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EXISTENCE AND CONCENTRATION RESULT FOR KIRCHHOFF EQUATIONS WITH CRITICAL EXPONENT AND HARTREE NONLINEARITY

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Abstract This paper is concerned with the following Kirchhoff-type equations

$$\begin{cases} -\big(\varepsilon^2 a+\varepsilon b\displaystyle\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x\big)\Delta u+V(x)u+\mu\phi|u|^{p-2}u=f(x,u), & \text{in } \mathbb{R}^3,\\ (-\Delta)^{\frac{\alpha}{2}}\phi=\mu|u|^p,\ u>0, & \text{in } \mathbb{R}^3, \end{cases}$$

where $f(x,u)=\lambda K(x)|u|^{q-2}u+Q(x)|u|^4u$, $a>0$, b, $\mu\geq 0$ are constants, $\alpha\in(0,3)$, $p\in[2,3)$, $q\in[2p,6)$ and ε, $\lambda>0$ are parameters. Under some mild conditions on $V(x)$, $K(x)$ and $Q(x)$, we prove that the above system possesses a ground state solution u_ε with exponential decay at infinity for $\lambda>0$ and ε small enough. Furthermore, u_ε concentrates around a global minimum point of $V(x)$ as $\varepsilon\to 0$. The methods used here are based on minimax theorems and the concentration-compactness principle of Lions. Our results generalize and improve those in Liu and Guo (Z Angew Math Phys 66: 747-769, 2015), Zhao and Zhao (Nonlinear Anal 70: 2150-2164, 2009) and some other related literature.

Keywords Kirchhoff equations, critical Sobolev exponent, Hartree-type nonlinearity, concentration-compactness principle.

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1. Introduction

This paper deals with the existence and concentration of positive ground state solutions to the following Kirchhoff-type equations:

$$\begin{cases} -\big(\varepsilon^2 a+\varepsilon b\displaystyle\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x\big)\Delta u+V(x)u+\mu\phi|u|^{p-2}u=f(x,u), & \text{in } \mathbb{R}^3,\\ (-\Delta)^{\frac{\alpha}{2}}\phi=\mu|u|^p,\ u>0, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $f(x,u)=\lambda K(x)|u|^{q-2}u+Q(x)|u|^4u$, $a>0$, b, $\mu\geq 0$ are constants, $\alpha\in(0,3)$, $p\in[2,3)$, $q\in[2p,6)$ and ε, $\lambda>0$ are parameters.

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When $\alpha = p = 2$, $b = 0$ and $\varepsilon = a = 1$, system (1.1) becomes the following classical Schrödinger-Poisson system:

$$\begin{cases} -\Delta u + V(x)u + \mu\phi u = \lambda K(x)|u|^{q-2}u + Q(x)|u|^4 u, & \text{in } \mathbb{R}^3, \\ -\Delta\phi = \mu u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.2}$$

which describes a charged wave interacting with its own electrostatic field [4]. In the past decades, with the aid of variational methods, there are lots of results on elliptic equations, see [2–4, 6, 8–11, 15, 17, 18, 26–28, 30, 31, 34] and the references therein. For instance, Sun et al. [31] obtained the existence and concentration of nontrivial solutions for system (1.2) with $q \in (3,4)$, $K(x) \equiv 1$ and $Q(x) \equiv 0$. By using the variant fountain theorem established by Zou [37], Xu et al. [34] established the existence of multiple negative energy solutions for system (1.2) when $\lambda K(x)|u|^{q-2}u+Q(x)|u|^4u$ and μ are replaced by $H(x)f(x,u)$ and $K(x)$, respectively.

When $\mu = 0$, system (1.1) is reduced to the following Kirchhoff-type equation:

$$-(\varepsilon^2 a + \varepsilon b\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x)\Delta u + V(x)u = \lambda K(x)|u|^{q-2}u + Q(x)|u|^4 u, \quad \text{in } \mathbb{R}^3, \tag{1.3}$$

which was related to the stationary analogue of the following equation:

$$\rho\frac{\partial^2 u}{\partial^2 t} - (\frac{P_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|^2 dx)\frac{\partial^2 u}{\partial^2 x} = 0. \tag{1.4}$$

Problem (1.4) arises in many mathematical physics context, which was presented by Kirchhoff [19] as an extension of the classical D'Alembert wave equation for free vibrations of elastic strings. Recently many attentions have been paid to Eq. (1.3), especially on the existence of positive solutions, ground state solutions, sign-changing solutions and multiple solutions, see [16, 24, 25, 29, 35] and the references therein. For example, Liu et al. [24] investigated the existence of positive ground state solution u_ε for Eq. (1.3) with exponential decay at infinity for $\lambda > 0$ and ε small enough. Moreover, they assumed that $V(x)$ satisfies the following assumption:
(V_1) $V \in C(\mathbb{R}^3, \mathbb{R}^+)$ and $V_\infty := \liminf_{n\to\infty} V(x) > V_0 = \inf_{x\in\mathbb{R}^3} V(x) > 0$.

He et al. [13] obtained the multiplicity and concentration of nontrivial solutions for problem (1.3) when the nonlinearity is $f(u)+u^5$ and the potential $V(x)$ is a locally Hölder continuous function which satisfies the condition (V_2) as follows:
(V_2) $V(x) \geq V_0 > 0$ for all $x \in \mathbb{R}^3$ and $\inf_{x\in B} V(x) < \min_{x\in\partial B} V(x)$ for some open and bounded set $B \subset \mathbb{R}^N$.

Very recently, Li et al. [21] studied the following more generalized Kirchhoff-type system:

$$\begin{cases} -(a + b\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x)\Delta u + \lambda V(x)u + \phi|u|^{p-2}u = f(u), & \text{in } \mathbb{R}^3, \\ (-\Delta)^{\frac{\alpha}{2}}\phi = l|u|^p, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.5}$$

where $a > 0$, b, $l \geq 0$, $\alpha \in (0,3)$, $p \in [2, 3+2\alpha)$ and $\lambda > 0$ is a parameter. By the minimization argument on the sign-changing Nehari manifold and a quantitative deformation lemma, they proved system (1.5) has a sign-changing solution. Moreover, the concentration behaviors of sign-changing solutions were obtained when the potential function tends to infinity. In addition, note that the second equation in system (1.5) is a fractional differential equation and $\phi = I * |u|^p$, where $I : \mathbb{R}^3 \to \mathbb{R}$ is the Riesz potential defined by $I(x) = \frac{l\Gamma((3-\alpha)/2)}{\Gamma(\alpha/2)\pi^{\frac{3}{2}}2^\alpha}\frac{1}{|x|^{3-\alpha}}$, $x \in \mathbb{R}^3 \setminus \{0\}$ and $*$ is a notation for the convolution of two functions in $\mathbb{R}^3$. It is easy to see $\phi|u|^{p-2}u$ is a Hartree-type nonlinearity and when $\alpha = p = 2$, system (1.5) reduces to the Kirchhoff-Schrödinger-Poisson system.

Inspired by the above works, in the present paper, we will consider the existence and concentration of positive ground state solutions for system (1.1). As far as we know, it seems that there is almost no work on the existence and concentration of positive ground state solution for system (1.1), which is just our aim.

In this paper, in addition to the condition (V_1), we assume that functions $V(x)$, $K(x)$ and $Q(x)$ satisfy the following hypotheses:

(H_1) $K \in C(\mathbb{R}^3, \mathbb{R}^+)$, $\lim\limits_{|x|\to\infty} K(x) = K_\infty \in (0,\infty)$ and $K(x) \geq K_\infty$ for $x \in \mathbb{R}^3$;

(H_2) $Q \in C(\mathbb{R}^3, \mathbb{R}^+)$, $\lim\limits_{|x|\to\infty} Q(x) = Q_\infty \in (0,\infty)$ and $Q(x) \geq Q_\infty$ for $x \in \mathbb{R}^3$;

(H_3) there exist $\nu > 0$ and $\beta > 0$ such that $Q(x) - Q(x^*) \leq \nu|x - x_0|^\beta$ for $|x - x^*| < \delta$, where $\beta \in [1,3)$ and $Q(x^*) = \max\limits_{x\in\mathbb{R}^3} Q(x)$;

(H_4) $\Lambda \cap \Lambda_1 \cap \Lambda_2 \neq \emptyset$, where $\Lambda = \{x \in \mathbb{R}^3 : V(x) = V_0\}$, $\Lambda_1 = \{x \in \mathbb{R}^3 : K(x) = K_0 := \max\limits_{x\in\mathbb{R}^3} K(x)\}$ and $\Lambda_2 = \{x \in \mathbb{R}^3 : Q(x) = Q_0 := \max\limits_{x\in\mathbb{R}^3} Q(x)\}$.

It is obvious that $Q(x)$ and $K(x)$ are bounded continuous functions. Similar hypotheses have been introduced by Che et al. [7] and Liu et al. [24] in their studies of nonlinear quasilinear Schrödinger equations and Kirchhoff equations. Without loss of generality, we may assume that $0 \in \Lambda \cap \Lambda_1 \cap \Lambda_2$.

Our main results are the following.

Theorem 1.1. *Suppose that conditions* (V_1) *and* $(H_1) - (H_4)$ *hold. Then we have the following results.*
(i) If $p \in [2, \frac{3+\alpha}{2})$ *and* $q \in (4,6)$*, then there exists* $\varepsilon_0 > 0$ *such that problem (1.1) has a positive solution* u_ε *for any* $\lambda > 0$ *and* $\varepsilon \in (0, \varepsilon_0)$.
(ii) If $\frac{3+\alpha}{2} \leq p < 3$ *or* $q = 4$*, then there exits* $\lambda_0 > 0$ *and* $\varepsilon_0 > 0$ *such that problem (1.1) has a positive solution* u_ε *for any and* $\lambda > \lambda_0$ *and* $\varepsilon \in (0, \varepsilon_0)$.

Theorem 1.2. *Let* u_ε *be the positive solution obtained in Theorem 1.1, then* u_ε *concentrates around a point* x_ε *in* $\mathbb{R}^3$ *such that, up to a subsequence,* $x_\varepsilon \to x_0$ *as* $\varepsilon \to 0$ *and* $\omega_\varepsilon(x) = u_\varepsilon(\varepsilon x + x_\varepsilon)$ *converges to a position ground state solution of*

$$\begin{cases} -\big(a + b\displaystyle\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\big)\Delta u + V_0 u + \mu\phi|u|^{p-2}u = \\ = \lambda K_0|u|^{q-2}u + Q_0|u|^4 u, & \text{in } \mathbb{R}^3, \\ (-\Delta)^{\frac{\alpha}{2}}\phi = \mu|u|^p, \ u > 0, & \text{in } \mathbb{R}^3. \end{cases} \tag{1.7}$$

Remark 1.1. It is easy to see that the Schrödinger equation, the Kirchhoff equation and the Schrödinger-Poisson equation are all the special situations of problem (1.1). Specially, problem (1.1) becomes the Kirchhoff equation when $\mu = 0$, the Schrödinger-Poisson equation when $b = 0$ and $\alpha = p = 2$, and the general Schrödinger equation when $b = \mu = 0$. Therefore, the problem (1.1) unifies the above three kinds of equations and our results also cover these cases.

Remark 1.2. From the results in He et al. [14] and Wang et al. [32], we know that the maximum point x_ε of the positive solution u_ε satisfies $\lim\limits_{\varepsilon\to 0^+} x_\varepsilon = y \in \Lambda$. While, in this paper, under our assumptions, we have the furthermore result: $\lim\limits_{\varepsilon\to 0^+} x_\varepsilon = y \in \Lambda \cap \Lambda_1 \cap \Lambda_2$ due to the appearance of the nonnegative functions $K(x)$ and $Q(x)$. Hence, our results not only unify but also generalize the previous results.

