Journal of Applied Analysis and Computation
Volume 10, Number 6, December 2020, 2459–2475
Website:http://www.jaac-online.com
DOI:10.11948/20190356

# POSITIVE SOLUTIONS OF FRACTIONAL DIFFERENTIAL EQUATION BOUNDARY VALUE PROBLEMS AT RESONANCE*

Yongqing Wang[1,†] and Yonghong Wu[2]

**Abstract** In this article, we study a class of fractional differential equations with resonant boundary value conditions. Some sufficient conditions for the existence of positive solutions are considered by means of the spectral theory of linear operator and the fixed point index theory.

**Keywords** Fractional differential equation, resonance, fixed point index, positive solution.

**MSC(2010)** 26A33, 34B15.

## 1. Introduction

In this paper, we investigate the following fractional differential equation with resonant boundary value conditions:

$$\begin{cases} D_{0+}^{\alpha}u(t)+f(t,u(t))=0, \quad 0<t<1, \\ u(0)=0, D_{0+}^{\beta}u(1)=\eta D_{0+}^{\beta}u(\xi), \end{cases} \tag{1.1}$$

where $1<\alpha<2$, $0<\beta<\alpha-1$, $\eta>0$, $0<\xi<1$ with $\eta\xi^{\alpha-\beta-1}=1$, $D_{0+}^{\alpha}$ is the standard Riemann-Liouville derivative. It is easy to see that $\lambda=0$ is an eigenvalue of the the corresponding linear problem

$$\begin{cases} -D_{0+}^{\alpha}u=\lambda u, \quad 0<t<1, \\ u(0)=0, D_{0+}^{\beta}u(1)=\eta D_{0+}^{\beta}u(\xi), \end{cases} \tag{1.2}$$

and $ct^{\alpha-1}, c\in\mathbb{R}$ is the corresponding eigenfunction, which implies that the problem (1.1) happens to be resonant.

Due to the accurate effect in describing various phenomena in widespread fields of science and engineering, various of fractional differential equations have been generally studied during the last few decades. In the recent years, a great deal of

†The corresponding author. Email address:wyqing9801@163.com(Y. Wang)
1School of Mathematical Sciences, Qufu Normal University, Qufu 273165, China
2Department of Mathematics and Statistics, Curtin University, Perth, WA 6845, Australia
*The authors were supported by the Natural Science Foundation of Shandong Province of China (ZR2017MA036), the National Natural Science Foundation of China (11871302), a Project of Shandong Province Higher Educational Science and Technology Program (J18KA217) and the International Cooperation Program of Key Professors by Qufu Normal University.

results dealing with the fractional differential equation boundary value problems (FBVPs) were established. Many authors focused on the existence of solutions to FBVPs by means of the techniques of nonlinear analysis; see [6, 10, 20, 23, 27, 32, 35, 39, 40] and references cited therein. The nonlocal boundary value problems have particularly attracted a great deal of attention [1–4, 8, 9, 15, 19, 25, 28, 33, 38].

In [38], Zhang and Zhong considered the following non-resonant boundary value problem:

$$\begin{cases} D_{0+}^{\alpha}u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \quad D_{0+}^{\beta}u(1) = \displaystyle\int_0^{\eta} a(t) D_{0+}^{\gamma}u(t)dt. \end{cases} \tag{1.3}$$

The existence of triple positive solutions of (1.3) were obtained by using the Leggett-Williams and Krasnosel'skii fxed point theorems.

When $\eta\xi^{\alpha-\beta-1} \neq 1$, FBVP (1.1) is non-resonant. Li et al. [15] established the existence of positive solutions for the non-resonant case of (1.1) by means of the fixed point theorem. Xu and Fei [33] obtained the existence of positive solutions for the non-resonant case of (1.1) by using the Schauder fixed-point theorem.

Some recent papers have studied the existence of positive solutions to non-resonant FBVPs under conditions concerning the first eigenvalue of the relevant linear operator, for example [5, 17, 18, 29, 36, 37]. By the theory of $u_0$-positive operator, Cui [5] investigated the uniqueness results of solution to

$$\begin{cases} D_{0+}^{\gamma}x(t) + a_1(t)f(t, x(t)) + a_2(t) = 0, \quad 0 < t < 1, \\ x(0) = x'(0) = 0, \quad x(1) = 0. \end{cases}$$

Zhang and Zhong [37] established the uniqueness results of solution to (1.3) by using the Banach contraction map principle and and the theory of $u_0$-positive linear operator. The main novelty of [5] and [37] was that the Lipschitz constant was associated with the first eigenvalues of the relevant linear operator.

While there are a lot of works dealing with non-resonant FBVPs, the results considering resonant FBVPs are relatively scarce. The main tool used to seek solutions for resonant boundary value problems is the coincidence degree theory (see [11–14, 22, 34]). To the best of our knowledge, research on positive solutions for resonant boundary value problems is quite rarely seen. For the case that $\alpha$ is an integer, Liang [16] and Webb [31] established existence results of positive solutions for second order boundary value problems at resonance by considering equivalent non-resonant perturbed problems. In [24], we obtained necessary conditions of the existence of positive solutions to the following fractional boundary value problem at resonance:

$$\begin{cases} D_{0+}^{\alpha}u(t) + f(t, u(t), D_{0+}^{\alpha-1}u(t)) = 0, \quad 0 < t < 1, \\ u(0) = 0, u(1) = \displaystyle\sum_{i=1}^{m-2} \eta_i u(\xi_i). \end{cases}$$

In [26], we established the existence and uniqueness results of positive solutions to the following resonant FBVP:

$$\begin{cases} D_{0+}^{\alpha}u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = 0, u(1) = \eta u(\xi). \end{cases}$$

The cone which is usually deduced from properties of the Green's function plays an important role in seeking positive solutions. It should be noted that the cone used in [26] does not suitable for the method of [31].

Motivated by the above works, in this paper we aim to establish the existence of positive solutions to the FBVP (1.1). Our work presented in this paper has the following features. Firstly, we consider the case that the resonant boundary condition involves an arbitrary fractional derivative which has been seldom studied due to the difficulties in direct sum decomposition for the relevant linear space. Secondly, some new properties of the Green function for this case have been obtained to deduce a suitable cone. Thirdly, the results on the existence of positive solutions concern the behavior of the nonlinearity at 0 and at $\infty$.

## 2. Basic definitions and preliminaries

**Definition 2.1** ( [21]). The fractional integral of order $\alpha > 0$ of a function $u : (0, +\infty) \to R$ is given by

$$I_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds$$

provided that the right-hand side is point-wise defined on $(0, +\infty)$.

