LIE SYMMETRY ANALYSIS TO FISHER'S EQUATION WITH TIME FRACTIONAL ORDER*

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Abstract The aim of this letter is to apply the Lie group analysis method to the Fisher's equation with time fractional order. We considered the symmetry analysis, explicit solutions to the time fractional Fisher's(TFF) equations with Riemann-Liouville (R-L) derivative. The time fractional Fisher's is reduced to respective nonlinear ordinary differential equation(ODE) of fractional order. We solve the reduced fractional ODE using an explicit power series method.

Keywords Lie symmetry analysis, time fractional Fisher's equation, Riemann-Liouville derivative, Erdelyi-Kober operators, explicit solutions.

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1. Introduction

Many phenomena in fluid mechanics, viscoelasticity, biology, physics, engineering and other areas of science can be successfully modeled by the fractional partial differential equations (FPDEs) in recent years. Several efficient methods have been presented to solve fractional partial differential equations of physical interest. It is necessary to point out that some methods to nonlinear FPDEs for constructing numerical, exact and explicit solutions, variational iteration method, fractional difference method, differential transform method, homotopy perturbation method, transform method, sub-equation method, Adomian decomposition method [1,2,4,5,8,9,12,14–17,20,21,32,35,37,38] and so on. Recently, in [3,6,27,29,39], the Lie symmetry analysis is effectively applied to FPDEs, and some investigations are derived. Schrödinger equations, dynamics and rogue waves problems [22–25,36] are also hot topic recently. In [33], the author studies the invariance properties of the time fractional generalized fifth-order KdV equations by using the Lie group analy-

sis method. At the same time, article [33] show that the FPDEs can be transformed into a nonlinear ODE of fractional order.

The motivation of this paper is to extend the application of Lie group analysis method to the nonlinear time-fractional Fisher's equation

$$D_t^\alpha u = u_{xx} + 6u(1-u), \tag{1.1}$$

where $0 < \alpha \le 1, D_t^\alpha = \partial^\alpha u / \partial t^\alpha$. When $\alpha = 1$, Eq.(1.1) can be reduced to Fisher's equation of general meaning. Eq.(1) represents the evolution of the population due to the two competing physical processes, diffusion and nonlinear local multiplication. And it also describes a prototype mode for a spreading name and a model equation for the finite domain evolution of neutron population in a nuclear reactor. S. Momani and Z. Odibat derived the numerical solution of Eq.(1.1) by using Homotopy perturbation method in [18].

The paper is organized as follows. Riemann and Liouville definitions and formulas are given in section 2. In Section 3, we give an account of Lie symmetry analysis method for TFF briefly. In Section 4, we perform Lie group classification on the TFF equation, and investigate the symmetry reductions of the TFF equation. Through the symmetry reduction, we transform the FPDE into the fractional ordinary differential equations (FODE) with a new independent variable. In the meantime, some exact solutions are obtained. In Section 5 contains discussion of the obtained results.

2. Preliminaries

For the fractional derivative operators, there exist various definitions which are not necessarily equivalent to each other. In this paper, we consider the most common definition named after Riemann and Liouville, which is the natural generalization of the Cauchy formula for the n-fold primitive of a function $f(x)$. The Riemann-Liouville(R-L) fractional derivative is defined as [28]:

$$D_t^\alpha f = \begin{cases} \frac{d^n f}{dt^n} I^{n-\alpha} f(t), & 0 \le n-1 < \alpha < n, \\ \frac{d^n f}{dt^n}, & \alpha = n, \end{cases} \tag{2.1}$$

where $n \in \mathbb{N}$, $I^\mu f(t)$ is the R-L fractional integral of order μ , namely,

$$I^\mu f(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-\xi)^{\mu-\alpha} f(\xi) d\xi, \ \mu > 0$$
$$I^0 f(t) = f(t),$$

and $\Gamma(z)$ is the standard Gamma function.