The main obstacles to deal with the existence of positive ground state solutions for problem (1.1) lie in two aspects. Firstly, the appearance of the nonlocal term $(\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x)\Delta u$ makes it difficult to verify the (PS) condition. Precisely, for any (PS) sequences $\{u_n\}$, if $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$, we do not know whether there holds

$$\int_{\mathbb{R}^3} |\nabla u_n|^2 \mathrm{d}x \int_{\mathbb{R}^3} \nabla u_n \nabla v \mathrm{d}x \to \int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x \int_{\mathbb{R}^3} \nabla u \nabla v \mathrm{d}x, \quad \forall\, v \in H^1(\mathbb{R}^3).$$

Secondly, the difficulty is caused by the lack of compactness due to the unboundness domain $\mathbb{R}^3$ and the nonlinearity with the critical Sobolev growth. In particular, the nonlinear term is nonautonomous, which makes it much more complicated to recover the compactness. To deal with the difficulty caused by the noncompactness, some arguments are in order. Firstly, a standard method is adopted to show that the energy functional possesses a mountain pass energy level. Secondly, we borrow an idea from Brezis et al. [5] to show that mountain pass energy level is less than some critical level (Lemma 3.4) and is even less than the least energy level of the limit problem of (Lemma 3.6). At last, by employing the concentration-compactness principle of Lions [22, 23], we prove that the Palais-Smale condition holds at the mountain pass energy level. Hence, the mountain pass critical value exists.

Notation. Throughout this paper, we shall denote by $|\cdot|_r$, $1 \le r \le +\infty$, the L^r-norm and C various positive generic constants, which may vary from line to line. $S = \inf\limits_{u\in D^{1,2}(\mathbb{R}^3)} \frac{\|u\|^2_{D^{1,2}}}{|u|_6^2}$ is the best Sobolev constant from the embedding of $D^{1,2}(\mathbb{R}^3)$ into $L^6(\mathbb{R}^3)$. Also if we take a subsequence of a sequence $\{u_n\}$, we shall denote it again by $\{u_n\}$.

The remainder of this paper is as follows. In Section 2, some preliminary results are presented. In Section 3, by verifying the mountain pass value is under the energy level, we prove the existence of a positive ground state solution. Section 4 is devoted to the concentration of the positive ground state solution.

2. Preliminaries

In this section, we outline the variational framework for system (1.1) and give some preliminary lemmas.

Let $H^1(\mathbb{R}^3)$ be the usual Hilbert space with the inner product and the norm

$$\langle u, v\rangle_{H^1} = \int_{\mathbb{R}^3} \big(\nabla u \nabla v + uv\big)\mathrm{d}x, \quad \|u\|_{H^1} = \langle u, v\rangle^{\frac{1}{2}}_{H^1},$$

and denote the norm of $D^{1,2}(\mathbb{R}^3)$ by

$$\|u\|_{D^{1,2}} = \left(\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\right)^{\frac{1}{2}}.$$

Make the change of variable $\varepsilon z = x$, then we can rewrite problem (1.1) as

$$\begin{cases} -\big(a+b\displaystyle\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x\big)\Delta u + V(\varepsilon x)u + \mu\phi|u|^{p-2}u \\ = \lambda K(\varepsilon x)|u|^{q-2}u + Q(\varepsilon x)|u|^4 u, & \text{in } \mathbb{R}^3, \\ (-\Delta)^{\frac{\alpha}{2}}\phi = \mu|u|^p,\ u>0, & \text{in } \mathbb{R}^3. \end{cases} \tag{2.1}$$

For any $\varepsilon > 0$, let

$$E_\varepsilon = \big\{u\in H^1(\mathbb{R}^3)\big|\int_{\mathbb{R}^3}V(\varepsilon x)u^2\mathrm{d}x < \infty\big\}$$

with the inner product and the norm

$$\langle u,v\rangle_\varepsilon = \int_{\mathbb{R}^3}\big(a\nabla u\nabla v + V(\varepsilon x)uv\big)\mathrm{d}x \quad\text{and}\quad \|u\|_\varepsilon^2 = \langle u,u\rangle_\varepsilon.$$

For any $u\in E_\varepsilon$, let

$$\begin{aligned} I_\varepsilon(u) = \frac{1}{2}\|u\|_\varepsilon^2 + \frac{b}{4}\big(\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x\big)^2 + \frac{\mu}{2p}\int_{\mathbb{R}^3}(I*|u|^p)|u|^p\mathrm{d}x \\ -\frac{\lambda}{q}\int_{\mathbb{R}^3}K(\varepsilon x)|u|^q\mathrm{d}x - \frac{1}{6}\int_{\mathbb{R}^3}Q(\varepsilon x)|u|^6\mathrm{d}x. \end{aligned} \tag{2.2}$$

Then I is well defined and of class $C^1(E_\varepsilon,\mathbb{R})$ (see [24]) and that

$$\begin{aligned} \big\langle I_\varepsilon'(u),v\big\rangle = \langle u,v\rangle_\varepsilon &+ b\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x\int_{\mathbb{R}^3}\nabla u\nabla v\mathrm{d}x \\ &+\mu\int_{\mathbb{R}^3}(I*|u|^p)|u|^{p-2}uv\mathrm{d}x - \int_{\mathbb{R}^3}\lambda K(\varepsilon x)|u|^{q-2}uv\mathrm{d}x \\ &-\int_{\mathbb{R}^3}Q(\varepsilon x)|u|^4uv\mathrm{d}x. \end{aligned} \tag{2.3}$$

Define

$$\mathcal{N}_\varepsilon := \{u\in E_\varepsilon\setminus\{0\} : \langle I_\varepsilon'(u),u\rangle = 0\}.$$

Then $\mathcal{N}_\varepsilon$ is a Nehari manifold associated to I_ε. In view of the Implicit Function Theorem, we know that $\mathcal{N}_\varepsilon$ is a manifold of C^1. Furthermore, it is not difficult to verify that I_ε is bounded from below on $\mathcal{N}_\varepsilon$. Thus we can consider the following minimization problem:

$$c_\varepsilon^* := \inf_{u\in\mathcal{N}_\varepsilon} I_\varepsilon(u).$$

Next we state some properties of I_ε, $\mathcal{N}_\varepsilon$ and c_ε^*. By a standard argument as [18], we have the following lemma.

Lemma 2.1. *Suppose that conditions* (V_1) *and* (H_1-H_4) *hold, if* $q\in[4,6)$, *then we have the following results.*
(i) If $\{u_n\}$ *is a* $(PS)_c$ *sequence in* E_ε, *then there exists* $u\in E_\varepsilon$ *such that* $u_n \rightharpoonup u$ *and* $I_\varepsilon'(u)=0$.
(ii) For every $u\in E_\varepsilon\setminus\{0\}$, *there exists a unique* $t_u>0$ *such that* $t_uu\in E_\varepsilon$ *and* $I_\varepsilon(t_uu) = \max_{t\geq 0} I(tu)$.

(iii) For any $u \in \mathcal{N}_\varepsilon$*, there exists* $C_1 > 0$ *such that* $\|u\|_\varepsilon \geq C_1$*.*
(iv) Let $\{u_n\} \subset E_\varepsilon$ *be a sequence satisfying* $\langle I'_\varepsilon(u_n), u_n\rangle \to 0$ *and* $\int_{\mathbb{R}^3}\big(K(\varepsilon x)|u_n|^q + Q(\varepsilon x)|u_n|^6\big)\mathrm{d}x \to \eta > 0$ *as* $n \to \infty$*, where* η *is some positive constant, then there exists* $t_n > 0$ *such that* $\langle I'_\varepsilon(t_n u_n), t_n u_n\rangle = 0$ *and* $t_n \to 1$ *as* $n \to \infty$*.*

In a standard way (see [12]), one can check that the energy functional I_ε satisfies the mountain pass geometry.

Lemma 2.2. *The functional* I_ε *possesses the following properties:*
(i) there exist $\alpha_0,\ \rho > 0$ *such that* $I_\varepsilon(u) \geq \alpha_0$ *for* $\|u\|_\varepsilon = \rho$*.*
(ii) there exists $e \in E_\varepsilon$ *such that* $I_\varepsilon(e) < 0$*.*

It follows from Lemma 2.2 and the Mountain Pass Theorem [12] that there exists a $(PS)_{c_\varepsilon}$ sequence $\{u_n\} \subset E_\varepsilon$ such that $I_\varepsilon(u_n) \to c_\varepsilon$ and $I'_\varepsilon(u_n) \to 0$ as $n \to \infty$, where c_ε is equal to the minimax level value $\inf\limits_{\gamma\in\Gamma}\max\limits_{t\geq 0} I_\varepsilon(\gamma(t))$, where

$$\Gamma = \big\{\gamma \in C^1([0,1], E_\varepsilon) : I_\varepsilon(\gamma(0)) = 0, I_\varepsilon(\gamma(1)) < 0\big\}.$$

Define $c_\varepsilon^{**} = \inf\limits_{u\in E_\varepsilon\setminus\{0\}}\max\limits_{t\geq 0} I_\varepsilon(tu)$, then similar to Theorem 4.2 in [33], we can obtain

$$c_\varepsilon = c_\varepsilon^* = c_\varepsilon^{**}. \tag{2.4}$$

Here, we need say something to explain the relation (2.4). In fact, from Lemma 2.1(ii) that $c_\varepsilon^* = c_\varepsilon^{**}$. Observe that for any $u \in E_\varepsilon \setminus \{0\}$, there exits some $t_0 > 0$ such that $I_\varepsilon(t_0 u) < 0$. Define a path $\gamma : [0,1] \to E_\varepsilon$ by $\gamma(t) = tt_0 u$. Obviously, $\gamma \in \Gamma$ and consequently, $c_\varepsilon \leq c_\varepsilon^{**}$. Analogous to the arguments in [1,33], we obatain that $c_\varepsilon \geq c_\varepsilon^{**}$. Thus (2.4) holds. □

In this paper, we shall make use of the Hardy-Littlewood-Sobolev inequality from [21].

Lemma 2.3 (Hardy-Littlewood-Sobolev inequality)**.** *Let* $r,\ s \in (0,\infty)$ *and* $\mu \in (0, N)$ *with* $\frac{1}{r} + \frac{\mu}{N} + \frac{1}{s} = 2$*. Then there exists a sharp constant* $C(r, N, \mu, s)$ *such that for all* $f \in L^r(\mathbb{R}^N)$ *and* $g \in L^s(\mathbb{R}^N)$*,*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{f(x)g(y)}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y \leq C(r,N,\mu,s)|f|_r|g|_s.$$

The sharp constant satisfies

$$C(r,N,\mu,s) \leq \frac{N}{N-\mu}\frac{1}{rs}\alpha(N)^{\frac{\mu}{N}}\left[\Big(\frac{\mu/N}{1-1/r}\Big)^{\mu/N} + \Big(\frac{\mu/N}{1-1/s}\Big)^{\mu/N}\right],$$

where $\alpha(N)$ *is the volume of unit sphere in* $\mathbb{R}^N$*. If* $r = s = 2N/(2N-\mu)$*, then*

$$C(N,\mu) = \pi^{\frac{\mu}{2}}\frac{\Gamma(N/2-\mu/2)}{\Gamma(N-\mu/2)}\Big[\frac{\Gamma(N/2)}{\Gamma(N)}\Big]^{-1+\mu/N}.$$

Set

$$D(u) = \int_{\mathbb{R}^3}\big(I * |u|^p\big)|u|^p\mathrm{d}x = \frac{\mu\Gamma((3-\alpha)/2)}{\Gamma(\alpha/2)\pi^{\frac{3}{2}}2^\alpha}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^p|u(y)|^p}{|x-y|^{3-\alpha}}\mathrm{d}x\mathrm{d}y.$$

Then from Lemma 2.3, we have the estimate of $D(u)$ as follows:

$$|D(u)| \leq C(\alpha)\bigg(\int_{\mathbb{R}^3}|u|^{\frac{6p}{3+\alpha}}\mathrm{d}x\bigg)^{\frac{3+\alpha}{3}} = C(\alpha)|u|^{2p}_{\frac{6p}{3+\alpha}}, \tag{2.5}$$

where $\frac{6p}{3+\alpha} \in (2,6)$ since $\alpha \in (0,3)$ and $p \in [2,3)$. Since $E_\varepsilon \hookrightarrow H^1(\mathbb{R}^3)$ and $H^1(\mathbb{R}^3) \hookrightarrow L^r(\mathbb{R}^3)$, $r \in [2,6]$ are continuous, then $D(u)$ is well defined in E_ε. Furthermore, similar to the proof of Lemma 2.3 in [36], we can get that $D(u) \in C^1(E_\varepsilon, \mathbb{R})$.