**Definition 2.2** ( [21]). The Riemann-Liouville fractional derivative of order $\alpha$ of a function $u : (0, +\infty) \to R$ is given by

$$D_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} u(s) ds,$$

where $n = [\alpha] + 1$, $[\alpha]$denotes the integer part of number $\alpha$, provided that the right-hand side is point-wise defined on $(0, +\infty)$.

Clearly, we know the function

$$g(x) := \frac{\alpha - \beta - 2}{\Gamma(\alpha - \beta - 1)} + \sum_{k=1}^{+\infty} \frac{x^k}{\Gamma((k+1)\alpha - \beta - 2)}$$

has a unique positive root $M_*$, that is, $g(M_*) = 0$.

It is obvious that (1.1) is equivalent to the following FBVP:

$$\begin{cases} -D_{0+}^{\alpha} u(t) + M_* u(t) = f(t, u(t)) + M_* u(t), \quad 0 < t < 1, \\ u(0) = 0, D_{0+}^{\beta} u(1) = \eta D_{0+}^{\beta} u(\xi), \end{cases}$$

Throughout this article, we always assume that the following assumption holds:

$(H_1)$ $f : [0, 1] \times [0, +\infty) \to \mathbb{R}$ is continuous and

$$f(t, x) \geq -M_* x.$$

For convenience, we use the following notations:

$$h(t) = t^{\alpha-1} E_{\alpha,\alpha}(M_* t^{\alpha}),$$
$$H(t) = t^{\alpha-\beta-1} E_{\alpha,\alpha-\beta}(M_* t^{\alpha}),$$

$$\overline{h}(t) = \begin{cases} h(t), & t \geq 0, \\ 0, & t < 0, \end{cases}$$

$$\overline{H}(t) = \begin{cases} H(t), & t \geq 0, \\ 0, & t < 0, \end{cases}$$

$$K(t,s) = \frac{1}{H(1)} \begin{cases} h(t)H(1-s), & 0 \leq t \leq s \leq 1, \\ h(t)H(1-s) - h(t-s)H(1), & 0 \leq s \leq t \leq 1, \end{cases}$$

$$K_0(t,s) = \frac{1}{H(1)} \begin{cases} H(t)H(1-s), & 0 \leq t \leq s \leq 1, \\ H(t)H(1-s) - H(t-s)H(1), & 0 \leq s \leq t \leq 1, \end{cases}$$

$$G(t,s) = K(t,s) + \frac{\eta K_0(\xi,s)h(t)}{H(1) - \eta H(\xi)},$$

$$G^*(t,s) = t^{2-\alpha}G(t,s),$$

where $E_{\tau,\nu}(x) = \sum_{k=0}^{+\infty} \frac{x^k}{\Gamma(k\tau+\nu)}$ is the Mittag-Leffler function.

It is easy to check that

$$\begin{cases} H(t) = D_{0+}^{\beta}h(t), & h(t) = I_{0+}^{\beta}H(t), \\ \overline{H}(t) = D_{0+}^{\beta}\overline{h}(t), & \overline{h}(t) = I_{0+}^{\beta}\overline{H}(t). \end{cases} \tag{2.1}$$

**Lemma 2.1** ( [15]). *Assume that $y \in L[0,1]$ and $\alpha > 1 \geq \beta \geq 0$. Then*

$$D_{0+}^{\beta} \int_0^t (t-s)^{\alpha-1} y(s)ds = \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} y(s)ds. \tag{2.2}$$

**Lemma 2.2.** *Assume that $y \in L[0,1]$. Then the FBVP*

$$\begin{cases} -D_{0+}^{\alpha}u(t) + M_* u(t) = y(t), & 0 < t < 1, \\ u(0) = 0, D_{0+}^{\beta}u(1) = \eta D_{0+}^{\beta}u(\xi), \end{cases} \tag{2.3}$$

*has a unique solution written as*

$$u(t) = \int_0^1 G(t,s)y(s)ds.$$

**Proof.** By $0 < \xi < 1$ and $\eta\xi^{\alpha-\beta-1} = 1$, we can get

$$H(1) - \eta H(\xi) = E_{\alpha,\alpha-\beta}(M_*) - E_{\alpha,\alpha-\beta}(M_*\xi^{\alpha}) > 0.$$

From [21], the solution of (2.3) can be expressed by

$$u(t) = -\int_0^t h(t-s)y(s)ds + c_1 h(t) + c_2 h'(t).$$

It follows from $u(0) = 0$ that $c_2 = 0$. Hence

$$u(t) = -\int_0^t h(t-s)y(s)ds + c_1 h(t). \tag{2.4}$$

Noticing (2.1) and (2.2), by direct calculation, we have

$$\begin{aligned} D_{0+}^{\beta}u(t) &= -\sum_{k=0}^{+\infty}\frac{\int_0^t M_*^k(t-s)^{(k+1)\alpha-\beta-1}y(s)ds}{\Gamma(k\alpha+\alpha-\beta)} + c_1H(t) \\ &= -\int_0^t H(t-s)y(s)ds + c_1H(t). \end{aligned}$$

Therefore,

$$\begin{aligned} D_{0+}^{\beta}u(1) &= -\int_0^1 H(1-s)y(s)ds + c_1H(1), \\ D_{0+}^{\beta}u(\xi) &= -\int_0^{\xi} H(\xi-s)y(s)ds + c_1H(\xi). \end{aligned}$$

It follows from $D_{0+}^{\beta}u(1) = \eta D_{0+}^{\beta}u(\xi)$ that

$$\begin{aligned} c_1 =& \frac{\int_0^1 H(1-s)y(s)ds - \eta\int_0^{\xi} H(\xi-s)y(s)ds}{H(1)-\eta H(\xi)} \\ =& \frac{\eta H(\xi)\int_0^1 H(1-s)y(s)ds - \eta H(1)\int_0^{\xi} H(\xi-s)y(s)ds}{H(1)[H(1)-\eta H(\xi)]} \\ &+ \frac{\int_0^1 H(1-s)y(s)ds}{H(1)}. \end{aligned}$$

Substituting into (2.4), we have the unique solution of (2.3) is

$$\begin{aligned} u(t) =& -\int_0^t h(t-s)y(s)ds + \frac{\int_0^1 H(1-s)y(s)ds}{H(1)}h(t) \\ &+ \frac{\eta H(\xi)\int_0^1 H(1-s)y(s)ds - \eta H(1)\int_0^{\xi} H(\xi-s)y(s)ds}{H(1)[H(1)-\eta H(\xi)]}h(t) \\ =& \int_0^1 K(t,s)y(s)ds + \frac{\int_0^1 \eta K_0(\xi,s)y(s)ds}{H(1)-\eta H(\xi)}h(t) \\ =& \int_0^1 G(t,s)y(s)ds. \end{aligned}$$

