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PERIODIC SOLUTION OF A STOCHASTIC SIQR EPIDEMIC MODEL INCORPORATING MEDIA COVERAGE*

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Abstract In this paper, we propose a stochastic SIQR epidemic model with periodic parameters and media coverage. Firstly, we study that the stochastic non-autonomous periodic system has a unique global positive solution. Secondly, by using the Khasminskii's theory, we prove that this stochastic periodic system has a nontrivial positive periodic solution. Then, we obtain the sufficient condition for extinction of the disease. Finally, numerical simulations are employed to illustrate our theoretical analysis.

Keywords Stochastic model, periodic, extinction, media coverage, SIQR.

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1. Introduction

In the study of epidemiology, mathematical models play an important role. Various epidemic models have been proposed and explored extensively, and great progress has been achieved in the studies of disease control and prevention [1,2,7,8]. Recently, mathematical models have been widely used to analyze the mechanisms of infectious diseases, such as polio, diphtheria, tuberculosis, tetanus, pertussis, measles, hepatitis B, etc [3, 11, 12, 14, 27, 30, 40], and various epidemic models of population dynamics have been proposed [6, 20, 28, 31, 37, 39]. For example, Nistal et al. [40] studied the stability and equilibrium points of multistaged SI(n)R epidemic models. Zhang et al. [39] investigated the asymptotic behavior of global positive solution to a stochastic SIRS epidemic model incorporating media coverage and saturated incidence rate. Ma et al. [20] considered an SIQR epidemic model with standard

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incidence rate and their model can be expressed as follows

$$\begin{cases} \frac{dS}{dt} = \Lambda - \beta \frac{SI}{N} - \mu S, \\ \frac{dI}{dt} = \beta \frac{SI}{N} - (\mu + \alpha + \delta + \gamma) I, \\ \frac{dQ}{dt} = \delta I - (\mu + \alpha + \epsilon) Q, \\ \frac{dR}{dt} = \gamma I + \epsilon Q - \mu R, \end{cases} \tag{1.1}$$

where S, I, R denote the number of susceptible, infective and removed, respectively, Q denotes the number of quarantined, $N = S + I + Q + R$ denotes the number of total population individuals. The parameter Λ denotes the recruitment rate of S corresponding to births and immigration, β is the disease transmission coefficient between compartments S and I, μ denotes the natural death rate, γ and ϵ are the recover rates from groups I, Q to R, δ represents the removal rate from I, α denotes the disease-caused death rate of I and Q. All parameters are assumed to be nonnegative and $\mu, \Lambda > 0$. Motivated by the system (1.1), liu et al. [15] developed a stochastic multigroup SIQR epidemic model with standard incidence rates and studied the existence of a stationary distribution of the positive solutions to the model, and established sufficient conditions for extinction of the disease.

When an infectious disease emerges and prevails in a region, the primary task of disease control units is to exert all efforts to prevent the spread of this disease. One of the important prevention measures is educating people with the correct preventive knowledge of the disease through mass media and other platforms at the first opportunity [5]. Mass media including television, radio, newspaper, networks and so on potentially affect the behavior of the people, which can be used to deliver preventive healthcare messages for precaution and avoidance of negative behavior as a result of panic and to present updated information about the disease. Thus, media coverage is an urgent issue that needs attention [4, 21, 34]. And in recent years, a significant number of epidemic models incorporating media coverage have been proposed and discussed [4, 5, 17, 22]. Cui et al. [4] developed an SIS model to consider the impact of media and eduction on the spread of infectious disease. Liu and Li [22] proposed a drug model to discuss the impact of media coverage on the spread and control of drug addiction. In Ref. [17], Liu and Zhang consider a SIS epidemic model on two patches incorporating media coverage. Recently, many mathematical models have been proposed to investigate the impact of media coverage on the transmission dynamics of infectious disease. Especially, Cui et al. [4], Tchuenche et al. [32] incorporated a nonlinear function of the number infective individuals in their transmission term to investigate the effects of media coverage on the transmission dynamic where β_1 is the contact rate before media alert, the terms $\beta_2 I/(m + I)$ measure the effect of reduction of the contact rate when infectious individuals are reported in the media. Because the coverage report cannot prevent disease from spreading completely, we have $\beta_1 \geq \beta_2 > 0$. The half-saturation constant $m > 0$ reflects the impact of media coverage on the contact transmission. The function $I/(m + I)$ is a continuous bounded function that takes into account disease saturation or psychological effects [29]. Hence, considering the effects of media coverage on the transmission dynamic, model (1.1) can be modified as follows

$$\begin{cases} \frac{dS}{dt} &= \Lambda - \left(\beta_1 - \frac{\beta_2 I}{m+I}\right)\frac{SI}{N} - \mu S, \\ \frac{dI}{dt} &= \left(\beta_1 - \frac{\beta_2 I}{m+I}\right)\frac{SI}{N} - (\mu + \alpha + \delta + \gamma)I, \\ \frac{dQ}{dt} &= \delta I - (\mu + \alpha + \epsilon)Q, \\ \frac{dR}{dt} &= \gamma I + \epsilon Q - \mu R. \end{cases} \tag{1.2}$$

In epidemiology models, many authors only considered the constant coefficients in models and neglected the time-dependent factors. However, the time-dependent factors play a very important role in the spread of infectious disease and the fluctuation has often been observed in the incidence of many infectious diseases. In particular, the periodic fluctuations are very common in the transmission of infectious diseases. Therefore, it is more realistic to assume that the coefficients are time-dependent or periodic (see [10, 36]).

In addition, real life is full of randomness and stochasticity, epidemic models are always affected by the environmental noise in an ecosystem. Therefore, numerous scholars have used stochastic differential equations to study the dynamic behaviors of stochastic biological mathematical models (see [13, 18, 19, 25, 26, 33, 35, 38]). For example, scholars obtained thresholds of the stochastic system which determine the extinction and persistence of the epidemic in [25, 33]. Lin et al. [19] prove that there is one nontrivial positive periodic solution of this stochastic model. Based on the discussion above, in this paper, we consider a stochastic non-autonomous SIQR model with periodic coefficients

$$\begin{cases} dS(t) &= \Big[\Lambda(t) - \Big(\beta_1(t) - \frac{\beta_2(t)I(t)}{m(t)+I(t)}\Big)\frac{S(t)I(t)}{N} - \mu(t)S(t)\Big]dt + \sigma_1(t)S(t)dB_1(t), \\ dI(t) &= \Big[\Big(\beta_1(t) - \frac{\beta_2(t)I(t)}{m(t)+I(t)}\Big)\frac{S(t)I(t)}{N} - (\mu(t) + \alpha(t) + \delta(t) + \gamma(t))I(t)\Big]dt \\ & \quad +\sigma_2(t)I(t)dB_2(t), \\ dQ(t) &= \Big[\delta(t)I(t) - (\mu(t) + \alpha(t) + \epsilon(t))Q(t)\Big]dt + \sigma_3(t)Q(t)dB_3(t), \\ dR(t) &= \Big[\gamma(t)I(t) + \epsilon(t)Q(t) - \mu(t)R(t)\Big]dt + \sigma_4(t)R(t)dB_4(t). \end{cases} \tag{1.3}$$

Where $B_i(t)(i = 1, 2, 3, 4)$ are independent Brownian motions and $\sigma_i(t)(i = 1, 2, 3, 4)$ are the coefficients of the effects of environmental stochastic perturbations on $S(t)$, $I(t), Q(t), R(t)$. The parameter functions $\Lambda(t), \beta_1(t), \beta_2(t), m(t), \mu(t), \alpha(t), \delta(t), \gamma(t)$, $\epsilon(t)$ and $\sigma_i(t)(i = 1, 2, 3, 4)$ are positive and continuous periodic functions with positive periodic $\mathbf{T}$.

