Journal of Applied Analysis and Computation
Volume 10, Number 5, October 2020, 2068–2084
Website:http://jaac-online.com
DOI:10.11948/20190325

THE BOUNDEDNESS FOR SOLUTIONS OF A CERTAIN TWO-DIMENSIONAL FRACTIONAL DIFFERENTIAL SYSTEM WITH DELAY*

Yue Yang[1] and Fanwei Meng[1,†]

Abstract In this paper, we study the components-wise upper bounds for solutions of two-dimensional fractional differential system with delay. Prior to the main results, we derive some results on two-dimensional nonlinear integral inequalities, then we investigate upper bounds of solutions basing on the obtained inequalities, finally, an example is given to illustrate the theoretical results.

Keywords Fractional differential system, delay, integral inequality, components-wise upper bounds.

MSC(2010) 26D15, 34A08, 34C11.

1. Introduction

Recently, fractional differential equations, which are regarded as the generalization of the traditional differential equations dealing with nonnegative integer order, have drawn more and more attention due to their widespread application. Numerous numerical and analytical results have been given for various differential equations with physical background [8, 9, 16, 18, 19, 23], biological [24] or ecological economic [17] implications. The study of the qualitative properties for solutions of fractional differential systems has become a very vital branch of the theory of differential equations [1, 3, 5, 12].

Integral inequalities play a fundamental role in the qualitative study of various differential equations and integral equations [6, 7, 10, 14, 15, 20], especially Gronwall-Bellman inequality. There has been an increasing interest in this area of research to satisfy the needs of colorful applications of these inequalities. Many authors have paid considerable attention to integral inequalities with weakly singular kernels and obtained some inspiring results [4, 13, 22].

On the other hand, due to the transmission of the signal or the mechanical transmission, fractional differential systems with delay have gained scholar's attention [2, 21, 25, 26]. Čermák et al. [2] investigated stability and asymptotic properties of the following equation

$$D^{\beta}y(t) = ay(t) + by(t-\tau).$$

[†]The corresponding author. Email address:fwmeng@qfnu.edu.cn(F. Meng)
[1]School of Mathematical Sciences, Qufu Normal University, Jing-xuan West Road, 273165, China
*The authors were supported by National Natural Science Foundation of China (Grant 11671227) and National Science Foundation of Shandong Province (Grant ZR2019MA034).

Ye and Gao [21] researched Henry-Gronwall type retarded integral inequalities and their applications to Caputo fractional differential equations with delay

$$\begin{cases} D^{\beta}x(t) = f\Big(t, x(t-r)\Big), \quad t \in [t_0, T), \\ x(t) = \varphi(t), \quad t \in [t_0 - r, t_0] \end{cases}$$

and

$$\begin{cases} D^{\beta}y(t) = f\Big(t, y(t), y(t-r)\Big), \quad t \in [t_0, T), \\ y(t) = \varphi(t), \quad t \in [t_0 - r, t_0]. \end{cases}$$

Zhao and Meng [25] studied properties for solutions of Riemann-Liouville fractional differential system with delay

$$\begin{cases} D^{\alpha}x(t) = f\Big(t, x(t-\tau), y(t-\tau)\Big), \\ D^{\alpha}y(t) = g\Big(t, x(t-\tau), y(t-\tau)\Big), \quad t \in [t_0, +\infty), \\ D^{\alpha-1}x(t) = \xi, \\ D^{\alpha-1}y(t) = \eta, \quad t \in [t_0 - \tau, t_0]. \end{cases}$$

Motivated by the work in [21] and [25], in this paper, we deal with the following nonlinear two-dimensional fractional differential system with delay

$$\begin{cases} D^{\alpha}x(t) = f\Big(t, x(t), y(t), x(t-r), y(t-r)\Big), \\ D^{\alpha}y(t) = g\Big(t, x(t), y(t), x(t-r), y(t-r)\Big), \quad t \in [t_0, +\infty), \\ x(t) = \varphi(t), \\ y(t) = \psi(t), \quad t \in [t_0 - r, t_0], \end{cases} \tag{1.1}$$

where $f,\ g \in C([t_0, +\infty) \times R^4, R)$. Besides, D^{α} is the fractional derivative (in the sense of Caputo) of order $\alpha > 0$, and $\varphi,\ \psi$ are known continuously differentiable functions on $[t_0 - r, t_0]$ up to order $n (n = -[-\alpha])$. In what follows, we denote $M_1 = max_{t\in[t_0-r,t_0]}|\varphi(t)|, M_2 = max_{t\in[t_0-r,t_0]}|\psi(t)|$ and $\varphi^{(k)}(t_0) = m_k,\ \psi^{(k)}(t_0) = n_k$, $k = 0, 1, 2, ..., n-1$ and

$$\begin{bmatrix} m(t) \\ n(t) \end{bmatrix}^{p_1}_{p_2} = \begin{bmatrix} m^{p_1}(t) \\ n^{p_2}(t) \end{bmatrix},\ p_1, p_2 \in R.$$

This paper is organized as follows: In Section 2, some basic definitions and useful lemmas of two-dimensional nonlinear integral inequalities are provided. In Section 3, we discuss the upper bounds for solutions of two-dimensional fractional differential system with delay. In Section 4, an example is given to illustrate our results.

2. Preliminaries

In this section, we recall and set the following lemmas which will be used in our proof.

Lemma 2.1 (Lemma 2.1, [10]). *Let $a \geq 0, p \geq q \geq 0$ and $p \neq 0$, then*

$$a^{\frac{q}{p}} \leq \frac{q}{p} K^{\frac{q-p}{p}} a + \frac{p-q}{p} K^{\frac{q}{p}}$$

for any $K > 0$.

Lemma 2.2 ([11]). *Let $n \in N$, and let $x_1, ..., x_n$ be non-negative real numbers. Then for $\sigma > 1$,*

$$\Big(\sum_{i=1}^{n} x_i\Big)^{\sigma} \leq n^{\sigma-1} \sum_{i=1}^{n} x_i^{\sigma}.$$

Lemma 2.3. *Let a_i, $b_{ij}(j = 1,2)$ and $c_{ij}(j = 3,4) \in C([t_0, \infty), R_+), i = 1,2$, $R_+ = [0, +\infty)$; $\phi_i \in C([t_0 - r, t_0], R_+)$, $a_i(t_0) = \phi_i(t_0)$, $r > 0$ be a constant.*
If $u_i \in C([t_0 - r, +\infty), R_+)$ and

$$\begin{cases} u_1^{p_1}(t) \leq a_1(t) + \int_{t_0}^{t} \Big[b_{11}(s) u_1^{q_{11}}(s) + b_{12}(s) u_2^{q_{12}}(s) \\ \qquad\qquad + c_{13}(s) u_1^{q_{13}}(s-r) + c_{14}(s) u_2^{q_{14}}(s-r)\Big] ds, \\ u_2^{p_2}(t) \leq a_2(t) + \int_{t_0}^{t} \Big[b_{21}(s) u_1^{q_{21}}(s) + b_{22}(s) u_2^{q_{22}}(s) \\ \qquad\qquad + c_{23}(s) u_1^{q_{23}}(s-r) + c_{24}(s) u_2^{q_{24}}(s-r)\Big] ds, \quad t \in [t_0, +\infty), \\ u_1^{p_1}(t) \leq \phi_1(t), \\ u_2^{p_2}(t) \leq \phi_2(t), \quad t \in [t_0 - r, t_0), \end{cases}$$

where p_i and q_{ij} are constants satisfying $p_1, p_2 \geq q_{ij} > 0$ and $p_i \neq 0$, then for any $K_j > 0 (j = 1,2,3,4)$, we have

$$\begin{bmatrix} u_1^{p_1}(t) \\ u_2^{p_2}(t) \end{bmatrix} \leq B(t) + G(t), \quad t \in [t_0, +\infty),$$

that is to say

$$\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \leq \Big[B(t) + G(t)\Big]^{\frac{1}{p_1}}_{\frac{1}{p_2}}, \quad t \in [t_0, +\infty),$$

where

$$G(t) = \begin{cases} \exp\Big\{\int_{t_0+r}^{t} \Big[H_1(\tau) + H_2(\tau)\Big] d\tau\Big\} \int_{t_0}^{t_0+r} \exp\Big\{\int_{s}^{t_0+r} H_1(\tau) d\tau\Big\} \Big[H_1(s) B(s) \\ + L_1(s) + \Phi(s)\Big] ds + \int_{t_0+r}^{t} \exp\Big\{\int_{s}^{t} [H_1(\tau) + H_2(\tau)] d\tau\Big\} \Big[H_1(s) B(s) \\ + H_2(s) B(s-r) + L_1(s) + L_2(s)\Big] ds, \ t \in [t_0 + r, +\infty), \\ \int_{t_0}^{t} \exp\Big\{\int_{s}^{t} H_1(\tau) d\tau\Big\} \Big[H_1(s) B(s) + L_1(s) + \Phi(s)\Big] ds, \quad t \in [t_0, t_0 + r], \end{cases} \tag{2.1}$$

$$B(t)=\begin{bmatrix} a_1(t)\\ a_2(t)\end{bmatrix}, \tag{2.2}$$

$$H_1(t)=\begin{bmatrix} b_{11}(t)\frac{q_{11}}{p_1}K_1^{\frac{q_{11}-p_1}{p_1}} & b_{12}(t)\frac{q_{12}}{p_2}K_2^{\frac{q_{12}-p_2}{p_2}}\\ b_{21}(t)\frac{q_{21}}{p_1}K_1^{\frac{q_{21}-p_1}{p_1}} & b_{22}(t)\frac{q_{22}}{p_2}K_2^{\frac{q_{22}-p_2}{p_2}}\end{bmatrix}, \tag{2.3}$$

