Journal of Applied Analysis and Computation
Volume 10, Number 5, October 2020, 2024–2035
Website:http://jaac-online.com
DOI:10.11948/20190319

INFINITELY MANY LOW- AND HIGH-ENERGY SOLUTIONS FOR A CLASS OF ELLIPTIC EQUATIONS WITH VARIABLE EXPONENT*

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Abstract This paper is concerned with the $p(x)$-Laplacian equation of the form

$$\begin{cases} -\Delta_{p(x)} u = Q(x)|u|^{r(x)-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{0.1}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $1 < p^- = \min_{x\in\overline{\Omega}} p(x) \le p(x) \le \max_{x\in\overline{\Omega}} p(x) = p^+ < N$, $1 \le r(x) < p^*(x) = \frac{Np(x)}{N-p(x)}$, $r^- = \min_{x\in\overline{\Omega}} r(x) < p^-$, $r^+ = \max_{x\in\overline{\Omega}} r(x) > p^+$ and $Q : \overline{\Omega} \to \mathbb{R}$ is a nonnegative continuous function. We prove that (0.1) has infinitely many small solutions and infinitely many large solutions by using the Clark's theorem and the symmetric mountain pass lemma.

Keywords $p(x)$-Laplacian, variable exponent, infinitely many solutions, Clark's theorem, symmetric mountain pass lemma.

MSC(2010) 35J20, 35J60, 35B33, 46E30.

1. Introduction and main results

In recent years, the following nonlinear elliptic equation

$$\begin{cases} -\Delta_{p(x)} u = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

has received considerable attention due to the fact that it can be applied to fluid mechanics and the field of image processing (see [7, 23]), where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $p : \overline{\Omega} \to \mathbb{R}$ is a continuous function satisfying $1 < p^- =$

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*C. Chu is supported by National Natural Science Foundation of China (No. 11861021). H. Liu is supported National Natural Science Foundation of China (Nos. 11701220, 11926334, 11926335).

$\min_{x\in\overline{\Omega}} p(x) \le p(x) \le \max_{x\in\overline{\Omega}} p(x) = p^+ < N$ and $f : \overline{\Omega}\times\mathbb{R} \to \mathbb{R}$ is a suitable function.

In 2003, Fan and Zhang in [10] gave several sufficient conditions for the existence and multiplicity of nontrivial solutions for problem (1.1). These conditions include either the sublinear growth condition

$$|f(x,t)| \le C\left(1+|t|^{\beta}\right), \quad \text{for } x\in\Omega \text{ and } t\in\mathbb{R}$$

or Ambrosetti-Rabinowitz type superlinear condition ((AR)-condition, for short)

$$f(x,t)t \ge \theta F(x,t) > 0, \quad \text{for } x\in\Omega \text{ and } |t| \text{ sufficiently large},$$

where $C > 0$, $1 \le \beta < p^-$, $\theta > p^+$ and $F(x,t) = \int_0^t f(x,s)\,ds$. Subsequently, Chabrowski and Fu in [6] discussed problem (1.1) in a more general setting than that in [10]. It is well known that (AR)-condition is important to guarantee the boundedness of Palais-Smale sequence of the Euler-Lagrange functional which plays a crucial pole in applying the critical point theory. However, it excludes many cases of nonlinearity (see [4, 13, 14, 22, 25, 27–29]). In fact, either the uniform superlinear growth condition or the uniform sublinear growth condition was still imposed on $f(x,t)$. In addition, some papers discussed problem (1.1) with concave-convex nonlinearities (see [3, 12, 18, 20, 26]).

For the case $f(x,t) = Q(x)|t|^{r(x)-2}t$, problem (1.1) reduces to

$$\begin{cases} -\Delta_{p(x)}u = Q(x)|u|^{r(x)-2}u, & \text{in } \Omega,\\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $Q, r : \overline{\Omega} \to \mathbb{R}$ are nonnegative continuous functions. The sets $\Omega_0 = \{x\in\Omega \,|\, r(x) = p(x)\}$, $\Omega_- = \{x\in\Omega \,|\, r(x) < p(x)\}$ and $\Omega_+ = \{x\in\Omega \,|\, r(x) > p(x)\}$ can have positive measure at the same time. This situation is new and closely related to the existence of variable exponents since we can't meet such a phenomenon in the constant exponent case(see [1–4]). Mihăilescu and Rădulescu in [19] have considered problem (1.2) with $Q(x) \equiv \lambda$ under the basic assumption $1 < r^- = \min_{x\in\overline{\Omega}} r(x) < p^- < r^+ = \max_{x\in\overline{\Omega}} r(x)$ and proved that there exists $\lambda_0 > 0$ such that any $\lambda \in (0,\lambda_0)$ is an eigenvalue for problem (1.2). Subsequently, Fan in [8] extended the main results of [19] in the case $\Omega = \Omega_-$ (but $r^+ < p^-$ does not hold) and in the case $\Omega = \Omega_+$ (but $r^- > p^+$ does not hold), respectively. Their results implied that for any positive constant $C > 0$ there exists $u_0 \in W_0^{1,p(x)}(\Omega)$ such that