Throughout this paper, we assume that $(\Omega, \{\mathbf{F}\}_{t\geq 0}, \mathbb{P})$ is a complete probability space with a filtration $\{\mathbf{F}\}_{t\geq 0}$ satisfying the usual conditions. Let $B_i(t)(i = 1, 2, 3, 4)$ be Brownian motions defined on this probability space. Also, let $\mathbb{R}^4_+ = \{\mathbf{X} \in \mathbb{R}^4, x_i > 0, 1 \leq i \leq 4\}$ and $\mathbf{X}(t) = (S(t), I(t), Q(t), R(t))$. If $f(t)$ is an integral function on $[0, +\infty)$, define $\langle f\rangle_t = \frac{1}{t}\int_0^t f(s)ds, t > 0$. If $f(t)$ is a bounded function on $[0, +\infty)$, define $f^l = \inf_{t\in[0,+\infty)} f(t)$ and $f^u = \sup_{t\in[0,+\infty)} f(t)$.

The objectives of this paper are as follows. In this paper, we will study the influence of media coverage on the spread of infectious disease by investigating a stochastic SIQR epidemic model incorporating media coverage. From a mathematical point of view, the existence of unique global positive solution of this stochastic system will be studied to show that the system is meaningful in biology. And then, in order to obtain the conditions for the disease to persist or extinct, we will s-tudy the existence of nontrivial periodic solutions and the extinct condition of the disease.

The rest of the paper is organized as follows. In Section 2, we show that there exists a unique global positive solution of system (1.3). In Section 3, we verify that there is a nontrivial positive periodic solution of system (1.3). In Section 4, we establish sufficient conditions for extinction of system (1.3). In Section 5, we give two examples to support the theoretical prediction.

2. Existence and uniqueness of the global positive solution

In this section, we use the Lyapunov function method to prove that the solution of system (1.3) is global and positive.

Theorem 2.1. *For any initial value $(S(0), I(0), Q(0), R(0)) \in \mathbb{R}^4_+$, there is a unique positive solution $(S(t), I(t), Q(t), R(t))$ of system (1.3) on $t \geq 0$ and the solution will remain in $\mathbb{R}^4_+$ with probability one, namely, $(S(t), I(t), Q(t), R(t)) \in \mathbb{R}^4_+$ for all $t \geq 0$ almost surely.*

Proof. Note that the coefficients of the model (1.3) are locally Lipschitz conditions, then for any given initial value $(S(0), I(0), Q(0), R(0)) \in \mathbb{R}^4_+$, there is a unique positive local solution $(S(t), I(t), Q(t), R(t))$ on $t \in [0, \tau_e)$, where τ_e is the explosion time [23]. To demonstrate that this solution is global, we only need to prove that $\tau_e = \infty$ a.s.

Let $k_0 > 0$ be sufficiently large for any initial value $S(0), I(0), Q(0)$ and $R(0)$ lying within the interval $[1/k_0, k_0]$. For each integer $k \geq k_0$, define the following stopping time

$$\tau_k = \inf\left\{t \in [0, \tau_e) : \min\{S(t), I(t), Q(t), R(t)\} \leq \frac{1}{k} \text{or} \max\{S(t), I(t), Q(t), R(t)\} \geq k\right\}$$

where we set $\inf \emptyset = \infty$ (as usual $\emptyset$ denotes the empty set). Clearly, τ_k is increasing as $k \to \infty$. Let $\tau_\infty = \lim_{k\to\infty} \tau_k$, hence $\tau_\infty \leq \tau_e$ a.s. Next, we only need to verify $\tau_\infty = \infty$ a.s. If this statement is false, then there exist two constants $T > 0$ and $\varepsilon \in (0, 1)$ such that $\mathbb{P}\{\tau_\infty \leq T\} > \varepsilon$. Thus there is an integer $k_1 \geq k_0$ such that $\mathbb{P}\{\tau_k \leq T\} \geq \varepsilon$, for all $k \geq k_1$.

Define a C^2-function $V : \mathbb{R}^4_+ \to \mathbb{R}_+$ as follows

$$V(S, I.Q, R) = S - a - a \ln \frac{S}{a} + I - 1 - \ln I + Q - 1 - \ln Q + R - 1 - \ln R,$$

the nonnegativity of this function can be obtained from $x - 1 - \ln x \geq 0, x > 0$, and the parameter a will be determined later.

Applying Itô's formula yields

$$\begin{aligned} dV(S, I, Q, R) = LV dt &+ (S - a)\sigma_1(t) dB_1(t) + (I - 1)\sigma_2(t) dB_2(t) \\ &+ (Q - 1)\sigma_3(t) dB_3(t) + (R - 1)\sigma_4(t) dB_4(t), \end{aligned}$$

where

$$
\begin{aligned}
LV =& (1-\frac{a}{S})\Big[\Lambda(t)-\Big(\beta_1(t)-\frac{\beta_2(t)I(t)}{m(t)+I(t)}\Big)\frac{S(t)I(t)}{N}-\mu(t)S(t)\Big]+\frac{a\sigma_1^2(t)}{2}\\
&+(1-\frac{1}{I})\Big[\Big(\beta_1(t)-\frac{\beta_2(t)I(t)}{m(t)+I(t)}\Big)\frac{S(t)I(t)}{N}-(\mu(t)+\alpha(t)+\delta(t)+\gamma(t))I(t)\Big]\\
&+\frac{\sigma_2^2(t)}{2}+(1-\frac{1}{Q})\Big[\delta(t)I(t)-(\mu(t)+\alpha(t)+\epsilon(t))Q(t)\Big]+\frac{\sigma_3^2(t)}{2}\\
&+(1-\frac{1}{R})\Big[\gamma(t)I(t)+\epsilon(t)Q(t)-\mu(t)R(t)\Big]+\frac{\sigma_4^2(t)}{2},
\end{aligned}
$$

which implies that

$$
\begin{aligned}
LV =& \Lambda(t)-\Big(\beta_1(t)-\frac{\beta_2(t)I(t)}{m(t)+I(t)}\Big)\frac{S(t)I(t)}{N}-\mu(t)S(t)+\frac{aI(t)}{N}\times\\
&\Big(\beta_1(t)-\frac{\beta_2(t)I(t)}{m(t)+I(t)}\Big)+a\mu(t)-\frac{a\Lambda(t)}{S}+\Big(\beta_1(t)-\frac{\beta_2(t)I(t)}{m(t)+I(t)}\Big)\frac{S(t)I(t)}{N}\\
&-(\mu(t)+\alpha(t)+\delta(t)+\gamma(t))I(t)-\Big(\beta_1(t)-\frac{\beta_2(t)I(t)}{m(t)+I(t)}\Big)\frac{S(t)}{N}\\
&+(\mu(t)+\alpha(t)+\delta(t)+\gamma(t))-(\mu(t)+\alpha(t)+\epsilon(t))Q(t)+\delta(t)I(t)-\frac{I(t)\delta(t)}{Q(t)}\\
&+(\mu(t)+\alpha(t)+\epsilon(t))+\gamma(t)I(t)+\epsilon(t)Q(t)-\mu(t)R(t)+\mu(t)-\frac{I(t)\gamma(t)}{R(t)}\\
&-\frac{Q(t)\epsilon(t)}{R(t)}+\frac{a\sigma_1^2(t)}{2}+\frac{\sigma_2^2(t)}{2}+\frac{\sigma_3^2(t)}{2}+\frac{\sigma_4^2(t)}{2}\\
\le& \Lambda(t)+\frac{aI(t)}{N}\beta_1(t)+a\mu(t)-(\mu(t)+\alpha(t))I(t)+3\mu(t)+2\alpha(t)+\epsilon(t)\\
&+\gamma(t)+\delta(t)+\frac{a\sigma_1^2(t)}{2}+\frac{\sigma_2^2(t)}{2}+\frac{\sigma_3^2(t)}{2}+\frac{\sigma_4^2(t)}{2}\\
\le& \Lambda^u-(\mu^l+\alpha^l-\frac{a\beta_1^u}{N})I+a\mu^u+3\mu^u+2\alpha^u+\epsilon^u+\gamma^u+\delta^u\\
&+\frac{a\sigma_1^{2u}+\sigma_2^{2u}+\sigma_3^{2u}+\sigma_4^{2u}}{2}.
\end{aligned}
$$

Choose $a=\frac{N(\mu^l+\alpha^l)}{\beta_1^u}$ such that $\mu^l+\alpha^l-\frac{a\beta_1^u}{N}=0$, then

$$
LV\le\Lambda^u+a\mu^u+3\mu^u+2\alpha^u+\epsilon^u+\gamma^u+\delta^u+\frac{a\sigma_1^{2u}+\sigma_2^{2u}+\sigma_3^{2u}+\sigma_4^{2u}}{2}:=K,
$$

where K_1 is a positive constant.

