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BIFURCATION OF LIMIT CYCLES FROM A COMPOUND LOOP WITH FIVE SADDLES*

Lijuan Sheng[1] and Maoan Han[2,1,†]

Abstract We concern the number of limit cycles of a polynomial system with degree nine. We prove that under different conditions, the system can have 12 and 20 limit cycles bifurcating from a compound loop with five saddles. Our method relies upon the Melnikov function method and the method of stability-changing of a double homoclinic loop proposed by the authors[J. Yang, Y. Xiong and M. Han, *Nonlinear Anal-Theor.*, 2014, 95, 756–773].

Keywords Limit cycle, bifurcation, Melnikov function, homoclinic loop.

MSC(2010) 34C07.

1. Introduction

The week Hilbert's 16th problem [1] is related to the maximin number of limit cycles of the following near-Hamiltonian system

$$\begin{aligned} \dot{x} &= H_y(x,y) + \varepsilon P(x,y), \\ \dot{y} &= -H_x(x,y) + \varepsilon Q(x,y), \end{aligned} \tag{1.1}$$

where ε is a small parameter, $H(x,y)$ is a polynomial with degree m and $P(x,y)$, $Q(x,y)$ are polynomials with degree n. One of the main tools to study the number of limit cycles of (1.1) is called the Melnikov function method, finding the number of zeros of the so called Abelian integral or the first order Melnikov function as follows

$$M(h) = \int_{L_h} Q(x,y)dx - P(x,y)dy,$$

where L_h denotes a periodic orbit of system $(1.1)_{\varepsilon=0}$ defined by $H(x,y) = h$ for $h \in J$ with J an open interval. By studying the asymptotic expansion of the above function, one can estimate the number of its zeros, which is also a lower bound of the number of limit cycles of system (1.1). Some related works can be found in [3, 9, 14–18, 21–23, 25, 26] and references therein.

An alien limit cycle is a limit cycle which can not be detected by the Melnikov function. Alien limit cycles may appear when the unperturbed system $(1.1)_{\varepsilon=0}$ has a poly-cycle containing a heteroclinic loop, see [2, 4, 5]. The author of [6] presented

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[†] the corresponding author. Email address:mahan@shnu.edu.cn(M. Han)
[1] Department of Mathematics, Shanghai Normal University, Shanghai 200234, China
[2] Department of Mathematics, Zhejiang Normal University Jinhua, Zhejiang 321004, China

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a new way to study this problem by changing the stability of a homoclinic loop('the stability-changing method' for short). The method is further developed in [24] when the unperturbed system $(1.1)_{\varepsilon=0}$ has a double heteroclinic loop or a compound loop. Then by using this method, the authors of [20] study the number of limit cycles bifurcating from a compound loop with three saddles, which contains alien limit cycles. More results can be found in [2,4,5,8,12,13].

In this paper, we consider the following polynomial system with degree nine

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= kx(x^2-a)(x^2-b)(x^2-c)(x^2-d)+\varepsilon(b_0x+b_1x^3+g_0(x)y), \end{aligned} \tag{1.2}$$

where $g_0(x)=a_0+a_2x^2+a_4x^4+a_6x^6+a_8x^8+a_{10}x^{10}+a_{12}x^{12}+x^{14}$, and k,a,b,c,d are real coefficients with $kabcd \neq 0$. The Hamiltonian function of system (1.2) is

$$H(x,y)=\frac{1}{2}y^2-\int_0^x ks(s^2-a)(s^2-b)(s^2-c)(s^2-d)ds.$$

Assume $a>b>c>d>0$. By analyzing the level curves of $H(x,y)$, there are 14 different cases for the portraits of system $(1.2)_{\varepsilon=0}$, shown in Figures 1 and 2.

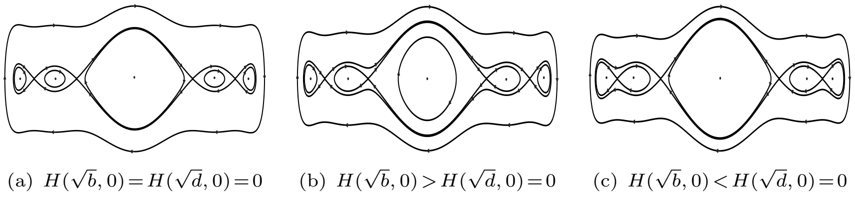

(a) $H(\sqrt{b},0)=H(\sqrt{d},0)=0$ (b) $H(\sqrt{b},0)>H(\sqrt{d},0)=0$ (c) $H(\sqrt{b},0)<H(\sqrt{d},0)=0$

Figure 1. The phase portraits of systems (1.2) for $k<0$.

For the case of $k>0$ and $H(\sqrt{c},0)=H(\sqrt{a},0)=0$, the unperturbed system $(1.2)_{\varepsilon=0}$ has a compound loop L_0 with five saddles, see Figure 2(a). By using the Melnikov function method and the stability-changing method, we have our main result on the number of limit cycles of system (1.2) bifurcating from L_0 as follows.

Theorem 1.1. *Let $b_0 \neq 0$ in system (1.2). If the unperturbed system $(1.2)_{\varepsilon=0}$ has a compound loop L_0 with five saddles as shown in Figure 2(a), we have the following two conclusions:*

(1) There exist 12 limit cycles of system (1.2) bifurcating from L_0 when $b_1=0$.

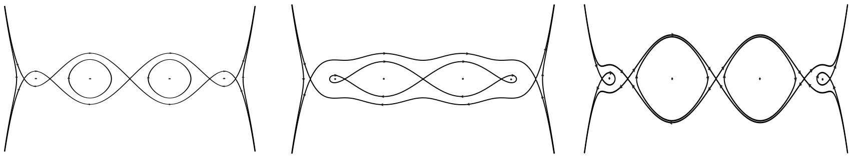
(2) There exist 20 limit cycles of system (1.2) bifurcating from L_0 when $-\frac{11}{8}b_1<b_0<-\frac{1}{2}b_1$ with $b_1>0$, or $-\frac{1}{2}b_1<b_0<-\frac{11}{8}b_1$ with $b_1<0$. Four of them are alien limit cycles.