□

**Lemma 2.3.** *The function $K_0(t,s)$ satisfies the following properties:*

*(i)* $K_0(t,s) > 0, \quad t,s\in(0,1)$*;*

*(ii)* $K_0(t,s) = K_0(1-s,1-t), \quad t,s\in[0,1]$*;*

*(iii)* $K_0(t,s) \le H(1)s(1-s)^{\alpha-\beta-1}t^{\alpha-\beta-2}, \quad s\in[0,1],\ t\in(0,1]$*;*

*(iv)* $K_0(t,s) \ge M_1 s(1-s)^{\alpha-\beta-1}(1-t)t^{\alpha-\beta-1}, \quad t,s\in[0,1],$ *where*

$$M_1 = \min\left\{\frac{1}{H(1)[\Gamma(\alpha-\beta)]^2}, \quad H(1)(\alpha-\beta-1)^2\right\}.$$

**Proof.** By the notation of $H(t)$, we can get

$$\frac{t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \leq H(t) \leq t^{\alpha-\beta-1}H(1), \quad t \in [0,1].$$

Noticing $g$ is strictly increasing on $(0,+\infty)$ and $g(M_*)=0$, by direct calculation, we obtain

$$\begin{aligned} H'(t) &= \sum_{k=0}^{+\infty} \frac{M_*^k t^{(k+1)\alpha-\beta-2}}{\Gamma((k+1)\alpha-\beta-1)} > 0, \quad t \in (0,1], \\ H''(t) &= t^{\alpha-\beta-3}\left[\frac{\alpha-\beta-2}{\Gamma(\alpha-\beta-1)} + \sum_{k=1}^{+\infty} \frac{M_*^k t^{k\alpha}}{\Gamma((k+1)\alpha-\beta-2)}\right] \\ &= t^{\alpha-\beta-3} g(M_* t^\alpha) < t^{\alpha-\beta-3} g(M_*) = 0, \quad t \in (0,1). \end{aligned}$$

Then we have that $H(t)$ is strictly increasing on $[0,1]$ and $H'(t)$ is strictly decreasing on $(0,1]$. In addition, it is clear that $H''(t)$ is strictly increasing on $(0,1]$.

Since the proof is similar to the Theorem 3.1 in [27], we omit it here. □

**Lemma 2.4.** *The function $K(t,s)$ admits the following properties:*

*(i)* $K(t,s) > 0, \quad t,s \in (0,1)$;

*(ii)* $K(t,s) \leq \frac{\Gamma(\alpha-\beta-1)}{\Gamma(\alpha-1)} H(1) s(1-s)^{\alpha-\beta-1} t^{\alpha-2}, \quad s \in [0,1],\ t \in (0,1]$;

*(iii)* $K(t,s) \geq \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} M_1 s(1-s)^{\alpha-\beta-1} t^{\alpha-1}\left[1-\frac{\alpha-\beta}{\alpha}t\right], \quad t,s \in [0,1]$.

**Proof.** By the notations, the functions $K_0$ and $K$ can be expressed by

$$\begin{aligned} K_0(t,s) &= \frac{H(t)H(1-s)-\overline{H}(t-s)H(1)}{H(1)}, \quad (t,s) \in [0,1]\times[0,1], \\ K(t,s) &= \frac{h(t)H(1-s)-\overline{h}(t-s)H(1)}{H(1)}, \quad (t,s) \in [0,1]\times[0,1]. \end{aligned}$$

Let $s_0 \in [0,1]$ be fixed. If $t \in [s_0,1]$, we have

$$\begin{aligned} I_{0+}^{\beta}\overline{H}(t-s_0) &= \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1}\overline{H}(\tau-s_0)d\tau \\ &= \frac{1}{\Gamma(\beta)} \int_{s_0}^t (t-\tau)^{\beta-1}H(\tau-s_0)d\tau \\ &= \sum_{k=0}^{+\infty} \frac{\int_{s_0}^t M_*^k (t-\tau)^{\beta-1}(\tau-s_0)^{(k+1)\alpha-\beta-1}d\tau}{\Gamma(\beta)\Gamma(k\alpha+\alpha-\beta)} \\ &= \sum_{k=0}^{+\infty} \frac{\int_0^{t-s_0} M_*^k (t-s_0-x)^{\beta-1}x^{(k+1)\alpha-\beta-1}dx}{\Gamma(\beta)\Gamma(k\alpha+\alpha-\beta)} \\ &= \sum_{k=0}^{+\infty} \frac{M_*^k (t-s_0)^{(k+1)\alpha-1}\int_0^1 (1-y)^{\beta-1}y^{(k+1)\alpha-\beta-1}dy}{\Gamma(\beta)\Gamma(k\alpha+\alpha-\beta)} \\ &= \sum_{k=0}^{+\infty} \frac{M_*^k (t-s_0)^{(k+1)\alpha-1}}{\Gamma((k+1)\alpha)} \end{aligned}$$

$$= \overline{h}(t - s_0).$$

If $t \in [0, s_0]$, it is obvious that

$$I_{0+}^{\beta} \overline{H}(t - s_0) = 0 = \overline{h}(t - s_0).$$

Therefore,

$$I_{0+}^{\beta} \overline{H}(t - s_0) = \overline{h}(t - s_0), \quad t \in [0, 1].$$

This combing with (2.1) implies that

$$K(t, s_0) = I_{0+}^{\beta} K_0(t, s_0), \quad t \in [0, 1].$$

Then $(i)$, $(ii)$ and $(iii)$ can be deduced from Lemma 2.3 directly. □

**Lemma 2.5.** *The function $G(t,s)$ has the following properties:*

*(i)* $G(t, s) \le M_2 s(1 - s)^{\alpha-\beta-1} t^{\alpha-2}, \quad s \in [0, 1],\ t \in (0, 1]$;

*(ii)* $G(t, s) \ge M_3 s(1 - s)^{\alpha-\beta-1} t^{\alpha-1}, \quad t, s \in [0, 1]$;

*(iii)* $G(t, s) \le M_4 (1 - s)^{\alpha-\beta-1} t^{\alpha-1}, \quad t, s \in [0, 1]$, *where*

$$M_2 = H(1) \left( \frac{\Gamma(\alpha - \beta - 1)}{\Gamma(\alpha - 1)} + \frac{\eta h(1) \xi^{\alpha-\beta-2}}{H(1) - \eta H(\xi)} \right),$$
$$M_3 = M_1 \left( \frac{\beta \Gamma(\alpha - \beta)}{\Gamma(\alpha + 1)} + \frac{1 - \xi}{\Gamma(\alpha)[H(1) - \eta H(\xi)]} \right),$$
$$M_4 = h(1) \left( 1 + \frac{H(1)}{H(1) - \eta H(\xi)} \right).$$

