqrt{1.-1.\phi}\phi + 13650.\mathrm{Ha}x^2\alpha^2\sqrt{1.-1.\phi}C_0^2 \\
+54600.x^4\alpha^2C_0 + 1412.67\mathrm{Re}^2x^2Y^{*2}\alpha^2C_0 - 1950.\mathrm{HaRe}x^2Y^*\alpha^3C_0 \\
+10920.x^2\alpha^4C_0 + 682.5\mathrm{Ha}^2x^2\alpha^4C_0 - \\
18200.x^4\alpha^4C_0 + 7280.x^6\alpha^4C_0 - 21840.\mathrm{Re}x^2Y^*\alpha\sqrt{1.-1.\phi}C_0 + \\
7800.\mathrm{Re}x^2Y^*\alpha^3\sqrt{1.-1.\phi}C_0 - 7280.\mathrm{Re}x^4Y^*\alpha^3\sqrt{1.-1.\phi}C_0 - \\
5460.\mathrm{Ha}x^2\alpha^4\sqrt{1.-1.\phi}C_0 + 4550.\mathrm{Ha}x^4\alpha^4\sqrt{1.-1.\phi}C_0 + \\
347.569\mathrm{Ha}^2\mathrm{Re}x^2Y^*\alpha^5\sqrt{1.-1.\phi}\phi^3C_0^2 - 201.992\mathrm{HaRe}^2x^2Y^{*2}\alpha^4\sqrt{1.-1.\phi}\phi C_0^2 + \\
1213.33\mathrm{HaRe}^2x^6Y^{*2}\alpha^4(1.-1.\phi)^{2.5}\phi^3C_0^2 - 577.121\mathrm{Re}^2x^2Y^{*2}\alpha^4\phi^4C_0^2 + \\
397.222\mathrm{HaRe}x^2Y^*\alpha^5\phi^4C_0^2 - 735.424\mathrm{Re}^3x^2Y^{*3}\alpha^3\sqrt{1.-1.\phi}\phi^4C_0^2 + \\
1009.96\mathrm{HaRe}^2x^2Y^{*2}\alpha^4\sqrt{1.-1.\phi}\phi^4C_0^2 - 347.569\mathrm{Ha}^2\mathrm{Re}x^2Y^*\alpha^5\sqrt{1.-1.\phi}\phi^4C_0^2 - \\
606.667\mathrm{HaRe}^2x^6Y^{*2}\alpha^4(1.-1.\phi)^{2.5}\phi^4C_0^2 + 115.424\mathrm{Re}^2x^2Y^{*2}\alpha^4\phi^5C_0^2 - \\
79.4444\mathrm{HaRe}x^2Y^*\alpha^5\phi^5C_0^2 + 441.255\mathrm{Re}^3x^2Y^{*3}\alpha^3\sqrt{1.-1.\phi}\phi^5C_0^2 - \\
605.977\mathrm{HaRe}^2x^2Y^{*2}\alpha^4\sqrt{1.-1.\phi}\phi^5C_0^2 + 208.542\mathrm{Ha}^2\mathrm{Re}x^2Y^*\alpha^5\sqrt{1.-1.\phi}\phi^5C_0^2 + \\
121.333\mathrm{HaRe}^2x^6Y^{*2}\alpha^4(1.-1.\phi)^{2.5}\phi^5C_0^2 - 147.085\mathrm{Re}^3x^2Y^{*3}\alpha^3\sqrt{1.-1.\phi}\phi^6C_0^2 \\
+27.5758\mathrm{Re}^2x^{12}Y^{*2}\alpha^4(1.-1.\phi)^{5.}C_0^2 - 303.333\mathrm{HaRe}x^6Y^*\alpha^5(1.-1.\phi)^{5.}C_0^2 \\
+162.5\mathrm{HaRe}x^8Y^*\alpha^5(1.-1.\phi)^{5.}C_0^2 - 101.111\mathrm{HaRe}x^{10}Y^*\alpha^5(1.-1.\phi)^{5.}C_0^2 \\
+97.0667\mathrm{Re}^3x^6Y^{*3}\alpha^3(1.-1.\phi)^{7.5}C_0^2 + \cdots
\end{array}\right).
\end{aligned}
\tag{5.7}
$$

and so on. To get more accurate results, one needs to continue iterating. In order to find optimum values of C_0, C_1, we can use the following resual

$$
\begin{aligned}
Res(x;C_0,C_1) &= F\left((S_2)''', (S_2)', S_2\right) \\
&= S_2''' + 2\alpha ReY^*(1-\phi)^{2.5}S_2S_2' + (4-(1-\phi)^{2.5}Ha)\alpha^2S_2'
\end{aligned}
\tag{5.8}
$$

for second order iteration. Using the idea at the end of the section 3 with the following equation:

$$J(C_0,C_1) = \int_0^1 Res^2(x;C_0,C_1)dx \tag{5.9}$$

one gets $C_0 = 1.00546, C_2 = 0.800122$ for $\alpha = \frac{\pi}{36}$, $Ha = 0, Re = 50$ and $\phi = 0$. Replacing the obtained paramters into the Eq. (5.7) results in the second order

OIPM solution. It should be noted here that, one can also use only one single convergence-control parameter to get approximations. However, CPU times spent for multiple parameters are less than single parameter.