$$C\int_\Omega |u_0|^{r(x)}dx \ge \int_\Omega |\nabla u_0|^{p(x)}dx.$$

Therefore, we have to overcome new difficulties in dealing with (1.2).

Different from the concave-convex nonlinearities, the main feature of problem (1.2) is that $Q(x)|t|^{r(x)}$ has both local superlinear growth and local sublinear growth. Due to this, it is difficult to prove the boundedness of Palais-Smale sequence of the Euler-Lagrange functional. To the best of our knowledge, we only realize that Aouaoui [1] obtainded at least three nontrivial solutions of problem (1.2) with $\Omega = \mathbb{R}^N$ by perturbation method. Motivated by [5] and [16], we are concerned with the existence of infinitely many small solutions and infinitely many large solutions for problem (1.2) under the assumption $1 \le r(x) < p^*(x) = \frac{Np(x)}{N-p(x)}$ and $r^- < p^- \le p^+ < r^+$. The main results of this paper read as follows.

Theorem 1.1. *Suppose that* $1 \le r(x) < p^*(x)$, $r^- < p^- \le p^+ < N$, $Q : \overline{\Omega} \to \mathbb{R}$ *is a nonnegative continuous function and there exists a point* $x_1 \in \Omega^- = \{x \in \Omega \,|\, r(x) < p^-\}$ *such that* $Q(x_1) > 0$. *Then, problem* (1.2) *has infinitely many solutions* $\{u_k\}$ *with the property* $\|u_k\|_{L^\infty(\Omega)} \to 0$ *as* $k \to \infty$.

Theorem 1.2. *Suppose that* $1 < p^- \le p^+ < N$, $1 \le r(x) < p^*(x)$, $r^+ > p^+$, $Q : \overline{\Omega} \to \mathbb{R}$ *is a nonnegative continuous function and there exists a point* $x_2 \in \Omega^+ = \{x \in \Omega \,|\, r(x) > p^+\}$ *such that* $Q(x_2) > 0$. *Either* $r^- > p^+$, *or* $1 < r^- \le p^+$ *and there exists* $\varepsilon > 0$ *such that* $Q(x) \equiv 0$ *in* $\Omega_\varepsilon = \{x \in \Omega \,|\, p^- - \varepsilon < r(x) < p^+ + \varepsilon\}$. *Then, problem* (1.2) *has infinitely many solutions* $\{v_k\}$ *such that* $\|v_k\| \to \infty$ *as* $k \to \infty$.

As a corollary of Theorems 1.1 and 1.2, we have

Corollary 1.3. *Suppose that* $1 < p^- \le p^+ < N$, $1 \le r(x) < p^*(x)$, $r^- < p^-$, $r^+ > p^+$, $Q(x)$ *is a nonnegative continuous function and there exist* $\varepsilon > 0$, $x_1 \in \Omega^- = \{x \in \Omega \,|\, r(x) < p^-\}$, $x_2 \in \Omega^+ = \{x \in \Omega \,|\, r(x) > p^+\}$ *such that* $Q(x_1)$, $Q(x_2) > 0$ *and* $Q(x) \equiv 0$ *in* $\Omega_\varepsilon = \{x \in \Omega \,|\, p^- - \varepsilon < r(x) < p^+ + \varepsilon\}$. *Then, problem* (1.2) *has infinitely many small solutions* $\{u_k\}$ *and infinitely many large solutions* $\{v_k\}$.

In this paper, the letters C and C_j stand for positive constants. $\|u\|_s$ denotes the standard norms of $L^s(\Omega)$ $(s \ge 1)$. The paper is organized as follows. In Section 2, we give some basic properties of the variable exponent Lebesgue space and Sobolev space. In Sections 3 and 4, we prove Theorems 1.1 and 1.2 by the Clark's theorem and the symmetric mountain pass lemma, respectively.