The remainder of the proof follows as that in [24]. The proof is completed. □

3. Existence of nontrivial T-periodic solution

In this section, we verify that the model (1.3) admits at least one nontrivial positive **T**-periodic solution.

Definition 3.1 ([9]). A stochastic process $r(t) = r(t, \omega)(-\infty < t < +\infty)$ is said to be periodic with period $\mathbf{T}$ if for every finite sequence of numbers $t_1, t_2, \cdots, t_n$ the joint distribution of random variables $r(t_1+h), r(t_2+h), \cdots, r(t_n+h)$ is independent of h, where $h = k\mathbf{T}, k = \pm1, \pm2, \cdots$.

Consider the following periodic stochastic equation

$$dx(t) = f(t, x(t))dt + g(t, x(t))dB(t), x \in \mathbb{R}^n, \tag{3.1}$$

where function $f(t)$ and $g(t)$ are $\mathbf{T}$-periodic in t.

Lemma 3.1 ([9]). *Assume that system* (3.1) *admits a unique global solution. Suppose further that there exists a function $V(t, x) \in C^2$ in $\mathbb{R}$ which is $\boldsymbol{T}$-periodic in t and satisfies the following conditions*

(A_1): $\inf_{\|x\|>\mathbb{R}} V(t, x) \to \infty$ *as* $\mathbb{R} \to \infty$;

(A_2): $LV(t, x) \leq -1$ *outside some compact set, where the operator L is given by*

$$LV(t, x) = V_t(t, x) + V_x(t, x)f(t, x) + \frac{1}{2}trace(g^T(t, x)V_{xx}(t, x)g(t, x)).$$

Then the system (3.1) *has a $\boldsymbol{T}$-periodic solution.*

Define a parameter

$$\Re_1 = \frac{\langle\Lambda(\beta_1 - \beta_2)\rangle_{\mathbf{T}}}{N\langle\mu + \frac{\sigma_1^2}{2}\rangle_{\mathbf{T}}\langle\mu + \alpha + \delta + \gamma + \frac{\sigma_2^2}{2}\rangle_{\mathbf{T}}}.$$

Theorem 3.1. *If $\Re_1 > 1$, then there exists a nontrivial positive $\boldsymbol{T}$-periodic solution of model* (1.3).

Proof. Define a C^2-function$V : [0, +\infty) \times \mathbb{R}^4_+ \to \mathbb{R}$:

$$\begin{aligned}
&V(S, I, Q, R) = M(V_1(S, I) + \omega(t)) + V_2(S, I, Q, R) + V_3(S) + V_4(Q) + V_5(R),\\
&V_1(S, I) = -C_1 \ln S - C_2 \ln I, V_2(S, I, Q, R) = \frac{1}{\theta + 1}(S + I + Q + R)^{\theta+1},\\
&V_3(S) = -\ln S, V_4(Q) = -\ln Q, V_5(R) = -\ln R,
\end{aligned}$$

where

$$\begin{aligned}
&C_1 = \frac{\langle\Lambda\rangle_{\mathbf{T}}}{\langle\mu + \frac{\sigma_1^2}{2}\rangle_{\mathbf{T}}}, C_2 = \frac{\langle\Lambda\rangle_{\mathbf{T}}}{\langle\mu + \alpha + \delta + \gamma + \frac{\sigma_2^2}{2}\rangle_{\mathbf{T}}},\\
&0 < \theta < \min\Big\{1, \frac{2\mu^l}{(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)^u}\Big\},
\end{aligned}$$

and $K_2 > 0$ satisfies the following condition

$$-M\lambda + C \leq -2,$$

where

$$\lambda = 2\langle\Lambda\rangle_{\mathbf{T}}(\Re_1^{\frac{1}{2}} - 1),$$

and

$$C = \sup_{(S,I,Q,R)\in R^4_+} \Big\{ -\frac{1}{2}(\mu^l - \frac{1}{2}\theta(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)^u)(S^{\theta+1} + I^{\theta+1} + Q^{\theta+1} + R^{\theta+1})$$
$$+D + 3\mu^u + \alpha^u + \epsilon^u + \frac{\sigma_1^{2u} + \sigma_3^{2u} + \sigma_4^{2u}}{2}\Big\},$$

where

$$D = \sup_{(S,I,Q,R)\in R^4_+} \Big\{\Lambda^u(S+I+Q+R)^\theta - \frac{1}{2}(\mu^l - \frac{1}{2}\theta(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)^u)$$
$$\times(S+I+Q+R)^{\theta+1}\Big\}.$$

Obviously, $V(S,I,Q,R)$ is a **T**-periodic function in t and satisfies

$$\liminf_{k\to\infty,(S,I,Q,R)\in R^4_+\setminus U_k} V(S,I,Q,R) = \infty,$$

where $U_k = (1/k,k)\times(1/k,k)\times(1/k,k)\times(1/k,k)$ and $k>1$ is a sufficiently large number. Therefore, the condition (A_1) in the Lemma 3.1 holds.

Next we prove that the condition (A_2) in Lemma 3.1 holds. By the Itô's formula, we obtain

$$\begin{aligned} LV_1 = &-\frac{C_1}{S}\Big[\Lambda(t) - \Big(\beta_1(t) - \frac{\beta_2(t)I}{m(t)+I}\Big)\frac{SI}{N} - \mu(t)S\Big] + \frac{C_1\sigma_1^2(t)}{2} \\ &-\frac{C_2}{I}\Big[\Big(\beta_1(t) - \frac{\beta_2(t)I}{m(t)+I}\Big)\frac{SI}{N} - (\mu(t)+\alpha(t)+\delta(t)+\gamma(t))I\Big] + \frac{C_2\sigma_2^2(t)}{2} \\ \le &-\frac{C_1\Lambda(t)}{S} - \frac{C_2S}{N}(\beta_1(t)-\beta_2(t)) + \frac{C_1\beta_1(t)I}{N} + C_1(\mu(t) + \frac{\sigma_1^2(t)}{2}) \\ &+C_2(\mu(t)+\alpha(t)+\delta(t)+\gamma(t)+\frac{\sigma_2^2(t)}{2}) \\ \le &-2\sqrt{\frac{C_1C_2\Lambda(t)}{N}(\beta_1(t)-\beta_2(t))} + \frac{C_1\beta_1(t)I}{N} + C_1(\mu(t)+\frac{\sigma_1^2(t)}{2}) \\ &+C_2(\mu(t)+\alpha(t)+\delta(t)+\gamma(t)+\frac{\sigma_2^2(t)}{2}) \\ \triangleq &\ B_0(t) + \frac{C_1\beta_1(t)I}{N}, \end{aligned}$$

where $B_0(t) = -2\sqrt{\frac{C_1C_2\Lambda(t)}{N}(\beta_1(t)-\beta_2(t))} + C_1(\mu(t)+\frac{\sigma_1^2(t)}{2}) + C_2(\mu(t)+\alpha(t)+\delta(t)+\gamma(t)+\frac{\sigma_2^2(t)}{2})$.