The paper is organized as follows: in section 2, we find the conditions for a centrally symmetrical system to have 12 and 20 limit cycles, see Theorem 2.1. In section 3, We prove Theorem 1.1 by using Theorem 2.1.

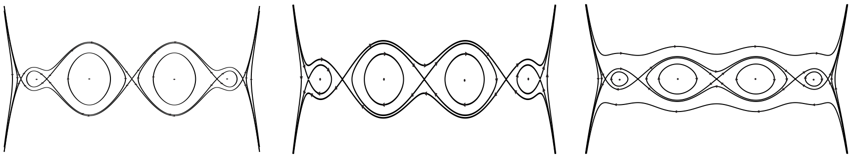
2. Preliminary lemmas

Consider the following centrally symmetrical near-Hamiltonian system

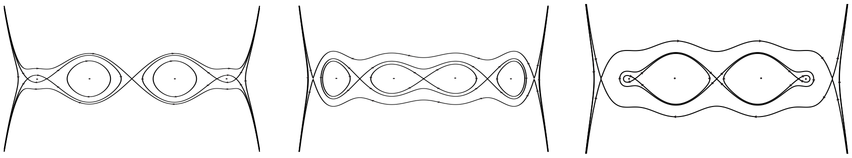
$$\begin{aligned} \dot{x} &= H_y(x,y)+\varepsilon p(x,y,\varepsilon,\delta), \\ \dot{y} &= -H_x(x,y)+\varepsilon q(x,y,\varepsilon,\delta), \end{aligned} \tag{2.1}$$



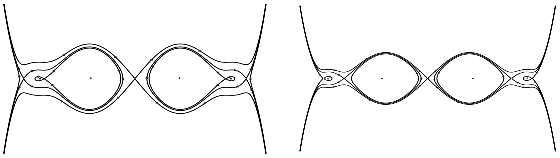
(a) $H(\sqrt{c},0)=H(\sqrt{a},0)=0$ (b) $H(\sqrt{a},0)>H(\sqrt{c},0)=0$ (c) $H(\sqrt{a},0)<H(\sqrt{c},0)=0$



(d) $H(\sqrt{c},0)<H(\sqrt{a},0)=0$ (e) $H(\sqrt{c},0)>\max\{H(\sqrt{a},0),0\}$ (f) $H(\sqrt{a},0)=H(\sqrt{c},0)>0$



(g) $H(\sqrt{a},0)=H(\sqrt{c},0)<0$ (h) $H(\sqrt{a},0)>H(\sqrt{c},0)>0$ (i) $H(\sqrt{a},0)>0>H(\sqrt{c},0)$



(j) $H(\sqrt{c},0)<H(\sqrt{a},0)<0$ (k) $H(\sqrt{a},0)<H(\sqrt{c},0)<0$

Figure 2. The phase portraits of systems (1.2) for $k>0$.

where ε is a small parameter, $H(x,y)$, $p(x,y,\varepsilon,\delta)$ and $q(x,y,\varepsilon,\delta)$ are C^∞ functions and $\delta=(\delta_1,\ldots,\delta_m)\in D\subset\mathbb{R}^m$ with D bounded.

Assume that the unperturbed system $(2.1)_{\varepsilon=0}$ has five hyperbolic saddles $S_{i0}(x_i^s,0)$, $\bar{S}_{i0}(-x_i^s,0)$, $S_{30}(0,0)$, and four centers $C_i(x_i^c,0)$, $\bar{C}_i(-x_i^c,0)$, where $x_i^s,x_i^c>0$, $i=1,2$. By the symmetry of system $(2.1)_{\varepsilon=0}$, we have

$$H(x_i^s,0)=H(-x_i^s,0),\quad H(x_i^c,0)=H(-x_i^c,0),\quad i=1,2.$$

Suppose system $(2.1)_{\varepsilon=0}$ has a compound loop L_0 passing through the saddles S_{i0}, $\bar{S}_{i0}(i=1,2)$, S_{30} such that $L_0=L_0^+\cup L_0^-$ with $L_0^+=\bigcup_{i,j=1}^2 L_{ij}$ and $L_0^-=\bigcup_{i,j=1}^2 \bar{L}_{ij}$, see Figure 3. Then, one can find easily that

$$H(\pm x_i^s,0)=H(0,0),\quad i=1,2.$$

Without loss of generality, we further assume that

$$H(0,0)=0,\ H(\pm x_i^c,0)=h_i<0,\ i=1,2,$$

from which we know L_0 is in clockwise orientation.