2. Preliminaries

The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is defined by

$$L^{p(x)}(\Omega) = \left\{ u \mid u : \Omega \to \mathbb{R} \text{ is measurable, } \int_\Omega |u|^{p(x)}\, dx < \infty \right\}$$

with the norm

$$|u|_{p(x)} = \inf \left\{ \lambda > 0 \;\Big|\; \int_\Omega \left| \frac{u}{\lambda} \right|^{p(x)} dx \le 1 \right\}.$$

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) \mid |\nabla u| \in L^{p(x)}(\Omega) \right\}$$

with the norm

$$\|u\|_{1,p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}.$$

Define $W_0^{1,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. The spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces if $1 < p^- \le p^+ < \infty$ (see [10]). Moreover, there is a constant $C > 0$ such that

$$|u|_{p(x)} \le C |\nabla u|_{p(x)}, \quad \text{for any } u \in W_0^{1,p(x)}(\Omega).$$

Therefore, $\|u\| = |\nabla u|_{p(x)}$ and $\|u\|_{1,p(x)}$ are equivalent norms on $W_0^{1,p(x)}(\Omega)$. We will use $\|u\|$ to replace $\|u\|_{1,p(x)}$ in the following discussions.

Lemma 2.1 ([10]). *If $q \in C(\overline{\Omega})$ satisfies $1 \leq q(x) < p^*(x)$ for $x \in \overline{\Omega}$, then the imbedding from $W^{1,p(x)}(\Omega)$ to $L^{q(x)}(\Omega)$ is compact and continuous.*

Lemma 2.2 ([10,11]). *Set*

$$\rho(u) = \int_\Omega |u|^{p(x)}\,dx, \quad \text{for } u \in L^{p(x)}(\Omega).$$

If $u \in L^{p(x)}(\Omega)$ and $\{u_k\}_{k\in\mathbb{N}} \subset L^{p(x)}(\Omega)$, then we have
(i) $|u|_{p(x)} < 1\ (= 1;\ > 1) \Leftrightarrow \rho(u) < 1\ (= 1;\ > 1)$;
(ii) $|u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^-} \leq \rho(u) \leq |u|_{p(x)}^{p^+}$;
(iii) $|u|_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p^+} \leq \rho(u) \leq |u|_{p(x)}^{p^-}$;
(iv) $\lim_{k\to\infty} |u_k - u|_{p(x)} = 0 \Leftrightarrow \lim_{k\to\infty} \rho(u_k - u) = 0 \Leftrightarrow u_k \to u$ in measure in Ω and $\lim_{k\to\infty} \rho(u_k) = \rho(u)$.

Similar to Lemma 2.2, we have

Lemma 2.3. *Set*

$$L(u) = \int_\Omega |\nabla u|^{p(x)}\,dx, \quad \text{for } u \in W_0^{1,p(x)}(\Omega).$$

If $u \in W_0^{1,p(x)}(\Omega)$ and $\{u_k\}_{k\in\mathbb{N}} \subset W_0^{1,p(x)}(\Omega)$, we have
(i) $\|u\| < 1\ (= 1;\ > 1) \Leftrightarrow L(u) < 1\ (= 1;\ > 1)$;
(ii) $\|u\| > 1 \Rightarrow \|u\|^{p^-} \leq L(u) \leq \|u\|^{p^+}$;
(iii) $\|u\| < 1 \Rightarrow \|u\|^{p^+} \leq L(u) \leq \|u\|^{p^-}$;
(iv) $\|u_k\| \to 0 \Leftrightarrow L(u_k) \to 0$; $\|u_k\| \to \infty \Leftrightarrow L(u_k) \to \infty$.

Definition 2.4. $u \in W_0^{1,p(x)}(\Omega)$ is called a weak solution of problem (1.2) if

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla\phi\,dx = \int_\Omega Q(x)|u|^{r(x)-2}u\phi\,dx$$

for all $\phi \in W_0^{1,p(x)}(\Omega)$.

3. Infinitely many small solutions

In this section, we use a truncation technique and the Clark's theorem to get a sequence of solutions converging to zero. We first introduce a variant of the Clark's theorem.