Define the **T**-periodic function $\omega(t)$

$$\omega'(t) = \langle B_0\rangle_{\mathbf{T}} - B_0(t).$$

Therefore

$$\begin{aligned} L(V_1+\omega(t)) &\le \langle B_0\rangle_{\mathbf{T}} + \frac{C_1\beta_1(t)I}{N} \\ &\le -2\langle\Lambda\rangle_{\mathbf{T}}\Big(\Big(\frac{\langle\Lambda(\beta_1-\beta_2)\rangle_{\mathbf{T}}}{N\langle\mu+\frac{\sigma_1^2}{2}\rangle_{\mathbf{T}}\langle\mu+\alpha+\delta+\gamma+\frac{\sigma_2^2}{2}\rangle_{\mathbf{T}}}\Big)^{\frac{1}{2}} - 1\Big) + \frac{C_1\beta_1(t)I}{N} \end{aligned}$$

$$
\begin{aligned}
&= -2\langle\Lambda\rangle_{\mathbf{T}}(\mathfrak{R}_1^{\frac{1}{2}} - 1) + \frac{C_1\beta_1(t)I}{N} \\
&\triangleq -\lambda + \frac{C_1\beta_1^u I}{N}.
\end{aligned}
$$

Similarly, we can obtain

$$
\begin{aligned}
LV_2 &= (S+I+Q+R)^{\theta}\Big[\Lambda(t) - \mu(t)S - (\mu(t)+\alpha(t))(I+Q) - \mu(t)R\Big] \\
&\quad + \frac{1}{2}\theta(S+I+Q+R)^{\theta-1}(\sigma_1^2(t)S^2 + \sigma_2^2(t)I^2 + \sigma_3^2(t)Q^2 + \sigma_4^2(t)R^2) \\
&\le \Lambda(t)(S+I+Q+R)^{\theta} - \mu(t)(S+I+Q+R)^{\theta+1} + \frac{1}{2}\theta(S+I+Q+R)^{\theta+1} \\
&\quad \times(\sigma_1^2(t)\vee\sigma_2^2(t)\vee\sigma_3^2(t)\vee\sigma_4^2(t)) \\
&= \Lambda(t)(S+I+Q+R)^{\theta} - \Big(\mu(t) - \frac{1}{2}\theta(\sigma_1^2(t)\vee\sigma_2^2(t)\vee\sigma_3^2(t)\vee\sigma_4^2(t))\Big) \\
&\quad \times(S+I+Q+R)^{\theta+1} \\
&\le D - \frac{1}{2}\Big(\mu^l - \frac{1}{2}\theta(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2)^u\Big)(S^{\theta+1} + I^{\theta+1} + Q^{\theta+1} + R^{\theta+1}), \\
LV_3 &= -\frac{1}{S}\Big[\Lambda(t) - \Big(\beta_1(t) - \frac{\beta_2(t)I}{m(t)+I}\Big)\frac{SI}{N} - \mu(t)S\Big] + \frac{\sigma_1^2(t)}{2} \\
&\le -\frac{\Lambda(t)}{S} + \frac{\beta_1(t)I}{N} + \mu(t) + \frac{\sigma_1^2(t)}{2} \\
&\le -\frac{\Lambda^l}{S} + \frac{\beta_1^u I}{N} + \mu^u + \frac{\sigma_1^{2u}}{2}, \\
LV_4 &= -\frac{1}{Q}\Big[\delta(t)I(t) - (\mu(t)+\alpha(t)+\epsilon(t))Q(t)\Big] + \frac{\sigma_3^2(t)}{2} \\
&= -\frac{\delta(t)I}{Q} + (\mu(t)+\alpha(t)+\epsilon(t)) + \frac{\sigma_3^2(t)}{2} \\
&\le -\frac{\delta^l I}{Q} + \mu^u + \alpha^u + \epsilon^u + \frac{\sigma_3^{2u}}{2},
\end{aligned}
$$

and

$$
\begin{aligned}
LV_5 &= -\frac{1}{R}\Big[\gamma(t)I(t) + \epsilon(t)Q(t) - \mu(t)R(t)\Big] + \frac{\sigma_4^2(t)}{2} \\
&= -\frac{\gamma(t)I}{R} - \frac{\epsilon(t)Q}{R} + \mu(t) + \frac{\sigma_4^2(t)}{2} \\
&\le -\frac{\gamma^l I}{R} + \mu^u + \frac{\sigma_4^{2u}}{2}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
LV(S,I,Q,R) &= ML(V_1 + \omega(t)) + LV_2 + LV_3 + LV_4 + LV_5 \\
&\le M(-\lambda + \frac{C_1\beta_1^u I}{N}) - \frac{\Lambda^l}{S} + \frac{\beta_1^u I}{N} + \mu^u + \frac{\sigma_1^{2u}}{2} - \frac{\delta^l I}{Q} + D \\
&\quad -\frac{1}{2}\Big(\mu^l - \frac{1}{2}\theta(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2)^u\Big)(S^{\theta+1} + I^{\theta+1} + Q^{\theta+1} + R^{\theta+1}) \\
&\quad -\frac{\gamma^l I}{R} + \mu^u + \frac{\sigma_4^{2u}}{2} + \frac{\sigma_3^{2u}}{2} + \mu^u + \alpha^u + \epsilon^u
\end{aligned}
$$

$$
\begin{aligned}
&= -M\lambda + \frac{\beta_1^u I}{N}(MC_1+1) - \frac{\Lambda^l}{S} + 3\mu^u + \frac{\sigma_1^{2u}}{2} - \frac{\delta^l I}{Q} + \alpha^u + \epsilon^u + \frac{\sigma_3^{2u}}{2} \\
&\quad - \frac{1}{2}\Big(\mu^l - \frac{1}{2}\theta(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)^u\Big)(S^{\theta+1} + I^{\theta+1} + Q^{\theta+1} + R^{\theta+1}) \\
&\quad - \frac{\gamma^l I}{R} + \frac{\sigma_4^{2u}}{2} + D.
\end{aligned}
$$

Now, we construct a compact subset U such that (A_2) in Lemma 3.1 holds. Define the following bounded closed set

$$
U = \Big\{(S, I, Q, R) \in R_+^4 : \varepsilon \le S \le \frac{1}{\varepsilon}, \varepsilon \le I \le \frac{1}{\varepsilon}, \varepsilon \le Q \le \frac{1}{\varepsilon}, \varepsilon \le R \le \frac{1}{\varepsilon}\Big\},
$$

where $\varepsilon > 0$ is a sufficiently small number. In the set $R_+^4 \setminus U$, we can choose ε sufficiently small such that

$$
-\frac{\Lambda^l}{\varepsilon} + E \le -1, \tag{3.2}
$$

$$
-M\lambda + \frac{\beta_1^u \varepsilon}{N}(MC_1 + 1) + C \le -1, \tag{3.3}
$$

$$
-\frac{\delta^l}{\varepsilon} + E \le -1, \tag{3.4}
$$

$$
-\frac{\gamma^l}{\varepsilon} + E \le -1, \tag{3.5}
$$

$$
-\frac{1}{4}\Big(\mu^l - \frac{1}{2}\theta(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)^u\Big)\frac{1}{\varepsilon^{\theta+1}} + F \le -1, \tag{3.6}
$$

$$
-\frac{1}{4}\Big(\mu^l - \frac{1}{2}\theta(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)^u\Big)\frac{1}{\varepsilon^{\theta+1}} + G \le -1, \tag{3.7}
$$

$$
-\frac{1}{4}\Big(\mu^l - \frac{1}{2}\theta(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)^u\Big)\frac{1}{\varepsilon^{2(\theta+1)}} + H \le -1, \tag{3.8}
$$

$$
-\frac{1}{4}\Big(\mu^l - \frac{1}{2}\theta(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)^u\Big)\frac{1}{\varepsilon^{2(\theta+1)}} + J \le -1, \tag{3.9}
$$

where E, C, F, G, H, J are positive constants which can be found below. For the sake of convenience, we divide into eight domains

$$
\begin{aligned}
&U_1 = \Big\{(S, I, Q, R) \in R_+^4 : 0 < S < \varepsilon\Big\}, \quad U_2 = \Big\{(S, I, Q, R) \in R_+^4 : 0 < I < \varepsilon\Big\}, \\
&U_3 = \Big\{(S, I, Q, R) \in R_+^4 : I > 0, 0 < Q < \varepsilon^2\Big\}, \\
&U_4 = \Big\{(S, I, Q, R) \in R_+^4 : I > 0, 0 < R < \varepsilon^2\Big\}, \\
&U_5 = \Big\{(S, I, Q, R) \in R_+^4 : S > \frac{1}{\varepsilon}\Big\}, \quad U_6 = \Big\{(S, I, Q, R) \in R_+^4 : I > \frac{1}{\varepsilon}\Big\}, \\
&U_7 = \Big\{(S, I, Q, R) \in R_+^4 : Q > \frac{1}{\varepsilon^2}\Big\}, \quad U_8 = \Big\{(S, I, Q, R) \in R_+^4 : R > \frac{1}{\varepsilon^2}\Big\}.
\end{aligned}
$$

Next we will prove that $LV(S, I, Q, R) \le -1$ on $R_+^4 \setminus U$, which is equivalent to proving it on the above eight domains.