Then we have the following Brézis-Lieb type Lemma for the nonlocal term $D(u)$.

Lemma 2.4. *Let $\alpha \in (0,3)$, $p \in [2, 3+\alpha)$, if $\{u_n\}$ is a bounded sequence such that $u_n \to u$ almost everywhere in $\mathbb{R}^3$ as $n \to \infty$, then the following hold.*
(i) $D(u_n - u) = D(u_n) - D(u) + o(1)$;
(ii) $D'(u_n - u) = D'(u_n) - D'(u) + o(1)$.

Proof. The proof is analogous to Lemma 3.3 in [16], we omit it here. □

As we shall see, it is important to compare c_ε with the minimax level of the following limit problem:

$$\begin{cases} -\big(a + b\displaystyle\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\big)\Delta u + V_\infty u + \mu\phi|u|^{p-2}u & \\ = \lambda K_\infty |u|^{q-2}u + Q_\infty |u|^4 u, & \text{in } \mathbb{R}^3, \\ (-\Delta)^{\frac{\alpha}{2}}\phi = \mu|u|^p, \ u > 0, & \text{in } \mathbb{R}^3. \end{cases} \tag{2.6}$$

The corresponding energy functional associated with problem (2.6) is defined by

$$\begin{aligned} I_\infty(u) = & \frac{1}{2}\int_{\mathbb{R}^3} \big(a|\nabla u|^2 + V_\infty u^2\big)\mathrm{d}x + \frac{b}{4}\big(\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\big)^2 \\ & + \frac{\mu}{2p}\int_{\mathbb{R}^3} (I * |u|^p)|u|^p \mathrm{d}x - \frac{\lambda}{q}\int_{\mathbb{R}^3} K_\infty |u|^q \mathrm{d}x \\ & - \frac{1}{6}\int_{\mathbb{R}^3} Q_\infty |u|^6 \mathrm{d}x. \end{aligned} \tag{2.7}$$

Define

$$\mathcal{N}_\infty := \{u \in H^1(\mathbb{R}^3) \setminus \{0\} : \langle I'_\infty(u), u\rangle = 0\} \quad \text{and} \quad c_\infty := \inf_{u \in \mathcal{N}_\infty} I_\infty(u).$$

In fact, it is obvious that c_∞ and $\mathcal{N}_\infty$ have some similar properties to those of c_ε and $\mathcal{N}_\varepsilon$.

3. Existence of positive ground state

In this section, we are devoted to showing that c_ε is achieved and the minimizer is a positive ground state solution to problem (1.1). In the following, we present some lemmas, which are useful to prove our result. To provide a precise description for the (PS) condition of I_ε, we shall apply the well known concentration-compactness principle of Lions.

Lemma 3.1 (see [22,23])**.** *Let $r > 0$ and $2 \le q < 2^*$. If $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$ and*

$$\lim_{n\to\infty} \sup_{y \in \mathbb{R}^N} \int_{B_r(y)} |u_n|^q = 0.$$

then $u_n \to 0$ in $L^s(\mathbb{R}^N)$ for $2 < s < 2^*$, where $2^* = (N-2)/2$, $N \ge 3$ and $2^* = \infty$, $N = 1, 2$.

Lemma 3.2 (see [22,23]). *Let $\{\rho_n\}$ be a sequence of nonnegative functions on $\mathbb{R}^N$ satisfying $\int_{\mathbb{R}^N}\rho_n(x)\mathrm{d}x = l$, where $l > 0$ is fixed. There exists a subsequence $\{\rho_n\}$ such that one of the following three possibilities holds:*
(i) (Compactness) there exists $y_n \in \mathbb{R}^N$ such that for any $\varepsilon > 0$, there exists $R > 0$ such that

$$\int_{B_R(y_n)} \rho_n(x)\mathrm{d}x \geq l - \varepsilon, \quad n = 1, 2, ...$$

(ii) (Vanishing) for all $R > 0$, there holds

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} \rho_n(x)\mathrm{d}x = 0.$$

(iii) (Dichotomy) there exists $\theta \in (0, l)$ and $\{y_n\} \subset \mathbb{R}^N$ such that for every $\varepsilon > 0$, there exists $n_0 \geq 1$, for all $r \geq n_0$ and $r' \geq r$, there holds

$$\limsup_{n\to+\infty} \left(\left|\theta - \int_{B_r(y_n)} \rho_n(x)\mathrm{d}x\right| + \left|(l-\theta) - \int_{\mathbb{R}^N\setminus B_{r'}(y_n)} \rho_n(x)\mathrm{d}x\right| \right) < \varepsilon.$$

Lemma 3.3. *Suppose that conditions (V_1) and (H_1)–(H_4) hold. If $c_\varepsilon < \min\{c_\infty, \Lambda^*\}$, then I_ε satisfies the (PS) condition for c_ε, where $\Lambda^* = \frac{abS^3}{4|Q|_\infty} + \frac{\left(b^2S^4+4aS|Q|_\infty\right)^{\frac{3}{2}}}{24|Q|_\infty^2} + \frac{b^3S^6}{24|Q|_\infty^2}$.*

Proof. Let $\{u_n\} \subset E_\varepsilon$ be a $(PS)_{c_\varepsilon}$ sequence with $c_\varepsilon < \min\{c_\infty, \Lambda^*\}$, i.e.

$$I_\varepsilon(u_n) \to c_\varepsilon, \qquad I'_\varepsilon(u_n) \to 0, \qquad \text{as } n \to \infty. \tag{3.1}$$

Thus for n sufficiently large, we obtain

$$\begin{aligned}
c_\varepsilon + 1 + \|u_n\|_\varepsilon &\geq I_\varepsilon(u_n) - \frac{1}{2p}\langle I'_\varepsilon(u_n), u_n\rangle \\
&= (\frac{1}{2} - \frac{1}{2p})\|u_n\|_\varepsilon^2 + (\frac{1}{4} - \frac{1}{2p})b\bigg(\int_{\mathbb{R}^3} |\nabla u_n|^2\mathrm{d}x\bigg)^2 \\
&\quad + (\frac{1}{2p} - \frac{1}{q})\lambda\int_{\mathbb{R}^3} K(\varepsilon x)|u_n|^q\mathrm{d}x + (\frac{1}{2p} - \frac{1}{6})\int_{\mathbb{R}^3} Q(\varepsilon x)|u_n|^6\mathrm{d}x \\
&\geq (\frac{1}{2} - \frac{1}{2p})\|u_n\|_\varepsilon^2,
\end{aligned}$$

which implies that $\{u_n\}$ is bounded in E_ε. Set $A = \lim\limits_{n\to\infty}\int_{\mathbb{R}^3} |\nabla u_n|^2\mathrm{d}x$ and

$$\begin{aligned}
\rho_n(x) = &\frac{2(q-2)a + (q-4)Ab}{4q}|\nabla u_n|^2 + \frac{(q-2)V(\varepsilon x)}{2q}|u_n|^2 \\
&+ \frac{(q-2p)}{2pq}\mu(I * |u_n|^p)|u_n|^p + \frac{(6-q)Q(\varepsilon x)}{6q}|u_n|^6 \in L^1(\mathbb{R}^3),
\end{aligned}$$

then ρ_n is bounded in $L^1(\mathbb{R}^3)$. Hence, by choosing a subsequence, we can assume that

$$\Psi(u_n) := |\rho_n|_1 \to l, \quad \text{as } n \to \infty.$$

Thus we derive $l > 0$, otherwise, $I_\varepsilon(u_n) \to 0$ as $n \to \infty$, a contradiction. Next we apply Lemma 3.2 to get the compactness of $\{\rho_n\}$.

If $\{\rho_n\}$ vanishes, then there exists $R>0$ such that

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^3}\int_{B_R(y)}|u_n|^2\mathrm{d}x=0.$$

It follows from Lemma 3.1 that $u_n\to 0$ in $L^s(\mathbb{R}^3),\ s\in(2,6)$. It follows from (2.5) that

$$\int_{\mathbb{R}^3}(I*|u_n|^p)|u_n|^p\mathrm{d}x\to 0,\quad \text{as } n\to\infty.$$

From (H_1) and $4\le q<6$, we derive

$$\int_{\mathbb{R}^3}K(\varepsilon x)|u_n|^q\mathrm{d}x\le |K|_\infty\int_{\mathbb{R}^3}|u_n|^q\mathrm{d}x\to 0,\quad \text{as } n\to\infty.$$

Therefore

$$I_\varepsilon(u_n)=\frac{1}{2}\|u_n\|_\varepsilon^2+\frac{b}{4}\big(\int_{\mathbb{R}^3}|\nabla u_n|^2\mathrm{d}x\big)^2-\frac{1}{6}\int_{\mathbb{R}^3}Q(\varepsilon x)|u_n|^6\mathrm{d}x+o(1),$$

and

$$\langle I'_\varepsilon(u_n),u_n\rangle=\|u_n\|_\varepsilon^2+b\big(\int_{\mathbb{R}^3}|\nabla u_n|^2\mathrm{d}x\big)^2-\int_{\mathbb{R}^3}Q(\varepsilon x)|u_n|^6\mathrm{d}x+o(1).$$

Hence, we may assume that there exist $l_i\ge 0 (i=1,2,3)$ such that

$$\|u_n\|_\varepsilon\to l_1,\quad b\big(\int_{\mathbb{R}^3}|\nabla u_n|^2\mathrm{d}x\big)^2\to l_2,\quad \int_{\mathbb{R}^3}Q(\varepsilon x)|u_n|^6\mathrm{d}x\to l_3,\quad \text{as } n\to\infty,$$

thus $l_3=l_1+l_2$. Moreover, it is obvious that $l_1>0$ and then $l_2,\ l_3>0$. It follows from (H_2) and the Sobolev inequality that

$$a^3\int_{\mathbb{R}^3}Q(\varepsilon x)|u_n|^6\mathrm{d}x\le a^3|Q|_\infty\left(S^{-1}\int_{\mathbb{R}^3}|\nabla u_n|^2\mathrm{d}x\right)^3\le S^{-3}|Q|_\infty\|u_n\|_\varepsilon^6,\tag{3.2}$$

and

$$b\left(\int_{\mathbb{R}^3}Q(\varepsilon x)|u_n|^6\mathrm{d}x\right)^{\frac{2}{3}}\le b|Q|_\infty^{\frac{2}{3}}\left(S^{-1}\int_{\mathbb{R}^3}|\nabla u_n|^2\mathrm{d}x\right)^2\le b|Q|_\infty^{\frac{2}{3}}S^{-2}\left(\int_{\mathbb{R}^3}|\nabla u_n|^2\mathrm{d}x\right)^2.\tag{3.3}$$

In view of (3.2), (3.3) and $l_3=l_1+l_2$, we obtain

$$l_1\ge aS|Q|_\infty^{\frac{-1}{3}}(l_1+l_2)^{\frac{1}{3}}\quad\text{and } l_2\ge bS^2|Q|_\infty^{\frac{-2}{3}}(l_1+l_2)^{\frac{2}{3}}.$$

It follows from (3.1) and Lemma 3.1 in [24] that

$$
\begin{aligned}
c_\varepsilon + o(1) &= I_\varepsilon(u_n) - \frac{1}{6}\langle I'_\varepsilon(u_n), u_n\rangle \\
&= \frac{1}{3}\|u_n\|_\varepsilon^2 + \frac{b}{12}\left(\int_{\mathbb{R}^3} |\nabla u_n|^2 \mathrm{d}x\right)^2 \\
&= \frac{l_1}{3} + \frac{l_2}{12} \\
&\geq \frac{1}{3}\frac{abS^3 + a\sqrt{b^2S^6 + 4a|Q|_\infty S^3}}{2|Q|_\infty} \\
&\quad + \frac{1}{12}\left(\frac{b^2S^3\sqrt{b^2S^6 + 4|Q|_\infty aS^3} + b^3S^6 + 2ab|Q|_\infty S^3}{2|Q|_\infty^2}\right) \\
&= \frac{ab}{4|Q|_\infty}S^3 + \frac{\left(b^2S^4 + 4aS|Q|_\infty\right)^{\frac{3}{2}}}{24|Q|_\infty^2} + \frac{b^3S^6}{24|Q|_\infty^2} := \Lambda^*,
\end{aligned}
\tag{3.4}
$$

which is a contradiction with the definition of c_ε. Thus, vanishing does not occur.