$$H_2(t)=\begin{bmatrix} c_{13}(t)\frac{q_{13}}{p_1}K_3^{\frac{q_{13}-p_1}{p_1}} & c_{14}(t)\frac{q_{14}}{p_2}K_4^{\frac{q_{14}-p_2}{p_2}}\\ c_{23}(t)\frac{q_{23}}{p_1}K_3^{\frac{q_{23}-p_1}{p_1}} & c_{24}(t)\frac{q_{24}}{p_2}K_4^{\frac{q_{24}-p_2}{p_2}}\end{bmatrix}, \tag{2.4}$$

$$L_1(t)=\begin{bmatrix} b_{11}(t)\frac{p_1-q_{11}}{p_1}K_1^{\frac{q_{11}}{p_1}}+b_{12}(t)\frac{p_2-q_{12}}{p_2}K_2^{\frac{q_{12}}{p_2}}\\ b_{21}(t)\frac{p_1-q_{21}}{p_1}K_1^{\frac{q_{21}}{p_1}}+b_{22}(t)\frac{p_2-q_{22}}{p_2}K_2^{\frac{q_{22}}{p_2}}\end{bmatrix}, \tag{2.5}$$

$$L_2(t)=\begin{bmatrix} c_{13}(t)\frac{p_1-q_{13}}{p_1}K_3^{\frac{q_{13}}{p_1}}+c_{14}(t)\frac{p_2-q_{14}}{p_2}K_4^{\frac{q_{14}}{p_2}}\\ c_{23}(t)\frac{p_1-q_{23}}{p_1}K_3^{\frac{q_{23}}{p_1}}+c_{24}(t)\frac{p_2-q_{24}}{p_2}K_4^{\frac{q_{24}}{p_2}}\end{bmatrix}, \tag{2.6}$$

$$\Phi(t)=\begin{bmatrix} c_{13}(t)\phi_1^{\frac{q_{13}}{p_1}}(t-r)+c_{14}(t)\phi_2^{\frac{q_{14}}{p_2}}(t-r)\\ c_{23}(t)\phi_1^{\frac{q_{23}}{p_1}}(t-r)+c_{24}(t)\phi_2^{\frac{q_{24}}{p_2}}(t-r)\end{bmatrix}. \tag{2.7}$$

Proof. For $t\in[t_0,+\infty)$ and $i=1,2$, let

$$z_i(t)=\int_{t_0}^{t}\Big[b_{i1}(s)u_1^{q_{i1}}(s)+b_{i2}(s)u_2^{q_{i2}}(s)+c_{i3}(s)u_1^{q_{i3}}(s-r)+c_{i4}(s)u_2^{q_{i4}}(s-r)\Big]ds,$$

then $z_i(t)\ge 0$ is nondecreasing,

$$z_i'(t)=b_{i1}(t)u_1^{q_{i1}}(t)+b_{i2}(t)u_2^{q_{i2}}(t)+c_{i3}(t)u_1^{q_{i3}}(t-r)+c_{i4}(t)u_2^{q_{i4}}(t-r) \tag{2.8}$$

and

$$u_i^{p_i}(t)\le a_i(t)+z_i(t),\quad u_i(t)\le\Big[a_i(t)+z_i(t)\Big]^{\frac{1}{p_i}}. \tag{2.9}$$

For $t\in[t_0+r,+\infty)$ and any $K_j>0(j=1,2,3,4)$, by (2.8), (2.9) and Lemma 2.1 we get

$$\begin{aligned} z_i'(t)\le& b_{i1}(t)\Big[a_1(t)+z_1(t)\Big]^{\frac{q_{i1}}{p_1}}+b_{i2}(t)\Big[a_2(t)+z_2(t)\Big]^{\frac{q_{i2}}{p_2}}\\ &+c_{i3}(t)\Big[a_1(t-r)+z_1(t-r)\Big]^{\frac{q_{i3}}{p_1}}+c_{i4}(t)\Big[a_2(t-r)+z_2(t-r)\Big]^{\frac{q_{i4}}{p_2}}\\ \le& b_{i1}(t)\Big\{\frac{q_{i1}}{p_1}K_1^{\frac{q_{i1}-p_1}{p_1}}\Big[a_1(t)+z_1(t)\Big]+\frac{p_1-q_{i1}}{p_1}K_1^{\frac{q_{i1}}{p_1}}\Big\}\\ &+b_{i2}(t)\Big\{\frac{q_{i2}}{p_2}K_2^{\frac{q_{i2}-p_2}{p_2}}\Big[a_2(t)+z_2(t)\Big]+\frac{p_2-q_{i2}}{p_2}K_2^{\frac{q_{i2}}{p_2}}\Big\}\\ &+c_{i3}(t)\Big\{\frac{q_{i3}}{p_1}K_3^{\frac{q_{i3}-p_1}{p_1}}\Big[a_1(t-r)+z_1(t-r)\Big]+\frac{p_1-q_{i3}}{p_1}K_3^{\frac{q_{i3}}{p_1}}\Big\}\\ &+c_{i4}(t)\Big\{\frac{q_{i4}}{p_2}K_4^{\frac{q_{i4}-p_2}{p_2}}\Big[a_2(t-r)+z_2(t-r)\Big]+\frac{p_2-q_{i4}}{p_2}K_4^{\frac{q_{i4}}{p_2}}\Big\} \end{aligned}$$

$$
\begin{aligned}
\leq & b_{i1}(t)\frac{q_{i1}}{p_1}K_1^{\frac{q_{i1}-p_1}{p_1}}a_1(t)+b_{i2}(t)\frac{q_{i2}}{p_2}K_2^{\frac{q_{i2}-p_2}{p_2}}a_2(t)\\
&+c_{i3}(t)\frac{q_{i3}}{p_1}K_3^{\frac{q_{i3}-p_1}{p_1}}a_1(t-r)+c_{i4}(t)\frac{q_{i4}}{p_2}K_4^{\frac{q_{i4}-p_2}{p_2}}a_2(t-r)\\
&+b_{i1}(t)\frac{p_1-q_{i1}}{p_1}K_1^{\frac{q_{i1}}{p_1}}+b_{i2}(t)\frac{p_2-q_{i2}}{p_2}K_2^{\frac{q_{i2}}{p_2}}\\
&+c_{i3}(t)\frac{p_1-q_{i3}}{p_1}K_3^{\frac{q_{i3}}{p_1}}+c_{i4}(t)\frac{p_2-q_{i4}}{p_2}K_4^{\frac{q_{i4}}{p_2}}\\
&+\Big[b_{i1}(t)\frac{q_{i1}}{p_1}K_1^{\frac{q_{i1}-p_1}{p_1}}+c_{i3}(t)\frac{q_{i3}}{p_1}K_3^{\frac{q_{i3}-p_1}{p_1}}\Big]z_1(t)\\
&+\Big[b_{i2}(t)\frac{q_{i2}}{p_2}K_2^{\frac{q_{i2}-p_2}{p_2}}+c_{i4}(t)\frac{q_{i4}}{p_2}K_4^{\frac{q_{i4}-p_2}{p_2}}\Big]z_2(t),\quad i=1,2.
\end{aligned}
\tag{2.10}
$$

Denote

$$W(t)=\begin{bmatrix} z_1(t)\\ z_2(t)\end{bmatrix},$$

then we derive from (2.10) that

$$W'(t)\leq H_1(t)B(t)+H_2(t)B(t-r)+L_1(t)+L_2(t)+\Big[H_1(t)+H_2(t)\Big]W(t),$$

where $B(t),H_1(t),H_2(t),L_1(t),L_2(t)$ are defined as (2.2)-(2.6).

Thus for $t\in[t_0+r,+\infty)$, we get

$$
\begin{aligned}
W(t)\leq & \exp\Big\{\int_{t_0+r}^{t}\Big[H_1(\tau)+H_2(\tau)\Big]d\tau\Big\}W(t_0+r)+\int_{t_0+r}^{t}\exp\Big\{\int_s^t\Big[H_1(\tau)\\
&+H_2(\tau)\Big]d\tau\Big\}\Big[H_1(s)B(s)+H_2(s)B(s-r)+L_1(s)+L_2(s)\Big]ds.
\end{aligned}
\tag{2.11}
$$

When $t\in[t_0,t_0+r]$, for $K_1,K_2>0$, we derive from (2.8), (2.9) and Lemma 2.1 that

$$
\begin{aligned}
z_i'(t)\leq & b_{i1}(t)\Big[a_1(t)+z_1(t)\Big]^{\frac{q_{i1}}{p_1}}+b_{i2}(t)\Big[a_2(t)+z_2(t)\Big]^{\frac{q_{i2}}{p_2}}\\
&+c_{i3}(t)\phi_1^{\frac{q_{i3}}{p_1}}(t-r)+c_{i4}(t)\phi_2^{\frac{q_{i4}}{p_2}}(t-r)\\
\leq & b_{i1}(t)\Big\{\frac{q_{i1}}{p_1}K_1^{\frac{q_{i1}-p_1}{p_1}}\Big[a_1(t)+z_1(t)\Big]+\frac{p_1-q_{i1}}{p_1}K_1^{\frac{q_{i1}}{p_1}}\Big\}\\
&+b_{i2}(t)\Big\{\frac{q_{i2}}{p_2}K_2^{\frac{q_{i2}-p_2}{p_2}}\Big[a_2(t)+z_2(t)\Big]+\frac{p_2-q_{i2}}{p_2}K_2^{\frac{q_{i2}}{p_2}}\Big\}\\
&+c_{i3}(t)\phi_1^{\frac{q_{i3}}{p_1}}(t-r)+c_{i4}(t)\phi_2^{\frac{q_{i4}}{p_2}}(t-r),\quad i=1,2.
\end{aligned}
$$

Denote $\Phi(t)$ as (2.7), then we obtain

$$W'(t)\leq H_1(t)B(t)+L_1(t)+\Phi(t)+H_1(t)W(t),\quad t\in[t_0,t_0+r],$$

it follows that

$$W(t)\leq\int_{t_0}^{t}\exp\Big\{\int_s^t H_1(\tau)d\tau\Big\}\Big[H_1(s)B(s)+L_1(s)+\Phi(s)\Big]ds,\quad t\in[t_0,t_0+r]. \tag{2.12}$$

Define $G(t)$ as (2.1), then by (2.9), (2.11) and (2.12) we get

$$\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \leq \Big[B(t)+G(t)\Big]^{\frac{1}{p_1}}_{\frac{1}{p_2}}, \quad t\in[t_0,+\infty).$$

This completes the proof of Lemma 2.3. □

3. Main results and proofs

In this section, we deal with the upper bounds for solutions of two-dimensional fractional differential system with delay (1.1). Firstly, we expose the result on the following retarded integral inequalities.

