Journal of Applied Analysis and Computation
Volume 10, Number 4, August 2020, 1666–1682
Website:http://jaac-online.com
DOI:10.11948/20190337

INTERNAL LAYERS FOR A QUASI–LINEAR SINGULARLY PERTURBED DELAY DIFFERENTIAL EQUATION*

Tao Feng[1] and Ming-kang Ni[1,†]

Abstract The current paper is mainly concerned with the internal layers for a quasi–linear singularly perturbed differential equation with time delays. By using the method of boundary layer functions and the theory of contrast structures, the existence of a uniformly valid smooth solution is proved, and the asymptotic expansion is constructed. As an application, a concrete example is presented to demonstrate the effectiveness of our result.

Keywords Delay, internal layers, boundary layer functions, contrast structures, asymptotic expansion.

MSC(2010) 34E20, 34B15.

1. Introduction

Delay differential equations (DDEs) arise naturally in just about every interaction of the real world, and the original motivations for studying DDEs mainly come from their application in feedback control theory. In the past several decades, DDEs have been extensively used in control theory [2, 3, 5–7, 13], population dynamics [8, 22] and many other scientific fields, where the delays naturally appear to account for a variety of situations. Among them, singularly perturbed DDEs are particularly relevant to describe lots of practical phenomena in many areas, see e.g., [1, 4, 10, 18, 19] and the references therein.

The effective method to investigate singularly perturbed DDEs include multi–scale method [9], the theory of boundary layer functions established by A. Vasil'eva [21], etc. By the theory of boundary layer functions, Vasil'eva [20] firstly discussed a kind of neutral type equation with small lag

$$\begin{cases} \dot{x}(t) = f(t, x(t), x(t-\tau), \dot{x}(t-\tau)), \\ x(t) = \varphi(t), \quad 0 \le t \le \tau, \end{cases} \tag{1.1}$$

wherein $\tau > 0$ is sufficiently small and $\varphi(t)$ is a given function, and the following results are obtained

†the corresponding author. Email address: xiaovikdo@163.com (M. Ni)
[1]School of Mathematical Sciences, East China Normal University, Shanghai, China
*The authors were supported by National Natural Science Foundation of China (11871217), supported in part by Science and Technology Commission of Shanghai Municipality (No. 18dz2271000).

(a) If $f(t,x,y,z)$ does not depend on z, then for $\tau \to 0$, the solution of (1.1) tends to a solution of initial problem

$$\dot{x}(t) = f(t, x(t)), \quad x(0) = \varphi(0);$$

(b) Otherwise, then the author declared that the result in (a) is valid if

$$\left| \frac{\partial f(t,x,y,z)}{\partial z} \right| \le a < 1,$$

for all (t,x,y,z) in the region defined by $\varphi(0)$.

Subsequently, Rožkov [14–17] generalized the results of singularly perturbed DDEs.

With the rapid development of boundary layer functions theory, it has been applied to deal with the related problems of uniformly valid asymptotic solution for Tikhonov systems [11, 12, 23, 24], among which, Ni & Lin [11] have studied a kind of singularly perturbed DDEs with internal layers

$$\begin{cases} \varepsilon^2 y'' = f\left(y(t), y(t-\tau), t\right), \quad 0 \le t \le T, \\ y(t) = \varphi(t), \quad -\tau \le t \le 0, \quad y(T) = y^T, \end{cases}$$

where $0 < \varepsilon \ll 1$ is a small parameter, $\varphi(t) \in C([-\tau, 0])$, $\tau > 0$ is deviation argument, $\tau < T < 2\tau$ is positive constant. The author proved the existence of uniformly valid asymptotic smooth solutions, and constructed their asymptotic expansion with an internal transition layer.

Motivated by the above discussions, in this paper, we consider the following quasi–linear singularly perturbed DDEs

$$\begin{cases} \varepsilon y'' = A(y(t), y(t-\sigma), t) y' + B(y(t), y(t-\sigma), t), \\ y(t) = \varphi(t), \quad -\sigma \le t \le 0, \quad y(T) = y^T, \end{cases} \tag{1.2}$$

where $0 < \varepsilon \ll 1$, $\sigma > 0$ is delay, $\varphi(t)$ defined in $[-\sigma, 0]$ is a smooth function, y^T is a given constant and for simplicity, we assume that $T \in (\sigma, 2\sigma]$. Furthermore, the functions A, B are assumed to be sufficiently smooth on region

$$\mathbb{D} = \{(t,y): \ 0 \le t \le T, \ y \in I_y\},$$

wherein I_y represents the admissible range of $y(t)$, and $\sigma \in (0,T)$ divides the domain $\mathbb{D}$ into two parts

$$\mathbb{D}_1 = \{(t,y): \ 0 \le t \le \sigma, \ y \in I_y\}, \quad \mathbb{D}_2 = \{(t,y): \ \sigma \le t \le T, \ y \in I_y\}.$$

In what follows, the authors are going to apply the boundary layer function method and the theory contrast structures to study (1.2).

Throughout this paper, the notations with superscript “ ′ ” stand for the derivative of functions on the corresponding variable, and we always make the following assumptions.

Assumption 1.1. There exists a unique solution $\bar{y} = \alpha_1(t)$ of the Cauchy problem on the interval $[0, \sigma]$

$$A(\bar{y}(t), \varphi(t-\sigma), t)\bar{y}' + B(\bar{y}(t), \varphi(t-\sigma), t) = 0, \quad \bar{y}(0) = \varphi(0),$$

while $\bar{y} = \alpha_2(t)$ is the unique solution of the Cauchy problem on the interval $[\sigma, T]$

$$A(\bar{y}(t), \alpha_1(t-\sigma), t)\bar{y}' + B(\bar{y}(t), \alpha_1(t-\sigma), t) = 0, \quad \bar{y}(T) = y^T.$$

Remark 1.1. Assumption 1.1 applies the initial–boundary value conditions for each cauchy problems, and the other cases will be discussed in further study.

Assumption 1.2. The following inequalities are guaranteed

$$A(\alpha_1(t), \varphi(t-\sigma), t) > 0, \quad t \in [0, \sigma],$$
$$A(\alpha_2(t), \alpha_1(t-\sigma), t) < 0, \quad t \in [\sigma, T].$$

In general, $\alpha_1(\sigma-) \neq \alpha_2(\sigma+)$, for the sake of definiteness, assume that $\alpha_2(\sigma) < \alpha_1(\sigma)$. Our focus is attracted to the study of existence of an asymptotic expansion to a solution of problem (1.2) in the class $C^1[0,T] \cap \left(C^2(0,\sigma) \cup C^2(\sigma,T)\right)$, with an internal transition layer localized in a neighbourhood of the point $t = \sigma$, namely, a solution close to the function $\alpha_1(t)$ on the left of this neighborhood and the function $\alpha_2(t)$ on the right of this neighborhood, which varying from $\alpha_1(t)$ to $\alpha_2(t)$ swiftly near $t = \sigma$, see Fig. 1. It is found from Assumption 1.1 that there is no boundary layers for system (1.2) on the considered domain.

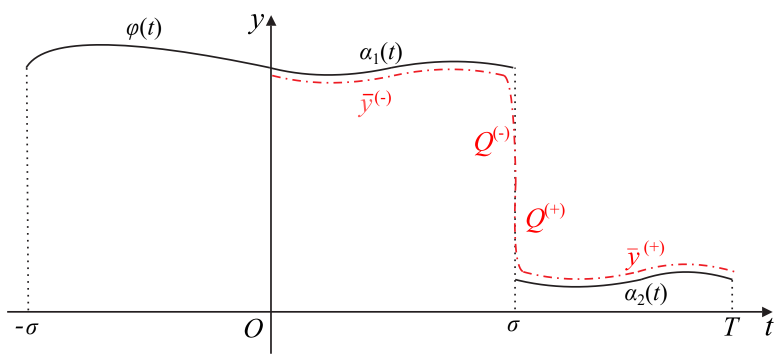

Figure 1. Internal transition layer phenomenon of system (1.2) localized in a neighbourhood of $t = \sigma$, herein, $\bar{y}^{(\mp)}$ denote the regular parts of the asymptotic solution and $Q^{(\mp)}$ denote the internal layers.