Case 1. If $(S, I, Q, R) \in U_1$, one can see that

$$
LV(S, I, Q, R) \le \frac{\beta_1^u I}{N}(MC_1 + 1) - \frac{\Lambda^l}{S} + 3\mu^u + \frac{\sigma_1^{2u}}{2} + \alpha^u + \epsilon^u + \frac{\sigma_3^{2u}}{2} + D + \frac{\sigma_4^{2u}}{2}
$$

$$
\begin{aligned}
&-\frac{1}{2}\Big(\mu^l-\frac{1}{2}\theta(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2)^u\Big)(S^{\theta+1}+I^{\theta+1}+Q^{\theta+1}+R^{\theta+1})\\
&\leq -\frac{\Lambda^l}{S}+E\\
&\leq -\frac{\Lambda^l}{\varepsilon}+E, \qquad (3.10)
\end{aligned}
$$

where

$$
\begin{aligned}
E=\sup_{(S,I,Q,R)\in R_+^4}&\Big\{\frac{\beta_1^u I}{N}(MC_1+1)+3\mu^u+\frac{\sigma_1^{2u}}{2}+\alpha^u+\epsilon^u+\frac{\sigma_3^{2u}}{2}+D+\frac{\sigma_4^{2u}}{2}\\
&-\frac{1}{2}\Big(\mu^l-\frac{1}{2}\theta(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2)^u\Big)(S^{\theta+1}+I^{\theta+1}+Q^{\theta+1}+R^{\theta+1})\Big\}.
\end{aligned}
$$

By (3.2), we have $LV\leq -1$ for all $(S,I,Q,R)\in U_1$.

Case 2. If $(S,I,Q,R)\in U_2$, one can see that

$$
\begin{aligned}
LV(S,I,Q,R)&\leq -M\lambda+\frac{\beta_1^u I}{N}(MC_1+1)+3\mu^u+\frac{\sigma_1^{2u}}{2}+\alpha^u+\epsilon^u+\frac{\sigma_3^{2u}}{2}+\frac{\sigma_4^{2u}}{2}+D\\
&\quad-\frac{1}{2}\Big(\mu^l-\frac{1}{2}\theta(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2)^u\Big)(S^{\theta+1}+I^{\theta+1}+Q^{\theta+1}+R^{\theta+1})\\
&\leq -M\lambda+\frac{\beta_1^u I}{N}(MC_1+1)+C\\
&\leq -M\lambda+\frac{\beta_1^u \varepsilon}{N}(MC_1+1)+C, \qquad (3.11)
\end{aligned}
$$

where

$$
\begin{aligned}
C=\sup_{(S,I,Q,R)\in R_+^4}&\Big\{-\frac{1}{2}(\mu^l-\frac{1}{2}\theta(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2)^u)(S^{\theta+1}+I^{\theta+1}+Q^{\theta+1}+R^{\theta+1})\\
&+D+3\mu^u+\alpha^u+\epsilon^u+\frac{\sigma_1^{2u}+\sigma_3^{2u}+\sigma_4^{2u}}{2}\Big\}.
\end{aligned}
$$

By (3.3), we have $LV\leq -1$ for all $(S,I,Q,R)\in U_2$.

Case 3. If $(S,I,Q,R)\in U_3$, one can see that

$$
\begin{aligned}
LV(S,I,Q,R)&\leq -\frac{\delta^l I}{Q}+\frac{\beta_1^u I}{N}(MC_1+1)+3\mu^u+\frac{\sigma_1^{2u}}{2}+\alpha^u+\epsilon^u+\frac{\sigma_3^{2u}}{2}+\frac{\sigma_4^{2u}}{2}+D\\
&\quad-\frac{1}{2}\Big(\mu^l-\frac{1}{2}\theta(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2)^u\Big)(S^{\theta+1}+I^{\theta+1}+Q^{\theta+1}+R^{\theta+1})\\
&\leq -\frac{\delta^l I}{Q}+E\\
&\leq -\frac{\delta^l}{\varepsilon}+E. \qquad (3.12)
\end{aligned}
$$

By (3.4), we have $LV\leq -1$ for all $(S,I,Q,R)\in U_3$.

Case 4. If $(S,I,Q,R)\in U_4$, one can see that

$$
\begin{aligned}
LV(S,I,Q,R)&\leq -\frac{\gamma^l I}{R}+\frac{\beta_1^u I}{N}(MC_1+1)+3\mu^u+\frac{\sigma_1^{2u}}{2}+\alpha^u+\epsilon^u+\frac{\sigma_3^{2u}}{2}+\frac{\sigma_4^{2u}}{2}+D\\
&\quad-\frac{1}{2}\Big(\mu^l-\frac{1}{2}\theta(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2)^u\Big)(S^{\theta+1}+I^{\theta+1}+Q^{\theta+1}+R^{\theta+1})
\end{aligned}
$$

$$\begin{aligned}&\leq -\frac{\gamma^l I}{R} + E\\&\leq -\frac{\gamma^l}{\varepsilon} + E.\end{aligned} \tag{3.13}$$

By (3.5), we have $LV \leq -1$ for all $(S, I, Q, R) \in U_4$.

Case 5. If $(S, I, Q, R) \in U_5$, one can see that

$$\begin{aligned}LV(S,I,Q,R) \leq\ & \frac{\beta_1^u I}{N}(MC_1+1) + 3\mu^u + \frac{\sigma_1^{2u}}{2} + \alpha^u + \epsilon^u + \frac{\sigma_3^{2u}}{2} + \frac{\sigma_4^{2u}}{2} + D\\&-\frac{1}{2}\Big(\mu^l - \frac{1}{2}\theta(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)^u\Big)(S^{\theta+1} + I^{\theta+1} + Q^{\theta+1} + R^{\theta+1})\\\leq\ & -\frac{1}{4}\Big(\mu^l - \frac{1}{2}\theta(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)^u\Big)S^{\theta+1} + \frac{\beta_1^u I}{N}(MC_1+1) + 3\mu^u\\&-\frac{1}{2}\Big(\mu^l - \frac{1}{2}\theta(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)^u\Big)(I^{\theta+1} + Q^{\theta+1}) + \alpha^u + \epsilon^u + D\\&-\frac{1}{4}\Big(\mu^l - \frac{1}{2}\theta(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)^u\Big)S^{\theta+1} + \frac{\sigma_1^{2u}}{2} + \frac{\sigma_3^{2u}}{2} + \frac{\sigma_4^{2u}}{2}\\\leq\ & -\frac{1}{4}\Big(\mu^l - \frac{1}{2}\theta(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)^u\Big)S^{\theta+1} + F\\\leq\ & -\frac{1}{4}\Big(\mu^l - \frac{1}{2}\theta(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)^u\Big)\frac{1}{\varepsilon^{\theta+1}} + F,\end{aligned} \tag{3.14}$$

where

$$\begin{aligned}F = \sup_{(S,I,Q,R)\in R_+^4} \Big\{&\frac{\beta_1^u I}{N}(MC_1+1) + 3\mu^u + \alpha^u + \epsilon^u + D + \frac{\sigma_1^{2u}}{2} + \frac{\sigma_3^{2u}}{2} + \frac{\sigma_4^{2u}}{2}\\&-\frac{1}{2}\Big(\mu^l - \frac{1}{2}\theta(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)^u\Big)(I^{\theta+1} + Q^{\theta+1})\\&-\frac{1}{4}\Big(\mu^l - \frac{1}{2}\theta(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)^u\Big)S^{\theta+1}\Big\}.\end{aligned}$$

By (3.6), we have $LV \leq -1$ for all $(S, I, Q, R) \in U_5$.