Theorem 3.1. *Let $a_i, b_{ij}, c_{ij}, \phi_i, p_i, q_{ij}$ be defined as in Lemma 2.3 and $\alpha>0$ be a constant. If $u_i\in C([t_0-r,+\infty),R_+)$ and*

$$\begin{cases} u_1^{p_1}(t)\leq a_1(t)+\int_{t_0}^t (t-s)^{\alpha-1}\Big[b_{11}(s)u_1^{q_{11}}(s)+b_{12}(s)u_2^{q_{12}}(s) \\ \qquad +c_{13}(s)u_1^{q_{13}}(s-r)+c_{14}(s)u_2^{q_{14}}(s-r)\Big]ds, \\ u_2^{p_2}(t)\leq a_2(t)+\int_{t_0}^t (t-s)^{\alpha-1}\Big[b_{21}(s)u_1^{q_{21}}(s)+b_{22}(s)u_2^{q_{22}}(s) \\ \qquad +c_{23}(s)u_1^{q_{23}}(s-r)+c_{24}(s)u_2^{q_{24}}(s-r)\Big]ds, \quad t\in[t_0,+\infty), \\ u_1^{p_1}(t)\leq \phi_1(t), \\ u_2^{p_2}(t)\leq \phi_2(t), \quad t\in[t_0-r,t_0), \end{cases}$$

then for any $K_j>0 (j=1,2,3,4)$, we have

(i) when $\alpha>\frac{1}{2}$, denote

$$\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} e^{\frac{t}{p_1}}\overline{u}_1^{\frac{1}{2}}(t) \\ e^{\frac{t}{p_2}}\overline{u}_2^{\frac{1}{2}}(t) \end{bmatrix}, \quad t\in[t_0,+\infty),$$

then we obtain

$$\begin{bmatrix} \overline{u}_1(t) \\ \overline{u}_2(t) \end{bmatrix} \leq \Big[\overline{B}(t)+\overline{G}(t)\Big]^{\frac{1}{p_1}}_{\frac{1}{p_2}}, \quad t\in[t_0,+\infty),$$

where

$$\overline{G}(t)=\begin{cases} \exp\Big\{\int_{t_0+r}^t \Big[\overline{H}_1(\tau)+\overline{H}_2(\tau)\Big]d\tau\Big\}\int_{t_0}^{t_0+r}\exp\Big\{\int_s^{t_0+r}\overline{H}_1(\tau)d\tau\Big\}\Big[\overline{H}_1(s)\overline{B}(s) \\ +\overline{L}_1(s)+\overline{\Phi}(s)\Big]ds+\int_{t_0+r}^t\exp\Big\{\int_s^t[\overline{H}_1(\tau)+\overline{H}_2(\tau)]d\tau\Big\}\Big[\overline{H}_1(s)\overline{B}(s) \\ +\overline{H}_2(s)\overline{B}(s-r)+\overline{L}_1(s)+\overline{L}_2(s)\Big]ds, t\in[t_0+r,+\infty), \\ \int_{t_0}^t\exp\Big\{\int_s^t\overline{H}_1(\tau)d\tau\Big\}\Big[\overline{H}_1(s)\overline{B}(s)+\overline{L}_1(s)+\overline{\Phi}(s)\Big]ds, \quad t\in[t_0,t_0+r], \end{cases}$$

$$\overline{B}(t) = 5e^{-2t}\begin{bmatrix} a_1^2(t) \\ a_2^2(t) \end{bmatrix},$$

$$\overline{H}_1(t) = \begin{bmatrix} \overline{b}_{11}(t)\frac{q_{11}}{p_1}K_1^{\frac{q_{11}-p_1}{p_1}} & \overline{b}_{12}(t)\frac{q_{12}}{p_2}K_2^{\frac{q_{12}-p_2}{p_2}} \\ \overline{b}_{21}(t)\frac{q_{21}}{p_1}K_1^{\frac{q_{21}-p_1}{p_1}} & \overline{b}_{22}(t)\frac{q_{22}}{p_2}K_2^{\frac{q_{22}-p_2}{p_2}} \end{bmatrix},$$

$$\overline{H}_2(t) = \begin{bmatrix} \overline{c}_{13}(t)\frac{q_{13}}{p_1}K_3^{\frac{q_{13}-p_1}{p_1}} & \overline{c}_{14}(t)\frac{q_{14}}{p_2}K_4^{\frac{q_{14}-p_2}{p_2}} \\ \overline{c}_{23}(t)\frac{q_{23}}{p_1}K_3^{\frac{q_{23}-p_1}{p_1}} & \overline{c}_{24}(t)\frac{q_{24}}{p_2}K_4^{\frac{q_{24}-p_2}{p_2}} \end{bmatrix},$$

$$\overline{L}_1(t) = \begin{bmatrix} \overline{b}_{11}(t)\frac{p_1-q_{11}}{p_1}K_1^{\frac{q_{11}}{p_1}} + \overline{b}_{12}(t)\frac{p_2-q_{12}}{p_2}K_2^{\frac{q_{12}}{p_2}} \\ \overline{b}_{21}(t)\frac{p_1-q_{21}}{p_1}K_1^{\frac{q_{21}}{p_1}} + \overline{b}_{22}(t)\frac{p_2-q_{22}}{p_2}K_2^{\frac{q_{22}}{p_2}} \end{bmatrix},$$

$$\overline{L}_2(t) = \begin{bmatrix} \overline{c}_{13}(t)\frac{p_1-q_{13}}{p_1}K_3^{\frac{q_{13}}{p_1}} + \overline{c}_{14}(t)\frac{p_2-q_{14}}{p_2}K_4^{\frac{q_{14}}{p_2}} \\ \overline{c}_{23}(t)\frac{p_1-q_{23}}{p_1}K_3^{\frac{q_{23}}{p_1}} + \overline{c}_{24}(t)\frac{p_2-q_{24}}{p_2}K_4^{\frac{q_{24}}{p_2}} \end{bmatrix},$$

$$\overline{\Phi}(t) = \begin{bmatrix} \overline{c}_{13}(t)\overline{\phi}_1^{\frac{q_{13}}{p_1}}(t-r) + \overline{c}_{14}(t)\overline{\phi}_2^{\frac{q_{14}}{p_2}}(t-r) \\ \overline{c}_{23}(t)\overline{\phi}_1^{\frac{q_{23}}{p_1}}(t-r) + \overline{c}_{24}(t)\overline{\phi}_2^{\frac{q_{24}}{p_2}}(t-r) \end{bmatrix},$$

$\overline{b}_{ij}(t) = \frac{5\Gamma(2\alpha-1)}{2^{2\alpha-1}}e^{2(\frac{q_{ij}}{p_j}-1)t}b_{ij}^2(t)$, $\overline{c}_{i3}(t) = \frac{5\Gamma(2\alpha-1)e^{-\frac{2q_{i3}r}{p_1}}}{2^{2\alpha-1}}e^{2(\frac{q_{i3}}{p_1}-1)t}c_{i3}^2(t)$, $\overline{c}_{i4}(t) = \frac{5\Gamma(2\alpha-1)e^{-\frac{2q_{i4}r}{p_2}}}{2^{2\alpha-1}}e^{2(\frac{q_{i4}}{p_2}-1)t}c_{i4}^2(t)$, $\overline{\phi}_i(t) = 5e^{-2t}\phi_i^2(t)$;