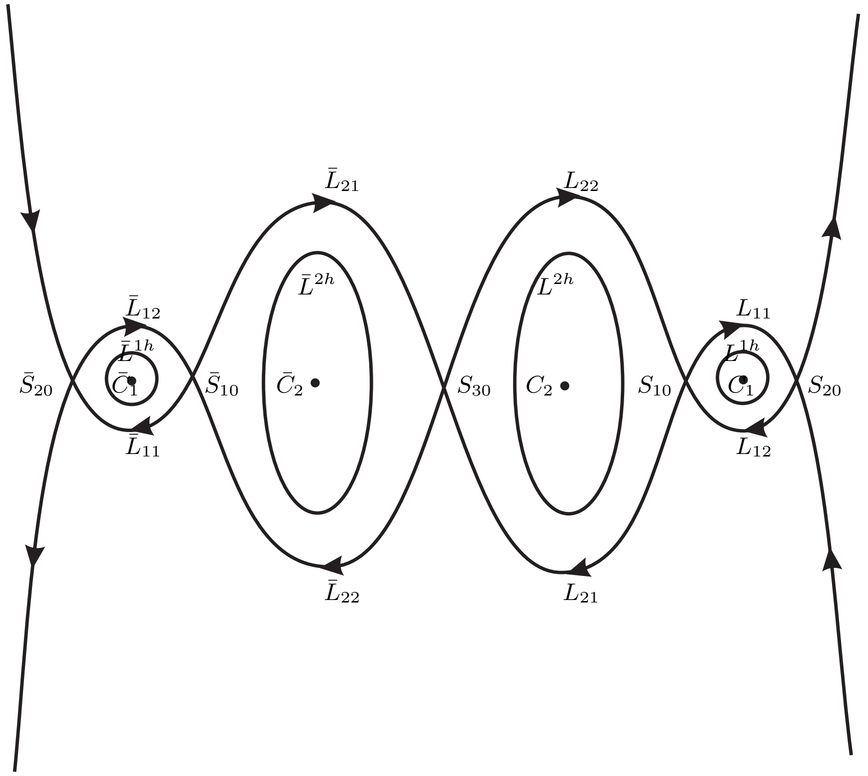


Figure 3. The phase portrait of the compound loop L_0

Let $L^{ih}, \bar{L}^{ih}$, $i = 1, 2$, be families of periodic orbits inside L_0 given by

$$\begin{aligned} &L^{1h}\ (\bar{L}^{1h}\text{ resp.}): \quad H(x,y) = h,\ h_1 < h < 0,\ x_1^s < x < x_2^s\ (x_1^s < -x < x_2^s\text{ resp.}),\\ &L^{2h}\ (\bar{L}^{2h}\text{ resp.}): \quad H(x,y) = h,\ h_2 < h < 0,\quad 0 < x < x_1^s\ (0 < -x < x_1^s\text{ resp.}) \end{aligned}$$

such that $L^{ih} \to L_{i1} \cup L_{i2}$, $\bar{L}^{ih} \to \bar{L}_{i1} \cup \bar{L}_{i2}$ as $h \to 0$, $i = 1, 2$.

Then there are four Melnikov functions of system (2.1) as follows

$$\begin{aligned} M_i(h,\delta) &= \oint_{L^{ih}} qdx - pdy\big|_{\varepsilon=0}, \quad h_i < h < 0,\ i = 1, 2,\\ \bar{M}_i(h,\delta) &= \oint_{\bar{L}^{ih}} qdx - pdy\big|_{\varepsilon=0}, \quad h_i < h < 0,\ i = 1, 2, \end{aligned} \tag{2.2}$$

where $M_i(h,\delta) = \bar{M}_i(h,\delta)(i = 1, 2)$ according to the symmetry of system (2.1).

We now introduce the following quantites

$$\begin{aligned}
&M_i(\delta)=M_i(0,\delta)=\sum_{j=1}^{2}M_{ij}(\delta),\ \text{where } M_{ij}(\delta)=\int_{L_{ij}}qdx-pdy\big|_{\varepsilon=0},\quad i,j=1,2,\\
&\mu_{1i}(\delta)=(p_x+q_y)(S_{i0},0,\delta),\ i=1,2,3,\\
&\mu_{21}(\delta)=\sum_{j=1}^{2}\int_{L_{1j}}(p_x+q_y)\Big|_{\mu_{11}=\mu_{12}=\varepsilon=0}dt,\\
&\mu_{22}(\delta)=\sum_{j=1}^{2}\int_{L_{2j}}(p_x+q_y)\Big|_{\mu_{11}=\mu_{13}=\varepsilon=0}dt.
\end{aligned}\tag{2.3}$$

For $i=1,2,3$, if S_{i0} is at the origin and $H(x,y)=\lambda_i xy+O(|(x,y)|^3)$ near the origin with $\lambda_i>0$, we further denote that

$$\begin{aligned}
\mu_{3i}(\delta)\Big|_{\mu_{1i}=0}=&-\frac{1}{2\lambda_i}\{(p_{xxy}+q_{xyy})-\frac{1}{\lambda_i}[H_{xyy}(p_{xx}+q_{xy})\\
&+H_{xxy}(p_{xy}+q_{yy})]\}\Big|_{x=y=\varepsilon=0}.
\end{aligned}\tag{2.4}$$

Then we have the following lemma from [24].

Lemma 2.1 ([24, Theorem 2.2]). *Suppose that system* $(2.1)_{\varepsilon=0}$ *has a double heteroclinic loop* L_0^+ *with* $L_0^+=\bigcup_{i,j=1}^{2}L_{ij}$ *shown in Figure 3. Let* M_i, M_{ij} *and* μ_{ij} *be defined in* (2.3). *If there exists* $\delta_0\in\mathbb{R}^m$ *such that*

$$\begin{aligned}
&M_1(\delta_0)=M_2(\delta_0)=0,\ M_{11}(\delta_0)M_{21}(\delta_0)>0,\ \mu_{ij}(\delta_0)=0,\ \textit{for } i=1,2,\ i+j\le 4,\\
&\mu_{31}(\delta_0)\neq 0,\quad \operatorname{rank}\frac{\partial(M_1,M_2,\mu_{11},\mu_{12},\mu_{13},\mu_{21},\mu_{22})}{\partial(\delta_1,\ldots,\delta_m)}(\delta_0)=7,
\end{aligned}$$

then there exist 10 limit cycles of system (2.1) *near* L_0^+ *for some* (ε,δ) *near* $(0,\delta_0)$.

By (2.3) and (2.4), we have from [7, 10, 11, 19] directly that

Lemma 2.2. *Suppose that system* $(2.1)_{\varepsilon=0}$ *has a compound loop* L_0 *with five saddles shown in Figure 3. The Melnikov functions* (2.2) *can be written as*

$$\begin{aligned}
M_1(h,\delta)&=c_0(\delta)+c_1(\delta)h\ln|h|+c_2(\delta)h+c_3(\delta)h^2\ln|h|+O(h^2),\ 0<-h\ll 1,\\
M_2(h,\delta)&=\bar{c}_0(\delta)+\bar{c}_1(\delta)h\ln|h|+\bar{c}_2(\delta)h+\bar{c}_3(\delta)h^2\ln|h|+O(h^2),\ 0<-h\ll 1,
\end{aligned}\tag{2.5}$$