Introducing auxiliary system (which is an analog of Tikhonov associated system, ref. [21, p21–27]) to the original problem by making a variable change $z = \frac{\mathrm{d}y}{\mathrm{d}t}$, then the following auxiliary system is obtained

$$\frac{\mathrm{d}\tilde{z}}{\mathrm{d}\tau} = A(\tilde{y}, \bar{y}_0, \sigma)\tilde{z}, \quad \frac{\mathrm{d}\tilde{y}}{\mathrm{d}\tau} = \tilde{z}, \quad -\infty < \tau < +\infty. \tag{1.3}$$

Obviously, the points $(\alpha_{1,2}(\sigma), 0)$ on the phase plane $(\tilde{y}, \tilde{z})$ are equilibrium points of system (1.3), and the characteristic equations

$$\lambda \left(A(\alpha_{1,2}, \bar{y}_0, \sigma) - \lambda\right) = 0$$

have eigenvalues

$$\lambda_1 = 0, \quad \lambda_2 = A(\alpha_{1,2}, \bar{y}_0, \sigma) \neq 0. \tag{1.4}$$

Dividing the first equation in (1.3) by the second equation, the first–order differential equation

$$\frac{\mathrm{d}\tilde{z}}{\mathrm{d}\tilde{y}} = A(\tilde{y}, \bar{y}_0, \sigma) \tag{1.5}$$

can be arrived for the functions $\tilde{z}(\tilde{y})$. According to (1.4) and (1.5), we find for $\tilde{y} \in (\alpha_2(\sigma), \alpha_1(\sigma))$,

(i) there exists an unstable manifold W^u leaving from the point $(\alpha_1(\sigma), 0)$ and

$$W^u: \ \tilde{z}^{(-)} = \int_{\alpha_1(\sigma)}^{\tilde{y}} A(u, \varphi(0), \sigma)\mathrm{d}u := \Phi^{(-)}(\tilde{y}), \tag{1.6}$$

(ii) there exists a stable manifold W^s entering into the point $(\alpha_2(\sigma), 0)$ and

$$W^s: \ \tilde{z}^{(+)} = \int_{\alpha_2(\sigma)}^{\tilde{y}} A(u, \alpha_1(0), \sigma)\mathrm{d}u := \Phi^{(+)}(\tilde{y}). \tag{1.7}$$

We suppose that the following condition hold

Assumption 1.3. On phase plane $(\tilde{y}, \tilde{z})$, there hold $\{\tilde{y} = p\} \cap \Phi^{(\mp)} \neq \emptyset$ for all $p \in (\alpha_2(\sigma), \alpha_1(\sigma))$.

Assumption 1.4. The following compatibility condition holds for σ

$$\int_{\alpha_2(\sigma)}^{\alpha_1(\sigma)} A(u, \varphi(0), \sigma)\mathrm{d}u = 0.$$

Remark 1.2. Assumption 1.4 needs certain requirements and it can always be realized in several cases, which can be seen in Example 4.1.

The remainder part of this paper is organized as follows. Sect. 2 is devoted to algorithm of construction for the asymptotic expansions which are established by the method of boundary layer functions and the theory of contrast structures. And the main result for the existence of solutions and estimation of remainder are presented in Sect. 3. Whereafter in Sect. 4, a concrete quasi–linear singularly perturbed delay differential equation and a series of numerical simulations for some given small parameter ε are carried out to demonstrate the effectiveness of our result.

2. Algorithm of Constructing the Asymptotic Expansions

The asymptotic expansion to a solution of problem (1.2) is constructed separately on $\mathbb{D}_{1,2}$ (namely, the left problem and the right problem respectively), denote it by $y^{(-)}(t,\varepsilon)$ on $\mathbb{D}_1$ and $y^{(+)}(t,\varepsilon)$ on $\mathbb{D}_2$ respectively, and the functions $y^{(-)}(t,\varepsilon)$ and $y^{(+)}(t,\varepsilon)$ are sewn smoothly, as well as their derivatives $z^{(-)} := y'^{(-)}$ and $z^{(+)} := y'^{(+)}$ at the point $t = \sigma$,

$$y^{(-)}(\sigma,\varepsilon) = y^{(+)}(\sigma,\varepsilon) = p(\varepsilon), \quad z^{(-)}(\sigma,\varepsilon) = z^{(+)}(\sigma,\varepsilon) = z(\varepsilon), \tag{2.1}$$

the values $p(\varepsilon)$ and $z(\varepsilon)$ are unknown and will be determined in the process of constructing the asymptotic expansions of solution of problem (1.2) and are sought in the form of expansions in powers of ε in the following

$$p(\varepsilon) = p_0 + \varepsilon p_1 + \cdots, \quad z(\varepsilon) = \varepsilon^{-1} z_{-1} + z_0 + \varepsilon z_1 + \cdots, \tag{2.2}$$

and problem (1.2) is separated into the left problem $P^{(-)}$: $(t, y) \in \mathbb{D}_1$,

$$\begin{cases} \varepsilon y''^{(-)} = A\big(y^{(-)}, y^{(-)}(t-\sigma), t\big) y'^{(-)} + B\big(y^{(-)}, y^{(-)}(t-\sigma), t\big), \\ y^{(-)}(0,\varepsilon) = \varphi(0), \quad y^{(-)}(\sigma,\varepsilon) = p(\varepsilon), \end{cases} \tag{2.3}$$

and the right problem $P^{(+)}$: $(t,y)\in\mathbb{D}_2$,

$$\begin{cases}\varepsilon y''^{(+)} = A\big(y^{(+)}, y^{(+)}(t-\sigma), t\big)y'^{(+)} + B\big(y^{(+)}, y^{(+)}(t-\sigma), t\big),\\ y^{(+)}(\sigma,\varepsilon) = p(\varepsilon), \quad y^{(+)}(T,\varepsilon) = y^T.\end{cases} \tag{2.4}$$

Problems (2.3)–(2.4) for second–order ordinary differential equations are equivalent to the following respective problems for systems of first–order equations

$$\begin{cases}\varepsilon z'^{(-)} = A\big(y^{(-)}, y^{(-)}(t-\sigma), t\big)z^{(-)} + B\big(y^{(-)}, y^{(-)}(t-\sigma), t\big), \quad y'^{(-)} = z^{(-)},\\ y^{(-)}(0,\varepsilon) = \varphi(0), \quad y^{(-)}(\sigma,\varepsilon) = p(\varepsilon),\end{cases} \tag{2.5}$$

and

$$\begin{cases}\varepsilon z'^{(+)} = A\big(y^{(+)}, y^{(+)}(t-\sigma), t\big)z^{(+)} + B\big(y^{(+)}, y^{(+)}(t-\sigma), t\big), \quad y'^{(+)} = z^{(+)},\\ y^{(+)}(\sigma,\varepsilon) = p(\varepsilon), \quad y^{(+)}(T,\varepsilon) = y^T.\end{cases} \tag{2.6}$$

To describe the behavior of solution in a neighborhood of the transition layer point in detail, we introduce a stretched variable $\tau = \frac{t-\sigma}{\varepsilon}$, and the asymptotic expansions of the solutions to systems (2.5)–(2.6) are constructed in the following forms

$$y^{(\mp)}(t,\varepsilon) = \bar{y}^{(\mp)}(t,\varepsilon) + Q^{(\mp)}y(\tau,\varepsilon), \quad z^{(\mp)}(t,\varepsilon) = \bar{z}^{(\mp)}(t,\varepsilon) + Q^{(\mp)}z(\tau,\varepsilon), \tag{2.7}$$

in which

$$\bar{y}^{(\mp)}(t,\varepsilon) = \bar{y}_0^{(\mp)}(t) + \varepsilon\bar{y}_1^{(\mp)}(t) + \cdots, \quad \bar{z}^{(\mp)}(t,\varepsilon) = \bar{z}_0^{(\mp)}(t) + \varepsilon\bar{z}_1^{(\mp)}(t) + \cdots, \tag{2.8}$$

represent the regular part of the asymptotic solution (2.7), which mainly reflects the behavior of the solution outside of the internal layers, and

$$\begin{aligned}&Q^{(\mp)}y(\tau,\varepsilon) = Q_0^{(\mp)}y(\tau) + \varepsilon Q_1^{(\mp)}y(\tau) + \cdots,\\ &Q^{(\mp)}z(\tau,\varepsilon) = \varepsilon^{-1}Q_{-1}^{(\mp)}z(\tau) + Q_0^{(\mp)}z(\tau) + \varepsilon Q_1^{(\mp)}z(\tau) + \cdots\end{aligned} \tag{2.9}$$

denote the internal layer part of the asymptotic solution (2.7). The internal layer functions are required to satisfy

$$\lim_{\tau\to\mp\infty} Q_k^{(\mp)}y(\tau) = 0, \quad k \ge 0,$$
$$\lim_{\tau\to\mp\infty} Q_k^{(\mp)}z(\tau) = 0, \quad k \ge -1.$$