(ii) when $0 < \alpha \le \frac{1}{2}$, *set* $p = 1+\alpha$, $q = 1+\frac{1}{\alpha}$ *and*

$$\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} e^{\frac{t}{p_1}}\widetilde{u}_1^{\frac{1}{q}}(t) \\ e^{\frac{t}{p_2}}\widetilde{u}_2^{\frac{1}{q}}(t) \end{bmatrix}, \quad t \in [t_0, +\infty),$$

then we obtain

$$\begin{bmatrix} \widetilde{u}_1(t) \\ \widetilde{u}_2(t) \end{bmatrix} \le \left[\widetilde{B}(t) + \widetilde{G}(t)\right]^{\frac{1}{p_1}}_{\frac{1}{p_2}}, \quad t \in [t_0, +\infty),$$

where

$$\widetilde{G}(t) = \begin{cases} \exp\left\{\int_{t_0+r}^t \left[\widetilde{H}_1(\tau)+\widetilde{H}_2(\tau)\right]d\tau\right\}\int_{t_0}^{t_0+r}\exp\left\{\int_s^{t_0+r}\widetilde{H}_1(\tau)d\tau\right\}\left[\widetilde{H}_1(s)\widetilde{B}(s)\right. \\ \left.+\widetilde{L}_1(s)+\widetilde{\Phi}(s)\right]ds + \int_{t_0+r}^t \exp\left\{\int_s^t\left[\widetilde{H}_1(\tau)+\widetilde{H}_2(\tau)\right]d\tau\right\}\left[\widetilde{H}_1(s)\widetilde{B}(s)\right. \\ \left.+\widetilde{H}_2(s)\widetilde{B}(s-r)+\widetilde{L}_1(s)+\widetilde{L}_2(s)\right]ds, t \in [t_0+r, +\infty), \\ \int_{t_0}^t \exp\left\{\int_s^t \widetilde{H}_1(\tau)d\tau\right\}\left[\widetilde{H}_1(s)\widetilde{B}(s)+\widetilde{L}_1(s)+\widetilde{\Phi}(s)\right]ds, \quad t \in [t_0, t_0+r], \end{cases}$$

$$\widetilde{B}(t) = 5^{q-1}e^{-qt}\begin{bmatrix} a_1^q(t) \\ a_2^q(t) \end{bmatrix},$$

$$\widetilde{H}_1(t) = \begin{bmatrix} \widetilde{b}_{11}(t)\frac{q_{11}}{p_1}K_1^{\frac{q_{11}-p_1}{p_1}} & \widetilde{b}_{12}(t)\frac{q_{12}}{p_2}K_2^{\frac{q_{12}-p_2}{p_2}} \\ \widetilde{b}_{21}(t)\frac{q_{21}}{p_1}K_1^{\frac{q_{21}-p_1}{p_1}} & \widetilde{b}_{22}(t)\frac{q_{22}}{p_2}K_2^{\frac{q_{22}-p_2}{p_2}} \end{bmatrix},$$

$$\widetilde{H}_2(t) = \begin{bmatrix} \widetilde{c}_{13}(t)\frac{q_{13}}{p_1}K_3^{\frac{q_{13}-p_1}{p_1}} & \widetilde{c}_{14}(t)\frac{q_{14}}{p_2}K_4^{\frac{q_{14}-p_2}{p_2}} \\ \widetilde{c}_{23}(t)\frac{q_{23}}{p_1}K_3^{\frac{q_{23}-p_1}{p_1}} & \widetilde{c}_{24}(t)\frac{q_{24}}{p_2}K_4^{\frac{q_{24}-p_2}{p_2}} \end{bmatrix},$$

$$\widetilde{L}_1(t) = \begin{bmatrix} \widetilde{b}_{11}(t)\frac{p_1-q_{11}}{p_1}K_1^{\frac{q_{11}}{p_1}} + \widetilde{b}_{12}(t)\frac{p_2-q_{12}}{p_2}K_2^{\frac{q_{12}}{p_2}} \\ \widetilde{b}_{21}(t)\frac{p_1-q_{21}}{p_1}K_1^{\frac{q_{21}}{p_1}} + \widetilde{b}_{22}(t)\frac{p_2-q_{22}}{p_2}K_2^{\frac{q_{22}}{p_2}} \end{bmatrix},$$

$$\widetilde{L}_2(t) = \begin{bmatrix} \widetilde{c}_{13}(t)\frac{p_1-q_{13}}{p_1}K_3^{\frac{q_{13}}{p_1}} + \widetilde{c}_{14}(t)\frac{p_2-q_{14}}{p_2}K_4^{\frac{q_{14}}{p_2}} \\ \widetilde{c}_{23}(t)\frac{p_1-q_{23}}{p_1}K_3^{\frac{q_{23}}{p_1}} + \widetilde{c}_{24}(t)\frac{p_2-q_{24}}{p_2}K_4^{\frac{q_{24}}{p_2}} \end{bmatrix},$$

$$\widetilde{\Phi}(t) = \begin{bmatrix} \widetilde{c}_{13}(t)\widetilde{\phi}_1^{\frac{q_{13}}{p_1}}(t-r) + \widetilde{c}_{14}(t)\widetilde{\phi}_2^{\frac{q_{14}}{p_2}}(t-r) \\ \widetilde{c}_{23}(t)\widetilde{\phi}_1^{\frac{q_{23}}{p_1}}(t-r) + \widetilde{c}_{24}(t)\widetilde{\phi}_2^{\frac{q_{24}}{p_2}}(t-r) \end{bmatrix},$$

$\widetilde{b}_{ij}(t) = \frac{5^{q-1}\Gamma^{\frac{1}{\alpha}}(\alpha^2)}{p^\alpha}e^{q(\frac{q_{ij}}{p_j}-1)t}b_{ij}^q(t)$, $\widetilde{c}_{i3}(t) = \frac{5^{q-1}\Gamma^{\frac{1}{\alpha}}(\alpha^2)e^{\frac{-qq_{i3}r}{p_1}}}{p^\alpha}e^{q(\frac{q_{i3}}{p_1}-1)t}c_{i3}^q(t)$, $\widetilde{c}_{i4}(t) = \frac{5^{q-1}\Gamma^{\frac{1}{\alpha}}(\alpha^2)e^{\frac{-qq_{i4}r}{p_2}}}{p^\alpha}e^{q(\frac{q_{i4}}{p_2}-1)t}c_{i4}^q(t)$, $\widetilde{\phi}_i(t) = 5^{q-1}e^{-qt}\phi_i^q(t)$.

Proof. (i) Assume $\alpha > \frac{1}{2}$. For $t \in [t_0, +\infty)$ and $i = 1, 2$, by Cauchy-Schwarz inequality we get

$$\begin{aligned} u_i^{p_i}(t) \le& a_i(t) + \int_{t_0}^t (t-s)^{\alpha-1}e^s e^{-s}b_{i1}(s)u_1^{q_{i1}}(s)ds \\ &+ \int_{t_0}^t (t-s)^{\alpha-1}e^s e^{-s}b_{i2}(s)u_2^{q_{i2}}(s)ds \\ &+ \int_{t_0}^t (t-s)^{\alpha-1}e^s e^{-s}c_{i3}(s)u_1^{q_{i3}}(s-r)ds \\ &+ \int_{t_0}^t (t-s)^{\alpha-1}e^s e^{-s}c_{i4}(s)u_2^{q_{i4}}(s-r)ds \\ \le& a_i(t) + \Big[\int_{t_0}^t (t-s)^{2(\alpha-1)}e^{2s}ds\Big]^{\frac{1}{2}}\Big[\int_{t_0}^t b_{i1}^2(s)e^{-2s}u_1^{2q_{i1}}(s)ds\Big]^{\frac{1}{2}} \\ &+ \Big[\int_{t_0}^t (t-s)^{2(\alpha-1)}e^{2s}ds\Big]^{\frac{1}{2}}\Big[\int_{t_0}^t b_{i2}^2(s)e^{-2s}u_2^{2q_{i2}}(s)ds\Big]^{\frac{1}{2}} \\ &+ \Big[\int_{t_0}^t (t-s)^{2(\alpha-1)}e^{2s}ds\Big]^{\frac{1}{2}}\Big[\int_{t_0}^t c_{i3}^2(s)e^{-2s}u_1^{2q_{i3}}(s-r)ds\Big]^{\frac{1}{2}} \\ &+ \Big[\int_{t_0}^t (t-s)^{2(\alpha-1)}e^{2s}ds\Big]^{\frac{1}{2}}\Big[\int_{t_0}^t c_{i4}^2(s)e^{-2s}u_2^{2q_{i4}}(s-r)ds\Big]^{\frac{1}{2}} \\ \le& a_i(t) + \Big[\frac{e^{2t}}{2^{2\alpha-1}}\Gamma(2\alpha-1)\Big]^{\frac{1}{2}}\Big[\int_{t_0}^t b_{i1}^2(s)e^{-2s}u_1^{2q_{i1}}(s)ds\Big]^{\frac{1}{2}} \\ &+ \Big[\frac{e^{2t}}{2^{2\alpha-1}}\Gamma(2\alpha-1)\Big]^{\frac{1}{2}}\Big[\int_{t_0}^t b_{i2}^2(s)e^{-2s}u_2^{2q_{i2}}(s)ds\Big]^{\frac{1}{2}} \end{aligned}$$

$$+\left[\frac{e^{2t}}{2^{2\alpha-1}}\Gamma(2\alpha-1)\right]^{\frac{1}{2}}\left[\int_{t_0}^{t}c_{i3}^2(s)e^{-2s}u_1^{2q_{i3}}(s-r)ds\right]^{\frac{1}{2}}$$
$$+\left[\frac{e^{2t}}{2^{2\alpha-1}}\Gamma(2\alpha-1)\right]^{\frac{1}{2}}\left[\int_{t_0}^{t}c_{i4}^2(s)e^{-2s}u_2^{2q_{i4}}(s-r)ds\right]^{\frac{1}{2}}.$$

Using Lemma 2.2 with $n=5,\sigma=2$ yields that

$$\begin{aligned}u_i^{2p_i}(t)\leq&5\Big[a_i^2(t)+\frac{e^{2t}}{2^{2\alpha-1}}\Gamma(2\alpha-1)\int_{t_0}^{t}b_{i1}^2(s)e^{-2s}u_1^{2q_{i1}}(s)ds\\&+\frac{e^{2t}}{2^{2\alpha-1}}\Gamma(2\alpha-1)\int_{t_0}^{t}b_{i2}^2(s)e^{-2s}u_2^{2q_{i2}}(s)ds\\&+\frac{e^{2t}}{2^{2\alpha-1}}\Gamma(2\alpha-1)\int_{t_0}^{t}c_{i3}^2(s)e^{-2s}u_1^{2q_{i3}}(s-r)ds\\&+\frac{e^{2t}}{2^{2\alpha-1}}\Gamma(2\alpha-1)\int_{t_0}^{t}c_{i4}^2(s)e^{-2s}u_2^{2q_{i4}}(s-r)ds\Big].\end{aligned}$$