**Proof.** Noticing

$$h(t) \le h(1) t^{\alpha-1}, \quad \forall t \ge 0,$$

it follows from $(iii)$ of Lemma 2.3 and $(ii)$ of Lemma 2.4 that

$$\begin{aligned}
G(t, s) =& K(t, s) + \frac{\eta K_0(\xi, s) h(t)}{H(1) - \eta H(\xi)} \\
\le& \frac{\Gamma(\alpha - \beta - 1)}{\Gamma(\alpha - 1)} H(1) s(1 - s)^{\alpha-\beta-1} t^{\alpha-2} + \frac{\eta H(1) s(1 - s)^{\alpha-\beta-1} \xi^{\alpha-\beta-2} h(t)}{H(1) - \eta H(\xi)} \\
\le& H(1) s(1 - s)^{\alpha-\beta-1} \left( \frac{\Gamma(\alpha - \beta - 1)}{\Gamma(\alpha - 1)} t^{\alpha-2} + \frac{\eta \xi^{\alpha-\beta-2} h(1)}{H(1) - \eta H(\xi)} t^{\alpha-1} \right) \\
\le& H(1) s(1 - s)^{\alpha-\beta-1} t^{\alpha-2} \left( \frac{\Gamma(\alpha - \beta - 1)}{\Gamma(\alpha - 1)} + \frac{\eta \xi^{\alpha-\beta-2} h(1)}{H(1) - \eta H(\xi)} \right) \\
=& M_2 s(1 - s)^{\alpha-\beta-1} t^{\alpha-2}.
\end{aligned}$$

Hence $(i)$ holds.

Noticing

$$h(t) \ge \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad \forall t \ge 0,$$

it follows from $(iv)$ of Lemma 2.3 and $(iii)$ of Lemma 2.4 that

$$G(t, s) \ge \frac{\Gamma(\alpha - \beta)}{\Gamma(\alpha)} M_1 s(1 - s)^{\alpha-\beta-1} t^{\alpha-1} \left[ 1 - \frac{\alpha - \beta}{\alpha} t \right]$$

$$
\begin{aligned}
&+\frac{\eta M_1 s(1-s)^{\alpha-\beta-1}(1-\xi)\xi^{\alpha-\beta-1}}{H(1)-\eta H(\xi)}h(t)\\
\geq&\frac{\beta\Gamma(\alpha-\beta)}{\alpha\Gamma(\alpha)}M_1 s(1-s)^{\alpha-\beta-1}t^{\alpha-1}\\
&+\frac{\eta M_1 s(1-s)^{\alpha-\beta-1}(1-\xi)\xi^{\alpha-\beta-1}}{H(1)-\eta H(\xi)}\times\frac{t^{\alpha-1}}{\Gamma(\alpha)}\\
=&M_3 s(1-s)^{\alpha-\beta-1}t^{\alpha-1}.
\end{aligned}
$$

Therefore $(ii)$ holds.

By the notations of $K$ and $K_0$, it is easy to get that

$$
\begin{aligned}
K(t,s)&\leq\frac{h(t)H(1-s)}{H(1)}\leq h(1)t^{\alpha-1}(1-s)^{\alpha-\beta-1},\\
K_0(t,s)&\leq\frac{H(t)H(1-s)}{H(1)}\leq H(1)t^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1}.
\end{aligned}
$$

Then

$$
\begin{aligned}
G(t,s)=&K(t,s)+\frac{\eta K_0(\xi,s)h(t)}{H(1)-\eta H(\xi)}\\
\leq&h(1)t^{\alpha-1}(1-s)^{\alpha-\beta-1}+\frac{\eta H(1)\xi^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1}h(t)}{H(1)-\eta H(\xi)}\\
\leq&h(1)t^{\alpha-1}(1-s)^{\alpha-\beta-1}+\frac{H(1)(1-s)^{\alpha-\beta-1}h(1)t^{\alpha-1}}{H(1)-\eta H(\xi)}\\
=&M_4(1-s)^{\alpha-\beta-1}t^{\alpha-1}.
\end{aligned}
$$

So $(iii)$ holds. □

From Lemma 2.5, we have the following Lemma:

**Lemma 2.6.** *The function $G^*(t,s):=t^{2-\alpha}G(t,s)$ satisfies:*

*(i)* $G^*(t,s)\leq M_2 s(1-s)^{\alpha-\beta-1},\quad t,s\in[0,1]$;

*(ii)* $G^*(t,s)\geq M_3 s(1-s)^{\alpha-\beta-1}t,\quad t,s\in[0,1]$;

*(ii)* $G^*(t,s)\leq M_4(1-s)^{\alpha-\beta-1}t,\quad t,s\in[0,1]$.

Let $E=C[0,1]$ be endowed with the maximum norm $\|u\|=\max_{0\leq t\leq 1}|u(t)|$, $\theta$ is the zero element of $E$, $B_r=\{u\in E:\|u\|<r\}$. Define two cones $P$ and $Q$ by

$$
\begin{aligned}
P&=\{u\in E:u(t)\geq 0,\quad t\in[0,1]\},\\
Q&=\left\{u\in E:\exists\ l_u>0,\ s.t.\ l_u t\geq u(t)\geq\frac{M_3\|u\|}{M_2}t,\quad t\in[0,1]\right\}.
\end{aligned}
$$

Define four operators as follows:

$$
\begin{aligned}
Au(t)&=\int_0^1 G(t,s)[f(s,u(s))+M_*u(s)]ds,\\
A^*u(t)&=\int_0^1 G^*(t,s)[f(s,s^{\alpha-2}u(s))+M_*s^{\alpha-2}u(s)]ds,
\end{aligned}
$$

$$Lu(t) = \int_0^1 G(t,s)u(s)ds,$$
$$L^*u(t) = \int_0^1 G^*(t,s)s^{\alpha-2}u(s)ds.$$

It is clear that the fixed point of the operator $A$ is a solution of the resonant FBVP (1.1). By Lemma 2.5 and Arzela-Ascoli theorem, we can get $A : P \to P$ is completely continuous, $L : P \to P$ is a completely continuous linear operator. By Lemma 2.6, we can get $L^* : Q \to Q$ is a completely continuous linear operator. Noticing that $\lambda = 0$ is the eigenvalue of the linear problem (1.2) and $t^{\alpha-1}$ is the corresponding eigenfunction, we have that the first eigenvalue of $L$ is $\lambda_1 = M_*$, and $\varphi_1(t) = t^{\alpha-1}$ is a positive eigenfunction corresponding to $\lambda_1$, that is $\varphi_1 = M_* L\varphi_1$. Therefore, the first eigenvalue of $L^*$ is $\lambda_1^* = M_*$, and $\varphi_1^*(t) = t$ is a positive eigenfunction corresponding to $\lambda_1^*$, that is, $\varphi_1^* = M_* L^*\varphi_1^*$.