Definition 2.1. The R-L fractional partial derivative is defined by

$$D_t^\alpha f = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \frac{\partial}{\partial t^n} \int_0^t (t-\xi)^{n-\alpha-1} u(\xi, x) d\xi, & 0 \le n-1 < \alpha < n, \\ \frac{\partial f}{\partial t^n}, & \alpha = n. \end{cases} \tag{2.2}$$

If it exists, where ∂_t^n is the usual partial derivative of integer order n [3, 33].

Some useful formulas and properties are given in [10], here we only motion the following:

$$D_t^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} t^{\gamma-\alpha}, \gamma > 0, \tag{2.3}$$

$$D_t^\alpha[u(t)v(t)] = u(t) \ D_t^\alpha v(t) + v(t) \ D_t^\alpha u(t), \tag{2.4}$$

$$D_t^\alpha[(f(u(t)))] = f_u'[u(t)]D_t^\alpha v(t) = D_u^\alpha f[u(t)](u_t')^\alpha. \tag{2.5}$$

Definition 2.2. The generalized Leibnitz rule [19, 26] defined by

$$D_t^\alpha[u(t)v(t)] = \sum_{n=0}^{\infty} \binom{\alpha}{n} D_t^{\alpha-n}u(t)D_t^n v(t), \ \alpha > 0, \tag{2.6}$$

where

$$\binom{\alpha}{n} = \frac{(-1)^{n-1}\alpha\Gamma(n-\alpha)}{\Gamma(1-\alpha)\Gamma(n+1)}. \tag{2.7}$$

Definition 2.3. In view of the generalization of the chain rule [13, 33] for composite functions

$$\frac{d^m f(g(t))}{dt^m} = \sum_{k=1}^{m}\sum_{r=0}^{k} \binom{k}{r} \frac{1}{k!}[-g(t)]^r \frac{d^m}{dt^m}[g(t)^{k-r}]\frac{d^k f(g)}{dt^k}. \tag{2.8}$$

3.   Lie symmetry analysis to FPDEs

In this section, we consider the time-fractional differential equations as the form

$$D_t^\alpha(u) = G(x,t,u,u_x,u_{xx},...), \ (0 < \alpha < 1), \tag{3.1}$$

where $u = u(x,t), u_x = \partial u/\partial x$, $D_t^\alpha u$ is a fractional derivative of u with respect to t. According to the Lie theory, if Eq.(3.1) is a invariant under a one parameter Lie group of point transformations

$$t^* = t + \varepsilon\tau(x,t,u) + \mathrm{O}(\varepsilon^2), x^* = x + \varepsilon\varsigma(x,t,u) + \mathrm{O}(\varepsilon^2), u^* = u + \varepsilon\eta(x,t,u) + \mathrm{O}(\varepsilon^2),$$

$$\frac{\partial u^*}{\partial t^*} = \frac{\partial^\alpha u}{\partial t^\alpha} + \varepsilon\eta_\alpha^0(x,t,u) + \mathrm{O}(\varepsilon^2), \frac{\partial u^*}{\partial x^*} = \frac{\partial u}{\partial x} + \varepsilon\eta^x(x,t,u) + \mathrm{O}(\varepsilon^2),$$

$$\frac{\partial^2 u^*}{\partial x^{*2}} = \frac{\partial^2 u}{\partial x^2} + \varepsilon\eta^{xx}(x,t,u) + \mathrm{O}(\varepsilon^2), \tag{3.2}$$

where $\varepsilon \ll 1$ is a small parameter, and

$$\eta^x = D_x(\eta) - u_x D_x(\varsigma) - u_t D_x(\tau),$$
$$\eta^{xx} = D_x(\eta^x) - u_{xt}D_x(\tau) - u_{xx}D_x(\varsigma),$$
$$\eta^{xxx} = D_x(\eta^{xx}) - u_{xxt}D_x(\tau) - u_{xxx}D_x(\varsigma),$$
$$\eta^{xxxx} = D_x(\eta^{xxx}) - u_{xxxt}D_x(\tau) - u_{xxxx}D_x(\varsigma). \tag{3.3}$$

Here D_x denotes total derivative

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + ..., \tag{3.4}$$

the vector field associated with the above group of transformations can be written as

$$\mathrm{V} = \varsigma(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u}. \tag{3.5}$$