Next, we prove that the dichotomy does not occur. Argue by contradiction that there exist $\theta \in (0, l)$ and $\{y_n\} \subset \mathbb{R}^3$ such that for every $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that for any $r > R_\varepsilon$ and $r' > R_\varepsilon$, there holds

$$
\liminf_{n\to\infty} \int_{B_{r_n}(y_n)} \rho_n(x)\mathrm{d}x \geq \theta - \varepsilon, \quad \liminf_{n\to\infty} \int_{B^c_{r'_n}(y_n)} \rho_n(x)\mathrm{d}x \geq (l-\theta) - \varepsilon. \tag{3.5}
$$

Choose $\varepsilon_n \to 0,\ r_n \to \infty$ and $r'_n = 4r_n$. Let $\xi \in C(\mathbb{R}^+, [0,1])$ be a cut-off function such that $\xi(s) = 0$ for $s \leq 1$ or $s \geq 4$, $\xi(s) = 1$ for $2 \leq s \leq 3$ and $|\xi'(s)| \leq 2$. Take $\xi_n = \xi(|x - y_n|/r_n)$, then from (3.5) and $\langle I'_\varepsilon(u_n), \xi_n u_n\rangle = o(1)$, we obtain

$$
\begin{aligned}
&\int_{B_{3r_n}(y_n)\backslash B_{2r_n}(y_n)} \left(a|\nabla u_n|^2 + A|\nabla u_n|^2 + V(\varepsilon x)|u_n|^2 + (I * |u_n|^p)|u_n|^p\right)\mathrm{d}x \\
&= \int_{B_{3r_n}(y_n)\backslash B_{2r_n}(y_n)} \left(K(\varepsilon x)|u_n|^q + Q(\varepsilon x)|u_n|^6\right)\mathrm{d}x + o(1) = o(1).
\end{aligned}
\tag{3.6}
$$

Take another cut-off function $\eta : \mathbb{R}^+ \to [0,1]$ be a cut-off function satisfying $\eta(s) \equiv 1$ for $s \leq 2$, $\eta(s) \equiv 0$ for $s \geq 3$ and $|\eta'(s)| \leq 2$. Define

$$
v_n(x) := \eta\left(\frac{x - y_n}{r_n}\right)u_n(x), \quad \omega_n(x) := \left(1 - \eta(\frac{x - y_n}{r_n})\right)u_n(x).
$$

Then

$$
\liminf_{n\to\infty} \Psi(v_n) \geq \theta \quad \text{and} \quad \liminf_{n\to\infty} \Psi(\omega_n) \geq l - \theta. \tag{3.7}
$$

Thus from (3.6), we derive

$$
l = \lim_{n\to\infty} \Psi(u_n) \geq \liminf_{n\to\infty} \Psi(v_n) + \liminf_{n\to\infty} \Psi(\omega_n) \geq l.
$$

Hence

$$
\lim_{n\to\infty} \Psi(v_n) = \theta, \lim_{n\to\infty} \Psi(\omega_n) = l - \theta. \tag{3.8}
$$

It follows from (3.1) and (3.6) that

$$o(1)=\langle I_\varepsilon'(u_n),u_n\rangle\geq\langle I_\varepsilon'(v_n),v_n\rangle+\langle I_\varepsilon'(\omega_n),\omega_n\rangle+o(1). \tag{3.9}$$

Next we will study the problem in two cases.

Case1. Up to a subsequence, we may assume that $\langle I_\varepsilon'(v_n),v_n\rangle\leq 0$ or $\langle I_\varepsilon'(\omega_n),\omega_n\rangle\leq 0$. Without loss of generality, we assume that $\langle I_\varepsilon'(v_n),v_n\rangle\leq 0$, then

$$\begin{aligned}&a\int_{\mathbb{R}^3}|\nabla v_n|^2\mathrm{d}x+\int_{\mathbb{R}^3}V(\varepsilon x)v_n^2\mathrm{d}x+b\Big(\int_{\mathbb{R}^3}|\nabla v_n|^2\mathrm{d}x\Big)^2\\&+\mu\int_{\mathbb{R}^3}(I*|v_n|^p)|v_n|^p\mathrm{d}x-\lambda\int_{\mathbb{R}^3}K(\varepsilon x)|v_n|^q\mathrm{d}x\\&-\int_{\mathbb{R}^3}Q(\varepsilon x)|v_n|^6\mathrm{d}x\leq 0.\end{aligned} \tag{3.10}$$

From Lemma 2.1, we know that for each $n\in\mathbb{N}$, there exists $t_n>0$ such that $t_nv_n\in\mathcal{N}_\varepsilon$ and $\langle I_\varepsilon'(t_nv_n),(t_nv_n)\rangle=0$, i.e.

$$\begin{aligned}&at_n^2\int_{\mathbb{R}^3}|\nabla v_n|^2\mathrm{d}x+t_n^2\int_{\mathbb{R}^3}V(\varepsilon x)v_n^2\mathrm{d}x+bt_n^4\Big(\int_{\mathbb{R}^3}|\nabla v_n|^2\mathrm{d}x\Big)^2\\&-\lambda t_n^q\int_{\mathbb{R}^3}K(\varepsilon x)|v_n|^q\mathrm{d}x+\mu t_n^{2p}\int_{\mathbb{R}^3}(I*|v_n|^p)|v_n|^p\mathrm{d}x\\&-t_n^6\int_{\mathbb{R}^3}Q(\varepsilon x)|v_n|^6\mathrm{d}x=0.\end{aligned} \tag{3.11}$$

It follows from (3.10) and (3.11) that

$$\begin{aligned}&(t_n^2-t_n^{2p})\|u_n\|_\varepsilon^2+b(t_n^4-t_n^{2p})\Big(\int_{\mathbb{R}^3}|\nabla v_n|^2\mathrm{d}x\Big)^2+\lambda(t_n^{2p}-t_n^q)\int_{\mathbb{R}^3}K(\varepsilon x)|v_n|^q\mathrm{d}x\\&+(t_n^{2p}-t_n^6)\int_{\mathbb{R}^3}Q(\varepsilon x)|v_n|^6\mathrm{d}x\geq 0,\end{aligned}$$

showing that $t_n\leq 1$ since $4\leq 2p\leq q<6$. Therefore, by (2.2), (2.3) and (3.8), for n large enough, we have

$$\begin{aligned}c_\varepsilon&\leq I_\varepsilon(t_nv_n)-\frac{1}{q}\langle I_\varepsilon'(t_nv_n),t_nv_n\rangle\\&=(\frac12-\frac1q)t_n^2\|v_n\|_\varepsilon^2+(\frac14-\frac1q)t_n^4b\Big(\int_{\mathbb{R}^3}|\nabla v_n|^2\mathrm{d}x\Big)^2\\&\quad+(\frac{1}{2p}-\frac1q)t_n^{2p}\mu\int_{\mathbb{R}^3}(I*|v_n|^p)|v_n|^p\mathrm{d}x+(\frac1q-\frac16)t_n^6\int_{\mathbb{R}^3}Q(\varepsilon x)|v_n|^6\mathrm{d}x\\&\leq(\frac12-\frac1q)\|v_n\|_\varepsilon^2+(\frac14-\frac1q)Ab\int_{\mathbb{R}^3}|\nabla v_n|^2\mathrm{d}x\\&\quad+(\frac{1}{2p}-\frac1q)\mu\int_{\mathbb{R}^3}(I*|v_n|^p)|v_n|^p\mathrm{d}x+(\frac1q-\frac16)\int_{\mathbb{R}^3}Q(\varepsilon x)|v_n|^6\mathrm{d}x\\&=\Psi(v_n)\to\theta<c_\varepsilon,\end{aligned}$$

which is a contradiction.

Case 2. Up to a subsequence, we may assume that $\langle I_\varepsilon'(v_n), v_n\rangle > 0$ and $\langle I_\varepsilon'(\omega_n), \omega_n\rangle > 0$. It follows from (3.9) that $\langle I_\varepsilon'(v_n), v_n\rangle \to 0$ and $\langle I_\varepsilon'(\omega_n), \omega_n\rangle \to 0$ as $n \to \infty$. It follows from (3.6) that

$$I_\varepsilon(u_n) \geq I_\varepsilon(v_n) + I_\varepsilon(\omega_n) + o(1). \tag{3.12}$$

If $\{y_n\} \subset \mathbb{R}^3$ is bounded, we can get a contradiction by comparing $I_\varepsilon(\omega_n)$ and c_∞. Indeed

$$\int_{\mathbb{R}^3} (K(\varepsilon x) - K_\infty)|\omega_n|^q \mathrm{d}x \leq \sup_{|x-y_n|\geq r_n} |K(\varepsilon x) - K_\infty||\omega_n|_q^q \to 0, \quad \text{as } n \to \infty.$$

Similarly

$$\int_{\mathbb{R}^3} (Q(\varepsilon x) - Q_\infty)|\omega_n|^6 \mathrm{d}x \to 0, \quad \text{as } n \to \infty.$$

Furthermore, it is easy to check that

$$\int_{\mathbb{R}^3} (V(\varepsilon x) - V_\infty)|\omega_n|^2 \mathrm{d}x \geq o(1).$$

Then

$$I_\varepsilon(\omega_n) \geq I_\infty(\omega_n) + o(1) \quad \text{and } o(1) = \langle I_\varepsilon'(\omega_n), \omega_n\rangle \geq \langle I_\infty'(\omega_n), \omega_n\rangle + o(1). \tag{3.13}$$