where

$$\begin{aligned}
&c_0(\delta)=M_1(\delta), && \bar{c}_0(\delta)=M_2(\delta),\\
&c_1(\delta)=-\frac{1}{\lambda_1}\mu_{11}(\delta)-\frac{1}{\lambda_2}\mu_{12}(\delta), && \bar{c}_1(\delta)=-\frac{1}{\lambda_1}\mu_{11}(\delta)-\frac{1}{\lambda_3}\mu_{13}(\delta),\\
&c_2(\delta)\big|_{\substack{\mu_{1i}(\delta)=0\\ i=1,2}}=\mu_{21}(\delta), && \bar{c}_2(\delta)\big|_{\substack{\mu_{1i}(\delta)=0\\ i=1,3}}=\mu_{22}(\delta),\\
&c_3(\delta)\big|_{\substack{\mu_{1i}(\delta)=0\\ i=1,2}}=-\frac{1}{\lambda_1}\mu_{31}(\delta)-\frac{1}{\lambda_2}\mu_{32}(\delta), && \bar{c}_3(\delta)\big|_{\substack{\mu_{1i}(\delta)=0\\ i=1,3}}=-\frac{1}{\lambda_1}\mu_{31}(\delta)-\frac{1}{\lambda_3}\mu_{33}(\delta).
\end{aligned}\tag{2.6}$$

With the help of the above lemmas, we can prove that

Theorem 2.1. *Suppose system* $(2.1)_{\varepsilon=0}$ *has a compound loop* L_0 *with five saddles like Figure 3. Let* M_i*,* M_{ij} *and* μ_{ij} *be defined in* (2.3)*. If there exists* $\delta_0 \in \mathbb{R}^m$ *such that*

$$\begin{gathered} M_i(\delta_0) = \mu_{ij}(\delta_0) = 0, \text{ for } i = 1,2, \ i+j \leq 4, \\ \operatorname{rank} \frac{\partial(M_1, M_2, \mu_{11}, \mu_{12}, \mu_{13}, \mu_{21}, \mu_{22})}{\partial(\delta_1, \dots, \delta_m)}(\delta_0) = 7, \end{gathered} \tag{2.7}$$

then for some (ε, δ) *near* $(0, \delta_0)$*,*

(i) there exist 12 limit cycles of system (2.1) *near* L_0 *when*

$$\frac{1}{\lambda_1}\mu_{31}(\delta_0) + \frac{1}{\lambda_{i+1}}\mu_{3\,i+1}(\delta_0) \neq 0, \text{ for } i = 1,2; \tag{2.8}$$

(ii) there exist 20 limit cycles of system (2.1) *near* L_0*, four of which are alien limit cycles, when*

$$M_{11}(\delta_0)M_{21}(\delta_0) > 0, \ \mu_{31}(\delta_0) \neq 0. \tag{2.9}$$

Proof. (i) Without loss of generality, we may suppose $m = 7$. From (2.6) and (2.7), for $1 \leq j \leq 7$ we have

$$\begin{aligned} \frac{\partial c_2}{\partial \delta_j}(\delta_0) &= \frac{\partial \mu_{21}}{\partial \delta_j}(\delta_0) + b_1 \frac{\partial \mu_{11}}{\partial \delta_j}(\delta_0) + b_2 \frac{\partial \mu_{12}}{\partial \delta_j}(\delta_0), \\ \frac{\partial \bar{c}_2}{\partial \delta_j}(\delta_0) &= \frac{\partial \mu_{22}}{\partial \delta_j}(\delta_0) + \bar{b}_1 \frac{\partial \mu_{11}}{\partial \delta_j}(\delta_0) + \bar{b}_2 \frac{\partial \mu_{13}}{\partial \delta_j}(\delta_0), \end{aligned}$$

where $b_i, \ \bar{b}_i (i = 1,2)$ are constants. Hence, it follows from (2.7) that

$$\operatorname{rank} \frac{\partial(c_0, \bar{c}_0, c_1, \bar{c}_1, c_2, \bar{c}_2)}{\partial(\delta_1, \dots, \delta_7)}(\delta_0) = 6,$$

which implies that one can solve $\delta = \phi(c_0, \bar{c}_0, c_1, \bar{c}_1, c_2, \bar{c}_2) = \delta_0 + O(|c_0, \bar{c}_0, c_1, \bar{c}_1, c_2, \bar{c}_2|)$ from $c_i = c_i(\delta)$ and $\bar{c}_i = \bar{c}_i(\delta)$, $i = 0,1,2$. Taking $\delta = \phi(c_0, \bar{c}_0, c_1, \bar{c}_1, c_2, \bar{c}_2)$ in (2.5) and (2.6), we have

$$\begin{aligned} &M_i(h, \delta) = M_i(h, \phi(c_0, \bar{c}_0, c_1, \bar{c}_1, c_2, \bar{c}_2)) \triangleq M_i^*(h, c), \ i = 1,2, \\ &c_3 = -\frac{1}{\lambda_1}\mu_{31}(\delta_0) - \frac{1}{\lambda_2}\mu_{32}(\delta_0) + O(|c|), \\ &\bar{c}_3 = -\frac{1}{\lambda_1}\mu_{31}(\delta_0) - \frac{1}{\lambda_3}\mu_{33}(\delta_0) + O(|c|), \end{aligned} \tag{2.10}$$

where $c = (c_0, \bar{c}_0, c_1, \bar{c}_1, c_2, \bar{c}_2) \in \mathbb{R}^6$. Then we take $c_i, \bar{c}_i (i = 0,1,2)$ as free parameters and prove the theorem in the following steps. From (2.8), for definiteness let $\frac{1}{\lambda_1}\mu_{31}(\delta_0) + \frac{1}{\lambda_{i+1}}\mu_{3\,i+1}(\delta_0) > 0, \ i = 1,2$.