Set

$$L_n^*u(t) = \int_{\frac{1}{n}}^1 G^*(t,s)s^{\alpha-2}u(s)ds.$$

It follows from [30] that the following Lemma holds.

**Lemma 2.7.** *The sequence of spectral radius $\{r(L_n^*)\}$ is increasing and converges to*

$$r(L^*) = \frac{1}{M_*}.$$

**Lemma 2.8** ( [7]). *Let $P$ be a cone in a Banach space $E$, and $\Omega$ be a bounded open set in $E$. $A : \overline{\Omega} \cap P \to P$ is a completely continuous operator. If there exists $u_0 \in P$ with $u_0 \neq \theta$ such that*

$$u - Au \neq \lambda u_0, \quad \forall\ \lambda \geq 0,\ u \in \partial\Omega \cap P,$$

*then $i(A, \Omega \cap P, P) = 0$.*

**Lemma 2.9** ( [7]). *Let $P$ be a cone in a Banach space $E$, $\Omega$ be a bounded open set in $E$ and $\theta \in \Omega$. Suppose that $A : \overline{\Omega} \cap P \to P$ is a completely continuous operator. If*

$$Au \neq \lambda u, \quad \forall\ \lambda \geq 1,\ u \in \partial\Omega \cap P,$$

*then $i(A, \Omega \cap P, P) = 1$.*

## 3. Main results

**Theorem 3.1.** *Assume that the following assumptions hold:*

$$\liminf_{x\to 0+} \min_{t\in[0,1]} \frac{f(t,x)}{x} > 0, \quad \limsup_{x\to +\infty} \max_{t\in[0,1]} \frac{f(t,x)}{x} < 0.$$

*Then (1.1) has at least one positive solution.*

**Proof.** The proof is similar to the Theorem 4.1 in [26], we omit it here. □

**Theorem 3.2.** *Assume that the following assumptions hold:*

$(H_2)$ *For any $R > r > 0$, there exists $\Psi_{r,R} \in L[0,1]$ such that*

$$t(1-t)^{\alpha-\beta-1}|f(t,x)| \leq \Psi_{r,R}(t), \ \forall \ t \in (0,1], \ x \in [rt^{\alpha-1}, Rt^{\alpha-2}].$$

$(H_3)$

$$\liminf_{x\to 0+} \inf_{t\in(0,1]} \frac{f(t,t^{\alpha-2}x)}{t^{\alpha-2}x} > 0, \tag{3.1}$$

$$\limsup_{x\to +\infty} \sup_{t\in(0,1]} \frac{f(t,t^{\alpha-2}x)}{t^{\alpha-2}x} < 0. \tag{3.2}$$

*Then* (1.1) *has at least one positive solution.*

**Proof.** By Lemma 2.6 and Arzela-Ascoli theorem, we can get $A^* : Q \to Q$ is completely continuous. It follows from (3.1) that there exists $r_1 > 0$ such that

$$f(t, t^{\alpha-2}x) \geq 0, \quad \forall \ (t,x) \in (0,1] \times [0, r_1].$$

For any $u \in \partial B_{r_1} \cap Q$, we have

$$\begin{aligned} A^*u(t) &= \int_0^1 G^*(t,s)[f(s, s^{\alpha-2}u(s)) + M_* s^{\alpha-2}u(s)]ds \\ &\geq \int_0^1 G^*(t,s) M_* s^{\alpha-2} u(s) ds \\ &= M_* L^* u(t). \end{aligned}$$

In the following, we will show that $A^*$ has at least one fixed point on $Q$.

Suppose that $A^*$ has no fixed points on $\partial B_{r_1} \cap Q$. Next, we will show that

$$u - A^*u \neq \mu\varphi_1^*, \quad \forall \ u \in \partial B_{r_1} \cap Q, \ \mu > 0, \tag{3.3}$$

where $\varphi_1^*$ is the positive eigenfunction of $L^*$. In fact, if there exist $\mu_0 > 0$ and $u_1 \in \partial B_{r_1} \cap Q$ such that

$$u_1 - A^*u_1 = \mu_0\varphi_1^*.$$

Then

$$u_1 = A^*u_1 + \mu_0\varphi_1^* \geq \mu_0\varphi_1^*.$$

Denote

$$\mu^* = \sup\{\mu : u_1 \geq \mu\varphi_1^*\}.$$

It follows from the definition of $Q$ that $\mu^* < +\infty$. Clearly we have $\mu^* \geq \mu_0$ and $u_1 \geq \mu^*\varphi_1^*$. Therefore,

$$L^*u_1 \geq \mu^* L^*\varphi_1^* = \frac{\mu^*}{M_*}\varphi_1^*.$$

Hence

$$u_1 = A^*u_1 + \mu_0\varphi_1^* \geq M_* L^* u_1 + \mu_0\varphi_1^* \geq (\mu^* + \mu_0)\varphi_1^*,$$

contradicts with the definition of $\mu^*$, that is (3.3) holds. By Lemma 2.8, we have

$$i(A^*, B_{r_1} \cap Q, Q) = 0. \tag{3.4}$$

From (3.2), there exist $\epsilon \in (0, M_*)$ and $r_2 > r_1$ such that

$$f(t, t^{\alpha-2}x) \leq -\epsilon t^{\alpha-2}x, \quad \forall\ (t,x) \in (0,1] \times [r_2, +\infty).$$

Set

$$W = \{u \in Q \backslash B_{r_1} :\ u = \mu A^* u,\ 0 \leq \mu \leq 1\}.$$

Next, we will show that $W$ is bounded.