Hence, similar to the arguments as in Case 1 and Lemma 2.1, there are two positive sequences $\{t_n\}$ and $\{s_n\}$ satisfying $t_n \leq 1$ and $s_n \to 1$ as $n \to \infty$, respectively, such that $t_n\omega_n \in \mathcal{N}_\infty$ and $s_n\omega_n \in \mathcal{N}_\varepsilon$. Then

$$\begin{aligned}
I_\varepsilon(\omega_n) &= I_\varepsilon(\omega_n) - \frac{1}{2p}\langle I_\varepsilon'(\omega_n), \omega_n\rangle + o(1) \\
&= (\frac{1}{2} - \frac{1}{2p})\|\omega_n\|_\varepsilon^2 + (\frac{1}{4} - \frac{1}{2p})b\Big(\int_{\mathbb{R}^3} |\nabla\omega_n|^2 \mathrm{d}x\Big)^2 \\
&\quad + (\frac{1}{2p} - \frac{1}{q})\lambda \int_{\mathbb{R}^3} K(\varepsilon x)|\omega_n|^q \mathrm{d}x + (\frac{1}{2p} - \frac{1}{6}) \int_{\mathbb{R}^3} Q(\varepsilon x)|\omega_n|^6 \mathrm{d}x \\
&\geq I_\infty(\omega_n) - \frac{1}{2p}\langle I_\infty'(\omega_n), \omega_n\rangle + o(1) \\
&\geq I_\infty(t_n\omega_n) - \frac{1}{2p}\langle I_\infty'(t_n\omega_n), t_n\omega_n\rangle + o(1) \\
&= I_\infty(t_n\omega_n) + o(1) \geq c_\infty
\end{aligned}$$

and

$$I_\varepsilon(v_n) = I_\varepsilon(s_n v_n) + o(1) \geq c_\varepsilon + o(1).$$

Thus, it follows from (3.12) that $c_\varepsilon \geq c_\varepsilon + c_\infty$, which is a contradiction. If $\{y_n\} \subset \mathbb{R}^3$ is unbounded, in a similar way, we can obtain a contradiction. Then the dichotomy does not happen. Therefore, the sequence $\{\rho_n\}$ is compact, i.e., there exists $\{y_n\} \subset \mathbb{R}^3$ such that for any $\varepsilon > 0$, there exists $R > 0$ such that $\int_{B_R^c(y_n)} \rho_n(x)\mathrm{d}x < \varepsilon$, which implies that

$$\int_{B_R^c(y_n)} \big(V(\varepsilon x)u_n^2 + \lambda K(\varepsilon x)|u_n|^q + Q(\varepsilon x)|u_n|^6\big)\mathrm{d}x < \varepsilon,$$

i.e., the sequences $\{V(\varepsilon x)u_n^2+\lambda K(\varepsilon x)|u_n|^q+Q(\varepsilon x)|u_n|^6\}$ is also compact. Thus, $\{y_n\}$ must be bounded. Otherwise

$$\int_{\mathbb{R}^3}(V(\varepsilon x)-V_\infty)|u_n|^2\mathrm{d}x\geq o(1),$$
$$\int_{\mathbb{R}^3}(K(\varepsilon x)-K_\infty)|u_n|^q\mathrm{d}x=\int_{\mathbb{R}^3}(Q(\varepsilon x)-Q_\infty)|u_n|^6\mathrm{d}x=o(1),$$

and then $I_\varepsilon(u_n)\geq I_\infty(u_n)+o(1)$ and $\langle I'_\infty(u_n),u_n\rangle\leq\langle I'_\varepsilon(u_n),u_n\rangle=o(1)$. By similar arguments as in Case 1, there exists a sequence $\{t_n\}$ satisfying $t_n\leq 1$ such that $t_nu_n\in\mathcal{N}_\infty$. Therefore

$$\begin{aligned}c_\varepsilon&=I_\varepsilon(u_n)-\frac{1}{2p}\langle I'_\varepsilon(u_n),u_n\rangle+o(1)\\&\geq I_\infty(u_n)-\frac{1}{2p}\langle I'_\infty(u_n),u_n\rangle+o(1)\\&\geq I_\infty(t_nu_n)-\frac{1}{2p}\langle I'_\infty(t_nu_n),t_nu_n\rangle+o(1)\\&=I_\infty(t_nu_n)+o(1)\geq c_\infty,\end{aligned}$$

which is a contradiction. Let $u_n\rightharpoonup u$ in E_ε. Since $\{y_n\}$ is bounded, then it is easy to see that $u_n\to u$ in $L^q(\mathbb{R}^3)$. Set $\chi_n=u_n-u$, it follows from Lemma 2.4 and Brézis-Lieb Lemma [33] that

$$\begin{aligned}I_\varepsilon(u_n)-I_\varepsilon(u)=&\frac{b}{4}\Big[\big(\int_{\mathbb{R}^3}|\nabla\chi_n|^2\mathrm{d}x\big)^2+2\int_{\mathbb{R}^3}|\nabla\chi_n|^2\mathrm{d}x\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x\Big]\\&+\frac{1}{2}\|\chi_n\|_\varepsilon^2-\frac{1}{6}\int_{\mathbb{R}^3}Q(\varepsilon x)|\chi_n|^6\mathrm{d}x+o(1)\end{aligned}\tag{3.14}$$

and

$$\begin{aligned}o(1)&=\langle I'_\varepsilon(u_n),u_n\rangle-\langle I'_\varepsilon(u),u\rangle\\&=\|\chi_n\|_\varepsilon^2+b\Big[\big(\int_{\mathbb{R}^3}|\nabla\chi_n|^2\mathrm{d}x\big)^2+2\int_{\mathbb{R}^3}|\nabla\chi_n|^2\mathrm{d}x\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x\Big]\\&\quad-\int_{\mathbb{R}^3}Q(\varepsilon x)|\chi_n|^6\mathrm{d}x+o(1).\end{aligned}\tag{3.15}$$

We may assume that there exist $a_i\geq 0(i=1,2,3)$ such that

$$\|\chi_n\|_\varepsilon^2\to a_1,\ b\Big[\big(\int_{\mathbb{R}^3}|\nabla\chi_n|^2\mathrm{d}x\big)^2+2\int_{\mathbb{R}^3}|\nabla\chi_n|^2\mathrm{d}x\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x\Big]\to a_2,$$

and

$$\int_{\mathbb{R}^3}Q(\varepsilon x)|\chi_n|^6\mathrm{d}x\to a_3$$

as $n\to\infty$. Thus, $a_3=a_1+a_2$. If $a_1>0$, then $a_2,\ a_3>0$. In view of $I_\varepsilon(u)\geq 0$, (3.14) and (3.15), we obtain

$$\begin{aligned}c_\varepsilon&\geq\frac{1}{3}\|\chi_n\|_\varepsilon^2+\frac{b}{12}\Big[\big(\int_{\mathbb{R}^3}|\nabla\chi_n|^2\mathrm{d}x\big)^2+2\int_{\mathbb{R}^3}|\nabla\chi_n|^2\mathrm{d}x\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x\Big]+o(1)\\&=\frac{a_1}{3}+\frac{a_2}{12}+o(1).\end{aligned}$$

Similar to the proof of (3.4), we have $c_\varepsilon \geq \Lambda^*$, which is a contradiction. Hence, $a_1 = 0$, i.e., $\|\chi_n\|_\varepsilon \to 0$, that is, $u_n \to u$ in E_ε and $I_\varepsilon(u) = c_\varepsilon$. □

We introduce the function $u_\varepsilon \in D^{1,2}(\mathbb{R}^3)$ defined by

$$u_\varepsilon := C_1 \frac{\varepsilon^{\frac{1}{4}}}{(\varepsilon + |x-x_0|^2)^{\frac{1}{2}}},$$

where C_1 is a normalizing constant and x_0 is chosen such that $K(x_0) = \max_{x\in\mathbb{R}^3} K(x)$ as in (H_4). Define $v_\varepsilon(x) = \psi(x-x_0)u_\varepsilon$, where $\psi \in C_0^\infty(B_{2r}(0))$ such that $\psi(x) = 1$ on $B_r(0)$ and $0 \leq \psi(x) \leq 1$. For $\varepsilon > 0$ small enough, from [33], we have

$$\int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2 \mathrm{d}x = K_1 + O(\varepsilon^{\frac{1}{2}}), \qquad \int_{\mathbb{R}^3} |v_\varepsilon|^6 \mathrm{d}x = K_2 + O(\varepsilon^{\frac{3}{2}}), \tag{3.16}$$

$$\int_{\mathbb{R}^3} |v_\varepsilon|^s \mathrm{d}x = \begin{cases} O(\varepsilon^{\frac{s}{4}}), & s \in [2,3), \\ O(\varepsilon^{\frac{3}{4}}|\ln\varepsilon|), & s = 3, \\ O(\varepsilon^{\frac{6-s}{4}}), & s \in (3,6), \end{cases} \tag{3.17}$$

where $K_1,\ K_2$ are positive constants and $S = \frac{K_1}{K_2^{\frac{1}{3}}}$. It follows from (3.16) and (3.17) that

$$\frac{\int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2 \mathrm{d}x}{\int_{\mathbb{R}^3} |v_\varepsilon|^6 \mathrm{d}x} = S + O(\varepsilon^{\frac{1}{2}}). \tag{3.18}$$

Lemma 3.4. *Suppose that the conditions* (V_1) *and* $(H_1)-(H_4)$ *hold. If* $2 \leq p < \frac{3+\alpha}{2}$ *and* $4 < q < 6$*, then* $c_\varepsilon < \Lambda^* = \frac{abS^3}{4|Q|_\infty} + \frac{\left(b^2S^4 + 4aS|Q|_\infty\right)^{\frac{3}{2}}}{24|Q|_\infty^2} + \frac{b^3S^6}{24|Q|_\infty^2}$*. Furthermore, if* $\frac{3+\alpha}{2} \leq p < 3$ *or* $q = 4$*, the above inequality still holds provided that* λ *is large enough.*

Proof. It follows from Lemma 2.1 and (2.4) that $c_\varepsilon \leq \max_{t\geq 0} I_\varepsilon(tv_\varepsilon)$. Define

$$g(t) := \frac{t^2}{2}\|v_\varepsilon\|_\varepsilon^2 + \frac{bt^4}{4}\left(\int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2 \mathrm{d}x\right)^2 - \frac{t^6}{6}\int_{\mathbb{R}^3} Q(x^*)|v_\varepsilon|^6 \mathrm{d}x.$$

Then

$$\begin{aligned} I_\varepsilon(tv_\varepsilon) = g(t) &+ \frac{\mu t^{2p}}{2p}\int_{\mathbb{R}^3} (I * |v_\varepsilon|^p)|v_\varepsilon|^p \mathrm{d}x - \frac{\lambda t^q}{q}\int_{\mathbb{R}^3} K(\varepsilon x)|v_\varepsilon|^q \mathrm{d}x \\ &+ \frac{t^6}{6}\int_{\mathbb{R}^3} \big(Q(x^*) - Q(\varepsilon x)\big)|v_\varepsilon|^6 \mathrm{d}x. \end{aligned} \tag{3.19}$$

By a direct computation, we can obtain

$$g(t) \leq \frac{abS^3}{4|Q|_\infty} + \frac{\left(b^2S^4 + 4aS|Q|_\infty\right)^{\frac{3}{2}}}{24|Q|_\infty^2} + \frac{b^3S^6}{24|Q|_\infty^2} + O(\varepsilon^{\frac{1}{2}}).$$

In view of (H_3) and the arguments in [15], we obtain

$$\int_{\mathbb{R}^3} \big(Q(x^*) - Q(\varepsilon x)\big)|v_\varepsilon|^6 \mathrm{d}x \leq C\varepsilon^{\frac{1}{2}}.$$

For $v_\varepsilon \in E_\varepsilon$, it follows from Lemma 2.1 that there exists $t_\varepsilon > 0$ such that $t_\varepsilon v_\varepsilon \in \mathcal{N}_\varepsilon$ and

$$\begin{aligned} c_\varepsilon \le I_\varepsilon(t_\varepsilon v_\varepsilon) \le & \frac{abS^3}{4|Q|_\infty} + \frac{\left(b^2S^4 + 4aS|Q|_\infty\right)^{\frac{3}{2}}}{24|Q|_\infty^2} + \frac{b^3S^6}{24|Q|_\infty^2} + O(\varepsilon^{\frac{1}{2}}) \\ & + C_1\varepsilon^{\frac{1}{2}} + C_2\left(\int_{\mathbb{R}^3} |v_\varepsilon|^{\frac{6p}{3+\alpha}} \mathrm{d}x\right)^{\frac{3+\alpha}{3}} - C_3\lambda \int_{\mathbb{R}^3} |v_\varepsilon|^q \mathrm{d}x, \end{aligned} \tag{3.20}$$

where C_i $(i = 1,\ 2,\ 3)$ are positive constants independent of ε. Thus, in order to complete the proof, it suffices to prove that