If the vector field Eq.(3.5) generates a symmetry of Eq.(3.1),then must satisfy the Lie's symmetry condition

$$\mathrm{Pr}^{(n)}\mathrm{V}\Delta\Big|_{\Delta=0} = 0, \tag{3.6}$$

where $\Delta = D_t^\alpha - G\left(x,t,u,u_x,u_{xx},...\right)$. Conversely, the corresponding group transformations Eq.(3.2) to known operator Eq.(3.6) are found by solving the Lie equations

$$\begin{aligned}
\frac{d\left(\bar{x}\left(\varepsilon\right)\right)}{d\varepsilon} &= \varsigma\left(\bar{x}\left(\varepsilon\right),\bar{t}\left(\varepsilon\right),\bar{u}\left(\varepsilon\right)\right),\ \bar{x}\left(0\right)=x, \\
\frac{d\left(\bar{u}\left(\varepsilon\right)\right)}{d\varepsilon} &= \eta\left(\bar{x}\left(\varepsilon\right),\bar{t}\left(\varepsilon\right),\bar{u}\left(\varepsilon\right)\right),\ \bar{u}\left(0\right)=u.
\end{aligned} \tag{3.7}$$

It is not different to see that Eq.(3.2) conserve the structure of fractional derivative infinitesimal operator Eq.(2.2). For the lower limit of the integral is fixed, it must be in variant with respect to Eq.(3.2). Thus, we can arrive at

$$\tau\left(x,t,u\right)|_{t=0} = 0. \tag{3.8}$$

For R-L fractional time derivative $[3,6,27,29,39]$, Eq.(3.8) can be changed into

$$\eta_\alpha^0 = D_t^\alpha\left(\eta\right) + \varsigma D_t^\alpha\left(u_x\right) - D_t^\alpha\left(\varsigma u_x\right) + D_t^\alpha\left(D_t\left(\tau\right)u\right) - D_t^{\alpha+1}\left(\tau u\right) + \tau D_t^{\alpha+1}\left(u\right), \tag{3.9}$$

with the help of the generalized Leibnitz rule Eq.(2.6), Eq.(3.9) can read as

$$\eta_\alpha^0 = D_t^\alpha\left(\eta\right) - \alpha D_t\left(\tau\right)\frac{\partial^\alpha u}{\partial t^\alpha} - \sum_{n=1}^{k}\binom{\alpha}{n}D_t^n(\varsigma)D_t^{\alpha-n}(u_x) - \sum_{n=1}^{\infty}\binom{\alpha}{n+1}D_t^{n+1}(\tau)D_t^{\alpha-n}(u). \tag{3.10}$$

Furthermore, using the chain rule Eq.(2.8) and the generalized Leibnitz rule Eq.(3.10) with $f(t) = 1$, we can arrive at

$$\eta_t^\alpha = \frac{\partial^\alpha \eta}{\partial t^\alpha} + \eta_u \frac{\partial^\alpha u}{\partial t^\alpha} - u\frac{\partial^\alpha \eta_u}{\partial t^\alpha} + \sum_{n=1}^{\infty}\binom{\alpha}{n}\frac{\partial^n \eta_u}{\partial t^n}D_t^{\alpha-n}(u) + \mu, \tag{3.11}$$

where

$$\mu = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=o}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\frac{t^{n-\alpha}}{\Gamma\left(n+1-\alpha\right)}(-u)^r\frac{\partial^m}{\partial t^m}\left(u^{k-r}\right)\frac{\partial^{n-m+k}\eta}{\partial t^{n-m}\partial u^k}. \tag{3.12}$$