Set $\overline{u}_i(t)=e^{-\frac{2}{p_i}t}u_i^2(t)$, $\overline{a}_i(t)=5e^{-2t}a_i^2(t)$, $\overline{b}_{ij}(t)=\frac{5\Gamma(2\alpha-1)}{2^{2\alpha-1}}e^{2(\frac{q_{ij}}{p_j}-1)t}b_{ij}^2(t)$, $\overline{c}_{i3}(t)=\frac{5\Gamma(2\alpha-1)e^{-\frac{2q_{i3}r}{p_1}}}{2^{2\alpha-1}}e^{2(\frac{q_{i3}}{p_1}-1)t}c_{i3}^2(t)$, $\overline{c}_{i4}(t)=\frac{5\Gamma(2\alpha-1)e^{-\frac{2q_{i4}r}{p_2}}}{2^{2\alpha-1}}e^{2(\frac{q_{i4}}{p_2}-1)t}c_{i4}^2(t)$, then for $t\in[t_0,+\infty)$ and $i=1,2$, we obtain

$$\begin{aligned}\overline{u}_i^{p_i}(t)\leq&\overline{a}_i(t)+\int_{t_0}^{t}\Big[\overline{b}_{i1}(s)\overline{u}_1^{q_{i1}}(s)+\overline{b}_{i2}(s)\overline{u}_2^{q_{i2}}(s)\\&+\overline{c}_{i3}(s)\overline{u}_1^{q_{i3}}(s-r)+\overline{c}_{i4}(s)\overline{u}_2^{q_{i4}}(s-r)\Big]ds.\end{aligned}\tag{3.1}$$

For $t\in[t_0-r,t_0)$ and $i=1,2$, we have

$$\overline{u}_i^{p_i}(t)=e^{-2t}u_i^{2p_i}(t)\leq 5e^{-2t}\phi_i^2(t).\tag{3.2}$$

Applying Lemma 2.3 to (3.1) and (3.2) obtains the desired conclusion.

(ii) Assume $0<\alpha\leq\frac{1}{2}$. For $t\in[t_0,+\infty)$ and $i=1,2$, by $H\ddot{o}lder's$ inequality with the index $p=1+\alpha$ for $\frac{1}{p}+\frac{1}{q}=1$, we get

$$\begin{aligned}u_i^{p_i}(t)\leq&a_i(t)+\int_{t_0}^{t}(t-s)^{\alpha-1}e^{s}e^{-s}b_{i1}(s)u_1^{q_{i1}}(s)ds\\&+\int_{t_0}^{t}(t-s)^{\alpha-1}e^{s}e^{-s}b_{i2}(s)u_2^{q_{i2}}(s)ds\\&+\int_{t_0}^{t}(t-s)^{\alpha-1}e^{s}e^{-s}c_{i3}(s)u_1^{q_{i3}}(s-r)ds\\&+\int_{t_0}^{t}(t-s)^{\alpha-1}e^{s}e^{-s}c_{i4}(s)u_2^{q_{i4}}(s-r)ds\\\leq&a_i(t)+\left[\int_{t_0}^{t}(t-s)^{p(\alpha-1)}e^{ps}ds\right]^{\frac{1}{p}}\left[\int_{t_0}^{t}b_{i1}^q(s)e^{-qs}u_1^{qq_{i1}}(s)ds\right]^{\frac{1}{q}}\\&+\left[\int_{t_0}^{t}(t-s)^{p(\alpha-1)}e^{ps}ds\right]^{\frac{1}{p}}\left[\int_{t_0}^{t}b_{i2}^q(s)e^{-qs}u_2^{qq_{i2}}(s)ds\right]^{\frac{1}{q}}\end{aligned}$$

$$
\begin{aligned}
&+\Big[\int_{t_0}^t (t-s)^{p(\alpha-1)}e^{ps}ds\Big]^{\frac{1}{p}}\Big[\int_{t_0}^t c_{i3}^q(s)e^{-qs}u_1^{qq_{i3}}(s-r)ds\Big]^{\frac{1}{q}}\\
&+\Big[\int_{t_0}^t (t-s)^{p(\alpha-1)}e^{ps}ds\Big]^{\frac{1}{p}}\Big[\int_{t_0}^t c_{i4}^q(s)e^{-qs}u_2^{qq_{i4}}(s-r)ds\Big]^{\frac{1}{q}}\\
\le& a_i(t)+\Big[\frac{e^{pt}}{p^{\alpha^2}}\Gamma(\alpha^2)\Big]^{\frac{1}{p}}\Big[\int_{t_0}^t b_{i1}^q(s)e^{-qs}u_1^{qq_{i1}}(s)ds\Big]^{\frac{1}{q}}\\
&+\Big[\frac{e^{pt}}{p^{\alpha^2}}\Gamma(\alpha^2)\Big]^{\frac{1}{p}}\Big[\int_{t_0}^t b_{i2}^q(s)e^{-qs}u_2^{qq_{i2}}(s)ds\Big]^{\frac{1}{q}}\\
&+\Big[\frac{e^{pt}}{p^{\alpha^2}}\Gamma(\alpha^2)\Big]^{\frac{1}{p}}\Big[\int_{t_0}^t c_{i3}^q(s)e^{-qs}u_1^{qq_{i3}}(s-r)ds\Big]^{\frac{1}{q}}\\
&+\Big[\frac{e^{pt}}{p^{\alpha^2}}\Gamma(\alpha^2)\Big]^{\frac{1}{p}}\Big[\int_{t_0}^t c_{i4}^q(s)e^{-qs}u_2^{qq_{i4}}(s-r)ds\Big]^{\frac{1}{q}}.
\end{aligned}
$$

Lemma 2.2 implies that

$$
\begin{aligned}
u_i^{qp_i}(t)\le& 5^{q-1}a_i^q(t)+5^{q-1}\Big[\frac{e^{qt}}{p^\alpha}\Gamma^{\frac{1}{\alpha}}(\alpha^2)\int_{t_0}^t b_{i1}^q(s)e^{-qs}u_1^{qq_{i1}}(s)ds\\
&+\frac{e^{qt}}{p^\alpha}\Gamma^{\frac{1}{\alpha}}(\alpha^2)\int_{t_0}^t b_{i2}^q(s)e^{-qs}u_2^{qq_{i2}}(s)ds\\
&+\frac{e^{qt}}{p^\alpha}\Gamma^{\frac{1}{\alpha}}(\alpha^2)\int_{t_0}^t c_{i3}^q(s)e^{-qs}u_1^{qq_{i3}}(s-r)ds\\
&+\frac{e^{qt}}{p^\alpha}\Gamma^{\frac{1}{\alpha}}(\alpha^2)\int_{t_0}^t c_{i4}^q(s)e^{-qs}u_2^{qq_{i4}}(s-r)ds\Big].
\end{aligned}
$$

Let $\widetilde{u}_i(t)=e^{-\frac{q}{p_i}t}u_i^q(t)$, $\widetilde{a}_i(t)=5^{q-1}e^{-qt}a_i^q(t)$, $\widetilde{b}_{ij}(t)=\frac{5^{q-1}\Gamma^{\frac{1}{\alpha}}(\alpha^2)}{p^\alpha}e^{q(\frac{q_{ij}}{p_j}-1)t}b_{ij}^q(t)$, $\widetilde{c}_{i3}(t)=\frac{5^{q-1}\Gamma^{\frac{1}{\alpha}}(\alpha^2)e^{\frac{-qq_{i3}r}{p_1}}}{p^\alpha}e^{q(\frac{q_{i3}}{p_1}-1)t}c_{i3}^q(t)$, $\widetilde{c}_{i4}(t)=\frac{5^{q-1}\Gamma^{\frac{1}{\alpha}}(\alpha^2)e^{\frac{-qq_{i4}r}{p_2}}}{p^\alpha}e^{q(\frac{q_{i4}}{p_2}-1)t}c_{i4}^q(t)$, then for $t\in[t_0,+\infty)$ and $i=1,2$, we obtain

$$
\begin{aligned}
\widetilde{u}_i^{p_i}(t)\le& \widetilde{a}_i(t)+\int_{t_0}^t\Big[\widetilde{b}_{i1}(s)\widetilde{u}_1^{q_{i1}}(s)+\widetilde{b}_{i2}(s)\widetilde{u}_2^{q_{i2}}(s)\\
&+\widetilde{c}_{i3}(s)\widetilde{u}_1^{q_{i3}}(s-r)+\widetilde{c}_{i4}(s)\widetilde{u}_2^{q_{i4}}(s-r)\Big]ds. \qquad (3.3)
\end{aligned}
$$

For $t\in[t_0-r,t_0)$ and $i=1,2$, we have

$$
\widetilde{u}_i^{p_i}(t)=e^{-qt}u_i^{p_iq}(t)\le 5^{q-1}e^{-qt}\phi_i^q(t). \qquad (3.4)
$$

Applying Lemma 2.3 to (3.3) and (3.4) obtains the conclusion and this proves the theorem. □

Next we consider the estimate of the components-wise for solutions of nonlinear two-dimensional fractional differential system with delay (1.1).