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{\frac{1}{2}}}\left[\left(\int_{\mathbb{R}^3} |v_\varepsilon|^{\frac{6p}{3+\alpha}} \mathrm{d}x\right)^{\frac{3+\alpha}{3}} - \lambda \int_{\mathbb{R}^3} |v_\varepsilon|^q \mathrm{d}x\right] = -\infty. \tag{3.21}$$

Indeed, since $p \in [2, 3)$ and $q \in [4, 6)$, then from (3.17), we have the following estimations as $\varepsilon \to 0$:

$$|v_\varepsilon|^{2p}_{\frac{6p}{3+\alpha}} - \lambda \int_{\mathbb{R}^3} |v_\varepsilon|^q \mathrm{d}x = \begin{cases} -C_4\lambda\varepsilon^{\frac{6-q}{4}} + C_5\varepsilon, & \text{if} \quad 2 \le p < \dfrac{3+\alpha}{2}, \\ -C_4\lambda\varepsilon^{\frac{6-q}{4}} + C_5\varepsilon^{\frac{p}{2}}|\ln\varepsilon|, & \text{if} \quad p = \frac{3+\alpha}{2}, \\ -C_4\lambda\varepsilon^{\frac{6-q}{4}} + C_5\varepsilon^{\frac{3+\alpha-p}{2}}, & \text{if} \quad \frac{3+\alpha}{2} < p < 3, \end{cases} \tag{3.22}$$

where C_i $(i = 4,\ 5)$ are positive constants independent of ε. If $2 \le p < \frac{3+\alpha}{2}$ and $q \in (4, 6)$, (3.21) follows from (3.22) for any $\lambda > 0$. If $\frac{3+\alpha}{2} \le p < 3$ or $q = 4$, in the above inequality, one can sress the parameter by choosing $\lambda = \varepsilon^{-\nu}$, $\nu > 0$, to obtain (3.21). The proof is complete. □

Lemma 3.5. *Suppose that conditions* (V_1) *and* $(H_1)-(H_4)$ *hold. If* $q \in (4, 6)$*, then* c_∞ *is achieved in* $H^1(\mathbb{R}^3)$*. If* $q = 4$*, then* c_∞ *is achieved in* $H^1(\mathbb{R}^3)$ *provided that* λ *is large enough.*

Proof. Similar to Lemma 2.2, we can check that I_∞ possesses the mountain pass geometry. Define

$$\Theta = \left\{\gamma \in C^1([0,1], H^1(\mathbb{R}^3)) : I_\infty(\gamma(0)) = 0, I_\infty(\gamma(1)) < 0\right\},$$

and

$$c_\infty^* = \inf_{\gamma\in\Theta} \max_{t\in[0,1]} I_\infty(\gamma(t)), \quad c_\infty^{**} = \inf_{u\in H^1(\mathbb{R}^3)\setminus\{0\}} \max_{t\ge 0} I_\infty(tu).$$

Similar to the proof of (2.4), we have

$$c_\infty = c_\infty^* = c_\infty^{**}. \tag{3.23}$$

Consequently, it follows from the Ekeland's variational principle [12] that there exists a sequence $\{u_n\} \subset H^1(\mathbb{R}^3)$ such that

$$I_\infty(u_n) \to c_\infty, \quad I'_\infty(u_n) \to 0, \quad \text{as } n \to \infty. \tag{3.24}$$

By a standard argument, we can prove that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Define $\omega_n(x) := u_n(x + y_n)$, where $x_n \in \mathbb{R}^3$. We claim that there exists $x_n \in \mathbb{R}^3$ such that

$\omega_n \rightharpoonup \omega \neq 0$ in $H^1(\mathbb{R}^3)$. Argue by contradiction that for any $y_n \in \mathbb{R}^3$, $\omega_n \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$. In this case, we claim that for any $r \in [2,6)$, there holds

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_1(y)} |u_n|^r \mathrm{d}x = 0. \tag{3.25}$$

If (3.24) is not true, then there exists $\sigma > 0$ and $r \in [2,6)$ such that

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_1(y)} |u_n|^r \mathrm{d}x > \sigma > 0. \tag{3.26}$$

Hence, it follows from (3.26) that $\lim\limits_{n\to\infty} \int_{B_1(y_n)} |u_n|^r \mathrm{d}x \geq \frac{\sigma}{2} > 0$. Then we obtain $\lim\limits_{n\to\infty} \int_{B_1(0)} |\omega_n|^r \mathrm{d}x \geq \frac{\sigma}{2} > 0$. Thus $\omega_n \rightharpoonup \omega \neq 0$, which is a contradiction. Therefore, (3.25) holds and it follows from Lemma 3.1 that $u_n \to 0$ in $L^r(\mathbb{R}^3)$ for $2 < r < 6$. Thus by the same arguments as in the proof of Lemma 3.3, we derive $c_\infty \geq \Lambda^*$, a contradiction. Then the claim holds. In view of (3.24), by a standard argument, we can obtain that $I'_\infty(\omega) = 0$, which implies

$$\begin{aligned}
c_\infty &= \lim_{n\to\infty} \left[I_\infty(\omega_n) - \frac{1}{2p}\langle I'_\infty(\omega_n), \omega_n\rangle \right] \\
&= \lim_{n\to\infty} \Bigg[(\frac{1}{2} - \frac{1}{2p})\|\omega_n\|_\varepsilon^2 + (\frac{1}{4} - \frac{1}{2p}) b \left(\int_{\mathbb{R}^3} |\nabla \omega_n|^2 \mathrm{d}x \right)^2 \\
&\quad + (\frac{1}{2p} - \frac{1}{q})\lambda \int_{\mathbb{R}^3} K(\varepsilon x)|\omega_n|^q \mathrm{d}x + (\frac{1}{2p} - \frac{1}{6}) \int_{\mathbb{R}^3} Q(\varepsilon x)|\omega_n|^6 \mathrm{d}x \Bigg] \\
&\geq (\frac{1}{2} - \frac{1}{2p})\|\omega\|_\varepsilon^2 + (\frac{1}{4} - \frac{1}{2p}) b \left(\int_{\mathbb{R}^3} |\nabla \omega|^2 \mathrm{d}x \right)^2 \\
&\quad + (\frac{1}{2p} - \frac{1}{q})\lambda \int_{\mathbb{R}^3} K(\varepsilon x)|\omega|^q \mathrm{d}x + (\frac{1}{2p} - \frac{1}{6}) \int_{\mathbb{R}^3} Q(\varepsilon x)|\omega|^6 \mathrm{d}x \\
&= I_\infty(\omega).
\end{aligned}$$

On the other hand, it follows from $\omega \in \mathcal{N}_\infty$ that $I_\infty(\omega) \geq c_\infty$. Thus, from (3.24), we know that $I_\infty(\omega) = c_\infty$. The proof is complete. □

Lemma 3.6. *Under the conditions of Theorem 1.1, there exists $\varepsilon_0 > 0$ such that $c_\varepsilon < c_\infty$ for all $\varepsilon \in (0, \varepsilon_0)$.*

Proof. It follows from the condition (V_1) that there exists a fixed $\zeta \in \mathbb{R}$ such that $V_0 < \zeta < V_\infty$. Define

$$\begin{aligned}
I_\zeta(u) &= \frac{a}{2}\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x + \frac{\zeta}{2}\int_{\mathbb{R}^3} u^2 \mathrm{d}x + \frac{b}{4}\Big(\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\Big)^2 + \frac{\mu}{2p}\int_{\mathbb{R}^3} (I * |u|^p)|u|^p \mathrm{d}x \\
&\quad - \frac{\lambda}{q}\int_{\mathbb{R}^3} K_\infty |u|^q \mathrm{d}x - \frac{1}{6}\int_{\mathbb{R}^3} Q_\infty |u|^6 \mathrm{d}x
\end{aligned}$$

and

$$\mathcal{N}_\zeta = \big\{ u \in H^1(\mathbb{R}^3) \setminus \{0\} : \langle I'_\zeta(u), u\rangle = 0 \big\}, \quad c_\zeta = \inf_{u\in\mathcal{N}_\zeta} I_\zeta(u).$$

From Lemma 3.5, we know that there exists $\zeta_0 \in \mathcal{N}_\zeta$ such that $c_\zeta = I_\zeta(\zeta_0)$. For any given $r > 0$, let $\psi_r \in C_0^\infty(\mathbb{R}^3, [0,1])$ be such that $\psi_r \equiv 1$ for all $|x| < r$ and $\psi_r \equiv 0$

for all $|x| \geq 2r$. Set $u_r(x) = \psi_r(x)\zeta_0(x)$ and take $t_r > 0$ such that $\bar{u}_r = t_r u_r \in \mathcal{N}_\zeta$. We claim that there exists $r_0 > 0$ such that $I_\zeta(\bar{u}) < c_\infty$ at $\bar{u} = \bar{u}_{r_0}$. Therefore

$$c_\infty = \liminf_{n\to\infty} I_\zeta(t_r u_r) = I_\zeta(\zeta_0) = c_\zeta < c_\infty,$$

which is impossible. Thus our claim is true. Since supp$\bar{u}$ is compact, then there exists $\varepsilon_0 > 0$ such that $V(\varepsilon x) \leq \zeta$ for all $0 < \varepsilon < \varepsilon_0$ and $x \in \text{supp}\bar{u}$. Therefore

$$\max_{t\geq 0} I_\varepsilon(t\bar{u}) \leq \max_{t\geq 0} I_\zeta(t\bar{u}) = I_\zeta(\bar{u}) = c_\zeta < c_\infty, \quad \text{for all } \varepsilon \in (0, \varepsilon_0).$$

In view of (2.4), we know that $c_\varepsilon < c_\infty$ for all $0 < \varepsilon < \varepsilon_0$. The proof is complete. □

Proof of Theorem 1.1. It follows from Lemma 2.2, Lemma 3.3, Lemma 3.4 and Lemma 3.6 that I_ε has a nontrivial solution $u \in E_\varepsilon$. In view of (2.4), u is a ground state solution of problem (1.1). If we replace I_ε by the following functional:

$$\begin{aligned} I_\varepsilon^+(u) = \frac{1}{2}\|u\|_\varepsilon^2 + \frac{b}{4}\Big(\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\Big)^2 + \frac{\mu}{2p}\int_{\mathbb{R}^3} (I * |u|^p)|u|^p \mathrm{d}x - \frac{\lambda}{q}\int_{\mathbb{R}^3} K(\varepsilon x)|u^+|^q \mathrm{d}x \\ - \frac{1}{6}\int_{\mathbb{R}^3} Q(\varepsilon x)|u^+|^6 \mathrm{d}x, \end{aligned}$$

where $u^\pm = \max\{\pm u, 0\}$. Then we see that all the above calculations can be repeated word by word. So I_ε^+ has a nontrivial ground state critical point $u \in E_\varepsilon$. Hence

$$\begin{aligned} 0 = \langle I_\varepsilon'(u), u^-\rangle &= \|u^-\|_\varepsilon^2 + b\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x \int_{\mathbb{R}^3} |\nabla u^-|^2 \mathrm{d}x + \mu\int_{\mathbb{R}^3} (I * |u|^p)|u^-|^p \mathrm{d}x \\ &\geq \|u^-\|_\varepsilon^2, \end{aligned}$$

where $u^\pm = \max\{\pm u, 0\}$, $u \geq 0$. It follows from the maximum principle that $u > 0$, i.e., u is a positive solution of problem (1.1). Similar to the proof of Theorem 1.1 in [20], by using the Nash-Moser method together with some careful estimations, we can obtain that $u \in L^\infty(\mathbb{R}^3)$ and there is $s > 1$, $r_0 = r_0(t) > 0$ such that for every $r \geq r_0$,

$$|u|_{\infty(|x|\geq r)} < M|u|_{s(|x|\geq \frac{r}{2})} < +\infty,$$

where M is a positive constant independent of r. Furthermore, $\lim_{|x|\to\infty} u(x) = 0$ and $u \in C_{loc}^{1,\gamma}(\mathbb{R}^3)$ for some $\gamma \in (0,1)$. The proof is complete. □