Rewriting the sewing conditions (2.1) with regards to the expansions (2.2), (2.8)–(2.9) give

$$\begin{aligned}&\bar{y}_0^{(-)}(\sigma) + \varepsilon\bar{y}_1^{(-)}(\sigma) + \cdots + Q_0^{(-)}y(0) + \varepsilon Q_1^{(-)}y(0) + \cdots\\ =&\bar{y}_0^{(+)}(\sigma) + \varepsilon\bar{y}_1^{(+)}(\sigma) + \cdots + Q_0^{(+)}y(0) + \varepsilon Q_1^{(+)}y(0) + \cdots\\ =&p_0 + \varepsilon p_1 + \cdots,\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}&\bar{z}_0^{(-)}(\sigma) + \varepsilon\bar{z}_1^{(-)}(\sigma) + \cdots + \varepsilon^{-1}Q_{-1}^{(-)}z(0) + Q_0^{(-)}z(0) + \varepsilon Q_1^{(-)}z(0) + \cdots\\ =&\bar{z}_0^{(+)}(\sigma) + \varepsilon\bar{z}_1^{(+)}(\sigma) + \cdots + \varepsilon^{-1}Q_{-1}^{(+)}z(0) + Q_0^{(+)}z(0) + \varepsilon Q_1^{(+)}z(0) + \cdots\\ =&\varepsilon^{-1}z_{-1} + z_0 + \varepsilon z_1 + \cdots.\end{aligned} \tag{2.11}$$

2.1. Regular parts

According to the standard Vasil'eva boundary layer functions method, the equations for the terms of the regular part can be obtained by substituting the expressions (2.8) for the functions $\bar{y}^{(\mp)}(t,\varepsilon)$ and $\bar{z}^{(\mp)}(t,\varepsilon)$ into the system of equations

$$\varepsilon z'^{(\mp)} = A\left(y^{(\mp)}, y^{(\mp)}(t-\sigma), t\right) z^{(\mp)} + B\left(y^{(\mp)}, y^{(\mp)}(t-\sigma), t\right), \quad y'^{(\mp)} = z^{(\mp)},$$

and into the additional conditions at $t = 0$ and $t = T$ in problems (2.5) and (2.6) respectively, and by equating the coefficients of like powers of ε on both sides of the obtaining equations. Then the functions $y_k^{(\mp)}(t)$ are determined for each $k = 0, 1, \cdots$ as the solutions of Cauchy problems for first–order differential equations, after which the expressions for their derivatives $z_k^{(\mp)}(t)$ can be found.

Indeed, the following Cauchy problem in the zeroth order expansion is obtained

$$0 = A\left(\bar{y}_0^{(-)}(t), \varphi(t-\sigma), t\right) \bar{y}_0'^{(-)} + B\left(\bar{y}_0^{(-)}(t), \varphi(t-\sigma), t\right), \quad \bar{y}_0^{(-)}(0) = \varphi(0),$$
$$0 = A\left(\bar{y}_0^{(+)}(t), \alpha_1(t-\sigma), t\right) \bar{y}_0'^{(+)} + B\left(\bar{y}_0^{(+)}(t), \alpha_1(t-\sigma), t\right), \quad \bar{y}_0^{(+)}(T) = y^T.$$

There exist solutions of these problems by Assumption 1.1. Taking

$$\bar{y}_0^{(-)}(t) = \alpha_1(t), \quad \bar{y}_0^{(+)}(t) = \alpha_2(t),$$

then from $y'^{(\mp)} = z^{\mp}$, we get

$$\bar{z}_0^{(-)}(t) = \alpha_1'(t), \quad \bar{z}_0^{(+)}(t) = \alpha_2'(t).$$

The regular terms $\bar{y}_k^{(-)}(t)$, $(k \geq 1)$ are determined as the solutions of the problems

$$\overline{A}(t)\frac{\mathrm{d}\bar{y}_k^{(-)}}{\mathrm{d}t} = -\left[\frac{\partial \overline{A}}{\partial y}(t)\alpha_1'(t) + \frac{\partial \overline{B}}{\partial y}(t)\right]\bar{y}_k^{(-)} + F_k^{(-)}(t), \quad \bar{y}_k^{(-)}(0) = 0,$$

therein the elements of $\overline{A}(t)$, $\frac{\partial \overline{A}}{\partial y}(t)$, $\frac{\partial \overline{B}}{\partial y}(t)$ are calculated at the point $(\alpha_1(t), \varphi(0), t)$, and the functions $F_k^{(-)}(t)$ are expressed recursively through $\bar{y}_j^{(-)}(t)$ and $\bar{z}_j^{(-)}(t)$, $0 \leq j \leq k-1$. Clearly, the following explicit expressions are obtained

$$\bar{y}_k^{(-)}(t) = \int_0^t \exp\left[-\int_s^t \left(\frac{\partial \overline{A}}{\partial y}(\eta)\alpha_1'(\eta) + \frac{\partial \overline{B}}{\partial y}(\eta)\right)\left(\overline{A}(\eta)\right)^{-1}\mathrm{d}\eta\right]\frac{F_k^{(-)}(s)}{\overline{A}(s)}\mathrm{d}s$$

for $\bar{y}_k^{(-)}(t)$ and then $\bar{z}_k^{(-)} = \bar{y}_k'^{(-)}$.

Similarly, the regular terms $\bar{y}_k^{(+)}(t)$, $(k \geq 1)$ are determined as the solutions of the problems

$$\widetilde{A}(t)\frac{\mathrm{d}\bar{y}_k^{(+)}}{\mathrm{d}t} = -\left[\frac{\partial \widetilde{A}}{\partial y}(t)\alpha_2'(t) + \frac{\partial \widetilde{B}}{\partial y}(t)\right]\bar{y}_k^{(+)} + F_k^{(+)}(t), \quad \bar{y}_k^{(+)}(T) = 0,$$

herein the elements of $\widetilde{A}(t)$, $\frac{\partial \widetilde{A}}{\partial y}(t)$, $\frac{\partial \widetilde{B}}{\partial y}(t)$ are calculated at the point $(\alpha_2(t), \alpha_1(0), t)$, and the functions $F_k^{(+)}(t)$ are expressed recursively through $\bar{y}_j^{(+)}(t)$ and $\bar{z}_j^{(+)}(t)$, $0 \leq j \leq k-1$. Clearly, the following explicit expressions are obtained

$$\bar{y}_k^{(+)}(t) = \int_T^t \exp\left[-\int_s^t \left(\frac{\partial \widetilde{A}}{\partial y}(\eta)\alpha_2'(\eta) + \frac{\partial \widetilde{B}}{\partial y}(\eta)\right)\left(\widetilde{A}(\eta)\right)^{-1}\mathrm{d}\eta\right]\frac{F_k^{(+)}(s)}{\widetilde{A}(s)}\mathrm{d}s$$

for $\bar{y}_k^{(+)}(t)$ and then $\bar{z}_k^{(+)} = \bar{y}_k'^{(+)}$.