Theorem 3.2. *If $f,g\in C([t_0,+\infty)\times R^4,R)$ and satisfy the following condition:*

$$
\begin{cases}
|f(t,m_1,m_2,m_3,m_4)|\le b_{11}(t)|m_1|^{k_{11}}+b_{12}(t)|m_2|^{k_{12}}+c_{13}(t)|m_3|^{k_{13}}+c_{14}(t)|m_4|^{k_{14}},\\
|g(t,m_1,m_2,m_3,m_4)|\le b_{21}(t)|m_1|^{k_{21}}+b_{22}(t)|m_2|^{k_{22}}+c_{23}(t)|m_3|^{k_{23}}+c_{24}(t)|m_4|^{k_{24}},
\end{cases} \qquad (3.5)
$$

where $b_{ij},\ c_{ij} \in C([t_0,+\infty),R_+)$, $k_{ij} \in (0,1]$ *are constants,* $i=1,2, j=1,2,3,4$, *then for any solution* $(x(t),y(t))$ *of the system* (1.1) *and* $K_j>0(j=1,2,3,4)$, *we have*

(*i*) *when* $\frac{1}{2}<\alpha\le 1$, *denote*

$$\begin{bmatrix} |x(t)| \\ |y(t)| \end{bmatrix} = e^t \begin{bmatrix} \overline{|x(t)|}^{\frac{1}{2}} \\ \overline{|y(t)|}^{\frac{1}{2}} \end{bmatrix}, \quad t\in[t_0,+\infty),$$

then we get

$$\begin{bmatrix} \overline{|x(t)|} \\ \overline{|y(t)|} \end{bmatrix} \le \overline{B}(t)+\overline{G}(t), \quad t\in[t_0,+\infty),$$

where

$$\overline{G}(t)=\begin{cases} \exp\left\{\int_{t_0+r}^{t}\left[\overline{H}_1(\tau)+\overline{H}_2(\tau)\right]d\tau\right\}\int_{t_0}^{t_0+r}\exp\left\{\int_s^{t_0+r}\overline{H}_1(\tau)d\tau\right\}\left[\overline{H}_1(s)\overline{B}(s)\right. \\ \left.+\overline{L}_1(s)+\overline{\Phi}(s)\right]ds+\int_{t_0+r}^{t}\exp\left\{\int_s^t\left[\overline{H}_1(\tau)+\overline{H}_2(\tau)\right]d\tau\right\}\left[\overline{H}_1(s)\overline{B}(s)\right. \\ \left.+\overline{H}_2(s)\overline{B}(s-r)+\overline{L}_1(s)+\overline{L}_2(s)\right]ds, t\in[t_0+r,+\infty), \\ \int_{t_0}^{t}\exp\left\{\int_s^t\overline{H}_1(\tau)d\tau\right\}\left[\overline{H}_1(s)\overline{B}(s)+\overline{L}_1(s)+\overline{\Phi}(s)\right]ds, \quad t\in[t_0,t_0+r], \end{cases}$$

$$\overline{B}(t)=5e^{-2t}\begin{bmatrix} M_1^2 \\ M_2^2 \end{bmatrix},$$

$$\overline{H}_1(t)=\begin{bmatrix} \overline{b}_{11}(t)k_{11}K_1^{k_{11}-1} & \overline{b}_{12}(t)k_{12}K_2^{k_{12}-1} \\ \overline{b}_{21}(t)k_{21}K_1^{k_{21}-1} & \overline{b}_{22}(t)k_{22}K_2^{k_{22}-1} \end{bmatrix},$$

$$\overline{H}_2(t)=\begin{bmatrix} \overline{c}_{13}(t)k_{13}K_3^{k_{13}-1} & \overline{c}_{14}(t)k_{14}K_4^{k_{14}-1} \\ \overline{c}_{23}(t)k_{23}K_3^{k_{23}-1} & \overline{c}_{24}(t)k_{24}K_4^{k_{24}-1} \end{bmatrix},$$

$$\overline{L}_1(t)=\begin{bmatrix} \overline{b}_{11}(t)(1-k_{11})K_1^{k_{11}}+\overline{b}_{12}(t)(1-k_{12})K_2^{k_{12}} \\ \overline{b}_{21}(t)(1-k_{21})K_1^{k_{21}}+\overline{b}_{22}(t)(1-k_{22})K_2^{k_{22}} \end{bmatrix},$$

$$\overline{L}_2(t)=\begin{bmatrix} \overline{c}_{13}(t)(1-k_{13})K_3^{k_{13}}+\overline{c}_{14}(t)(1-k_{14})K_4^{k_{14}} \\ \overline{c}_{23}(t)(1-k_{23})K_3^{k_{23}}+\overline{c}_{24}(t)(1-k_{24})K_4^{k_{24}} \end{bmatrix},$$

$$\overline{\Phi}(t)=\begin{bmatrix} \overline{c}_{13}(t)\overline{\phi}_1^{k_{13}}(t-r)+\overline{c}_{14}(t)\overline{\phi}_2^{k_{14}}(t-r) \\ \overline{c}_{23}(t)\overline{\phi}_1^{k_{23}}(t-r)+\overline{c}_{24}(t)\overline{\phi}_2^{k_{24}}(t-r) \end{bmatrix},$$

$\overline{b}_{ij}(t)=\frac{5\Gamma(2\alpha-1)}{2^{2\alpha-1}\Gamma^2(\alpha)}e^{2(k_{ij}-1)t}b_{ij}^2(t)$, $\overline{c}_{i3}(t)=\frac{5\Gamma(2\alpha-1)e^{-2k_{i3}r}}{2^{2\alpha-1}\Gamma^2(\alpha)}e^{2(k_{i3}-1)t}c_{i3}^2(t)$, $\overline{c}_{i4}(t)=\frac{5\Gamma(2\alpha-1)e^{-2k_{i4}r}}{2^{2\alpha-1}\Gamma^2(\alpha)}e^{2(k_{i4}-1)t}c_{i4}^2(t)$, $\overline{\phi}_i(t)=5e^{-2t}M_i^2$;

(ii) when $0<\alpha\le\frac{1}{2}$, *let* $p=1+\alpha, q=1+\frac{1}{\alpha}$ *and*

$$\begin{bmatrix}|x(t)|\\|y(t)|\end{bmatrix}=e^t\begin{bmatrix}\widetilde{|x(t)|}^{\frac{1}{q}}\\\widetilde{|y(t)|}^{\frac{1}{q}}\end{bmatrix},\quad t\in[t_0,+\infty),$$

then we have

$$\begin{bmatrix}\widetilde{|x(t)|}\\\widetilde{|y(t)|}\end{bmatrix}\le\widetilde{B}(t)+\widetilde{G}(t),\quad t\in[t_0,+\infty),$$

where

$$\widetilde{G}(t)=\begin{cases}\exp\Big\{\int_{t_0+r}^t\Big[\widetilde{H}_1(\tau)+\widetilde{H}_2(\tau)\Big]d\tau\Big\}\int_{t_0}^{t_0+r}\exp\Big\{\int_s^{t_0+r}\widetilde{H}_1(\tau)d\tau\Big\}\Big[\widetilde{H}_1(s)\widetilde{B}(s)\\+\widetilde{L}_1(s)+\widetilde{\Phi}(s)\Big]ds+\int_{t_0+r}^t\exp\Big\{\int_s^t\Big[\widetilde{H}_1(\tau)+\widetilde{H}_2(\tau)\Big]d\tau\Big\}\Big[\widetilde{H}_1(s)\widetilde{B}(s)\\+\widetilde{H}_2(s)\widetilde{B}(s-r)+\widetilde{L}_1(s)+\widetilde{L}_2(s)\Big]ds, t\in[t_0+r,+\infty),\\\int_{t_0}^t\exp\Big\{\int_s^t\widetilde{H}_1(\tau)d\tau\Big\}\Big[\widetilde{H}_1(s)\widetilde{B}(s)+\widetilde{L}_1(s)+\widetilde{\Phi}(s)\Big]ds,\quad t\in[t_0,t_0+r],\end{cases}$$

$$\widetilde{B}(t)=5^{q-1}e^{-qt}\begin{bmatrix}M_1^q\\M_2^q\end{bmatrix},$$

$$\widetilde{H}_1(t)=\begin{bmatrix}\widetilde{b}_{11}(t)k_{11}K_1^{k_{11}-1}&\widetilde{b}_{12}(t)k_{12}K_2^{k_{12}-1}\\\widetilde{b}_{21}(t)k_{21}K_1^{k_{21}-1}&\widetilde{b}_{22}(t)k_{22}K_2^{k_{22}-1}\end{bmatrix},$$

$$\widetilde{H}_2(t)=\begin{bmatrix}\widetilde{c}_{13}(t)k_{13}K_3^{k_{13}-1}&\widetilde{c}_{14}(t)k_{14}K_4^{k_{14}-1}\\\widetilde{c}_{23}(t)k_{23}K_3^{k_{23}-1}&\widetilde{c}_{24}(t)k_{24}K_4^{k_{24}-1}\end{bmatrix},$$

$$\widetilde{L}_1(t)=\begin{bmatrix}\widetilde{b}_{11}(t)(1-k_{11})K_1^{k_{11}}+\widetilde{b}_{12}(t)(1-k_{12})K_2^{k_{12}}\\\widetilde{b}_{21}(t)(1-k_{21})K_1^{k_{21}}+\widetilde{b}_{22}(t)(1-k_{22})K_2^{k_{22}}\end{bmatrix},$$

$$\widetilde{L}_2(t)=\begin{bmatrix}\widetilde{c}_{13}(t)(1-k_{13})K_3^{k_{13}}+\widetilde{c}_{14}(t)(1-k_{14})K_4^{k_{14}}\\\widetilde{c}_{23}(t)(1-k_{23})K_3^{k_{23}}+\widetilde{c}_{24}(t)(1-k_{24})K_4^{k_{24}}\end{bmatrix},$$

$$\widetilde{\Phi}(t)=\begin{bmatrix}\widetilde{c}_{13}(t)\widetilde{\phi}_1^{k_{13}}(t-r)+\widetilde{c}_{14}(t)\widetilde{\phi}_2^{k_{14}}(t-r)\\\widetilde{c}_{23}(t)\widetilde{\phi}_1^{k_{23}}(t-r)+\widetilde{c}_{24}(t)\widetilde{\phi}_2^{k_{24}}(t-r)\end{bmatrix},$$