4. Concentration of positive ground state

In this section, we are devoted to the concentration behavior of the positive ground state for problem (1.1). We need the following energy functional associated with problem (1.7)

$$\begin{aligned} I_0(u) = \frac{a}{2}\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3} V_0 u^2 \mathrm{d}x + \frac{b}{4}\Big(\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\Big)^2 \\ + \frac{\mu}{2p}\int_{\mathbb{R}^3} (I * |u|^p)|u|^p \mathrm{d}x - \frac{\lambda}{q}\int_{\mathbb{R}^3} K_0|u|^q \mathrm{d}x - \frac{1}{6}\int_{\mathbb{R}^3} Q_0|u|^6 \mathrm{d}x, \end{aligned} \tag{4.1}$$

and set

$$\mathcal{N}_0 = \{u \in H^1(\mathbb{R}^3) \setminus \{0\} : \langle I_0'(u), u\rangle = 0\}, \quad c_0 = \inf_{u\in\mathcal{N}_0} I_0(u).$$

In the following, we will introduce some properties for the energy functional I_0.

Lemma 4.1. *Let $\{u_n\} \subset \mathcal{N}_0$ be a sequence such that $I_0(u_n) \to c_0$ as $n \to \infty$, then either $\{u_n\}$ has a convergent subsequence in $H^1(\mathbb{R}^3)$ or there exists $\{y_n\} \subset \mathbb{R}^3$ such that the sequence $\omega_n(x) = u_n(x + y_n)$ converges strongly in $H^1(\mathbb{R}^3)$. In particular, there exists a minimizer of c_0.*

Proof. By a standard argument, we can obtain that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Then there exists $u_0 \in H^1(\mathbb{R}^3)$ such that $u_n \rightharpoonup u_0$ in $H^1(\mathbb{R}^3)$. Now we claim that there exist $R,\ \sigma > 0$ and a sequence $\{y_n\} \subset \mathbb{R}^3$ such that

$$\liminf_{n\to\infty} \int_{B_R(y_n)} |u_n|^2 \mathrm{d}x \geq \sigma > 0. \tag{4.2}$$

Argue by contradiction that for all $r > 0$, there holds

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_R(y)} |u_n|^2 \mathrm{d}x = 0.$$

Then it follows from Lemma 3.1 that $u_n \to 0$ in $L^r(\mathbb{R}^3)$ for all $2 < r < 6$. Hence, $\int_{\mathbb{R}^3} K_0|u|^q \mathrm{d}x \to 0$ as $n \to \infty$ and $\int_{\mathbb{R}^3}(I * |u_n|^p)|u_n|^p \mathrm{d}x \to 0,$ as $n \to \infty$ by (2.5). Furthermore, from $\langle I_0'(u_n), u_n\rangle = 0$, we obtain

$$a\int_{\mathbb{R}^3} |\nabla u_n|^2 \mathrm{d}x + \int_{\mathbb{R}^3} V_0 u_n^2 \mathrm{d}x + b\bigg(\int_{\mathbb{R}^3} |\nabla u_n|^2 \mathrm{d}x\bigg)^2 = \int_{\mathbb{R}^3} Q_0|u_n|^6 \mathrm{d}x + o(1). \tag{4.3}$$

Hence, we may assume that there exist $l_i \geq 0 (i = 1, 2, 3)$ such that

$$\int_{\mathbb{R}^3} \big(a|\nabla u_n|^2 + V_0 u_n^2\big)\mathrm{d}x \to l_1,\ b\Big(\int_{\mathbb{R}^3} |\nabla u_n|^2 \mathrm{d}x\Big)^2 \to l_2,\ \int_{\mathbb{R}^3} Q_0|u_n|^6 \mathrm{d}x \to l_3, \text{as } n \to \infty,$$

thus $l_3 = l_1 + l_2$. It is obvious that $l_1 > 0$ and then $l_2\ , l_3 > 0$. In view of (4.1), we have

$$\begin{aligned} m_0 + o(1) &= I_0(u_n) - \frac{1}{6}\langle I_0'(u_n), u_n\rangle \\ &= \frac{a}{3}\int_{\mathbb{R}^3} |\nabla u_n|^2 \mathrm{d}x + \frac{1}{3}\int_{\mathbb{R}^3} V_0 u_n|^2 \mathrm{d}x + \frac{b}{12}\bigg(\int_{\mathbb{R}^3} |\nabla u_n|^2 \mathrm{d}x\bigg)^2 + o(1) \\ &= \frac{1}{3} l_1 + \frac{1}{12} l_2. \end{aligned}$$

By the similar arguments as the proof of (3.4), we can obtain that $c_0 \geq \Lambda^*$. Since $c_\varepsilon < \Lambda^*$, then in order to prove (4.2), it suffices to prove that

$$c_\varepsilon \geq c_0 \ \text{ for all } \varepsilon \in (0, \varepsilon^*). \tag{4.4}$$

Argue by contradiction that there exists $\varepsilon_0 \in (0, \varepsilon^*)$ such that $c_{\varepsilon_0} < c_0$. It follows from the definition of c_0 that $\max_{t>0} I_0(tu_{\varepsilon_0}) \geq c_0$. Furthermore, from the definitions of $V_0,\ K_0$ and Q_0, we have

$$c_0 > \max_{t>0} I_{\varepsilon_0}(tu_{\varepsilon_0}) \geq \max_{t>0} I_0(tu_{\varepsilon_0}) \geq c_0,$$

which is a contradiction, then (4.2) holds. Moreover, in view of the Ekeland variational principle in [12], we may assume that $I_0(u_n) \to c_0$ and $I_0'(u_n) \to 0$ as $n \to \infty$. It follows from Lemma 2.1 that $\langle I_0'(u_0), u_0\rangle = 0$. If $u_0 \neq 0$, then from (4.1) and Fatou's Lemma, we obtain

$$\begin{aligned}
c_0 \le I_0(u_0) &= I_0(u_0) - \frac{1}{q}\langle I_0'(u_0), u_0\rangle \\
&= (\frac{1}{2} - \frac{1}{q}) \int_{\mathbb{R}^3} \big(a|\nabla u_0|^2 + V_0 u_0^2\big)\mathrm{d}x + (\frac{1}{4} - \frac{1}{q})\bigg(\int_{\mathbb{R}^3} |\nabla u_0|^2 \mathrm{d}x\bigg)^2 \\
&\quad + (\frac{1}{2p} - \frac{1}{q})\mu \int_{\mathbb{R}^3} (I * |u_0|^p)|u_0|^p \mathrm{d}x + (\frac{1}{q} - \frac{1}{6}) \int_{\mathbb{R}^3} Q_0 |u_0|^6 \mathrm{d}x \\
&\le \liminf_{n\to\infty} \bigg[(\frac{1}{2} - \frac{1}{q}) \int_{\mathbb{R}^3} \big(a|\nabla u_n|^2 + V_0 u_n^2\big)\mathrm{d}x + (\frac{1}{4} - \frac{1}{q})\bigg(\int_{\mathbb{R}^3} |\nabla u_n|^2 \mathrm{d}x\bigg)^2 \\
&\quad + (\frac{1}{2p} - \frac{1}{q})\mu \int_{\mathbb{R}^3} (I * |u_n|^p)|u_n|^p \mathrm{d}x + (\frac{1}{q} - \frac{1}{6}) \int_{\mathbb{R}^3} Q_0 |u_n|^6 \mathrm{d}x\bigg] \\
&= \liminf_{n\to\infty} \bigg(I_0(u_n) - \frac{1}{q}\langle I_0'(u_n), u_n\rangle\bigg) \le c_0,
\end{aligned}$$

which implies that $\lim\limits_{n\to\infty} \int_{\mathbb{R}^3} \big(a|\nabla u_n|^2 + V_0 u_n^2\big)\mathrm{d}x = \int_{\mathbb{R}^3} \big(a|\nabla u_0|^2 + V_0 u_0^2\big)\mathrm{d}x$. Thus, $u_n \to u_0$ in $H^1(\mathbb{R}^3)$. For the case $u_0 = 0$, let $v_n(x) = u_n(x + y_n)$, then $I_0(v_n) \to c_0$ and $I_0'(v_n) \to 0$ as $n \to \infty$. It follows from (4.2) that there exists $v \in H^1(\mathbb{R}^3)$ with $v \neq 0$ such that $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^3)$. Thus, the proof follows from the arguments used in the case of $u_0 \neq 0$. The proof is complete. □

Lemma 4.2. *The minimax level c_ε converges to c_0 as $\varepsilon \to 0^+$.*

Proof. For any $R > 0$, define $u_R(x) = \varphi_R(x)u_0$, here u_0 is a positive ground state solution of problem (1.7) and $\varphi_R = \psi(x/R)$, where $\psi \in C^\infty(\mathbb{R}^3, [0, 1])$ satisfying $\psi(x) = 1$ if $|x| \le \frac{1}{2}$ and $\psi(x) = 0$ if $|x| \ge 1$. It follows from Lebesgue Theorem that

$$u_R \to u_0, \quad \text{as } R \to \infty. \tag{4.5}$$

It follows from Lemma 2.1 that for each $\varepsilon,\ R > 0$, there exists $t_{\varepsilon,R} > 0$ such that

$$I_\varepsilon(t_{\varepsilon,R} u_R) = \max_{t\ge 0} I_\varepsilon(t u_R).$$

Then

$$\begin{aligned}
&\frac{1}{t_{\varepsilon,R}^{2p-2}} \int_{B_R(0)} \big(a|\nabla u_R|^2 + V(\varepsilon x)u_R^2\big)\mathrm{d}x + \frac{b}{t_{\varepsilon,R}^{2p-4}}\bigg(\int_{B_R(0)} |\nabla u_R|^2 \mathrm{d}x\bigg)^2 \\
&+ \mu \int_{B_R(0)} (I * |u_R|^p)|u_R|^p \mathrm{d}x \\
=& t_{\varepsilon,R}^{q-2p} \int_{B_R(0)} \lambda K(\varepsilon x)|u_R|^q \mathrm{d}x + t_{\varepsilon,R}^{6-2p} \int_{B_R(0)} Q(\varepsilon x)|u_R|^6 \mathrm{d}x.
\end{aligned} \tag{4.6}$$

From (4.6), we get

$$\begin{aligned}&\frac{1}{t_{\varepsilon,R}^{2p-2}}\int_{B_R(0)}\big(a|\nabla u_R|^2+|V|_{\infty(|x|<R)}u_R^2\big)\mathrm{d}x+\frac{b}{t_{\varepsilon,R}^{2p-4}}\bigg(\int_{B_R(0)}|\nabla u_R|^2\mathrm{d}x\bigg)^2\\&+\mu\int_{B_R(0)}(I*|u_R|^p)|u_R|^p\mathrm{d}x\\\geq& t_{\varepsilon,R}^{q-2p}\int_{B_R(0)}\lambda K_\infty|u_R|^q\mathrm{d}x+t_{\varepsilon,R}^{6-2p}\int_{B_R(0)}Q_\infty|u_R|^6\mathrm{d}x.\end{aligned}\tag{4.7}$$