Theorem 3.1 ([17], Theorem 1.1). *Let X be a Banach space, $\Phi \in C^1(X, \mathbb{R})$. Assume Φ satisfies the Palais-Smale condition ((PS) condition for short), is even and bounded from below, and $\Phi(0) = 0$. If for any $k \in \mathbb{N}$, there exists a k-dimensional subspace X^k of X and $\rho_k > 0$ such that $\sup_{X^k \cap S_{\rho_k}} \Phi < 0$, where $S_\rho = \{u \in X \mid \|u\| = \rho\}$, then at least one of the following conclusions holds.*
(i) There exists a sequence of critical points $\{u_k\}$ satisfying $\Phi(u_k) < 0$ for all k and $\|u_k\| \to 0$ as $k \to \infty$.
(ii) There exists $R > 0$ such that for any $0 < b < R$ there exists a critical point u such that $\|u\| = b$ and $\Phi(u) = 0$.

Recall that there is no restriction on r^+ in Theorem 1.1. In order to obtain infinitely many small solutions, we need to have a proper truncation of the nonlinear terms. Let $\phi \in C(\mathbb{R},\mathbb{R})$ be an even function satisfying $0 \le \phi(t) \le 1$, $\phi(t) = 1$ for $|t| \le \frac{1}{2}$ and $\phi(t) = 0$ for $|t| \ge 1$. Define $g : \overline{\Omega}\times\mathbb{R} \to \mathbb{R}$ by $g(x,t) := Q(x)\phi(t)|t|^{r(x)-2}t$ and consider the auxiliary problem

$$\begin{cases} -\Delta_{p(x)}u = g(x,u), \text{ in } \Omega, \\ u = 0, \qquad\qquad\quad \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

The energy functional $J : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ associated with (3.1) is defined by

$$J(u) = \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}\,dx - \int_\Omega G(x,u)\,dx,$$

where $G(x,t) = \int_0^t g(x,s)\,ds$. We will show that J satisfies the conditions of Theorem 3.1 and obtain infinitely many solutions $\{u_k\}$ of (3.1) such that $\|u_k\|_{L^\infty(\Omega)} \le \frac{1}{2}$ for large k. Then, for large k, there holds $g(x,u_k) = Q(x)|u_k|^{r(x)-2}u_k$, and so u_k becomes a solution of (1.2).

Proof of Theorem 1.1. From the properties of η, we see that there exists a constant $M > 0$ such that $|g(x,t)| \le M$ and $|G(x,t)| \le M$ for all $(x,t) \in \overline{\Omega}\times\mathbb{R}$. Set $X := W_0^{1,p(x)}(\Omega)$. Then it is easy to see that $J(0) = 0$, $J \in C^1(X,\mathbb{R})$ is even and bounded from below, and satisfies the (PS) condition.

Since $r(x_1) < p^-$ and $Q(x_1) > 0$, we see from the continuity of Q and r that there exist $\delta_1 > 0$, $Q_1 > 0$ and $r_1 < p^-$ such that

$$r^- \le r(x) < r_1 \ \text{ and } \ Q(x) > Q_1, \ \text{ for } x \in \Omega_1 \triangleq B(x_1,\delta_1)\cap\Omega. \tag{3.2}$$

By the definition of g and (3.2), we have

$$G(x,u) = \frac{Q(x)}{r(x)}|u|^{r(x)} \ge \frac{Q_1}{r_1}|u|^{r_1}, \ \text{ for } x \in \Omega_1 \text{ and } |u| \le \frac{1}{2}. \tag{3.3}$$

For $k \in \mathbb{N}$, choose $\{\varphi_j\}_{j=1}^k \subset C_0^\infty(\Omega)$ such that

$$\varphi_j \ne 0, \ \operatorname{supp}\varphi_j \subset \Omega_1, \ \operatorname{supp}\varphi_i \cap \operatorname{supp}\varphi_j = \emptyset \text{ for } i \ne j.$$

Let $X^k := \operatorname{span}\{\varphi_1,\varphi_2,\cdots,\varphi_k\}$. Then X^k is a k-dimensional subspace of X. Since any norms in a finite dimensional space are equivalent, there exist $a_k, b_k > 0$ such that

$$\|u\|_{r_1} \ge a_k\|u\|, \ \ \|u\| \ge b_k\|u\|_{L^\infty(\Omega)}, \ \text{ for any } u \in X^k. \tag{3.4}$$