It should be noted that we have $\mu = 0$ when the infinitesimal η is linear of the variable u, because of the existence of the derivatives $\frac{\partial^k \eta}{\partial u^k}, k \geq 2$ in above expression.Summarizing the reasonings above, we obtain the explicit form of $\eta^{\alpha,t}$

$$\eta^{\alpha,t} = \frac{\partial^\alpha \eta}{\partial t^\alpha} + (\eta_u - \alpha D_t(\tau)) \frac{\partial^\alpha u}{\partial t^\alpha} - u \frac{\partial^\alpha \eta_u}{\partial t^\alpha} + \mu$$
$$+ \sum_{n=1}^{\infty} [\binom{\alpha}{n} \frac{\partial^\alpha \eta_u}{\partial t^\alpha} - \binom{\alpha}{n+1} D_t^{n+1}(\tau)] D_t^{\alpha-n}(u) - \sum_{n=1}^{\infty} \binom{\alpha}{n} D_t^n(\varsigma) D_t^{\alpha-n}(u_x). \tag{3.13}$$

According to the Lie theory, we have

Theorem 3.1. *The function $u = \phi(x,t)$ is an invariant solution of Eq.(3.1) if and only if*
(i)$V\phi = 0 \Leftrightarrow \left(\varsigma(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u}\right)\phi = 0$,
(ii)$u = \phi(x,t)$ is the solution of FDPE Eq.(3.1).

4.   The time fractional Fisher's equation

In the preceding section, we have given some definitions and formulas of Lie symmetry analysis method on the FPDEs. In this section, we will deal with the invariance properties of the TFF equation. Then we give some exact and explicit solutions to the TFF equation.

4.1. Lie symmetry of time fractional Fisher's equation

By the Lie group theory, we can derive the corresponding system of symmetry equations as

$$\eta_\alpha^0 - \eta^{xx} - 6\eta + 12u\eta = 0. \tag{4.1}$$

Solving Eq.(3.1) with the help of Eq.(3.3), we can arrive at

$$\varsigma_u = \tau_u = \varsigma_t = \tau_x = \varsigma_{xx} = \eta_{uu} = 0, \ \frac{\partial^\alpha \eta}{\partial t^\alpha} - u\frac{\partial^\alpha \eta_u}{\partial t^\alpha} - \eta_{xx} - 12u\eta = 0,$$
$$2\varsigma_x - \alpha\tau_t = 0, \ \eta_{xu} - 2\varsigma_{xx} = 0, \tag{4.2}$$
$$\binom{\alpha}{n} \partial_t^n(\eta_u) - \binom{\alpha}{n+1} D_t^{n+1}(\tau) = 0, \ \text{for } (n = 1,2).$$

Then we can get

$$\varsigma = c_1 x + c_2, \tau = \frac{2c_1}{\alpha}t, \eta = -c_1 u, \tag{4.3}$$

where c_1 and c_2 are arbitrary constants. Furthermore, the corresponding operator can be arrived at

$$V = (c_1 x + c_2)\frac{\partial}{\partial x} + \frac{2c_1 t}{\alpha}\frac{\partial}{\partial t} + c_1 u\frac{\partial}{\partial u}. \tag{4.4}$$

Similarly, the Lie algebra of infinitesimal symmetries of Eq.(1.1) is spanned by the two vector fields

$$V_1 = \frac{\partial}{\partial x}, \ V_2 = x\frac{\partial}{\partial x} + \frac{2t}{\alpha}\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}. \tag{4.5}$$

It is easy to check that the vector fields are closed under the Lie bracket, respectively,

$$[V_1, V_2] = V_1, \ \ [V_2, V_1] = -V_1. \tag{4.6}$$

In order to get the similarity variables for V_2 , we have to solve the corresponding characteristic equations

$$\frac{dx}{x} = \frac{\alpha dt}{2t} = \frac{du}{-u}. \tag{4.7}$$

Thus, we derive group-invariant solution and group-invariant as follow

$$\theta = xt^{-\frac{\alpha}{2}}, \ u = t^{-\frac{\alpha}{2}}g(\theta). \tag{4.8}$$

It is not difficult to see that Eq.(1.1) is reduced to a nonlinear ordinary differential equation (NODE). We have a theorem as follow.