For $u \in W$, we have

$$f(t, t^{\alpha-2}u(t)) \leq -\epsilon t^{\alpha-2}u(t) + |f(t, t^{\alpha-2}\tilde{u}(t))|,$$

here $\tilde{u}(t) = \min\{u(t), r_2\}$. It is easy to see that

$$\frac{M_3 r_1}{M_2} t \leq \tilde{u}(t) \leq r_2.$$

Therefore

$$r_2 t^{\alpha-2} \geq t^{\alpha-2}\tilde{u}(t) \geq \frac{M_3 r_1}{M_2} t^{\alpha-1} := r_0 t^{\alpha-1}.$$

Then

$$\begin{aligned} u(t) = \mu A^* u(t) &\leq \int_0^1 G^*(t,s)[f(s, s^{\alpha-2}u(s)) + M_* s^{\alpha-2}u(s)]ds \\ &\leq (M_* - \varepsilon)L^* u(t) + \int_0^1 G^*(t,s)|f(s, s^{\alpha-2}\tilde{u}(s))|ds \\ &\leq (M_* - \varepsilon)L^* u(t) + \int_0^1 M_2 s(1-s)^{\alpha-\beta-1}|f(s, s^{\alpha-2}\tilde{u}(s))|ds \\ &\leq (M_* - \varepsilon)L^* u(t) + M, \end{aligned} \tag{3.5}$$

here

$$M = \int_0^1 M_2 \Psi_{r_0, r_2}(s) ds.$$

Let $I$ be the identity mapping, then the operator $I - (M_* - \varepsilon)L^*$ is inverse since $r(L^*) = \frac{1}{M_*}$. Thus (3.5) yields

$$u(t) \leq M \| (I - (M_* - \varepsilon)L^*)^{-1} \|,$$

which implies $W$ is bounded.

Let $R = r_2 + M\| (I - (M_* - \varepsilon)L^*)^{-1} \|$. It follows from Lemma 2.9 that

$$i(A^*, B_R \cap Q, Q) = 1. \tag{3.6}$$

By (3.4) and (3.6), we get

$$i(A^*, (B_R \backslash \bar{B}_{r_1}) \cap Q, Q) = 1.$$

Therefore $A^*$ has a fixed point $u^* \in (B_R \backslash \bar{B}_{r_1}) \cap Q$, that is,

$$u^*(t) = A^* u^*(t) = \int_0^1 G^*(t,s)[f(s, s^{\alpha-2}u^*(s)) + M_* s^{\alpha-2}u^*(s)]ds.$$

It is easy to see that $t^{\alpha-2}u^*(t)$ is a positive solution of FBVP (1.1). $\square$

**Theorem 3.3.** *Assume that $(H_2)$ and the following assumptions hold:*
$(H_4)$

$$\limsup_{x\to 0+}\sup_{t\in(0,1]}\frac{f(t,t^{\alpha-2}x)}{t^{\alpha-2}x}<0, \tag{3.7}$$

$$\liminf_{x\to +\infty}\inf_{t\in(0,1]}\frac{f(t,t^{\alpha-2}x)}{t^{\alpha-2}x}>0. \tag{3.8}$$

*Then (1.1) has at least one positive solution.*

**Proof.** It follows from (3.7) that there exists $r_3>0$ such that

$$f(t,t^{\alpha-2}x)\le 0,\quad \forall\ (t,x)\in(0,1]\times[0,r_3]. \tag{3.9}$$

Suppose that $A^*$ has no fixed points on $\partial B_{r_3}\cap Q$ (otherwise, the proof is finished). In the following, we will prove that

$$A^*u\neq \mu u,\quad \forall\ u\in\partial B_{r_3}\cap Q,\ \mu>1.$$

If otherwise, there exists $u_1\in\partial B_{r_3}\cap Q$ and $\mu_0>1$ such that $A^*u_1=\mu_0u_1$. It follows from (3.9) that

$$\mu_0u_1=A^*u_1\le M_*L^*u_1:=Tu_1.$$

Then

$$\mu_0^2u_1\le M_*L^*\mu_0u_1=T\mu_0u_1\le T^2u_1.$$

By induction, we can get

$$\mu_0^nu_1\le T^nu_1,\quad n=1,2,\cdots.$$

Hence

$$\|\mu_0^nu_1\|\le\|T^nu_1\|\le\|T^n\|\|u_1\|.$$

By the Gelfand's formula, we have

$$r(T)=\lim_{n\to+\infty}\sqrt[n]{\|T^n\|}\ge\mu_0>1,$$

contradicts with $r(T)=M_*r(L^*)=1$. Therefore, it follows from Lemma 2.9 that

$$i(A^*,B_{r_3}\cap Q,Q)=1. \tag{3.10}$$

From (3.8), there exist $\epsilon>0$ and $r_4>r_3$ such that

$$f(t,t^{\alpha-2}x)\ge\epsilon t^{\alpha-2}x,\quad\forall\ (t,x)\in(0,1]\times[r_4,+\infty).$$

By Lemma 2.7, there exists $m$ large enough such that

$$r(L_m^*)>\frac{1}{M_*+\varepsilon}.$$

Let

$$R=\frac{mM_2r_4}{M_3}+r_4.$$

For any $u \in \partial B_R \cap Q$, we have

$$u(t) \geq \frac{M_3\|u\|}{M_2} t \geq r_4, \quad \forall\, t \in \left[\frac{1}{m}, 1\right]. \tag{3.11}$$

By virtue of the Krein-Rutmann theorem, we know that there exists a positive eigenfunction $\psi_m$ corresponding to the first eigenvalue of $L_m^*$, that is, $L_m^* \psi_m = r(L_m^*)\psi_m$. For any $u \in \partial B_R \cap Q$. It follows from (3.11) that

$$\begin{aligned} A^* u(t) &\geq \int_{\frac{1}{m}}^{1} G^*(t,s)[f(s, s^{\alpha-2}u^*(s)) + M_* s^{\alpha-2}u^*(s)]ds \\ &\geq \int_{\frac{1}{m}}^{1} G^*(t,s)[\epsilon s^{\alpha-2}u^*(s) + M_* s^{\alpha-2}u^*(s)]ds \\ &= (\epsilon + M_*)(L_m^* u)(t), \quad t \in [0,1]. \end{aligned}$$

We may suppose that $A^*$ has no fixed points on $\partial B_R \cap Q$ (if otherwise, the proof is finished). Next, we will proof that

$$u - A^* u \neq \mu \psi_m, \quad \forall\, u \in \partial B_R \cap Q,\ \mu > 0. \tag{3.12}$$