Case 6. If $(S, I, Q, R) \in U_6$, one can see that

$$\begin{aligned}LV(S,I,Q,R) \leq\ & \frac{\beta_1^u I}{N}(MC_1+1) + 3\mu^u + \frac{\sigma_1^{2u}}{2} + \alpha^u + \epsilon^u + \frac{\sigma_3^{2u}}{2} + \frac{\sigma_4^{2u}}{2} + D\\&-\frac{1}{2}\Big(\mu^l - \frac{1}{2}\theta(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)^u\Big)(S^{\theta+1} + I^{\theta+1} + Q^{\theta+1} + R^{\theta+1})\\\leq\ & -\frac{1}{4}\Big(\mu^l - \frac{1}{2}\theta(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)^u\Big)I^{\theta+1} + \frac{\beta_1^u I}{N}(MC_1+1) + 3\mu^u\\&-\frac{1}{2}\Big(\mu^l - \frac{1}{2}\theta(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)^u\Big)(S^{\theta+1} + R^{\theta+1}) + \alpha^u + \epsilon^u + D\\&-\frac{1}{4}\Big(\mu^l - \frac{1}{2}\theta(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)^u\Big)I^{\theta+1} + \frac{\sigma_1^{2u}}{2} + \frac{\sigma_3^{2u}}{2} + \frac{\sigma_4^{2u}}{2}\\\leq\ & -\frac{1}{4}\Big(\mu^l - \frac{1}{2}\theta(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)^u\Big)I^{\theta+1} + G\\\leq\ & -\frac{1}{4}\Big(\mu^l - \frac{1}{2}\theta(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)^u\Big)\frac{1}{\varepsilon^{\theta+1}} + G,\end{aligned} \tag{3.15}$$

where

$$G = \sup_{(S,I,Q,R)\in R_+^4} \Big\{\frac{\beta_1^u I}{N}(MC_1+1) + 3\mu^u + \alpha^u + \epsilon^u + D + \frac{\sigma_1^{2u}}{2} + \frac{\sigma_3^{2u}}{2} + \frac{\sigma_4^{2u}}{2}$$

$$-\frac{1}{2}\Big(\mu^l-\frac{1}{2}\theta(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2)^u\Big)(S^{\theta+1}+R^{\theta+1})$$
$$-\frac{1}{4}\Big(\mu^l-\frac{1}{2}\theta(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2)^u\Big)I^{\theta+1}\Big\}.$$

By (3.7), we have $LV\le -1$ for all $(S,I,Q,R)\in U_6$.

Case 7. If $(S,I,Q,R)\in U_7$, one can see that

$$\begin{aligned}
LV(S,I,Q,R)&\le \frac{\beta_1^u I}{N}(MC_1+1)+3\mu^u+\frac{\sigma_1^{2u}}{2}+\alpha^u+\epsilon^u+\frac{\sigma_3^{2u}}{2}+\frac{\sigma_4^{2u}}{2}+D\\
&\quad-\frac{1}{2}\Big(\mu^l-\frac{1}{2}\theta(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2)^u\Big)(S^{\theta+1}+I^{\theta+1}+Q^{\theta+1}+R^{\theta+1})\\
&\le -\frac{1}{4}\Big(\mu^l-\frac{1}{2}\theta(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2)^u\Big)Q^{\theta+1}+\frac{\beta_1^u I}{N}(MC_1+1)+3\mu^u\\
&\quad-\frac{1}{2}\Big(\mu^l-\frac{1}{2}\theta(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2)^u\Big)(S^{\theta+1}+I^{\theta+1})+\alpha^u+\epsilon^u+D\\
&\quad-\frac{1}{4}\Big(\mu^l-\frac{1}{2}\theta(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2)^u\Big)Q^{\theta+1}+\frac{\sigma_1^{2u}}{2}+\frac{\sigma_3^{2u}}{2}+\frac{\sigma_4^{2u}}{2}\\
&\le -\frac{1}{4}\Big(\mu^l-\frac{1}{2}\theta(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2)^u\Big)Q^{\theta+1}+H\\
&\le -\frac{1}{4}\Big(\mu^l-\frac{1}{2}\theta(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2)^u\Big)\frac{1}{\varepsilon^{2(\theta+1)}}+H,
\end{aligned}\tag{3.16}$$

where

$$H=\sup_{(S,I,Q,R)\in R_+^4}\Big\{\frac{\beta_1^u I}{N}(MC_1+1)+3\mu^u+\alpha^u+\epsilon^u+D+\frac{\sigma_1^{2u}}{2}+\frac{\sigma_3^{2u}}{2}+\frac{\sigma_4^{2u}}{2}$$
$$-\frac{1}{2}\Big(\mu^l-\frac{1}{2}\theta(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2)^u\Big)(S^{\theta+1}+I^{\theta+1})$$
$$-\frac{1}{4}\Big(\mu^l-\frac{1}{2}\theta(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2)^u\Big)Q^{\theta+1}\Big\}.$$

By (3.8), we have $LV\le -1$ for all $(S,I,Q,R)\in U_7$.

Case 8. If $(S,I,Q,R)\in U_8$, one can see that

$$\begin{aligned}
LV(S,I,Q,R)&\le \frac{\beta_1^u I}{N}(MC_1+1)+3\mu^u+\frac{\sigma_1^{2u}}{2}+\alpha^u+\epsilon^u+\frac{\sigma_3^{2u}}{2}+\frac{\sigma_4^{2u}}{2}+D\\
&\quad-\frac{1}{2}\Big(\mu^l-\frac{1}{2}\theta(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2)^u\Big)(S^{\theta+1}+I^{\theta+1}+Q^{\theta+1}+R^{\theta+1})\\
&\le -\frac{1}{4}\Big(\mu^l-\frac{1}{2}\theta(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2)^u\Big)R^{\theta+1}+\frac{\beta_1^u I}{N}(MC_1+1)+3\mu^u\\
&\quad-\frac{1}{2}\Big(\mu^l-\frac{1}{2}\theta(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2)^u\Big)(S^{\theta+1}+I^{\theta+1})+\alpha^u+\epsilon^u+D\\
&\quad-\frac{1}{4}\Big(\mu^l-\frac{1}{2}\theta(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2)^u\Big)R^{\theta+1}+\frac{\sigma_1^{2u}}{2}+\frac{\sigma_3^{2u}}{2}+\frac{\sigma_4^{2u}}{2}\\
&\le -\frac{1}{4}\Big(\mu^l-\frac{1}{2}\theta(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2)^u\Big)R^{\theta+1}+J\\
&\le -\frac{1}{4}\Big(\mu^l-\frac{1}{2}\theta(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2)^u\Big)\frac{1}{\varepsilon^{2(\theta+1)}}+J,
\end{aligned}\tag{3.17}$$

where

$$J=\sup_{(S,I,Q,R)\in R_+^4}\Big\{\frac{\beta_1^u I}{N}(MC_1+1)+3\mu^u+\alpha^u+\epsilon^u+D+\frac{\sigma_1^{2u}}{2}+\frac{\sigma_3^{2u}}{2}+\frac{\sigma_4^{2u}}{2}$$

$$-\frac{1}{2}\Big(\mu^l-\frac{1}{2}\theta(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2)^u\Big)(S^{\theta+1}+I^{\theta+1})$$
$$-\frac{1}{4}\Big(\mu^l-\frac{1}{2}\theta(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2)^u\Big)R^{\theta+1}\Big\}.$$

By (3.9), we have $LV\leq -1$ for all $(S,I,Q,R)\in U_8$.

Therefore, we have proof that for a sufficiently small $\varepsilon>0$,

$$LV(S,I,Q,R)\leq -1,(S,I,Q,R)\in R_+^4\setminus U.$$

Hence, (A_2) in Lemma 3.1 holds. This completes the proof of Theorem 3.1. □

4. Extinction of model (1.3)

In this section, we investigate the conditions for the extinction of model (1.3).