$\widetilde{b}_{ij}(t)=\frac{5^{q-1}\Gamma^{\frac{1}{\alpha}}(\alpha^2)}{p^\alpha\Gamma^q(\alpha)}e^{q(k_{ij}-1)t}b_{ij}^q(t)$, $\widetilde{c}_{i3}(t)=\frac{5^{q-1}\Gamma^{\frac{1}{\alpha}}(\alpha^2)e^{-qk_{i3}r}}{p^\alpha\Gamma^q(\alpha)}e^{q(k_{i3}-1)t}c_{i3}^q(t)$, $\widetilde{c}_{i4}(t)=\frac{5^{q-1}\Gamma^{\frac{1}{\alpha}}(\alpha^2)e^{-qk_{i4}r}}{p^\alpha\Gamma^q(\alpha)}e^{q(k_{i4}-1)t}c_{i4}^q(t)$, $\widetilde{\phi}_i(t)=5^{q-1}e^{-qt}M_i^q$;

(iii) when $\alpha>1$, *denote*

$$\begin{bmatrix}|x(t)|\\|y(t)|\end{bmatrix}=e^t\begin{bmatrix}\overline{|x(t)|}^{\frac{1}{2}}\\\overline{|y(t)|}^{\frac{1}{2}}\end{bmatrix},\quad t\in[t_0,+\infty),$$

then we obtain

$$\begin{bmatrix} \overline{|x(t)|} \\ \overline{|y(t)|} \end{bmatrix} \le \overline{B}(t) + \overline{G}(t), \quad t \in [t_0, +\infty),$$

where

$$\overline{G}(t) = \begin{cases} \exp\Big\{\int_{t_0+r}^{t}\Big[\overline{H}_1(\tau)+\overline{H}_2(\tau)\Big]d\tau\Big\}\int_{t_0}^{t_0+r}\exp\Big\{\int_s^{t_0+r}\overline{H}_1(\tau)d\tau\Big\}\Big[\overline{H}_1(s)\overline{B}(s) \\ +\overline{L}_1(s)+\overline{\Phi}(s)\Big]ds + \int_{t_0+r}^{t}\exp\Big\{\int_s^t\Big[\overline{H}_1(\tau)+\overline{H}_2(\tau)\Big]d\tau\Big\}\Big[\overline{H}_1(s)\overline{B}(s) \\ +\overline{H}_2(s)\overline{B}(s-r)+\overline{L}_1(s)+\overline{L}_2(s)\Big]ds, t\in[t_0+r,+\infty), \\ \int_{t_0}^{t}\exp\Big\{\int_s^t\overline{H}_1(\tau)d\tau\Big\}\Big[\overline{H}_1(s)\overline{B}(s)+\overline{L}_1(s)+\overline{\Phi}(s)\Big]ds, \quad t\in[t_0,t_0+r], \end{cases}$$

$$\overline{B}(t) = 5e^{-2t}\begin{bmatrix} \Big[M_1 + \sum_{j=1}^{n-1}\frac{|m_j|}{j!}(t-t_0)^j\Big]^2 \\ \Big[M_2 + \sum_{j=1}^{n-1}\frac{|n_j|}{j!}(t-t_0)^j\Big]^2 \end{bmatrix},$$

$$\overline{H}_1(t) = \begin{bmatrix} \bar{b}_{11}(t)k_{11}K_1^{k_{11}-1} & \bar{b}_{12}(t)k_{12}K_2^{k_{12}-1} \\ \bar{b}_{21}(t)k_{21}K_1^{k_{21}-1} & \bar{b}_{22}(t)k_{22}K_2^{k_{22}-1} \end{bmatrix},$$

$$\overline{H}_2(t) = \begin{bmatrix} \bar{c}_{13}(t)k_{13}K_3^{k_{13}-1} & \bar{c}_{14}(t)k_{14}K_4^{k_{14}-1} \\ \bar{c}_{23}(t)k_{23}K_3^{k_{23}-1} & \bar{c}_{24}(t)k_{24}K_4^{k_{24}-1} \end{bmatrix},$$

$$\overline{L}_1(t) = \begin{bmatrix} \bar{b}_{11}(t)(1-k_{11})K_1^{k_{11}} + \bar{b}_{12}(t)(1-k_{12})K_2^{k_{12}} \\ \bar{b}_{21}(t)(1-k_{21})K_1^{k_{21}} + \bar{b}_{22}(t)(1-k_{22})K_2^{k_{22}} \end{bmatrix},$$

$$\overline{L}_2(t) = \begin{bmatrix} \bar{c}_{13}(t)(1-k_{13})K_3^{k_{13}} + \bar{c}_{14}(t)(1-k_{14})K_4^{k_{14}} \\ \bar{c}_{23}(t)(1-k_{23})K_3^{k_{23}} + \bar{c}_{24}(t)(1-k_{24})K_4^{k_{24}} \end{bmatrix},$$

$$\overline{\Phi}(t) = \begin{bmatrix} \bar{c}_{13}(t)\overline{\phi}_1^{k_{13}}(t-r) + \bar{c}_{14}(t)\overline{\phi}_2^{k_{14}}(t-r) \\ \bar{c}_{23}(t)\overline{\phi}_1^{k_{23}}(t-r) + \bar{c}_{24}(t)\overline{\phi}_2^{k_{24}}(t-r) \end{bmatrix},$$

$\bar{b}_{ij}(t) = \frac{5\Gamma(2\alpha-1)}{2^{2\alpha-1}\Gamma^2(\alpha)}e^{2(k_{ij}-1)t}b_{ij}^2(t)$, $\bar{c}_{i3}(t) = \frac{5\Gamma(2\alpha-1)e^{-2k_{i3}r}}{2^{2\alpha-1}\Gamma^2(\alpha)}e^{2(k_{i3}-1)t}c_{i3}^2(t)$, $\bar{c}_{i4}(t) = \frac{5\Gamma(2\alpha-1)e^{-2k_{i4}r}}{2^{2\alpha-1}\Gamma^2(\alpha)}e^{2(k_{i4}-1)t}c_{i4}^2(t)$, $\overline{\phi}_i(t) = 5e^{-2t}M_i^2$.

Proof. The system (1.1) is equivalent to the fractional integral system

$$\begin{cases} x(t) = \sum_{j=0}^{n-1}\frac{m_j}{j!}(t-t_0)^j + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}f\Big(s,x(s),y(s),x(s-r),y(s-r)\Big)ds, \\ y(t) = \sum_{j=0}^{n-1}\frac{n_j}{j!}(t-t_0)^j \\ \qquad + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}g\Big(s,x(s),y(s),x(s-r),y(s-r)\Big)ds, \quad t\in[t_0,+\infty), \\ x(t) = \varphi(t), \\ y(t) = \psi(t), \quad t\in[t_0-r,t_0]. \end{cases}$$

When $0 < \alpha \le 1$, we derive from (3.5) that

$$\begin{cases} |x(t)| \le M_1 + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \Big[b_{11}(s)|x(s)|^{k_{11}} + b_{12}(s)|y(s)|^{k_{12}} \\ \qquad\qquad +c_{13}(s)|x(s-r)|^{k_{13}} + c_{14}(s)|y(s-r)|^{k_{14}} \Big] ds, \\ |y(t)| \le M_2 + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \Big[b_{21}(s)|x(s)|^{k_{21}} + b_{22}(s)|y(s)|^{k_{22}} \\ \qquad\qquad +c_{23}(s)|x(s-r)|^{k_{23}} + c_{24}(s)|y(s-r)|^{k_{24}} \Big] ds, \quad t \in [t_0, +\infty), \\ |x(t)| \le M_1, \\ |y(t)| \le M_2, \quad t \in [t_0 - r, t_0]. \end{cases} \tag{3.6}$$

When $\alpha > 1$, we have

$$\begin{cases} |x(t)| \le M_1 + \sum_{j=1}^{n-1} \frac{|m_j|}{j!} (t-t_0)^j + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \Big[b_{11}(s)|x(s)|^{k_{11}} + b_{12}(s)|y(s)|^{k_{12}} \\ \qquad\qquad +c_{13}(s)|x(s-r)|^{k_{13}} + c_{14}(s)|y(s-r)|^{k_{14}} \Big] ds, \\ |y(t)| \le M_2 + \sum_{j=1}^{n-1} \frac{|n_j|}{j!} (t-t_0)^j + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \Big[b_{21}(s)|x(s)|^{k_{21}} + b_{22}(s)|y(s)|^{k_{22}} \\ \qquad\qquad +c_{23}(s)|x(s-r)|^{k_{23}} + c_{24}(s)|y(s-r)|^{k_{24}} \Big] ds, \quad t \in [t_0, +\infty), \\ |x(t)| \le M_1, \\ |y(t)| \le M_2, \quad t \in [t_0 - r, t_0]. \end{cases} \tag{3.7}$$

Applying Theorem 3.1 to (3.6) and (3.7) yields the desired conclusion. □

4. An illustrative example

Example 4.1. Consider the following fractional differential system

$$\begin{cases} D^{\frac{7}{8}} x(t) = f\Big(t, x(t), y(t), x(t-1), y(t-1)\Big), \\ D^{\frac{7}{8}} y(t) = g\Big(t, x(t), y(t), x(t-1), y(t-1)\Big), \quad t \in [3, +\infty), \\ x(t) = \Gamma(\frac{7}{8}), \\ y(t) = \Gamma(\frac{7}{8}), \quad t \in [2, 3], \end{cases} \tag{4.1}$$

where $f(t,x,y,z,w) = g(t,x,y,z,w) = \frac{2^{\frac{3}{8}}\Gamma(\frac{7}{8})}{\sqrt{5\Gamma(\frac{3}{4})}} t^{\frac{1}{2}} [e^{\frac{1}{2}t}(x^{\frac{1}{2}} + y^{\frac{1}{2}}) + e^{\frac{1}{2}(t+1)}(z^{\frac{1}{2}} + w^{\frac{1}{2}})]$.