2.2. Internal transition layer parts

The coefficients $Q_k^{(\mp)}y(\tau)$, $k = 0, 1, \cdots$, and $Q_k^{(\mp)}z(\tau)$, $k = -1, 0, 1, \cdots$, can be obtained by substituting the expansions (2.9) into the systems of equations

$$\begin{cases} \dfrac{\mathrm{d}Q^{(\mp)}z}{\mathrm{d}\tau} = & A\left(\bar{y}^{(\mp)}(\tau\varepsilon+\sigma) + Q^{(\mp)}y, \bar{y}^{(\mp)}(\tau\varepsilon), \tau\varepsilon+\sigma\right)\left(\bar{z}^{(\mp)}(\tau\varepsilon+\sigma) + Q^{(\mp)}z\right) \\ & - A\left(\bar{y}^{(\mp)}(\tau\varepsilon+\sigma), \bar{y}^{(\mp)}(\tau\varepsilon), \tau\varepsilon+\sigma\right)\bar{z}^{(\mp)}(\tau\varepsilon+\sigma) \\ & + B\big(\bar{y}^{(\mp)}(\tau\varepsilon+\sigma) + Q^{(\mp)}y, \bar{y}^{(\mp)}(\tau\varepsilon), \tau\varepsilon+\sigma\big) \\ & - B\big(\bar{y}^{(\mp)}(\tau\varepsilon+\sigma), \bar{y}^{(\mp)}(\tau\varepsilon), \tau\varepsilon+\sigma\big), \\ \dfrac{\mathrm{d}Q^{(\mp)}y}{\mathrm{d}\tau} = & \varepsilon Q^{(\mp)}z. \end{cases} \tag{2.12}$$

Equating the coefficients of ε^0 in the second equations in (2.12) and in conditions (2.10) as well as the coefficients of ε^{-1} in the second equations in (2.12), the following problems for the leading terms of the expansions (2.9) can be obtained

$$\begin{cases} \dfrac{\mathrm{d}Q_{-1}^{(-)}z}{\mathrm{d}\tau} = A\big(\alpha_1(\sigma) + Q_0^{(-)}y, \varphi(0), \sigma\big)Q_{-1}^{(-)}z, \quad \dfrac{\mathrm{d}Q_0^{(-)}y}{\mathrm{d}\tau} = Q_{-1}^{(-)}z, \\ Q_0^{(-)}y(0) = p_0 - \alpha_1(\sigma), \quad Q_0^{(-)}y(-\infty) = 0, \end{cases} \tag{2.13}$$

for $\tau \le 0$ and

$$\begin{cases} \dfrac{\mathrm{d}Q_{-1}^{(+)}z}{\mathrm{d}\tau} = A\big(\alpha_2(\sigma) + Q_0^{(+)}y, \alpha_1(0), \sigma\big)Q_{-1}^{(+)}z, \quad \dfrac{\mathrm{d}Q_0^{(+)}y}{\mathrm{d}\tau} = Q_{-1}^{(+)}z, \\ Q_0^{(+)}y(0) = p_0 - \alpha_2(\sigma), \quad Q_0^{(+)}y(+\infty) = 0. \end{cases} \tag{2.14}$$

for $\tau \ge 0$, respectively. Making variable changes

$$\tilde{y}_0^{(\mp)}(\tau) = \alpha_{1,2}(\sigma) + Q_0^{(\mp)}y(\tau), \tag{2.15}$$

then problems (2.13)–(2.14) are written as

$$\begin{cases} \dfrac{\mathrm{d}Q_{-1}^{(-)}z}{\mathrm{d}\tau} = A\big(\tilde{y}_0^{(-)}, \varphi(0), \sigma\big)Q_{-1}^{(-)}z, \quad \dfrac{\mathrm{d}\tilde{y}_0^{(-)}}{\mathrm{d}\tau} = Q_{-1}^{(-)}z, \quad \tau \le 0, \\ \tilde{y}_0^{(-)}(0) = p_0, \quad \tilde{y}_0^{(-)}(-\infty) = \alpha_1(\sigma), \end{cases} \tag{2.16}$$

and

$$\begin{cases} \dfrac{\mathrm{d}Q_{-1}^{(+)}z}{\mathrm{d}\tau} = A\big(\tilde{y}_0^{(+)}, \alpha_1(0), \sigma\big)Q_{-1}^{(+)}z, \quad \dfrac{\mathrm{d}\tilde{y}_0^{(+)}}{\mathrm{d}\tau} = Q_{-1}^{(+)}z, \quad \tau \ge 0, \\ \tilde{y}_0^{(+)}(0) = p_0, \quad \tilde{y}_0^{(-)}(+\infty) = \alpha_2(\sigma). \end{cases} \tag{2.17}$$

Up to notation, systems (2.16) and (2.17) coincide with the associated system (1.3), hence, as was shown in Sect. 1, the functions

$$Q_{-1}^{(-)}z(\tau) = \int_{\alpha_1(\sigma)}^{\tilde{y}_0^{(-)}} A(s, \varphi(0), \sigma)\mathrm{d}s, \quad Q_{-1}^{(+)}z(\tau) = \int_{\alpha_2(\sigma)}^{\tilde{y}_0^{(+)}} A(s, \alpha_1(0), \sigma)\mathrm{d}s \tag{2.18}$$

are defined for $\alpha_2(\sigma) \le \tilde{y}_0^{(\mp)} \le \alpha_1(\sigma)$. It follows from the sewing conditions in (2.11) in the order of ε^{-1} in view of the conditions at $\tau = 0$ in problems (2.16)–(2.17) that

$$\int_{\alpha_1(\sigma)}^{p_0} A(s, \varphi(0), \sigma)\mathrm{d}s = \int_{\alpha_2(\sigma)}^{p_0} A(s, \alpha_1(0), \sigma)\mathrm{d}s = z_{-1}. \tag{2.19}$$

It is worthy to be noted that p_0 is unknown which will be determined later.

Back to systems (2.16)–(2.17) and (2.18), the Cauchy problem for the functions $\tilde{y}_0^{(\mp)}(\tau)$ are obtained as

$$\frac{\mathrm{d}\tilde{y}_0^{(-)}}{\mathrm{d}\tau} = \int_{\alpha_1(\sigma)}^{\tilde{y}_0^{(-)}} A(u, \varphi(0), \sigma)\mathrm{d}u, \quad \tilde{y}_0^{(-)}(0) = p_0, \tag{2.20}$$

and

$$\frac{\mathrm{d}\tilde{y}_0^{(+)}}{\mathrm{d}\tau} = \int_{\alpha_2(\sigma)}^{\tilde{y}_0^{(+)}} A(u, \alpha_1(0), \sigma)\mathrm{d}u, \quad \tilde{y}_0^{(+)}(0) = p_0, \tag{2.21}$$

by virtue of Assumptions 1.2–1.3, there exist $\tilde{y}_0^{(\mp)}(\tau)$ as solutions of (2.20)–(2.21) respectively. Once the expressions for the functions $\tilde{y}_0^{(\mp)}(\tau)$ have been obtained, then $Q_0^{(\mp)}y(\tau)$ can be obtained by (2.15) and then $Q_{-1}^{(\mp)}z(\tau) = \frac{\mathrm{d}Q_0^{(\mp)}y(\tau)}{\mathrm{d}\tau}$.

Lemma 2.1. *Under Assumptions 1.1–1.3, $Q_0^{(\mp)}y(\tau)$ satisfy the exponential estimates*

$$\left|Q_0^{(\mp)}y(\tau)\right| \le C_1 \exp(-\kappa|\tau|), \quad \left|Q_{-1}^{(\mp)}z(\tau)\right| \le C_2 \exp(-\kappa|\tau|),$$

where C_1, C_2, $\kappa > 0$ are positive constants.

The proof of Lemma 2.1 is similar to the proof in [11], which is omitted here.

With regards to the functions $Q_k^{(-)}y(\tau)$ and $Q_{k-1}^{(-)}z(\tau)$, $k \ge 1$, can be obtained by the following systems

$$\begin{cases} \dfrac{\mathrm{d}Q_{k-1}^{(-)}z}{\mathrm{d}\tau} = A\left(\alpha_1(\sigma) + Q_0^{(-)}y, \varphi(0), \sigma\right) Q_{k-1}^{(-)}z(\tau) \\ \qquad\qquad + A_y\left(\alpha_1(\sigma) + Q_0^{(-)}y, \varphi(0), \sigma\right) Q_{-1}^{(-)}z(\tau) Q_k^{(-)}y(\tau) + G_{k-1}^{(-)}(\tau), \\ \dfrac{\mathrm{d}Q_k^{(-)}y}{\mathrm{d}\tau} = Q_{k-1}^{(-)}z(\tau), \\ Q_k^{(-)}y(-\infty) = 0, \quad Q_k^{(-)}y(0) = p_k - \bar{y}_k(\sigma), \end{cases} \tag{2.22}$$

wherein $G_{k-1}^{(-)}(\tau)$ are known functions with respect to $Q_j^{(-)}y(\tau)$, $0 \le j \le k-1$ and $Q_j^{(-)}z(\tau)$, $-1 \le j \le k-2$.