Step 1. Let $c_i = \bar{c}_i = 0, \ i = 0,1,2$. It follows from (2.5) and (2.10) that

$$\begin{aligned} M_1^*(h, c) &= c_3 h^2 \ln|h| + O(h^2), \\ &= \left(-\frac{1}{\lambda_1}\mu_{31}(\delta_0) - \frac{1}{\lambda_2}\mu_{32}(\delta_0) + O(|c|) \right) h^2 \ln|h| + O(h^2), \ 0 < -h \ll 1, \\ M_2^*(h, c) &= \bar{c}_3 h^2 \ln|h| + O(h^2) \end{aligned}$$

$$= \left(-\frac{1}{\lambda_1}\mu_{31}(\delta_0) - \frac{1}{\lambda_3}\mu_{33}(\delta_0) + O(|c|) \right) h^2 \ln|h| + O(h^2),\ 0 < -h \ll 1.$$

Then it is easy to see $M_i^*(h, c) > 0,\ i = 1, 2$.

Step 2. Let $c_i = \bar{c}_i = 0,\ i = 0, 1$. Vary $c_2, \bar{c}_2$ near zero such that $0 < c_2 \ll \frac{1}{\lambda_1}\mu_{31}(\delta_0) + \frac{1}{\lambda_2}\mu_{32}(\delta_0)$ and $0 < \bar{c}_2 \ll \frac{1}{\lambda_1}\mu_{31}(\delta_0) + \frac{1}{\lambda_3}\mu_{33}(\delta_0)$. Then

$$M_1^*(h, c) = c_2 h + c_3 h^2 \ln|h| + O(h^2) < 0,\ \text{for } 0 < -h \ll 1,$$
$$M_2^*(h, c) = \bar{c}_2 h + \bar{c}_3 h^2 \ln|h| + O(h^2) < 0,\ \text{for } 0 < -h \ll 1.$$

From step 1, we see that the sign of $M_i^*(h, c)(i = 1, 2)$ changes from positive to negative, which derives that there exist some $h_i < 0$ such that $M_i^*(h_i, c) = 0,\ i = 1, 2$.

Step 3. Keeping $c_0 = \bar{c}_0 = 0$, we vary $c_1, \bar{c}_1$ as $0 < c_1 \ll |c_2|$ and $0 < \bar{c}_1 \ll |\bar{c}_2|$, from which and (2.5) one can find that

$$M_1^*(h, c) = c_1 h \ln|h| + c_2 h + c_3 h^2 \ln|h| + O(h^2) > 0,\ \text{for } 0 < -h \ll 1,$$
$$M_2^*(h, c) = \bar{c}_1 h \ln|h| + \bar{c}_2 h + \bar{c}_3 h^2 \ln|h| + O(h^2) > 0,\ \text{for } 0 < -h \ll 1.$$

Hence we can find two more zeros $h_{i+2}(i = 1, 2)$ satisfying that

$$M_i^*(h_{i+2}, c) = 0 \text{ with } 0 < -h_{i+2} \ll |h_i|,\ i = 1, 2.$$

Step 4. Let $0 < -c_0 \ll |c_1|$ and $0 < -\bar{c}_0 \ll |\bar{c}_1|$. Similar to the above steps, we can obtain $M_i^*(h, c) < 0(i = 1, 2)$ so that

$$M_i^*(h_{i+4}, c) = 0,\ \text{for some } 0 < -h_{i+4} \ll |h_{i+2}|,\ i = 1, 2.$$

Till now, we have already found six isolated zeros of the functions $M_i^*(h, c), i = 1, 2$ when $c = (c_0, \bar{c}_0, c_1, \bar{c}_1, c_2, \bar{c}_2)$ is chosen as follows

$$0 < -c_0 \ll c_1 \ll c_2 \ll \frac{1}{\lambda_1}\mu_{31}(\delta_0) + \frac{1}{\lambda_2}\mu_{32}(\delta_0), \tag{2.11}$$
$$0 < -\bar{c}_0 \ll \bar{c}_1 \ll \bar{c}_2 \ll \frac{1}{\lambda_1}\mu_{31}(\delta_0) + \frac{1}{\lambda_3}\mu_{33}(\delta_0). \tag{2.12}$$

For $i = 1, 2$, if $\frac{1}{\lambda_1}\mu_{31}(\delta_0) + \frac{1}{\lambda_{i+1}}\mu_{3\,i+1}(\delta_0) < 0$, we can still find six zeros of $M_i(h, \delta)$ by varying the free parameters as follows

$$0 < c_0 \ll -c_1 \ll -c_2 \ll \left| \frac{1}{\lambda_1}\mu_{31}(\delta_0) + \frac{1}{\lambda_2}\mu_{32}(\delta_0) \right|, \tag{2.13}$$
$$0 < \bar{c}_0 \ll -\bar{c}_1 \ll -\bar{c}_2 \ll \left| \frac{1}{\lambda_1}\mu_{31}(\delta_0) + \frac{1}{\lambda_3}\mu_{33}(\delta_0) \right|. \tag{2.14}$$

If $\frac{1}{\lambda_1}\mu_{31}(\delta_0) + \frac{1}{\lambda_2}\mu_{32}(\delta_0) > 0(< 0$, resp.) and $\frac{1}{\lambda_1}\mu_{31}(\delta_0) + \frac{1}{\lambda_2}\mu_{32}(\delta_0) < 0(> 0$, resp.), six zeros of $M_i(h, \delta)(i = 1, 2)$ can be found when $c_i, \bar{c}_i(i = 0, 1, 2)$ are chosen to satisfy (2.11)((2.13), resp.) and (2.14)((2.12), resp.).

Finally, let us recall that $\bar{M}_i(h, \delta) = M_i(h, \delta)(i = 1, 2)$, where $\bar{M}_i(h, \delta)$ are defined in (2.2). Then from the above proof, we find 12 isolated zeros of the Melnikov functions (2.2), each of which has three zeros. That is to say, system (2.1) has 12 limit cycles inside L_0. See Figure 4 for the distribution of the12 limit cycles.