It is obvious that $|f(t,x,y,z,w)| = |g(t,x,y,z,w)| \le \frac{2^{\frac{3}{8}}\Gamma(\frac{7}{8})}{\sqrt{5\Gamma(\frac{3}{4})}} t^{\frac{1}{2}} [e^{\frac{1}{2}t}(|x|^{\frac{1}{2}} + |y|^{\frac{1}{2}}) + e^{\frac{1}{2}(t+1)}(|z|^{\frac{1}{2}} + |w|^{\frac{1}{2}})]$, $t \in [3, +\infty)$. From (4.1) and Theorem 3.2 combining with the arbitrariness of $K_j (j = 1,2,3,4)$, let $K_j = 1$, then we obtain $\overline{b}_{ij}(t) = \overline{c}_{ij}(t) = t, \overline{\phi}_i(t) = 5e^{-2t}\Gamma^2(\frac{7}{8}), i = 1,2, j = 1,2,3,4$. Thus $\overline{H}_1(t) = \overline{H}_2(t) = \frac{1}{2}t \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \overline{L}_1(t) = \overline{L}_2(t) = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \overline{\Phi}(t) = 2\sqrt{5}\Gamma(\frac{7}{8}) t e^{-(t-1)} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Set

$$\begin{bmatrix} |x(t)| \\ |y(t)| \end{bmatrix} = e^t \begin{bmatrix} \overline{|x(t)|}^{\frac{1}{2}} \\ \overline{|y(t)|}^{\frac{1}{2}} \end{bmatrix}, \quad t \in [3, +\infty),$$

then we have

$$\begin{bmatrix} \overline{|x(t)|} \\ \overline{|y(t)|} \end{bmatrix} \leq \overline{B}(t) + \overline{G}(t), \quad t \in [3, +\infty),$$

where

$$\overline{B}(t) = 5e^{-2t}\Gamma^2(\frac{7}{8}) \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

for $t \in [3, 4]$,

$$\begin{aligned} \overline{G}(t) &= \int_3^t \exp\Big\{ \int_s^t \overline{H}_1(\tau)d\tau \Big\} \Big[\overline{H}_1(s)\overline{B}(s) + \overline{L}_1(s) + \overline{\Phi}(s) \Big] ds \\ &\leq \Big[\frac{1}{2}\Gamma^2(\frac{7}{8}) + 21\sqrt{5}\Gamma(\frac{7}{8}) + 4 \Big] e^{\frac{7}{4}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \end{aligned}$$

for $t \in [4, +\infty)$,

$$\begin{aligned} \overline{G}(t) = &\exp\Big\{ \int_4^t \Big[\overline{H}_1(\tau) + \overline{H}_2(\tau) \Big] d\tau \Big\} \int_3^4 \exp\Big\{ \int_s^4 \overline{H}_1(\tau)d\tau \Big\} \Big[\overline{H}_1(s)\overline{B}(s) \\ &+ \overline{L}_1(s) + \overline{\Phi}(s) \Big] ds + \int_4^t \exp\Big\{ \int_s^t \Big[\overline{H}_1(\tau) + \overline{H}_2(\tau) \Big] d\tau \Big\} \Big[\overline{H}_1(s)\overline{B}(s) \\ &+ \overline{H}_2(s)\overline{B}(s-1) + \overline{L}_1(s) + \overline{L}_2(s) \Big] ds \\ \leq &\Big[21\Gamma^2(\frac{7}{8}) + 42\sqrt{5}\Gamma(\frac{7}{8}) + 12 \Big] e^{\frac{1}{2}t^2 - \frac{25}{4}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \end{aligned}$$

Acknowledgements

The authors are grateful to the editor and the reviewers for their constructive comments and suggestions for the improvement of the paper.

References

[1] N. Aguila-Camacho and M. A. Duarte-Mermoud, *Boundedness of the solutions for certain classes of fractional differential equations with application to adaptive systems*, ISA. Trans., 2016, 60, 82–88.

[2] J. Čermák, Z. Došlá and T. Kisela, *Fractional differential equations with a constant delay: stability and asymptotics of solutions*, Appl. Math. Comput., 2017, 298, 336–350.

[3] N. Cong, H. Tuan and H. Trinh, *On asymptotic properties of solutions to fractional differential equations*, J. Math. Anal. Appl., 2020, 484(2), 123759.

[4] Z. Denton and A. S. Vatsala, *Fractional integral inequalities and applications*, Comput. Math. Appl., 2010, 59(3), 1087–1094.

[5] R. Douaifia, S. Abdelmalek and S. Bendoukha, *Asymptotic stability conditions for autonomous time–fractional reaction–diffusion systems*, Commun. Nonlinear. Sci. Numer. Simul., 2020, 80, 104982.

[6] Q. Feng, F. Meng and B. Zheng, *Gronwall–Bellman type nonlinear delay integral inequalities on time scales*, J. Math. Anal. Appl., 2011, 382(2), 772–784.

[7] J. Gu and F. Meng, *Some new nonlinear Volterra–Fredholm type dynamic integral inequalities on time scales*, Appl. Math. Comput., 2014, 245, 235–242.

[8] J. He, X. Zhang, L. Liu et al., *Existence and asymptotic analysis of positive solutions for a singular fractional differential equation with nonlocal boundary conditions*, Bound. Value. Probl., 2018, 2018(1), 189.

[9] J. He, X. Zhang, L. Liu et al., *A singular fractional Kelvin–Voigt model involving a nonlinear operator and their convergence properties*, Bound. Value. Probl., 2019, 2019(1), 1–19.

[10] F. Jiang and F. Meng, *Explicit bounds on some new nonlinear integral inequalities with delay*, J. Comput. Appl. Math., 2007, 205(1), 479–486.

[11] M. Kuczma, *An introduction to the theory of functional equations and inequalities: Cauchy's equation and Jensen's inequality*, Springer Science & Business Media, 2009.

[12] Y. Liu, *Existence and uniqueness of solutions for a class of initial value problems of fractional differential systems on half lines*, Bull. Sci. Math., 2013, 137(8), 1048–1071.

[13] Q. Ma and J. Pečarić, *Some new explicit bounds for weakly singular integral inequalities with applications to fractional differential and integral equations*, J. Math. Anal. Appl., 2008, 341(2), 894–905.

[14] F. Meng, Q. Feng and B. Zheng, *Explicit bounds to some new Gronwall-Bellman-type delay integral inequalities in two independent variables on time scales*, J. Appl. Math., 2011, 2011.

[15] F. Meng and J. Shao, *Some new Volterra–Fredholm type dynamic integral inequalities on time scales*, Appl. Math. Comput., 2013, 223, 444–451.

[16] T. Ren, S. Li, X. Zhang and L. Liu, *Maximum and minimum solutions for a nonlocal p-Laplacian fractional differential system from eco-economical processes*, Bound. Value. Probl., 2017, 2017(1), 1–15.

[17] T. Ren, H. Xiao, Z. Zhou et al., *The iterative scheme and the convergence analysis of unique solution for a singular fractional differential equation from the eco-economic complex system's co-evolution process*, Complex., 2019, 2019.

[18] J. Wu, X. Zhang, L. Liu et al., *The convergence analysis and error estimation for unique solution of a p-Laplacian fractional differential equation with singular decreasing nonlinearity*, Bound. Value. Probl., 2018, 2018(1), 82.

[19] J. Wu, X. Zhang, L. Liu et al., *Convergence analysis of iterative scheme and error estimation of positive solution for a fractional differential equation*, Math. Model. Anal., 2018, 23(4), 611–626.

[20] R. Xu, F. Meng and C. Song, *On some integral inequalities on time scales and their applications*, J. Inequal. Appl., 2010, 2010, 1–13.

[21] H. Ye and J. Gao, *Henry–Gronwall type retarded integral inequalities and their applications to fractional differential equations with delay*, Appl. Math. Comput., 2011, 218(8), 4152–4160.

[22] H. Ye, J. Gao and Y. Ding, *A generalized Gronwall inequality and its application to a fractional differential equation*, J. Math. Anal. Appl., 2007, 328(2), 1075–1081.

[23] X. Zhang, L. Liu and Y. Wu, *Variational structure and multiple solutions for a fractional advection–dispersion equation*, Comput. Math. Appl., 2014, 68(12), 1794–1805.

[24] X. Zhang, C. Mao, L. Liu and Y. Wu, *Exact iterative solution for an abstract fractional dynamic system model for bioprocess*, Qual. Theory. Dyn. Syst., 2017, 16(1), 205–222.

[25] J. Zhao and F. Meng, *Some properties for solutions of Riemann-Liouville fractional differential systems with a delay*, J. Funct. Spaces., 2018, 2018.

[26] Y. Zhou, *Existence and uniqueness of fractional functional differential equations with unbounded delay*, Int. J. Dyn. Syst. Differ. Eqs., 2008, 1(4), 239–244.