If otherwise, there exist $u_1 \in \partial B_R \cap Q$ and $\mu_0 > 0$ such that

$$u_1 - A^* u_1 = \mu_0 \psi_m.$$

Denote

$$\mu^* = \sup\{\mu : u_1 \geq \mu \psi_m\}.$$

It is clear that $\mu_0 \leq \mu^* < +\infty$ and $u_1 \geq \mu^* \psi_m$. Then

$$\begin{aligned} u_1 &= A^* u_1 + \mu_0 \psi_m \\ &\geq (\epsilon + M_*) L_m^* u_1 + \mu_0 \psi_m \\ &\geq (\epsilon + M_*) L_m^* \mu^* \psi_m + \mu_0 \psi_m \\ &= (\epsilon + M_*) \mu^* r(L_m^*) \psi_m + \mu_0 \psi_m \\ &\geq (\mu^* + \mu_0) \varphi_m, \end{aligned}$$

contradicts with the definition of $\mu^*$. Hence (3.12) holds. By Lemma 2.9, we have

$$i(A^*, B_R \cap Q, Q) = 0. \tag{3.13}$$

Then (3.10) and (3.13) yields

$$i(A^*, (B_R \backslash \bar{B}_{r_3}) \cap Q, Q) = -1,$$

which implies that FBVP (1.1) has at least one positive solution on $(B_R \backslash \bar{B}_{r_3}) \cap Q$. □

# 4. An Example

**Example 4.1.** Consider the following resonant FBVP:

$$\begin{cases} D_{0+}^{\frac{7}{4}} u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = 0, D_{0+}^{\frac{1}{4}} u(1) = 3 D_{0+}^{\frac{1}{4}} u(\frac{1}{9}), \end{cases} \tag{4.1}$$

where

$$f(t,x)=\begin{cases} -\dfrac{x}{5}, & t=0, \quad x\in[0,+\infty), \\ -\dfrac{x}{5}, & t\in(0,1], \quad x\in[0,10t^{\alpha-2}], \\ x-12t^{\alpha-2}, & t\in(0,1], \quad x\in(10t^{\alpha-2},15t^{\alpha-2}], \\ \dfrac{x}{5}, & t\in(0,1], \quad x\in(15t^{\alpha-2},+\infty). \end{cases}$$

For any $x\in[0,+\infty)$, we have

$$\begin{aligned} g(x) &= -\frac{1}{2\sqrt{\pi}}+\frac{x}{\Gamma(\frac{5}{4})}+\sum_{k=2}^{+\infty}\frac{x^k}{\Gamma(\frac{7}{4}k-\frac{1}{2})} \\ &\le -\frac{1}{2\sqrt{\pi}}+\frac{x}{\Gamma(\frac{5}{4})}+\sum_{k=2}^{+\infty}\frac{x^k}{\Gamma(k)} \\ &= -\frac{1}{2\sqrt{\pi}}+\frac{x}{\Gamma(\frac{5}{4})}+x(e^x-1), \end{aligned}$$

and

$$\begin{aligned} g(x) &= -\frac{1}{2\sqrt{\pi}}+\frac{x}{\Gamma(\frac{5}{4})}+\sum_{k=2}^{+\infty}\frac{x^k}{\Gamma(\frac{7}{4}k-\frac{1}{2})} \\ &\ge -\frac{1}{2\sqrt{\pi}}+\frac{x}{\Gamma(\frac{5}{4})}+\sum_{k=2}^{+\infty}\frac{x^k}{\Gamma(2k-1)} \\ &= -\frac{1}{2\sqrt{\pi}}+\frac{x}{\Gamma(\frac{5}{4})}+x\left[\frac{e^{\sqrt{x}}+e^{-\sqrt{x}}}{2}-1\right]. \end{aligned}$$

Therefore we gey $g(\frac{1}{5})<-0.0171$ and $g(\frac{1}{4})>0.0256$, which implies $M_*\in\left(\frac{1}{5},\frac{1}{4}\right)$.

It is clear that $f:[0,1]\times[0,+\infty)\to\mathbb{R}$ is continuous and $f(t,x)\ge-\frac{x}{5}$, that is $(H_1)$ holds. For any $R>r>0$, we have

$$t(1-t)^{\frac{1}{2}}|f(t,x)|\le\max\left\{3,\frac{R}{5}\right\}\sqrt{t(1-t)}, \quad \forall\ t\in(0,1],\ x\in[rt^{\alpha-1},Rt^{\alpha-2}].$$

So $(H_2)$ holds. Moreover,

$$\limsup_{x\to0+}\sup_{t\in(0,1]}\frac{f(t,t^{\alpha-2}x)}{t^{\alpha-2}x}=-\frac{1}{5}, \quad \liminf_{x\to+\infty}\inf_{t\in(0,1]}\frac{f(t,t^{\alpha-2}x)}{t^{\alpha-2}x}=\frac{1}{5},$$

which implies $(H_4)$ holds. Then Theorem 3.3 ensures that (4.1) has at least one positive solution.

## 5. Conclusions

In this paper, we discuss a class of FBVPs at resonance by considering equivalent non-resonant perturbed problems with the same boundary conditions. The main novelty of the problem we discussed lies in that an arbitrary fractional derivative is involved in the resonant boundary condition. Some interesting properties of the Green's function have been deduced to construct a suitable cone. By using the fixed point index theory on the cone, some existence results of positive solutions to the resonant FBVPs are obtained. The main results of this paper is valid to the multi-point boundary value problem at resonance:

$$\begin{cases} D_{0+}^{\alpha}u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = 0, D_{0+}^{\beta}u(1) = \sum_{i=1}^{m-2} \eta_i D_{0+}^{\beta}u(\xi_i), \end{cases}$$

where $0 < \xi_1 < \cdots < \xi_{m-2} < 1$, $\eta_i > 0$ with $\sum_{i=1}^{m-2} \eta_i \xi_i^{\alpha-\beta-1} = 1$.

## Acknowledgements

The author would like to thank the referees for their pertinent comments and valuable suggestions.

## References

[1] B. Ahmad and R. Luca, *Existence of solutions for sequential fractional integro-differential equations and inclusions with nonlocal boundary conditions*, Appl. Math. Comput., 2018, 339, 516–534.

[2] B. Ahmad and S. K. Ntouyas, *Existence results for a coupled system of Caputo type sequential fractional differential equations with nonlocal integral boundary conditions*, Appl. Math. Comput., 2015, 266, 615–622.

[3] A. Cabada and Z. Hamdi, *Nonlinear fractional differential equations with integral boundary value conditions*, Appl. Math. Comput., 2014, 228, 251–257.

[4] A. Cabada and G. Wang, *Positive solutions of nonlinear fractional differential equations with integral boundary value conditions*, J. Math. Anal. Appl., 2012, 389, 403–411.

[5] Y. Cui, *Uniqueness of solution for boundary value problems for fractional differential equations*, Appl. Math. Lett., 2016, 51, 48–54.

[6] Y. Cui, W. Ma, Q. Sun et al., *New uniqueness results for boundary value problem of fractional differential equation*, Nonlinear Anal: Model. Control., 2018, 23, 31–39.

[7] D. Guo, *Nonlinear Functional Analysis*, Shandong Science and Technology Press, Jinan, 1985.