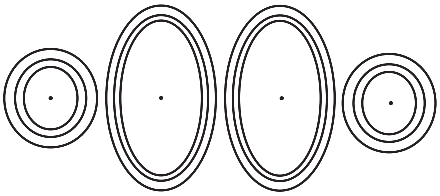


Figure 4. The distribution of 12 limit cycles of system (2.1)

(ii) The second conclusion can be obtained directly from the symmetry of system (2.1) and Lemma 2.1. In fact, according to the proof of Lemma 2.1 in [24, Theorem 2.2], when (2.7) and (2.9) hold, system (2.1) can have two double homoclinic loops $L_\varepsilon = L_\varepsilon^1 \cup L_\varepsilon^2$, $\bar{L}_\varepsilon = \bar{L}_\varepsilon^1 \cup \bar{L}_\varepsilon^2$ as shown in Figure 5 such that

$$L_\varepsilon^i \to L_{i1} \cup L_{i2},\ \bar{L}_\varepsilon^i \to \bar{L}_{i1} \cup \bar{L}_{i2},\quad i = 1, 2, \qquad \text{as} \qquad \varepsilon \to 0.$$

Then by varing the free parameters $M_1, M_2, \mu_{11}, \mu_{12}, \mu_{13}, \mu_{21}, \mu_{22}$ near zero, we can change the stability of the homoclinic loops L_ε^i, $\bar{L}_\varepsilon^i$ and the double homoclinic loops L_ε, $\bar{L}_\varepsilon$ several times. By Poincaré-Bendixson Theorem, 20 limit cycles, shown in Figure 6, can be produced one by one. We can see from Figure 6 that there are four big limit cycles surrounding eight small limit cycles and the big limit cycles can not be detected by the Melnikov functions (2.2). That is to say, they are alien limit cycles. □

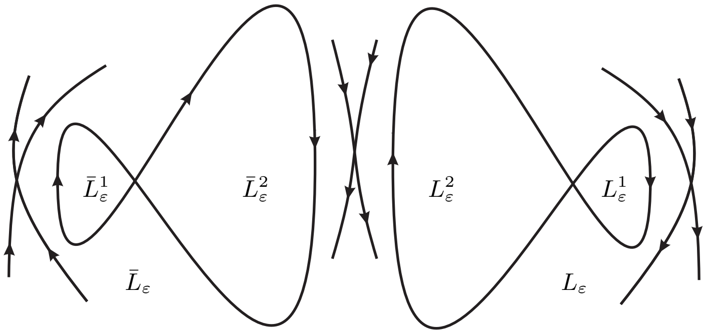


Figure 5. The double homoclinic loops of system (2.1)

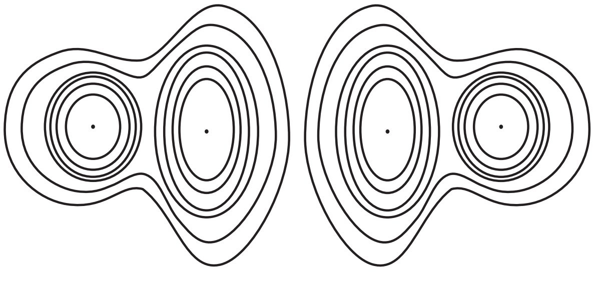


Figure 6. The distribution of 20 limit cycles of system (2.1)

3. Proof of Theorem 1.1

Now take $a=\frac{7}{4}, b=\frac{7}{5}, c=1, d=\frac{1}{4}, k=10$ in system (1.2) and consider the number of limit cycles for the following Liénard system

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= -\frac{49}{8}x+\frac{77}{2}x^3-\frac{531}{8}x^5+44x^7-10x^9+\varepsilon(b_0x+b_1x^3+g_0(x)y),\end{aligned}\tag{3.1}$$

where $g_0(x)=a_0+a_2x^2+a_4x^4+a_6x^6+a_8x^8+a_{10}x^{10}+a_{12}x^{12}+x^{14}$, $b_0\neq 0$. The unperturbed system $(3.1)_{\varepsilon=0}$ has five saddles $S_{10}=(1,0), \bar{S}_{10}=(-1,0), S_{20}=(\frac{\sqrt{7}}{2},0), \bar{S}_{20}=(-\frac{\sqrt{7}}{2},0), S_{30}=(0,0)$ and four centers $(\pm\frac{\sqrt{35}}{5},0), (\pm\frac{1}{2},0)$. It has the following Hamiltonian function

$$H(x,y)=\frac{1}{2}y^2-x^2(x^2-1)^2(x^2-\frac{7}{4})^2.$$

The equation $H(x,y)=0$ defines a compound loop L_0 with five saddles.

By (2.3), we have the following integrals along the curves $L_{1j}(L_{2j}, \text{resp.}): y=y_j(x)=(-1)^jx(x^2-1)(x^2-\frac{7}{4}), 1\le x\le\frac{\sqrt{7}}{2}(0\le x\le 1, \text{resp.}),\ j=1,2,$

$$\begin{aligned}M_{1j} &= \int_{L_{1j}} qdx-pdy=\int_{L_{1j}}(b_0x+b_1x^3+g_0(x)y)dx\\ &= \int_1^{\frac{\sqrt{7}}{2}} g_0(x)x(x^2-1)(\frac{7}{4}-x^2)dx+(-1)^{j+1}(\frac{3}{8}b_0+\frac{33}{64}b_1)\\ &= \sum_{\substack{l=2i\\ 0\le i\le 6}} a_lI_l+I_{14}+(-1)^{j+1}(\frac{3}{8}b_0+\frac{33}{64}b_1)\end{aligned}\tag{3.2}$$

where

$$I_l=\int_1^{\frac{\sqrt{7}}{2}} x^{l+1}(x^2-1)(\frac{7}{4}-x^2)dx,\ l=2i,\ 0\le i\le 7$$

and

$$\begin{aligned}M_{2j} &= \int_{L_{2j}} qdx-pdy=\int_{L_{2j}}(b_0x+b_1x^3+g_0(x)y)dx\\ &= \int_0^1 g_0(x)x(x^2-1)(x^2-\frac{7}{4})dx+(-1)^j(\frac{1}{2}b_0+\frac{1}{4}b_1)\\ &= \sum_{\substack{l=2i\\ 0\le i\le 6}} a_lJ_l+J_{14}+(-1)^j(\frac{1}{2}b_0+\frac{1}{4}b_1)\end{aligned}\tag{3.3}$$

where

$$J_l=\int_0^1 x^{l+1}(x^2-1)(x^2-\frac{7}{4})dx,\ l=2i,\ 0\le i\le 7.$$

By calculating the integrals, we summarize the values of I_l and J_l in Table 1.