Set

$$\rho_k = \min\left\{\frac{1}{2}, \frac{b_k}{2}, \left(\frac{p^-Q_1a_k^{r_1}}{2r_1}\right)^{\frac{1}{p^--r_1}}\right\}.$$

It follows from (3.3), (3.4) and Lemma 2.3 that, for any $u \in X^k \cap S_{\rho_k}$,

$$J(u) \le \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}\,dx - \frac{Q_1}{r_1}\int_{\Omega_1}|u|^{r_1}dx \le \frac{1}{p^-}\|u\|^{p^-} - \frac{Q_1a_k^{r_1}}{r_1}\|u\|^{r_1} < 0.$$

According to Theorem 3.1, J has a sequence of nontrivial critical points $\{u_k\}$ satisfying $J(u_k) \le 0$ for all k and $\|u_k\| \to 0$ as $k \to \infty$. Since $g(x,t)$ is bounded in $\overline{\Omega} \times \mathbb{R}$, the weak solutions $\{u_k\}$ belong to $C^{1,\mu}(\overline{\Omega})$ for some $\mu \in (0,1)$ and they are bounded in this space (see [9]). Here μ is independent of k and $C^{1,\mu}(\overline{\Omega})$ denotes the set of all $C^1(\overline{\Omega})$ functions whose derivatives are Hölder continuous with exponent μ. Since $C^{1,\mu}(\overline{\Omega})$ is compactly embedded in $C^1(\overline{\Omega})$, there is a subsequence of $\{u_k\}$, still denoted by itself, such that $u_k \to u_\infty$ in $C^1(\overline{\Omega})$. Since $u_k \to 0$ in X, u_∞ must be zero. By the uniqueness of the limit u_∞, we can show that $\{u_k\}$ itself (without extracting a subsequence) converges to zero in $C^1(\overline{\Omega})$. Then $\|u_k\|_{L^\infty(\Omega)} \le \frac{1}{2}$ for large k and so u_k is a solution of (1.2). The proof is complete. □

4. Infinitely many large solutions

In this section, we will apply the symmetric mountain pass lemma (see [21, Theorem 9.12]) to get a sequence of large solutions. As in Section 3, we denote $X = W_0^{1,p(x)}(\Omega)$. The energy functional $I : X \to \mathbb{R}$ associated with (1.2) is defined by

$$I(u) = \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}\, dx - \int_\Omega \frac{Q(x)}{r(x)} |u|^{r(x)}\, dx.$$

First of all, we prove that the functional I satisfies (PS) condition.

Lemma 4.1. *Under the assumption of theorem 1.2, the functional I satisfies (PS) condition.*

Proof. Let $\{u_n\} \subset X$ be a (PS) sequence of the functional I. Then there exists a constant $C > 0$ such that

$$I(u_n) \le C, \quad I'(u_n) \to 0 \text{ in } X^*, \tag{4.1}$$

where X^* denotes the dual space of X.

We first prove that $\{u_n\}$ is bounded in X. If $r^- > p^+$, then it follows from $Q(x) \ge 0$ and Lemma 2.3 that

$$\begin{aligned} r^- I(u_n) - \langle I'(u_n), u_n\rangle &= \int_\Omega \frac{r^- - p(x)}{p(x)} |\nabla u|^{p(x)}\, dx + \int_\Omega \frac{r(x) - r^-}{r(x)} Q(x)|u_n|^{r(x)} dx \\ &\ge \frac{r^- - p^+}{p^+} \min\{\|u_n\|^{p^-}, \|u_n\|^{p^+}\}. \end{aligned}$$

From (4.1) and $1 < p^- \le p^+$, we see that $\{u_n\}$ is bounded in X. If $1 < r^- \le p^+$, we set $\Omega_{\varepsilon^-} := \{x \in \Omega \,|\, r(x) \le p^- - \varepsilon\}$ and $\Omega_{\varepsilon^+} := \{x \in \Omega \,|\, r(x) \ge p^+ + \varepsilon\}$. From $Q(x) \ge 0$ and Lemma 2.1, we have

$$\begin{aligned} \int_{\Omega_{\varepsilon^-}} \frac{p^+ + \varepsilon - r(x)}{r(x)} Q(x)|u_n|^{r(x)} dx &\le \frac{p^+ + \varepsilon}{r^-} \sup_{x\in\Omega} Q(x) \int_\Omega (|u_n|^{p^- - \varepsilon} + 1) dx \\ &\le C_\varepsilon \|u_n\|^{p^- - \varepsilon} + C_\varepsilon \end{aligned} \tag{4.2}$$