[8] L. Guo, L. Liu and Y. Wu, *Existence of positive solutions for singular higher-order fractional differential equations with infinite-points boundary conditions*, Bound. Value. Probl., 2016, DOI:10.1186/s13661–016–0621–8.

[9] X. Hao, H. Wang, L. Liu et al., *Positive solutions for a system of nonlinear fractional nonlocal boundary value problems with parameters and p-Laplacian operator*, Bound. Value. Probl., 2017, DOI:10.1186/s13661–017–0915–5.

[10] J. Henderson and R. Luca, *Existence of positive solutions for a singular fractional boundary value problem*, Nonlinear Anal: Model. Control., 2017, 22, 99–114.

[11] G. Infantea and M. Zima, *Positive solutions of multi-point boundary value problems at resonance*, Nonlinear Anal., 2008, 69, 2458–2465.

[12] W. Jiang, *The existence of solutions to boundary value problems of fractional differential equations at resonance*, Nonlinear Anal., 2011, 74, 1987–1994.

[13] N. Kosmatov, *A boundary value problem of fractional order at resonance*, Electronic. J. Diff. Equ., 2010, 135, 1–10.

[14] N. Kosmatov and W. Jiang, *Resonant functional problems of fractional order*, Chaos. Soliton. Fract., 2016, 91, 573–579.

[15] C. Li, X. Luo and Y. Zhou, *Existence of positive solutions of the boundary value problem for nonlinear fractional differential equations*, Comput.Math.Appl., 2010, 59, 1363–1375.

[16] S. Liang and L. Mu, *Multiplicity of positive solutions for singular three-point boundary value problems at resonance*, Nonlinear Anal., 2009, 71, 2497–2505.

[17] S. Liu, H. Li, Q. Dai et al., *Existence and uniqueness results for nonlocal integral boundary value problems for fractional differential equations*, Adv. Differ. Equ., 2016, DOI:10.1186/s13662–016–0847–x.

[18] X. Liu, L. Liu and Y. Wu, *Existence of positive solutions for a singular nonlinear fractional differential equation with integral boundary conditions involving fractional derivatives*, Bound. Value. Probl., 2018, DOI:10.1186/s13661–018–0943–9.

[19] S. Meng and Y. Cui, *Multiplicity Results to a Conformable Fractional Differential Equations Involving Integral Boundary Condition*, Complexity., 2019, 2019, 1–8.

[20] X. Meng and M. Stynes, *The Green's function and a maximum principle for a Caputo two-point boundary value problem with a convection term*, J. Math. Anal. Appl., 2018, 461, 198–218.

[21] I. Podlubny, *Fractional Differential Equations, in: Mathematics in Science and Engineering*, vol. 198, Academic Press, New York, London, Toronto, 1999.

[22] Q. Sun, S. Meng and Y. Cui, *Existence results for fractional order differential equation with nonlocal Erdelyi-Kober and generalized Riemann-Liouville type integral boundary conditions at resonance*, Adv. Differ. Equ., 2018, DOI:10.1186/s13662–018–1668–x.

[23] G. Wang, B. Ahmad and L. Zhang, *Existence results for nonlinear fractional differential equations with closed boundary conditions and impulses*, Adv. Differ. Equ., 2012, DOI:10.1186/1687–1847–2012–169.

[24] Y. Wang, *Necessary conditions for the existence of positive solutions to fractional boundary value problems at resonance*, Appl. Math. Lett., 2019, 97, 34–40.

[25] Y. Wang,*Existence and multiplicity of positive solutions for a class of singular fractional nonlocal boundary value problems*, Bound. Value. Probl., 2019, DOI:10.1186/s13661–019–1205–1.

[26] Y. Wang and L. Liu, *Positive solutions for a class of fractional 3-point boundary value problems at resonance*, Adv. Differ. Equ., 2017, DOI:10.1186/s13662–016–1062–5.

[27] Y. Wang and L. Liu, *Positive properties of the Green function for two-term fractional differential equations and its application*, J. Nonlinear Sci. Appl., 2017, 10, 2094–2102.

[28] Y. Wang and L. Liu, *Positive solutions for a class of fractional infinite-point boundary value problems*, Bound. Value. Probl., 2018, DOI:10.1186/s13661–018–1035–6.

[29] Y. Wang, L. Liu and Y. Wu, *Positive solutions for a nonlocal fractional differential equation*, Nonlinear Anal., 2011, 74, 3599–3605.

[30] J. Webb and K. Lan, *Eigenvalue criteria for existence of multiple positive solutions of nonlinear boundary value problems of local and nonlocal type*, Topol. Method. Nonl. An., 2006, 27, 91–116.

[31] J. Webb and M. Zima, *Multiple positive solutions of resonant and non-resonant nonlocal boundary value problems*, Nonlinear Anal., 2009, 71, 1369–1378.

[32] J. Wu, X. Zhang, L. Liu et al., *The convergence analysis and error estimation for unique solution of a p-Laplacian fractional differential equation with singular decreasing nonlinearity*, Bound. Value. Probl., 2018, DOI:10.1186/s13661–018–1003–1.

[33] X. Xu and X. Fei, *The positive properties of Green's function for three point boundary value problems of nonlinear fractional differential equations and its applications*, Commun Nonlinear Sci., 2012, 17, 1555–1565.

[34] L. Yang and C. Shen, *On the existence of positive solution for a kind of multi-point boundary value problem at resonance*, Nonlinear Anal., 2010, 72, 4211–4220.

[35] X. Zhang, L. Liu and Y. Wu, *The uniqueness of positive solution for a fractional order model of turbulent flow in a porous medium*, Appl. Math. Lett., 2014, 37, 26–33.

[36] X. Zhang, L. Wang and Q. Sun, *Existence of positive solutions for a class of nonlinear fractional differential equations with integral boundary conditions and a parameter*, Appl. Math. Comput., 2014, 226, 708–718.

[37] X. Zhang and Q. Zhong, *Uniqueness of solution for higher-order fractional differential equations with conjugate type integral conditions*, Fract. Calc. Appl. Anal., 2017, 20, 1471–1484.

[38] X. Zhang and Q. Zhong, *Triple positive solutions for nonlocal fractional differential equations with singularities both on time and space variables*, Appl. Math. Lett., 2018, 80, 12–19.

[39] B. Zhu, L. Liu and Y. Wu et al., *Local and global existence of mild solutions for a class of semilinear fractional integro-differential equations*, Fract. Calc. Appl. Anal., 2017, 20, 1338–1355.

[40] Y. Zou, and G. He, *On the uniqueness of solutions for a class of fractional differential equations*, Appl. Math. Lett., 2017, 74, 68–73.