Applying (2.3) again, we can obtain μ_{1j}, $j=1,2,3$ that

$$\begin{aligned}\mu_{11}(\delta) &= 1+a_0+a_2+a_4+a_6+a_8+a_{10}+a_{12},\\ \mu_{12}(\delta) &= \frac{823543}{16384}+a_0+\frac{7}{4}a_2+\frac{49}{16}a_4+\frac{343}{64}a_6+\frac{2401}{256}a_8+\frac{16807}{1024}a_{10}+\frac{117649}{4096}a_{12},\\ \mu_{13}(\delta) &= a_0,\end{aligned}\tag{3.4}$$

Table 1. values of I_l and J_l

	$l=0$	$l=2$	$l=4$	$l=6$	$l=8$	$l=10$	$l=12$	$l=14$
I_l	$\frac{9}{256}$	$\frac{99}{2048}$	$\frac{2763}{40960}$	$\frac{7821}{81920}$	$\frac{314109}{2293760}$	$\frac{1459953}{7340032}$	$\frac{1225971}{4194304}$	$\frac{4552911}{10485760}$
J_l	$\frac{17}{48}$	$\frac{5}{48}$	$\frac{23}{480}$	$\frac{13}{480}$	$\frac{29}{1680}$	$\frac{1}{84}$	$\frac{5}{576}$	$\frac{19}{2880}$

where $\delta = (a_0, a_2, a_4, a_6, a_8, a_{10}, a_{12})$.

From (3.4), solve $\mu_{11}(\delta) = 0$ and $\mu_{12}(\delta) = 0$ to get

$$\begin{aligned} a_0 &= \frac{264957}{4096} + \frac{7}{4}a_4 + \frac{77}{16}a_6 + \frac{651}{64}a_8 + \frac{5005}{256}a_{10} + \frac{36827}{1024}a_{12}, \\ a_2 &= -\frac{269053}{4096} - \frac{11}{4}a_4 - \frac{93}{16}a_6 - \frac{715}{64}a_8 - \frac{5261}{256}a_{10} - \frac{37851}{1024}a_{12}, \end{aligned} \tag{3.5}$$

From (2.3) and (3.1), we have

$$\begin{aligned} \mu_{21}(\delta) &= \sum_{j=1}^{2} \int_{L_{1j}} (q_y)|_{\mu_{11}=\mu_{12}=0} dt = 2\int_1^{\frac{\sqrt{7}}{2}} \frac{g_0(x)|_{\mu_{11}=\mu_{12}=0}}{x(x^2-1)(\frac{7}{4}-x^2)} dx \\ &= \sum_{\substack{l=2i \\ 2\le i\le 6}} a_l K_l + K_{14}, \end{aligned} \tag{3.6}$$

where $K_l = \frac{\partial \mu_{21}(\delta)}{\partial a_l}$, $l = 2i$, $2 \le i \le 6$. Then (3.6) gives the form of K_l as

$$K_l = \begin{cases} 2\displaystyle\int_1^{\frac{\sqrt{7}}{2}} \frac{\frac{\partial (g_0(x)|_{\mu_{11}=\mu_{12}=0})}{\partial a_l}}{x(x^2-1)(\frac{7}{4}-x^2)} dx, & l = 2i,\ 2 \le i \le 6, \\ 2\displaystyle\int_1^{\frac{\sqrt{7}}{2}} \frac{g_0(x)|_{\substack{\mu_{11}=\mu_{12}=0 \\ a_j=0,\ j=2i, 2\le i\le 6}}}{x(x^2-1)(\frac{7}{4}-x^2)} dx, & l = 14. \end{cases} \tag{3.7}$$

Putting (3.5) into (3.7), we have values of K_l in Table 2.

From (3.4), solve $\mu_{11}(\delta) = 0$ and $\mu_{13}(\delta) = 0$ to get

$$\begin{aligned} a_0 &= 0, \\ a_2 &= -1 - a_4 - a_6 - a_8 - a_{10} - a_{12}. \end{aligned} \tag{3.8}$$

Inserting (3.8) into (2.3), we have

$$\begin{aligned} \mu_{22}(\delta) &= \sum_{j=1}^{2} \int_{L_{2j}} (q_y)|_{\mu_{11}=\mu_{13}=0} dt = 2\int_0^{1} \frac{g_0(x)|_{\mu_{11}=\mu_{13}=0}}{x(x^2-1)(x^2-\frac{7}{4})} dx \\ &= \sum_{\substack{l=2i \\ 2\le i\le 6}} a_l L_l + L_{14}, \end{aligned} \tag{3.9}$$

where

$$L_l = 2\int_0^1 \frac{x(x^{l-2}-1)}{(x^2-1)(x^2-\frac{7}{4})} dx,\ l = 2i,\ 2 \le i \le 7$$

whose values are listed in Table 2.