Theorem 4.1. *The TFF equation Eq.(1.1) can be reduced into a NODE of fractional order by the transformation Eq.(4.8) as follow*

$$\left(P_{\frac{2}{\alpha}}^{1-\frac{3\alpha}{2},\alpha}g\right)(\theta) = g_{\theta\theta} + 6g - 6g^2, \tag{4.9}$$

with the Erdelyi-Kober (EK) fractional differential operator $P_\beta^{\tau,\alpha}$ of order [36]

$$\left(P_\beta^{\tau^2,\alpha}g\right) := \prod_{j=0}^{n-1}\left(\tau^2 + j - \frac{1}{\beta}\theta\frac{d}{d\theta}\right)\left(K_\beta^{\tau^2+\alpha,n-\alpha}g\right)(\theta), \tag{4.10}$$

$$n = \begin{cases} [\alpha] + 1, & \alpha \neq \mathbb{N}, \\ \alpha & \alpha \in \mathbb{N}, \end{cases} \tag{4.11}$$

where

$$\left(K_\beta^{\tau^2,\alpha}g\right) := \begin{cases} \frac{1}{\Gamma(\alpha)}\int_1^\infty (u-1)^{\alpha-1}u^{-(\tau^2+\alpha)}g\left(\theta u^{\frac{1}{\beta}}\right)du, & \alpha > 0, \\ g(\theta), & \alpha = 0. \end{cases} \tag{4.12}$$

is the EK fractional integral operator [11, 34].

The theorem 2 has been proved in [33], here omit.

4.2. Exact and explicit solutions of the time-fractional Fisher's equation

We investigate the exact analytic solutions via power series method [7] and symbolic computations [31] for Eq.(1.1). Furthermore, we analyze the convergence of the power series solutions. Set

$$g(\theta) = \sum_{n=0}^\infty a_n\theta^n, \tag{4.13}$$

from Eq.(4.12), we can have

$$g' = \sum_{n=0}^{\infty} n a_n \theta^{n-1}, \quad g'' = \sum_{n=0}^{\infty} n(n-1) a_n \theta^{n-2}. \tag{4.14}$$

Substituting Eqs.(4.13) and (4.14) into Eq.(4.9), we obtain

$$\eta^{\alpha,t} = \frac{\partial^\alpha \eta}{\partial t^\alpha} + (\eta_u - \alpha D_t(\tau)) \frac{\partial^\alpha u}{\partial t^\alpha} - u \frac{\partial^\alpha \eta_u}{\partial t^\alpha} + \mu$$

$$+ \sum_{n=1}^{\infty} \left[\binom{\alpha}{n} \frac{\partial^\alpha \eta_u}{\partial t^\alpha} - \binom{\alpha}{n+1} D_t^{n+1}(\tau) \right] D_t^{\alpha-n}(u) - \sum_{n=1}^{\infty} \binom{\alpha}{n} D_t^n(\varsigma) D_t^{\alpha-n}(u_x). \tag{4.15}$$

$$\sum_{n=0}^{\infty} \frac{\Gamma\left(2 + (-\frac{\alpha}{2}) + \frac{n\alpha}{2}\right)}{\Gamma\left(2 - \alpha + (-\frac{\alpha}{2}) + \frac{n\alpha}{2}\right)} a_n \theta^n - \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} \theta^n$$

$$- 6 \sum_{n=0}^{\infty} a_n \theta^n + 6 \sum_{n=0}^{\infty} a_n \theta^n \sum_{n=0}^{\infty} a_n \theta^n = 0. \tag{4.16}$$

Comparing coefficients in Eq.(4.15) when $n = 0$, we get

$$a_2 = \frac{1}{2} \left(\frac{\Gamma\left(2 + (-\frac{\alpha}{2})\right)}{\Gamma\left(2 - \alpha + (-\frac{\alpha}{2})\right)} a_0 - 6 a_0 + 6 a_0^2 \right), \tag{4.17}$$

when $n \geq 1$,we have

$$a_{n+2} = \frac{1}{(n+2)(n+1)} \left\{ \frac{\Gamma\left(2 + (-\frac{\alpha}{2}) + \frac{n\alpha}{2}\right)}{\Gamma\left(2 - \alpha + (-\frac{\alpha}{2}) + \frac{n\alpha}{2}\right)} a_n + 6 \sum_{k=0}^{\infty} a_k a_{n-k} - 6 a_n \right\}. \tag{4.18}$$