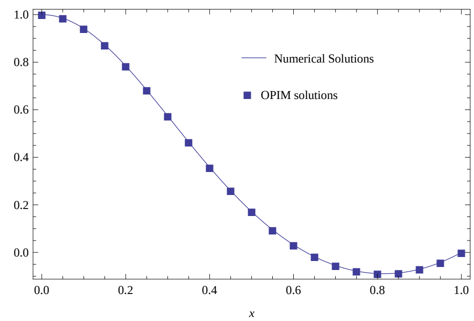

Figure 4. Numerical results and the second order approximation for $\alpha = \frac{\pi}{20}$, $Ha = 0$, $\phi = 0.02$ $Re = 110$.

It is clear from the figure 2 that, new approximate solutions agree very well with the numerical results. Even for the fifth order OIPM solutions, absolute errors are very less. Furthermore, as it is shown in Figures 3–9, numerical data for optimal iterative perturbation technique is compared with Runge–Kutta Method for different Reynold number, Hartman number, ϕ and α. According to these figures, one can conclude that OIPM can be selected as a reference method to solve the the Jeffery–Hamel flow with high magnetic field and nanoparticle. One can also easily analyze the effects of Reynolds number and steep angle of the channel on velocity profile of fluid.

6. Results and discussion

In this research paper, we modify the classical perturbation iteration method by adding multiple parameters into the iterations. Then, we apply optimal iterative perturbation technique to deal with the third order nonlinear differential equation that governs the Jeffery–Hamel flow with high magnetic field and nanoparticle problem. In comparison with the other well known numerical techniques, we see that

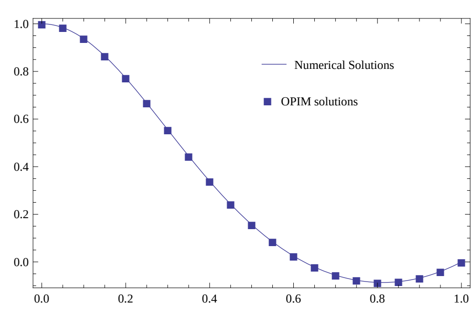

Figure 5. Numerical results and the third order approximation for $Re = 130$, $\phi = 0.015$ and $\alpha = \frac{\pi}{20}$, $Ha = 100$

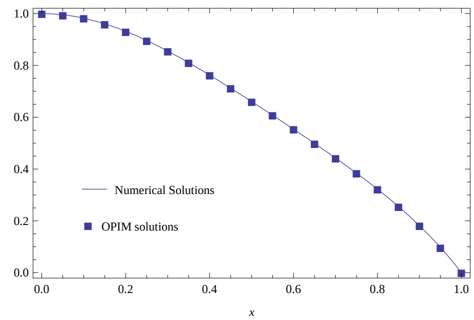

Figure 6. Numerical results and the third order approximation for $Re = 130$, $\phi = 0.015$ and $\alpha = \dfrac{\pi}{20}$, $Ha = 1000$

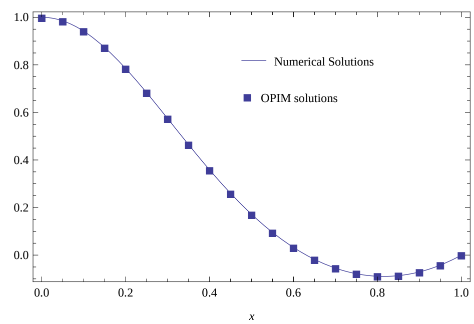

Figure 7. Numerical results and the second order approximation for $Re = 110$, $\phi = 0.01$, $Ha = 0$, $\alpha = \dfrac{\pi}{20}$.

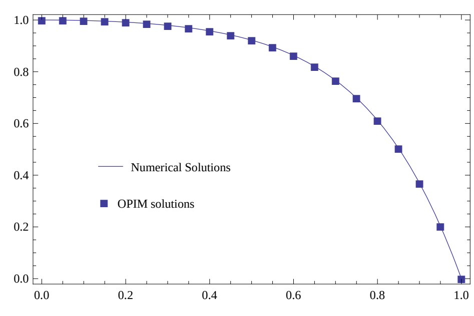

Figure 8. Numerical results and the second order approximation for $Re = 100$, $Ha = 1000$, $\phi = 0.05$, $\alpha = -\dfrac{\pi}{36}$

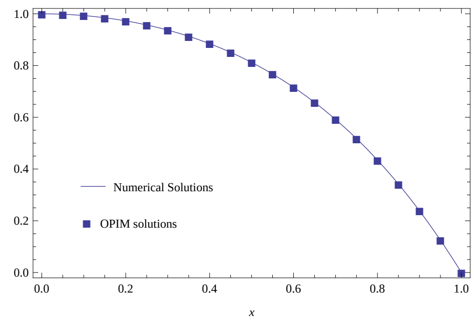

Figure 9. Numerical results and the second order approximation for $Re = 100$, $Ha = 1000$, $\phi = 0.02$, $\alpha = -\frac{\pi}{120}$

the OIPM yields better results and can be implemented without any restrictive assumptions. Figures also proves the accuracy of the proposed method.With the help of these graphics, we show that increasing Reynolds numbers leads to adverse pressure gradient which cause velocity reduction near the walls. Moreover, it is also seen that increasing Hartmann number will lead to backflow reduction and high Hartmann number (Ha) is needed to reduction of backflow in greater angles or Reynolds numbers (Re). Finally, we can say that the results obtained in this work affirm the notion that the OIPM is an effective and powerful technique for finding approximate solutions for nonlinear differential equations which have great significance in different fields of science and engineering.

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