Table 2. values of K_l and L_l

	$l=4$	$l=6$	$l=8$
K_l	$2\ln 2-\ln 7$	$-\frac{3}{4}+\frac{11}{4}(2\ln 2-\ln 7)$	$-\frac{99}{32}+\frac{93}{16}(2\ln 2-\ln 7)$
L_l	$\ln 3-\ln 7$	$1+\frac{11}{4}(\ln 3-\ln 7)$	$\frac{13}{4}+\frac{93}{16}(\ln 3-\ln 7)$
	$l=10$	$l=12$	$l=14$
K_l	$-\frac{1107}{128}+\frac{715}{64}(2\ln 2-\ln 7)$	$-\frac{20955}{1024}+\frac{5261}{256}(2\ln 2-\ln 7)$	$-\frac{905697}{20480}+\frac{37851}{1024}(2\ln 2-\ln 7)$
L_l	$\frac{361}{48}+\frac{715}{64}(\ln 3-\ln 7)$	$\frac{2927}{192}+\frac{5261}{256}(\ln 3-\ln 7)$	$\frac{37071}{1280}+\frac{37851}{1024}(\ln 3-\ln 7)$

In order to calculate $\mu_{31}(\delta)$, we make the following transformation

$$x=u-v+1,\ y=\frac{3\sqrt{2}}{2}(u+v),$$

which carries system (3.1) into

$$\begin{aligned}\dot{u}&=-\frac{\sqrt{2}}{12}\Big(2H_x\big(u-v+1,\frac{3\sqrt{2}}{2}(u+v)\big)-9u-9v\Big)+\varepsilon\hat{f}(u,v),\\ \dot{v}&=-\frac{\sqrt{2}}{12}\Big(2H_x\big(u-v+1,\frac{3\sqrt{2}}{2}(u+v)\big)+9u+9v\Big)+\varepsilon\hat{f}(u,v),\end{aligned}$$

where

$$\hat{f}(u,v)=\frac{\sqrt{2}}{6}\Big(b_0(u-v+1)+\frac{3\sqrt{2}}{2}(u+v)g_0(u-v+1)\Big).$$

The Hamiltonian function of the new system is

$$\begin{aligned}\hat{H}(u,v)&=\frac{\sqrt{2}}{6}H(u-v+1,\frac{3\sqrt{2}}{2}(u+v))\\ &=\frac{3\sqrt{2}}{8}(u+v)^2-\frac{\sqrt{2}}{6}(u-v+1)^2(u-v)^2(u-v+2)^2\Big((u-v+1)^2-\frac{7}{4}\Big)^2\\ &=\frac{3}{\sqrt{2}}uv+O\big(|(u,v)|^3\big).\end{aligned}$$

Thus, we have from (2.4) that

$$\begin{aligned}\mu_{31}(\delta)&=-\frac{3}{2\sqrt{2}}\Big\{(\hat{f}_{uuv}+\hat{f}_{uvv})-\frac{3}{\sqrt{2}}\Big[\hat{H}_{uvv}(\hat{f}_{uu}+\hat{f}_{uv})+\hat{H}_{uuv}(\hat{f}_{uv}+\hat{f}_{vv})\Big]\Big\}\Big|_{u=v=\varepsilon=0}\\ &=\frac{4\sqrt{2}}{3}(35+2a_2+5a_4+9a_6+14a_8+20a_{10}+27a_{12}).\end{aligned}\tag{3.10}$$

In a similar way, from (2.4) we also have

$$\begin{aligned}\mu_{32}(\delta)&=\frac{\sqrt{2}}{2688}(18432a_2+60928a_4+150528a_6+329280a_8+672280a_{10}\\ &\quad+1310946a_{12}+2470629),\end{aligned}\tag{3.11}$$

$$\mu_{33}(\delta)=\frac{2\sqrt{2}}{7}a_2.$$

By (2.3), (3.2)–(3.4), (3.6) and (3.9), solving the equations $M_1 = M_2 = \mu_{11} = \mu_{12} = \mu_{13} = \mu_{21} = \mu_{22} = 0$ gives

$$\begin{aligned}&\hat{a}_0 = 0,\ \hat{a}_2 = \frac{343}{320},\ \hat{a}_4 = -\frac{539}{64},\ \hat{a}_6 = \frac{7301}{320},\\ &\hat{a}_8 = -\frac{9537}{320},\ \hat{a}_{10} = \frac{1639}{80},\ \hat{a}_{12} = -\frac{143}{20}.\end{aligned}$$

Thus, we can take $\delta_0 = (\hat{a}_0, \hat{a}_2, \hat{a}_4, \hat{a}_6, \hat{a}_8, \hat{a}_{10}, \hat{a}_{12})$. In this case, we have

$$\begin{aligned}&M_1(\delta_0) = M_2(\delta_0) = \mu_{11}(\delta_0) = \mu_{12}(\delta_0) = \mu_{13}(\delta_0) = \mu_{21}(\delta_0) = \mu_{22}(\delta_0) = 0,\\ &\det \frac{\partial(M_1, M_2, \mu_{11}, \mu_{12}, \mu_{13}, \mu_{21}, \mu_{22})}{\partial(a_0, a_2, a_4, a_6, a_8, a_{10}, a_{12})}(\delta_0) = -\frac{28588707}{21990232555520} \neq 0.\end{aligned}$$

Furthermore, from (3.10) and (3.11) we have

$$\mu_{31}(\delta_0) = -\frac{9}{40}\sqrt{2}, \quad \mu_{32}(\delta_0) = -\frac{441}{640}\sqrt{2}, \quad \mu_{33}(\delta_0) = \frac{49}{160}\sqrt{2}.$$

Then the conditions (2.7) and (2.8) are satisfied.

Next, in order to apply Theorem 2.1, we need to classify b_1 in two cases:

Case 1: $b_1 = 0$.

From (3.2) and (3.3), we have

$$M_{11}(\delta_0)M_{21}(\delta_0) = -\frac{3}{16}b_0^2 < 0.$$

Then (2.9) can not be satisfied here. From Theorem 2.1(i), there exist 12 limit cycles of system (3.1) for some (ε, δ) near $(0, \delta_0)$.

Case 2: $b_1 \neq 0$.

It is easy to see $M_{11}(\delta_0)M_{21}(\delta_0) > 0$ if and only if

$$-\frac{11}{8}b_1 < b_0 < -\frac{1}{2}b_1 \quad \text{with} \quad b_1 > 0,$$

or

$$-\frac{1}{2}b_1 < b_0 < -\frac{11}{8}b_1 \quad \text{with} \quad b_1 < 0.$$

Then by Theorem 2.1(ii), there exist 20 limit cycles of system (3.1) for some (ε, δ) near $(0, \delta_0)$, four of which are alien limit cycles. This completes the proof.

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