Since

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}\tau}\left[A\left(\alpha_1(\sigma) + Q_0^{(-)}y, \varphi(0), \sigma\right) Q_k^{(-)}y\right] \\ =&A\left(\alpha_1(\sigma) + Q_0^{(-)}y, \varphi(0), \sigma\right) \frac{\mathrm{d}Q_k^{(-)}y}{\mathrm{d}\tau} \\ &+ A_y\left(\alpha_1(\sigma) + Q_0^{(-)}y, \varphi(0), \sigma\right) \frac{\mathrm{d}\left(\alpha_1(\sigma) + Q_0^{(-)}y\right)}{\mathrm{d}\tau} Q_k^{(-)}y \end{aligned}$$

$$
\begin{aligned}
=&A\left(\alpha_1(\sigma)+Q_0^{(-)}y,\varphi(0),\sigma\right)Q_{k-1}^{(-)}z(\tau)\\
&+A_y\left(\alpha_1(\sigma)+Q_0^{(-)}y,\varphi(0),\sigma\right)Q_{-1}^{(-)}z(\tau)Q_k^{(-)}y(\tau),
\end{aligned}
$$

thereby, we have

$$
\frac{\mathrm{d}Q_{k-1}^{(-)}z}{\mathrm{d}\tau}=\frac{\mathrm{d}}{\mathrm{d}\tau}\left[A\left(\alpha_1(\sigma)+Q_0^{(-)}y,\varphi(0),\sigma\right)Q_k^{(-)}y\right]+G_{k-1}^{(-)}(\tau).
$$

That is to say, the functions $\{Q_k^{(-)}y, Q_{k-1}^{(-)}z\}$ are satisfied

$$
\begin{cases}
\dfrac{\mathrm{d}Q_{k-1}^{(-)}z}{\mathrm{d}\tau}=\dfrac{\mathrm{d}}{\mathrm{d}\tau}\left[A\left(\alpha_1(\sigma)+Q_0^{(-)}y,\varphi(0),\sigma\right)Q_k^{(-)}y\right]+G_{k-1}^{(-)}(\tau),\\
\dfrac{\mathrm{d}Q_k^{(-)}y}{\mathrm{d}\tau}=Q_{k-1}^{(-)}z(\tau),\\
Q_k^{(-)}y(-\infty)=0,\quad Q_k^{(-)}y(0)=p_k-\bar{y}_k^{(-)}(\sigma).
\end{cases}
$$

By Liouville formula and variation of constants formula, it follows that

$$
Q_k^{(-)}y(\tau)=\left(p_k-\bar{y}_k^{(-)}(\sigma)\right)\frac{\tilde{z}_{-1}^{(-)}(\tau)}{\tilde{z}_{-1}^{(-)}(0)}+\tilde{z}_{-1}^{(-)}(\tau)\int_0^\tau\frac{J^{(-)}(\nu)}{[\tilde{z}_{-1}^{(-)}(\nu)]^2\vartheta^{(-)}(\nu)}\mathrm{d}\nu, \tag{2.23}
$$

herein

$$
\begin{aligned}
J^{(-)}(\nu)&=\int_{-\infty}^{\nu}\tilde{z}_{-1}^{(-)}(s)\vartheta^{(-)}(s)G_{k-1}^{(-)}(s)\mathrm{d}s,\\
\vartheta^{(-)}(\tau)&=\exp\left[-\int_0^\tau A\left(\alpha_1(\sigma)+Q_0^{(-)}y(\nu),\varphi(0),\sigma\right)\mathrm{d}\nu\right].
\end{aligned}
$$

Homoplastically, we have

$$
Q_k^{(+)}y(\tau)=\left(p_k-\bar{y}_k^{(+)}(\sigma)\right)\frac{\tilde{z}_{-1}^{(+)}(\tau)}{\tilde{z}_{-1}^{(+)}(0)}+\tilde{z}_{-1}^{(+)}(\tau)\int_0^\tau\frac{J^{(+)}(\nu)}{[\tilde{z}_{-1}^{(+)}(\nu)]^2\vartheta^{(+)}(\nu)}\mathrm{d}\nu, \tag{2.24}
$$

where

$$
\begin{aligned}
J^{(+)}(\nu)&=\int_{\infty}^{\nu}\tilde{z}_{-1}^{(+)}(s)\vartheta^{(+)}(s)G_{k-1}^{(+)}(s)\mathrm{d}s,\\
\vartheta^{(+)}(\tau)&=\exp\left[-\int_0^\tau A\left(\alpha_2(\sigma)+Q_0^{(+)}y(\nu),\alpha_1(0),\sigma\right)\mathrm{d}\nu\right].
\end{aligned}
$$

Similarly, we have

Lemma 2.2. *Under Assumptions 1.1–1.3,* $Q_k^{(\mp)}y(\tau)$ *satisfy the exponential estimates*

$$
\left|Q_k^{(\mp)}y(\tau)\right|\le C_{k1}\exp(-\kappa|\tau|),\quad \left|Q_{k-1}^{(\mp)}z(\tau)\right|\le C_{k2}\exp(-\kappa|\tau|),
$$

where C_{k1}, C_{k2}, $\kappa>0$ *are some positive constants.*

2.3. Coefficients p_k in expansion (2.3)

From (2.19), we have

$$\int_{\alpha_1(\sigma)}^{p_0} A(s,\varphi(0),\sigma)\mathrm{d}s = \int_{\alpha_2(\sigma)}^{p_0} A(s,\alpha_1(0),\sigma)\mathrm{d}s. \tag{2.25}$$

Hence

$$\begin{aligned} G(p) \equiv & \int_{\alpha_2(\sigma)}^{p_0} A(s,\varphi(0),\sigma)\mathrm{d}s - \int_{\alpha_1(\sigma)}^{p_0} A(s,\alpha_1(0),\sigma)\mathrm{d}s \\ = & \int_{\alpha_2(\sigma)}^{\alpha_1(\sigma)} A(s,\varphi(0),\sigma)\mathrm{d}s = 0, \end{aligned}$$

this actually is Assumption 1.4 with respect to σ and hence we cannot obtain p_0 by $G(p) = 0$. Thus we pay attention to $\{Q_0^{(-)}z,\ Q_1^{(-)}y\}$, it is obtained that

$$\begin{cases} \dfrac{\mathrm{d}Q_0^{(-)}z}{\mathrm{d}\tau} = A\left(\alpha_1(\sigma) + Q_0^{(-)}y, \varphi(0), \sigma\right) Q_0^{(-)}z + f_1^{(-)}(\tau), \\ \dfrac{\mathrm{d}Q_1^{(-)}y}{\mathrm{d}\tau} = Q_0^{(-)}z, \\ Q_1^{(-)}y(0) = p_1 - \bar{y}_1^{(-)}(\sigma), \quad Q_1^{(-)}y(-\infty) = 0, \end{cases} \tag{2.26}$$

wherein

$$\begin{aligned} f_1^{(-)}(\tau) = & \Big\{ A_y(\alpha_1(\sigma) + Q_0^{(-)}y, \varphi(0), \sigma)\big[\alpha_1'(\sigma)\tau + \bar{y}_1^{(-)}(\sigma) + Q_0^{(-)}y\big] \\ & + \big[A_\varphi(\alpha_1(\sigma) + Q_0^{(-)}y, \varphi(0), \sigma)\varphi'(0) + A_t(\alpha_1(\sigma) + Q_0^{(-)}y, \varphi(0), \sigma)\big]\tau \Big\} Q_{-1}^{(-)}z \\ & + \big[A(\alpha_1(\sigma) + Q_0^{(-)}y, \varphi(0), \sigma) - A(\alpha_1(\sigma), \varphi(0), \sigma)\big]\bar{z}_0^{(-)}(\sigma) \\ & + B(\alpha_1(\sigma) + Q_0^{(-)}y, \varphi(0), \sigma) - B(\alpha_1(\sigma), \varphi(0), \sigma). \end{aligned}$$