Lemma 4.1 ([16]). *Let $M=\{M_t\}_t\geq 0$ be a real-valued continuous local martingale vanishing $t=0$. Then*

$$\lim_{t\to\infty}\langle M,M\rangle_t=\infty\quad a.s.\Rightarrow\quad \lim_{t\to\infty}\frac{M_t}{\langle M,M\rangle_t}=0\quad a.s.$$

and also

$$\limsup_{t\to\infty}\frac{\langle M,M\rangle_t}{t}<\infty\quad a.s.\Rightarrow\quad \lim_{t\to\infty}\frac{M_t}{t}=0\quad a.s.$$

Define a parameter

$$\Re_2=\frac{\Lambda^u\beta_1^u}{N\mu^l\langle\mu+\alpha+\delta+\gamma+\frac{\sigma_2^2}{2}\rangle_{\mathbf{T}}}.$$

Theorem 4.1. *Let$(S(t),I(t),Q(t),R(t)$ be a solution of model (1.3) with initial value $(S(0),I(0),Q(0),R(0)\in R_+^4$. If $\Re_2<1$, then the disease I goes to extinction exponentially with probability one, i.e.,*

$$\lim_{t\to\infty}I(t)=0\quad a.s.$$

and also

$$\lim_{t\to\infty}\langle S\rangle_t\leq\frac{\Lambda^u}{\mu^l},\quad \lim_{t\to\infty}Q(t)=\lim_{t\to\infty}R(t)=0.\quad a.s.$$

Proof. From model (1.3), we have

$$\frac{S(t)-S(0)}{t}=\langle\Lambda\rangle_t-\langle(\beta_1-\frac{\beta_2 I}{m+I})\frac{SI}{N}\rangle_t-\langle\mu S\rangle_t+\frac{\int_0^t\sigma_1(s)S(s)dB_1(s)}{t},$$

and

$$\frac{I(t)-I(0)}{t}=\langle(\beta_1-\frac{\beta_2 I}{m+I})\frac{SI}{N}\rangle_t-\langle(\mu+\alpha+\delta+\gamma)I\rangle_t+\frac{\int_0^t\sigma_2(s)I(s)dB_2(s)}{t}.$$

Then

$$\begin{aligned}\frac{I(t)-I(0)}{t}+\frac{S(t)-S(0)}{t} &= \langle\Lambda\rangle_t-\langle\mu S\rangle_t-\langle(\mu+\alpha+\delta+\gamma)I\rangle_t\\ &\quad+\frac{\int_0^t\sigma_1(s)S(s)dB_1(s)}{t}+\frac{\int_0^t\sigma_2(s)I(s)dB_2(s)}{t}\\ &\le \Lambda^u-\mu^l\langle S\rangle_t-(\mu^l+\alpha^l+\delta^l+\gamma^l)\langle I\rangle_t\\ &\quad+\frac{\sigma_1^u\int_0^t S(s)dB_1(s)}{t}+\frac{\sigma_2^u\int_0^t I(s)dB_2(s)}{t}.\end{aligned}$$

It is easy to obtain

$$\langle S\rangle_t\le\frac{\Lambda^u}{\mu^l}-\frac{\mu^l+\alpha^l+\delta^l+\gamma^l}{\mu^l}\langle I\rangle_t+H(t), \tag{4.1}$$

where

$$H(t)=\frac{\frac{\sigma_1^u\int_0^t S(s)dB_1(s)}{t}}{\mu^l}+\frac{\frac{\sigma_2^u\int_0^t I(s)dB_2(s)}{t}}{\mu^l}-\frac{\frac{I(t)-I(0)}{t}+\frac{S(t)-S(0)}{t}}{\mu^l}.$$

According to Lemma 4.1, we have

$$\lim_{t\to\infty}H(t)=0\quad a.s. \tag{4.2}$$

By the It$\hat{o}$'s formula, we obtain

$$\begin{aligned}d\ln I(t) &= \Big\{\frac{1}{I}\Big[\Big(\beta_1(t)-\frac{\beta_2(t)I(t)}{m(t)+I(t)}\Big)\frac{S(t)I(t)}{N}-(\mu(t)+\alpha(t)+\delta(t)+\gamma(t))I(t)\Big]\\ &\qquad-\frac{\sigma_2^2(t)}{2}\Big\}dt+\sigma_2(t)dB_2(t)\\ &\le\Big(\frac{\beta_1(t)S}{N}-(\mu(t)+\alpha(t)+\delta(t)+\gamma(t))-\frac{\sigma_2^2(t)}{2}\Big)dt+\sigma_2(t)dB_2(t).\end{aligned} \tag{4.3}$$

Integrating (4.3) from 0 to t and dividing t on both sides, we get

$$\begin{aligned}\frac{\ln I(t)-\ln I(0)}{t} &\le \frac{\langle\beta_1(t)S\rangle_t}{N}-\langle\mu+\alpha+\delta+\gamma+\frac{\sigma_2^2}{2}\rangle_t+\frac{\int_0^t\sigma_2(s)dB_2(s)}{t}\\ &\le\frac{\beta_1^u\langle S\rangle_t}{N}-\langle\mu+\alpha+\delta+\gamma+\frac{\sigma_2^2}{2}\rangle_t+\frac{\int_0^t\sigma_2(s)dB_2(s)}{t}.\end{aligned}$$

Together with (4.1), we have

$$\begin{aligned}\frac{\ln I(t)}{t} &\le \frac{\beta_1^u}{N}\Big[\frac{\Lambda^u}{\mu^l}-\frac{\mu^l+\alpha^l+\delta^l+\gamma^l}{\mu^l}\langle I\rangle_t+H(t)\Big]-\langle\mu+\alpha+\delta+\gamma+\frac{\sigma_2^2}{2}\rangle_t\\ &\quad+\frac{\int_0^t\sigma_2(s)dB_2(s)}{t}+\frac{\ln I(0)}{t}\\ &\le\frac{\beta_1^u\Lambda^u}{N\mu^l}+\frac{\beta_1^u H(t)}{N}-\langle\mu+\alpha+\delta+\gamma+\frac{\sigma_2^2}{2}\rangle_t+\frac{\int_0^t\sigma_2(s)dB_2(s)}{t}\\ &\quad+\frac{\ln I(0)}{t}.\end{aligned} \tag{4.4}$$

Taking the limit superior of both of (4.4) and using Lemma 4.1, which together with (4.2), we can obtain

$$
\begin{aligned}
\limsup_{t\to+\infty}\frac{\ln I(t)}{t} &\le \frac{\beta_1^u\Lambda^u}{N\mu^l}-\langle\mu+\alpha+\delta+\gamma+\frac{\sigma_2^2}{2}\rangle_{\mathbf{T}}\\
&=\langle\mu+\alpha+\delta+\gamma+\frac{\sigma_2^2}{2}\rangle_{\mathbf{T}}\Big(\frac{\beta_1^u\Lambda^u}{N\mu^l\langle\mu+\alpha+\delta+\gamma+\frac{\sigma_2^2}{2}\rangle_{\mathbf{T}}}-1\Big)\\
&=\langle\mu+\alpha+\delta+\gamma+\frac{\sigma_2^2}{2}\rangle_{\mathbf{T}}(\mathfrak{R}_2-1)\\
&<0.
\end{aligned}
$$

which implies $\lim_{t\to\infty} I(t)=0 \quad a.s.$

From (4.1), it is easy to get that

$$\lim_{t\to\infty}\langle S\rangle_t\le\frac{\Lambda^u}{\mu^l},$$

From the third and fourth equations of model (1.3), it is easy to obtain that

$$\lim_{t\to\infty}Q(t)=0,\lim_{t\to\infty}R(t)=0.$$

This completes the proof. □

5. Numerical simulations

In this section, we give two examples to support the theoretical prediction.