It follows from (4.7) that $(t_{\varepsilon,R})$ is bounded, i.e., for each $R>0$, there exists $t_R>0$ such that

$$0<\lim_{\varepsilon\to 0^+}t_{\varepsilon,R}=t_R.$$

Thus, passing the limit as $\varepsilon\to 0^+$ in (4.6), we have

$$\begin{aligned}&\frac{1}{t_R^2}\int_{B_R(0)}\big(a|\nabla u_R|^2+V(\varepsilon x)u_R^2\big)\mathrm{d}x+b\bigg(\int_{B_R(0)}|\nabla u_R|^2\mathrm{d}x\bigg)^2\\&+\mu\int_{B_R(0)}(I*|u_R|^p)|u_R|^p\mathrm{d}x\\=&t_R^{q-4}\int_{B_R(0)}\lambda K(\varepsilon x)|u_R|^q\mathrm{d}x+t_R^2\int_{B_R(0)}Q(\varepsilon x)|u_R|^6\mathrm{d}x.\end{aligned}\tag{4.8}$$

From (4.5) and (4.8), we can easily get that $\lim\limits_{R\to\infty}t_R=1$ and $I_0(t_Ru_R)=\max\limits_{t\geq 0}I_0(tu_R)$. Hence, it follows form (4.8) and $c_\varepsilon\leq\max\limits_{t\geq 0}I_\varepsilon(tu_R)=I_\varepsilon(t_Ru_R)$ that

$$\limsup_{\varepsilon\to 0^+}c_\varepsilon\leq I_0(t_Ru_R).$$

From (4.5), we deduce that $\limsup\limits_{\varepsilon\to 0^+}c_\varepsilon\leq c_0$. On the other hand, it follows from (4.4) that $\liminf\limits_{\varepsilon\to 0^+}c_\varepsilon\geq c_0$. Thus, $\lim\limits_{\varepsilon\to 0^+}c_\varepsilon=c_0$. The proof is complete. □

Lemma 4.3. *For the family $v_\varepsilon(x):=u_\varepsilon(\varepsilon x)$ satisfying $I_\varepsilon(v_\varepsilon)=c_\varepsilon$ and $I'_\varepsilon(v_\varepsilon)=0$, there exist $\varepsilon^*>0$, a family $\{y_\varepsilon\}\subset\mathbb{R}^3$ and constants $r,\sigma>0$ such that*

$$\int_{B_r(y_\varepsilon)}v_\varepsilon^2\geq\sigma,\quad\textit{for all }\varepsilon\in(0,\varepsilon^*).\tag{4.9}$$

Furthermore, the family $\{\varepsilon y_\varepsilon\}$ is bounded. In particular, if x_0 is the limit of the sequence $\{\varepsilon_n y_{\varepsilon_n}\}$ in the family $\{\varepsilon y_\varepsilon\}$, then we have $V(x_0)=V_0$ and $x_0\in\Lambda\cap\Lambda_1\cap\Lambda_2$.

Proof. Argue by contradiction that (4.9) does not hold. Then there exists a sequence ε_n converging to zero such that

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^3}\int_{B_r(y)}v_{\varepsilon_n}^2=0.$$

Similar to the proof of Lemma 4.1, we can get a contradiction. Thus, (4.9) holds. Set $y_n=y_{\varepsilon_n}$ and $v_n(x)=v_{\varepsilon_n}(x)$. Suppose by contradiction that $\varepsilon_n y_n\to\infty$ as $n\to\infty$. Set $\omega_n(x)=v_n(x+y_n)$. Then it follows from (4.9) that

$$\int_{B_r(0)}\omega_n^2\geq\sigma,\quad\text{for all }n\in\mathbb{N}.\tag{4.10}$$

Therefore, ω_n solves the following system

$$\begin{cases} -\big(a+b\int_{\mathbb{R}^3}|\nabla\omega_n|^2\mathrm{d}x\big)\Delta\omega_n+V(\varepsilon_nx+\varepsilon_ny)\omega_n+\mu\phi|\omega_n|^{p-2}\omega_n \\ =f(x,\omega_n), \qquad\qquad \text{in } \mathbb{R}^3, \\ (-\Delta)^{\frac{\alpha}{2}}\phi=\mu|\omega_n|^p,\ u>0, \quad \text{in } \mathbb{R}^3, \end{cases}$$

where $f(x,\omega_n)=\lambda K(\varepsilon_nx+\varepsilon_ny)|\omega_n|^{q-2}\omega_n+Q(\varepsilon_nx+\varepsilon_ny)|\omega_n|^4\omega_n$. It follows from the invariance of $\mathbb{R}^3$ by translations that $\|\omega_n\|=\|v_n\|$. By a standard argument, we can prove that ω_n is bounded in $H^1(\mathbb{R}^3)$. Then there exists $\omega\in H^1(\mathbb{R}^3)$ such that $\omega_n\rightharpoonup\omega$. It follows from (4.10) that $\omega\geq 0,\ \omega\neq 0$. It follows from Lemma 2.1 that there exists $t_n>0$ such that $\tilde\omega_n=t_n\omega_n\in\mathcal{N}_0$. From Lemma 4.2, we obtain

$$\begin{aligned} I_0(\tilde\omega_n)\leq{}&\frac12\int_{\mathbb{R}^3}\big(a|\nabla\tilde\omega_n|^2+V(\varepsilon_n(x+y_n)\tilde\omega_n^2\big)\mathrm{d}x+\frac b4\bigg(\int_{\mathbb{R}^3}|\nabla\tilde\omega_n|^2\mathrm{d}x\bigg)^2\\ &+\frac{\mu}{2p}\int_{\mathbb{R}^3}(I*|\tilde\omega_n|^p)|\tilde\omega_n|^p\mathrm{d}x-\frac1q\int_{\mathbb{R}^3}\lambda K(\varepsilon_n(x+y_n)|\tilde\omega_n|^q\mathrm{d}x\\ &-\frac16\int_{\mathbb{R}^3}Q(\varepsilon_n(x+y_n)|\tilde\omega_n|^6\mathrm{d}x\\ ={}&I_{\varepsilon_n}(t_nv_n)\leq I_{\varepsilon_n}(v_n)=c_0+o(1). \end{aligned}$$

It is obvious that $I_0(\tilde\omega_n)\geq c_0$. Hence, $\lim\limits_{n\to\infty}I_0(\tilde\omega_n)=c_0$. We first claim that $\{t_n\}$ is bounded. If not, then $t_n\to\infty$ as $n\to\infty$. We can easily prove that $I_0(t_n\omega_n)\to-\infty$ as $n\to\infty$, which is a contradiction with $I_0(t_n\omega_n)\geq c_0$ for all $n\in\mathbb{N}$. Up to a subsequence, we may assume that $t_n\to t\geq 0$. If $t=0$, then it follows from the boundedness of $\{\omega_n\}$ that $\tilde\omega_n\to 0$ in $H^1(\mathbb{R}^3)$, thus, $I_0(\tilde\omega_n)\to 0$ as $n\to\infty$, which contradicts with $c_0>0$. Then $t>0$ and the weak limit of $\tilde\omega_n$ is different zero. Let $\tilde\omega_n\rightharpoonup\tilde\omega$ in $H^1(\mathbb{R}^3)$ as $n\to\infty$. It follows from the uniqueness of the weak limit and the sequentially continuity of I_0' that $\tilde\omega=t\omega$ and $\tilde\omega\in\mathcal{N}_0$. From Lemma 4.1, we know that $\tilde\omega_n\to\tilde\omega$ in $H^1(\mathbb{R}^3)$ as $n\to\infty$, then $\omega_n\to\omega$ in $H^1(\mathbb{R}^3)$. Hence, it follows from Fatou's Lemma and $\tilde\omega_n\in\mathcal{N}_0$ that

$$\begin{aligned} c_0\leq I_0(\tilde\omega)&<I_\infty(\tilde\omega)-\frac1{2p}\langle I_0'(\tilde\omega),\tilde\omega\rangle\\ &=\int_{\mathbb{R}^3}\big[(\frac12-\frac1{2p})a|\nabla\tilde\omega|^2+(\frac{V_\infty}{2}-\frac{V_0}{2p})\tilde\omega^2\big]\mathrm{d}x\\ &\quad+(\frac14-\frac1{2p})b\bigg(\int_{\mathbb{R}^3}|\nabla\tilde\omega|^2\mathrm{d}x\bigg)^2\\ &\quad+\int_{\mathbb{R}^3}(\frac{\lambda}{2p}K_0-\frac{\lambda}{q}K_\infty)|\tilde\omega|^q\mathrm{d}x+\int_{\mathbb{R}^3}(\frac{Q_0}{2p}-\frac{Q_\infty}{6})|\tilde\omega|^6\mathrm{d}x \qquad (4.11)\\ &\leq\liminf_{n\to\infty}\bigg[\int_{\mathbb{R}^3}\big[(\frac12-\frac1{2p})a|\nabla\tilde\omega_n|^2+(\frac{V(\varepsilon_nx+\varepsilon_ny)}{2}-\frac{V_0}{2p})\tilde\omega_n^2\big]\mathrm{d}x\\ &\quad+(\frac14-\frac1{2p})b\bigg(\int_{\mathbb{R}^3}|\nabla\tilde\omega|^2\mathrm{d}x\bigg)^2\\ &\quad+\int_{\mathbb{R}^3}(\frac{\lambda}{2p}K_0-\frac{\lambda}{q}K(\varepsilon_nx+\varepsilon_ny))|\tilde\omega_n|^q\mathrm{d}x \end{aligned}$$

$$+\int_{\mathbb{R}^3}\big(\frac{Q_0}{2p}-\frac{Q(\varepsilon_n x+\varepsilon_n y)}{6}\big)|\tilde{\omega}_n|^6\mathrm{d}x\Big]$$
$$\leq \liminf_{n\to\infty} I_{\varepsilon_n}(t_n v_n)\leq \liminf_{n\to\infty} I_{\varepsilon_n}(v_n)=c_0,$$

which is a contradiction. Thus, $\{\varepsilon_n y_n\}$ is bounded, and then there exists $x_0 \in \mathbb{R}^3$ such that $\varepsilon_n y_n \to x_0$.

Define the functional I_{x_0} as follows:

$$I_{x_0}(u)=\frac{a}{2}\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}^3}V(x_0)u^2\mathrm{d}x+\frac{b}{4}\Big(\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x\Big)^2$$
$$+\frac{\mu}{2p}\int_{\mathbb{R}^3}(I*|u|^p)|u|^p\mathrm{d}x-\frac{\lambda}{q}\int_{\mathbb{R}^3}K(x_0)|u|^q\mathrm{d}x-\frac{1}{6}\int_{\mathbb{R}^3}Q(x_0)|u|^6\mathrm{d}x.$$

Hence, if $V(x_0) > V_0$, we can get a contradiction by using I_{x_0} to take the place of I_∞ in (4.11). Thus, $V(x_0) = V_0$, that is, $x_0 \in \Lambda$. If $x_0 \notin \Lambda_1 \cap \Lambda_2$, i.e., $K(x_0) < K_0$ or $Q(x_0) < Q_0$, then $I_0(\tilde{\omega}) < I_{x_0}(\tilde{\omega})$. Therefore, we can get a contradiction by repeating the same arguments used above. Thus, $x_0 \in \Lambda \cap \Lambda_1 \cap \Lambda_2$. The proof is complete. □

Proof of Theorem 1.2. From Lemma 4.3, we can easily prove Theorem 1.2. □

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