It is easy to find that $f_1^{(-)}(\tau)$ is irrelevant to $Q_0^{(-)}z, Q_1^{(-)}y, p_1$, and the first equation of (2.26) can be rewritten as

$$\frac{\mathrm{d}Q_0^{(-)}z}{\mathrm{d}\tau} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left[A\Big(\alpha_1(\sigma) + Q_0^{(-)}y, \varphi(0), \sigma\Big)Q_1^{(-)}y\right] + f_1^{(-)}(\tau),$$

consequently, it implied

$$Q_0^{(-)}z(\tau) = A\Big(\alpha_1(\sigma) + Q_0^{(-)}y, \varphi(0), \sigma\Big)Q_1^{(-)}y + \int_{-\infty}^{\tau} f_1^{(-)}(s)\mathrm{d}s,$$

sequentially, we get

$$Q_0^{(-)}z(0) = A\Big(p_0, \varphi(0), \sigma\Big)\big[p_1 - \bar{y}_1^{(-)}(\sigma)\big] + \int_{-\infty}^{0} f_1^{(-)}(s)\mathrm{d}s.$$

Similarly, it can be obtained

$$Q_0^{(+)}z(0) = A\Big(p_0, \alpha_1(0), \sigma\Big)\big[p_1 - \bar{y}_1^{(+)}(\sigma)\big] + \int_{+\infty}^{0} f_1^{(+)}(s)\mathrm{d}s,$$

where

$$f_1^{(+)}(\tau)=\Big\{A_y(\alpha_2(\sigma)+Q_0^{(+)}y,\alpha_1(0),\sigma)\big[\alpha_2'(\sigma)\tau+\bar{y}_1^{(+)}(\sigma)+Q_0^{(+)}y\big]$$
$$+\big[A_\varphi(\alpha_2(\sigma)+Q_0^{(+)}y,\alpha_1(0),\sigma)\alpha_1'(0)+A_t(\alpha_2(\sigma)+Q_0^{(+)}y,\alpha_1(0),\sigma)\big]\tau\Big\}Q_{-1}^{(+)}z$$
$$+\big[A(\alpha_2(\sigma)+Q_0^{(+)}y,\alpha_1(0),\sigma)-A(\alpha_2(\sigma),\alpha_1(0),\sigma)\big]\bar{z}_0^{(+)}(\sigma)$$
$$+B(\alpha_2(\sigma)+Q_0^{(+)}y,\alpha_1(0),\sigma)-B(\alpha_2(\sigma),\alpha_1(0),\sigma).$$

For the sake of convenience, define

$$\tilde{f}_1(s)=\begin{cases} f_1^{(-)}(s), & s\in(-\infty,0],\\ f_1^{(+)}(s), & s\in[0,+\infty),\end{cases}$$

according to (2.19), it can be achieved

$$\alpha_2'(\sigma)-\alpha_1'(\sigma)=\left[A\big(p_0,\varphi(0),\sigma\big)\big[p_1-\bar{y}_1^{(-)}(\sigma)\big]+\int_{-\infty}^{0}f_1^{(-)}(s)\mathrm{d}s\right]$$
$$-\left[A\big(p_0,\alpha_1(0),\sigma\big)\big[p_1-\bar{y}_1^{(+)}(\sigma)\big]+\int_{+\infty}^{0}f_1^{(+)}(s)\mathrm{d}s\right]$$
$$=A\big(p_0,\alpha_1(0),\sigma\big)\Big[\bar{y}_1^{(+)}(\sigma)-\bar{y}_1^{(-)}(\sigma)\Big]+\int_{-\infty}^{+\infty}\tilde{f}_1(s)\mathrm{d}s.$$

Let

$$H(p)=A\left(p,\alpha_1(0),\sigma\right)\Big[\bar{y}_1^{(+)}(\sigma)-\bar{y}_1^{(-)}(\sigma)\Big]$$
$$+\alpha_1'(\sigma)-\alpha_2'(\sigma)+\int_{-\infty}^{+\infty}\tilde{f}_1(s)\mathrm{d}s, \tag{2.27}$$

then $H(p)$ is a function with respect to p.

Below, the assumption about $H(p)$ is presented.

Assumption 2.1. There exists a solution $p=p_0$ for $H(p)=0$, and $\dfrac{\mathrm{d}H(p_0)}{\mathrm{d}p}\neq 0$.

The sewing conditions for determining coefficients p_k take the following form

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{y}_{k-1}^{(-)}(\sigma)+\frac{\mathrm{d}}{\mathrm{d}\tau}Q_k^{(-)}y(0)=\frac{\mathrm{d}}{\mathrm{d}t}\bar{y}_{k-1}^{(+)}(\sigma)+\frac{\mathrm{d}}{\mathrm{d}\tau}Q_k^{(+)}y(0),\quad k\geq 1,$$

in other word, the sewing conditions write

$$\bar{z}_{k-1}^{(-)}(\sigma)+Q_{k-1}^{(-)}z(0)=\bar{z}_{k-1}^{(+)}(\sigma)+Q_{k-1}^{(+)}z(0),\quad k\geq 1. \tag{2.28}$$

Recall (1.6)–(1.7), which yield

$$\Phi^{(-)}(p_0)=\int_{\alpha_1(\sigma)}^{p_0}A(u,\varphi(0),\sigma)\mathrm{d}u,\quad \Phi^{(+)}(p_0)=\int_{\alpha_2(\sigma)}^{p_0}A(u,\varphi(0),\sigma)\mathrm{d}u. \tag{2.29}$$

Combining Eqs. (2.23), (2.24), (2.28), (2.29) with $\frac{\mathrm{d}Q_k^{(\mp)}y}{\mathrm{d}\tau}=Q_{k-1}^{(\mp)}z(\tau)$, $\frac{\mathrm{d}\bar{y}_k^{(\mp)}}{\mathrm{d}t}=\bar{z}_k^{(\mp)}(t)$, which leads to

$$\left[\frac{\mathrm{d}\Phi^{(-)}(p_0)}{\mathrm{d}p_0}-\frac{\mathrm{d}\Phi^{(+)}(p_0)}{\mathrm{d}p_0}\right]p_k$$

$$
\begin{aligned}
=&\frac{\mathrm{d}\bar{y}_k^{(+)}}{\mathrm{d}t}(\sigma)-\frac{\mathrm{d}\bar{y}_k^{(-)}}{\mathrm{d}t}(\sigma)-\frac{\mathrm{d}\Phi^{(+)}(p_0)}{\mathrm{d}p_0}\bar{y}_k^{(+)}+\frac{\mathrm{d}\Phi^{(-)}(p_0)}{\mathrm{d}p_0}\bar{y}_k^{(-)}\\
&-\int_{-\infty}^{0}\tilde{z}_{-1}^{(-)}(0)\tilde{z}_{-1}^{(-)}(s)g_{k-1}^{(-)}(s)\mathrm{d}s+\int_{\infty}^{0}\tilde{z}_{-1}^{(+)}(0)\tilde{z}_{-1}^{(+)}(s)g_{k-1}^{(+)}(s)\mathrm{d}s\\
=&\bar{y}_k^{\prime(+)}(\sigma)-\bar{y}_k^{\prime(-)}(\sigma)+\Psi,
\end{aligned}
$$

that is

$$
\frac{\mathrm{d}H(p_0)}{\mathrm{d}p}p_k=\bar{y}_k^{\prime(+)}(\sigma)-\bar{y}_k^{\prime(-)}(\sigma)+\Psi,
$$

where Ψ is relevant to p_i $(i=0,1,\cdots,k-1)$, thus Assumption 2.1 implies the existence and uniqueness of p_k.

3. Main Result

This section is devoted to the existence of solutions and estimation of remainder, the main result is presented as follows.