Example 5.1. In model (1.3), let $\Lambda(t)=1+0.1\sin\pi t,\beta_1(t)=0.8+0.15\sin\pi t,$ $\beta_2(t)=0.4+0.05\sin\pi t, m(t)=1+\sin\pi t,\mu(t)=0.5+0.1\sin\pi t,\delta(t)=0.2+0.1\sin\pi t,\epsilon(t)=\alpha(t)=0.25+0.1\sin\pi t,\gamma(t)=0.15+0.1\sin\pi t,\sigma_1(t)=\sigma_2(t)=\sigma_3(t)=\sigma_4(t)=0.01+0.005\sin\pi t, N=1$ and the initial value are taken as $(S(0),I(0),Q(0),R(0))=(1.5,0.9,0.26,1.01)$.Then by calculation, we have

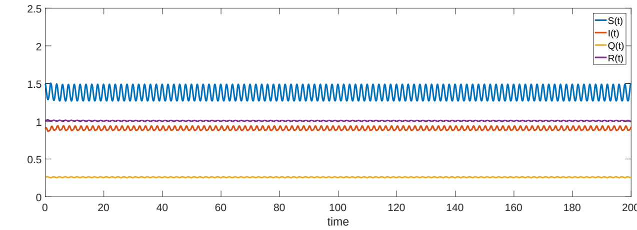
$$\mathfrak{R}_1=\frac{\langle\Lambda(\beta_1-\beta_2)\rangle_{\mathbf{T}}}{N\langle\mu+\frac{\sigma_1^2}{2}\rangle_{\mathbf{T}}\langle\mu+\alpha+\delta+\gamma+\frac{\sigma_2^2}{2}\rangle_{\mathbf{T}}}\approx 1.77>1.$$

In other words, the condition in Theorem 3.1 holds. Hence, the stochastic model (1.3) has a positive periodic solution (see Figure 1).

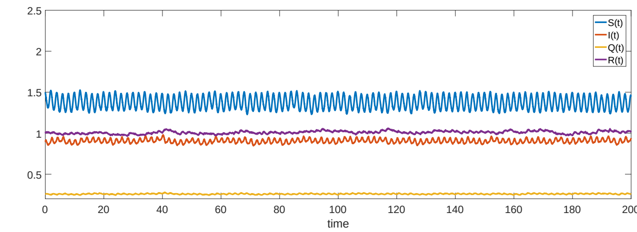
Example 5.2. In model (1.3), let $\Lambda(t)=1+0.1\sin\pi t,\beta_1(t)=0.3+0.1\sin\pi t,\beta_2(t)=0.1+0.05\sin\pi t, m(t)=1+\sin\pi t,\mu(t)=0.5+0.1\sin\pi t,\delta(t)=0.2+0.1\sin\pi t,\epsilon(t)=\alpha(t)=0.25+0.1\sin\pi t,\gamma(t)=0.15+0.1\sin\pi t,\sigma_1(t)=\sigma_2(t)=\sigma_3(t)=\sigma_4(t)=0.1+0.05\sin\pi t, N=10$ and the initial value are taken as $(S(0),I(0),Q(0),R(0))=(0.8,0.5,0.4,0.2)$. Note that

$$\mathfrak{R}_2=\frac{\Lambda^u\beta_1^u}{N\mu^l\langle\mu+\alpha+\delta+\gamma+\frac{\sigma_2^2}{2}\rangle_{\mathbf{T}}}\approx 0.05<1.$$

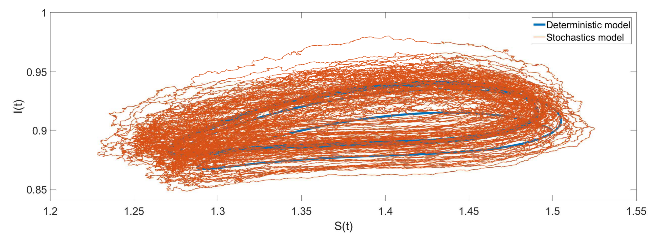
That is, the condition in Theorem 4.1 holds. Hence, the disease $I(t)$ will die out almost surely (see Figure 2).



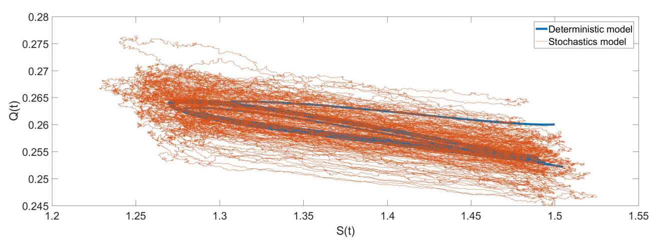
(a)



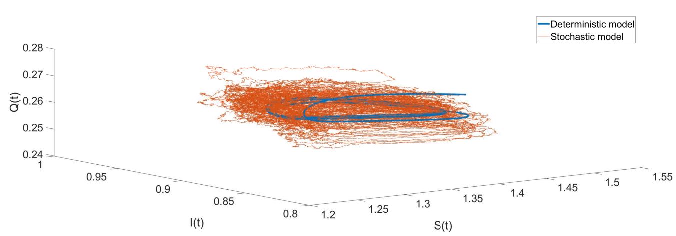
(b)



(c)

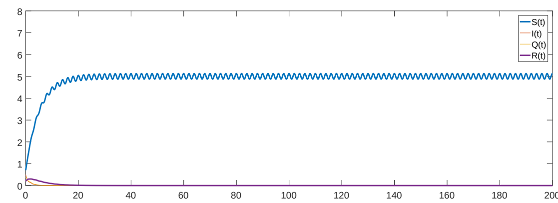


(d)

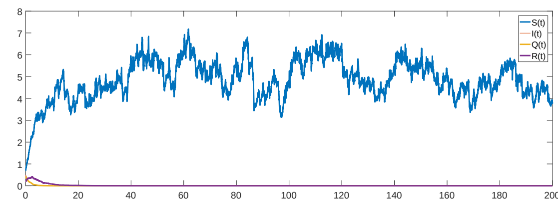


(e)

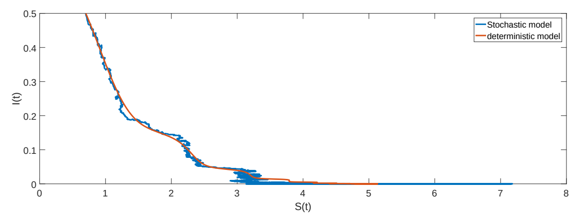
Figure 1. The stochastic model (1.3) has a positive periodic solution. (a) is the deterministic model (1.3), (b) is the stochastic model (1.3), (c) is phase portrait of $S(t)$ and $I(t)$ in the stochastic model,(d) is phase portrait of $S(t)$ and $Q(t)$ in the stochastic model, (e) stands for the phase portrait of (a) and (b).



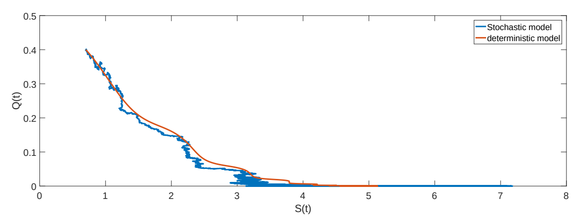
(a)



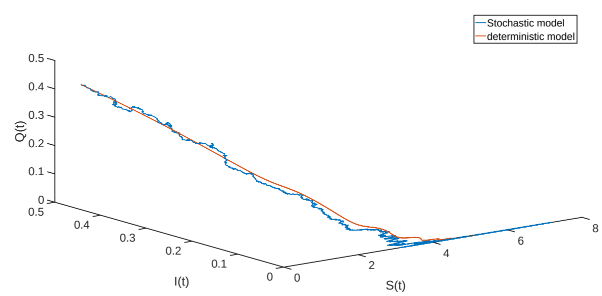
(b)



(c)



(d)



(e)

Figure 2. The disease $I(t)$ will die out almost surely. (a) is the deterministic model (1.3), (b) is the stochastic model (1.3), (c) is phase portrait of $S(t)$ and $I(t)$ in the stochastic model,(d) is phase portrait of $S(t)$ and $Q(t)$ in the stochastic model, (e) stands for the phase portrait of (a) and (b).

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