Thus, each coefficient $a_n (n \geq 1)$ for Eq.(4.13) are found by the arbitrary constants $a_i (i = 0, 1, 2)$. This means that the exact power series solution for Eq.(4.9) exists and its coefficients depend on the Eqs.(4.16) and (4.17).Therefore, it is obvious that the power series for Eq.(4.9) is an exact power series solution. Hence, the power series solution for Eq.(4.9) can be represented in the form:

$$g(\theta) = a_0 + a_1 \theta + a_2 \theta^2 + \sum_{n=1}^{\infty} a_{n+2} \theta^{n+2}$$

$$= a_0 + a_1 \theta + \frac{1}{2} \left(\frac{\Gamma\left(2 + (-\frac{\alpha}{2})\right)}{\Gamma\left(2 - \alpha + (-\frac{\alpha}{2})\right)} a_0 - 6 a_0 + 6 a_0^2 \right) \theta^2$$

$$+ \sum_{n=1}^{\infty} \frac{1}{(n+2)(n+1)} \left\{ \frac{\Gamma\left(2 + (-\frac{\alpha}{2}) + \frac{n\alpha}{2}\right)}{\Gamma\left(2 - \alpha + (-\frac{\alpha}{2}) + \frac{n\alpha}{2}\right)} a_n + 6 \sum_{k=0}^{\infty} a_k a_{n-k} - 6 a_n \right\} \theta^{n+2}.$$

Consequently, we acquire the explicit power series solution for Eq.(1.1) as

$$u(x,t) = a_0 t^{-\frac{\alpha}{2}} + a_1 x t^{-\alpha} + a_2 x^2 t^{-\frac{3\alpha}{2}} + \sum_{n=1}^{\infty} a_{n+2} x^{n+2} t^{-\frac{\alpha(n+3)}{2}}$$

$$= a_0 + a_1 x t^{-\alpha} + \frac{1}{2} \left(\frac{\Gamma\left(2 + (-\frac{\alpha}{2})\right)}{\Gamma\left(2 - \alpha + (-\frac{\alpha}{2})\right)} a_0 + 6 a_0 - 6 a_0^2 \right) x^2 t^{-\frac{3\alpha}{2}}$$

$$+ \sum_{n=1}^{\infty} \frac{1}{(n+2)(n+1)} \left\{ \frac{\Gamma\left(2 + (-\frac{\alpha}{2}) + \frac{n\alpha}{2}\right)}{\Gamma\left(2 - \alpha + (-\frac{\alpha}{2}) + \frac{n\alpha}{2}\right)} a_n + 6 \sum_{k=0}^{\infty} a_k a_{n-k} - 6a_n \right\}$$
$$\times x^{n+2} t^{-\frac{\alpha(n+3)}{2}}.$$

Remark 4.1. Above all, we could obtain power series solutions for some NFEDs. To the best of our knowledge, the solutions obtained in this paper have not been reported in previous literature. Thus, these solutions are new.

Remark 4.2. Besides the Riemann-Liouville definition of fractional derivatives, there are several other different definitions, such as the modified Riemann-Liouville(m-RL) derivative [6], the Grünwald-Letnikov(G-L) derivative and Caputo's fractional derivative [19, 26], and so on.

5.   Concluding remarks

In this research, we considered the symmetry analysis, explicit solutions to the time fractional Fisher's equations with Riemann-Liouville derivative. The time fractional Fisher's was reduced to a nonlinear ordinary differential equation(ODE) of fractional order. The reduced fractional ODE was solved using an explicit power series method. To summarize, Lie group analysis method is successfully to study the symmetry properties of Fisher's equation with time fractional order. However, the obtained point transformation groups of Eq.(1.1) are narrower than those for Fisher's equation for general meaning. It is shown that the technique introduced here is effective and easy to implement. This problem can be considered further.

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