Theorem 3.1. *Under Assumptions 1.1–2.1, Eqs.* (1.2) *has a smooth asymptotic solution, it takes the following expressions for* $0<\varepsilon\ll 1$ *and* $\tau=(t-\sigma)/\varepsilon$

$$
y(t,\varepsilon)=\begin{cases}\displaystyle\sum_{k=0}^{n+1}\varepsilon^k\Big[\bar{y}_k^{(-)}(t)+Q_k^{(-)}y(\tau)\Big]+O\big(\varepsilon^{n+2}\big), & 0\le t\le\sigma,\\ \displaystyle\sum_{k=0}^{n+1}\varepsilon^k\Big[\bar{y}_k^{(+)}(t)+Q_k^{(+)}y(\tau)\Big]+O\big(\varepsilon^{n+2}\big), & \sigma\le t\le T.\end{cases}
$$

Proof. The solution of equations (1.2) can be compounded by the smooth solution of the left problem $P^{(-)}$: $(t,y)\in\mathbb{D}_1$,

$$
\begin{cases}\varepsilon y''^{(-)}=A\big(y^{(-)},y^{(-)}(t-\sigma),t\big)y'^{(-)}+B\big(y^{(-)},y^{(-)}(t-\sigma),t\big),\\ y^{(-)}(0,\varepsilon)=\varphi(0),\quad y^{(-)}(\sigma,\varepsilon)=\bar{p}(\varepsilon),\end{cases}\tag{3.1}
$$

and the right problem $P^{(+)}$: $(t,y)\in\mathbb{D}_2$,

$$
\begin{cases}\varepsilon y''^{(+)}=A\big(y^{(+)},y^{(+)}(t-\sigma),t\big)y'^{(+)}+B\big(y^{(+)},y^{(+)}(t-\sigma),t\big),\\ y^{(+)}(\sigma,\varepsilon)=\bar{p}(\varepsilon),\quad y^{(+)}(T,\varepsilon)=y^T,\end{cases}\tag{3.2}
$$

here $\bar{p}(\varepsilon)=p_0+\varepsilon p_1+\varepsilon^2p_2+\cdots+\varepsilon^{n+1}(p_{n+1}+\delta)$ and δ is a parameter.

The solutions $y^{(\mp)}(t,\varepsilon)$ for (3.1)–(3.2) have already been constructed in Sect. 2 respectively, and $y^{(\mp)}(t,\varepsilon)$ take the following forms for $\tau=(t-\sigma)/\varepsilon$

$$
\begin{aligned}
y^{(-)}(t,\varepsilon)&=\sum_{k=0}^{n+1}\varepsilon^k\Big[\bar{y}_k^{(-)}(t)+Q_k^{(-)}y(\tau)\Big]+O\big(\varepsilon^{n+2}\big),\quad 0\le t\le\sigma,\\
y^{(+)}(t,\varepsilon)&=\sum_{k=0}^{n+1}\varepsilon^k\Big[\bar{y}_k^{(+)}(t)+Q_k^{(+)}y(\tau)\Big]+O\big(\varepsilon^{n+2}\big),\quad \sigma\le t\le T.
\end{aligned}
$$

From the boundary value conditions of systems (3.1)–(3.2), $y^{(\mp)}(t,\varepsilon)$ is found to be continuous at the point $t=\sigma$. To sew smoothly at $t=\sigma$, we need

$$y'^{(-)}(\sigma,\varepsilon) = y'^{(+)}(\sigma,\varepsilon).$$

To this end, taking

$$I(p,\varepsilon) = y'^{(-)}(\sigma,\varepsilon) - y'^{(+)}(\sigma,\varepsilon),$$

from Sect. 2, we get

$$\begin{aligned} I(p,\varepsilon) &= \varepsilon^{n+1}\left[\bar{y}_k'^{(-)}(\sigma) - \bar{y}_k'^{(+)}(\sigma) + \frac{\mathrm{d}}{\mathrm{d}\tau}Q_{k+1}^{(-)}y(0) - \frac{\mathrm{d}}{\mathrm{d}\tau}Q_{k+1}^{(+)}y(0)\right] + O(\varepsilon^{n+2}) \\ &= \varepsilon^{n+1}\delta[A(\alpha_1(\sigma),\varphi(0),\sigma) - A(\alpha_2(\sigma),\alpha_1(0),\sigma)] + O(\varepsilon^{n+2}), \end{aligned}$$

which shows that if δ takes opposite sign for $0<\varepsilon\ll 1$, then $I(p,\varepsilon)$ is. According to intermediate value theorem, there exists $p^*\in[p_{n+1}-\delta,\ p_{n+1}+\delta]$ such that $I(p^*,\varepsilon)=0$, which allows a smooth solution $y(t,\varepsilon)$ of system (1.2) compounded by $y^{(-)}(t,\varepsilon)$ and $y^{(+)}(t,\varepsilon)$. □

4. Application

A concrete example is presented to demonstrate the effectiveness of the main result.

Example 4.1. Consider a second–order quasi–linear singularly perturbed delay differential equation in the following form $(0<\varepsilon\ll 1)$

$$\begin{cases} \varepsilon y'' = \big[y + y(t-1) - 2t\big]y', \quad 0\le t\le 2, \\ y(t) = t+2, \quad -1\le t\le 0, \quad y(2) = -2. \end{cases} \tag{4.1}$$

In what follows, the first–order smooth asymptotic solution with internal transition layers are constructed and the effects of parameter ε are shown in several figures by choosing some fixed values of ε.

For $\varepsilon=0$, the first equation of (4.1) degenerates into

$$\big[y + y(t-1) - 2t\big]y' = 0.$$

By the method of steps, it can be found

$$\alpha_1(t) = 2, \qquad \text{for } 0\le t\le 1,$$

and

$$\alpha_2(t) = -2, \quad \text{for } 1\le t\le 2,$$

where $\alpha_{1,2}(t)$ are solutions of Cauchy problems in Assumption 1.1. It is found that there takes place an internal transition phenomenon for system (4.1) on the interval [0,2] (See Fig. 2) and easy to verify that system (4.1) satisfy Assumptions 1.2–1.3 well. From (2.27), with a simple computation, it can be obtained

$$H(p) = \int_{-\infty}^{+\infty}\tilde{f}_1(s)\mathrm{d}s,$$

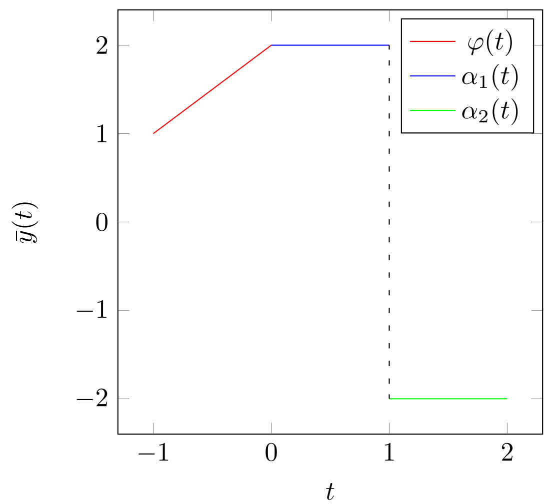

Figure 2. Internal transition phenomena of system (4.1).

where $\tilde{f}_1(s)$ has presented in Subsect. 2.3, which yields

$$\begin{aligned} H(p) &= \int_2^p \ln \frac{(2+s)(2-p)}{(2-s)(2+p)} \mathrm{d}s - \int_{-2}^p \ln \frac{(2+s)(2-p)}{(2-s)(2+p)} \mathrm{d}s \\ &= -\int_{-2}^2 \ln \frac{(2+s)(2-p)}{(2-s)(2+p)} \mathrm{d}s \\ &= -4 \ln \frac{2-p}{2+p}. \end{aligned}$$

Obviously, $H(p) = 0$ has a solution $p = 0$ and $\frac{\mathrm{d}H(0)}{\mathrm{d}p} = -1 \neq 0$, which means that Assumption 2.1 is fulfilled well.

Introducing a stretched variable $\tau = (t-1)/\varepsilon$, according to Theorem 3.1, system (4.1) has a uniformly valid asymptotic solution

$$y(t,\varepsilon) = \begin{cases} y^{(-)}(t,\varepsilon) = 2 + Q_0^{(-)}y(\tau) + O(\varepsilon), \quad t \in [0,1], \\ y^{(+)}(t,\varepsilon) = -2 + Q_0^{(+)}y(\tau) + O(\varepsilon), t \in [1,2], \end{cases} \tag{4.2}$$

where $Q_0^{(\mp)}y(\tau)$ stand for the zeroth order internal transition layers.

Taking $Q_{-1}^{(\mp)}z(\tau) = \frac{\mathrm{d}Q_0^{(\mp)}y(\tau)}{\mathrm{d}\tau}$, it can be found from (2.16)–(2.17) immediately

$$\begin{cases} \dfrac{\mathrm{d}Q_{-1}^{(-)}z}{\mathrm{d}\tau} = \left[2 + Q_0^{(-)}y\right] Q_{-1}^{(-)}z(\tau), \quad \dfrac{\mathrm{d}Q_0^{(-)}y}{\mathrm{d}\tau} = Q_{-1}^{(-)}z(\tau), \quad \tau \leq 0, \\ Q_0^{(-)}y(-\infty) = 0, \quad Q_0^{(-)}y(0) = -2, \end{cases}$$

and

$$\begin{cases} \dfrac{\mathrm{d}Q_{-1}^{(+)}z}{\mathrm{d}\tau} = \left[-2 + Q_0^{(+)}y\right] Q_{-1}^{(+)}z(\tau), \quad \dfrac{\mathrm{d}Q_0^{(+)}y}{\mathrm{d}\tau} = Q_{-1}^{(+)}z(\tau), \quad \tau \geq 0, \\ Q_0^{(+)}y(+\infty) = 0, \quad Q_0^{(+)}y(0) = 2. \end{cases}$$

It is obtained that

$$Q_0^{(-)}y(\tau) = -\frac{4\exp(2\tau)}{1+\exp(2\tau)}, \quad \tau \le 0,$$

and

$$Q_0^{(+)}y(\tau) = \frac{4}{1+\exp(2\tau)}, \quad \tau \ge 0.$$

Therefore, the asymptotic solution of (4.1) takes the following form

$$y(t,\varepsilon) = \begin{cases} 2 - \dfrac{4\exp(2(t-1)/\varepsilon)}{1+\exp(2(t-1)/\varepsilon)} + O(\varepsilon), & t \in [0,1], \\ -2 + \dfrac{4}{1+\exp(2(t-1)/\varepsilon)} + O(\varepsilon), & t \in [1,2]. \end{cases}$$

In the following, we give the graphics of the first–order smooth asymptotic solution $y_1(t,\varepsilon)$ for some given ε, see Fig. 3.

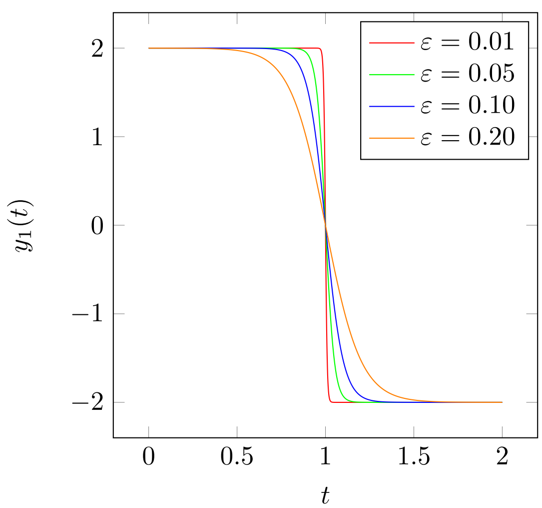

Figure 3. The graphics of the first–order smooth asymptotic solution $y_1(t,\varepsilon)$ for some given ε.

Acknowledgment

The authors are grateful to the editors and referees for their useful comments which improved the contents of this article.

References

[1] Z. Bartoszewski and A. Baranowska, *Solving boundary value problems for second order singularly perturbed delay differential equations by 'ε'–approximate fixed–point method*, Math. Model. Anal., 2015, 20(3), 369–381.

[2] Y. Cao, *Bifurcations in an internet congestion control system with distributed delay*, Appl. Math. Comput., 2019, 347, 54–63.

[3] È. Èl'sgol'c, *Qualitative Methods in Mathematical Analysis*, American Mathematical Society, Providence, 1964.

[4] F. Erdogan and Z. Cen, *A uniformly almost second order convergent numerical method for singularly perturbed delay differential equations*, J. Comput. Appl. Math., 2018, 333, 282–394.

[5] H. Gao, X. Meng and T. Chen, *Stabilization of networked control systems with a new delay characterization*, IEEE Trans. Automat. Control, 2008, 53(9), 2142–2148.

[6] S. Hu and Q. Zhu, *Stochastic optimal control and analysis of stability of networked control systems with long delay*, Automatica, 2003, 39(11), 1877–1884.

[7] H. Kargar, J. Zarei and R. Razavi–Far, *Robust fault detection filter design for nonlinear networked control systems with time–varying delays and packet dropout*, Circuits, Systems, Signal Process., 2019, 38(1), 63–84.

[8] Y. Kuang, *Delay Differential Equations with Applications in Population Dynamics*, Academic Press, Boston, 1993.

[9] C. Kuehn, *Multiple Time Scale Dynamics*, Applied Mathematical Sciences, Springer, 2015.

[10] C. Lange and R. Miura, *Singular perturbation analysis of boundary–value problems for differential–difference equations. VI. small shifts with rapid oscillations*, SIAM J. Appl. Math., 1994, 54(1), 273–283.

[11] M. Ni and W. Lin, *The asymptotic solutions of delay singularly perturbed differential difference equations* (in Chinese), Acta Math. Sci., 2010, 30A(6), 1413–1423. Available from: http://121.43.60.238/sxwlxbA/CN/Y2010/V30/I6/1413.

[12] M. Ni, *On the internal layer for a singularly perturbed system of second–order delay differential equations*, Diff. Equ., 2013, 49(8), 941–954.

[13] N. Olgac, H. Elmali, M. Hosek and M. Renzulli, *Active vibration control of distributed systems using delayed resonator with acceleration feedback*, J. Dyn. Syst., Measure., Control, 1997, 119(3), 380–389.

[14] V. Rožkov, *On the asymptotic behaviour of solutions of equations of neutral type with small time lag* (in Russian), Ž. Vyčisl. Mat. i Mat. Fiz., 1964, 4(4, suppl.), 161–175.

[15] V. Rožkov, *Asymptotic formulae in terms of a small time lag for the solution of an equation of neutral type with two retarded arguments* (in Russian), Ž. Vyčisl. Mat. i Mat. Fiz., 1964, 4(4, suppl.), 176–182.

[16] V. Rožkov, *Equations of neutral types with variable small retardation* (in Russian), Differencial'nye Uravnenija, 1966, 2, 407–416.

[17] V. Rožkov, *An equation of neutral type with a small retardation dependent on the unknown function* (in Russian), Izv. Vysš. Učebn. Zaved. Mat., 1968, 1968(6), 90–97.

[18] J. Ren and W. Ge, *Singularly perturbed boundary value problems for semi–linear retarded differential equations with nonlinear boundary conditions*, Appl. Math. Mech., 2003, 24(12), 1450–1455.

[19] H. Tian, *Asymptotic expansion for the solutions of singularly perturbed delay differential equations*, J. Math. Anal. Appl., 2003, 281(2), 678–696.

[20] A. Vasil'eva, *An equation of neutral type with small lag* (in Russian), Dokl. Akad. Nauk SSSR, 1962, 145, 495–497.

[21] A. Vasil'eva and V. Butuzov, *Asymptotic methods in the theory of singular perturbations* (in Russian), Vyssh. Shkola, Moscow, 1990.

[22] V. Volterra, *Fluctuations in the abundance of a species considered mathematically*, Nature, 1926, 118(2972), 558–560.

[23] A. Wang and M. Ni, *The interior layer for a nonlinear singularly perturbed differential–difference equation*, Acta Math. Sci., 2012, 32B(2), 695–709.

[24] N. Wang and M. Ni, *The interior layer phenomena for a class of singularly perturbed delay–differential equations*, Acta Math. Sci., 2013, 33B(2), 532–542.