and

$$\int_{\Omega_{\varepsilon^+}} \frac{p^+ + \varepsilon - r(x)}{r(x)} Q(x)|u_n|^{r(x)} dx \le 0, \tag{4.3}$$

where $C_\varepsilon > 0$. Recall that $Q(x) \equiv 0$ in Ω_ε. By (4.2), (4.3) and Lemma 2.3, we have

$$\begin{aligned}
&(p^+ + \varepsilon)I(u_n) - \langle I'(u_n), u_n\rangle \\
=&\int_\Omega \frac{p^+ + \varepsilon - p(x)}{p(x)}|\nabla u|^{p(x)}\,dx - \int_{\Omega_{\varepsilon-}} \frac{p^+ + \varepsilon - r(x)}{r(x)}Q(x)|u_n|^{r(x)}dx \\
&- \int_{\Omega_\varepsilon} \frac{p^+ + \varepsilon - r(x)}{r(x)}Q(x)|u_n|^{r(x)}dx - \int_{\Omega_{\varepsilon+}} \frac{p^+ + \varepsilon - r(x)}{r(x)}Q(x)|u_n|^{r(x)}dx \\
\geq&\frac{\varepsilon}{p^+}\min\{\|u_n\|^{p^-}, \|u_n\|^{p^+}\} - C_\varepsilon\|u_n\|^{p^- - \varepsilon} - C_\varepsilon.
\end{aligned}$$

Then, by (4.1), $\{u_n\}$ is bounded in X.

Up to a subsequence, we may assume that $u_n \rightharpoonup u$ and then $\langle I'(u_n), u_n - u\rangle \to 0$ as $n \to \infty$. Since the imbedding from X to $L^{r(x)}(\Omega)$ is compact, we obtain $u_n \to u$ in $L^{r(x)}(\Omega)$. Then

$$\left|\int_\Omega Q(x)|u_n|^{r(x)-2}u_n(u_n - u)dx\right| \leq \sup_{x\in\Omega} Q(x)\int_\Omega |u_n|^{r(x)-1}|u_n - u|dx \to 0.$$

Therefore, one has

$$\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \cdot \nabla(u_n - u)dx \to 0.$$

By [10, Theorem 3.1], we have $u_n \to u$ in X. Therefore, I satisfies (PS) condition. □

Since $r(x_2) > p^+$ and $Q(x_2) > 0$, we see from the continuity of Q and r that there exist $\delta_2 > 0$, $Q_2 > 0$ and $r_2 > p^+$ such that

$$r_2 < r(x) \leq r^+ \quad \text{and} \quad Q(x) \geq Q_2, \quad \text{for all } x \in \Omega_2 \triangleq B(x_2, \delta_2) \cap \Omega. \tag{4.4}$$

For $k \in \mathbb{N}$, choose $\{\psi_j\}_{j=1}^k \subset C_0^\infty(\Omega)$ such that

$$\psi_j \neq 0,\ \operatorname{supp}\psi_j \subset \Omega_2,\ \operatorname{supp}\psi_i \cap \operatorname{supp}\psi_j = \emptyset \text{ for } i \neq j.$$

Denote $Y^k := \operatorname{span}\{\psi_1, \psi_2, \cdots, \psi_k\}$.

Lemma 4.2. *Under the assumption of theorem 1.2, there exists $R_k > 0$ such that*

$$I(u) < 0, \quad \textit{for any } u \in Y^k \textit{ with } \|u\| \geq R_k. \tag{4.5}$$

Proof. By (4.4), we have

$$\frac{Q(x)}{r(x)}|u|^{r(x)} \geq \frac{Q_2}{r^+}|u|^{r_2}, \quad \text{for } x \in \Omega_2 \text{ and } |u| > 1,$$

which implies that

$$\frac{Q(x)}{r(x)}|u|^{r(x)} \geq \frac{Q_2}{r^+}(|u|^{r_2} - 1), \quad \text{for } x \in \Omega_2 \text{ and } u \in \mathbb{R}. \tag{4.6}$$

Since $\dim Y^k < \infty$, there exists $\tilde{a}_k > 0$ such that

$$\|u\|_{r_2} \geq \tilde{a}_k\|u\|, \quad \text{for any } u \in Y^k. \tag{4.7}$$

Using (4.6) and (4.7) we have, for any $u \in Y^k$,

$$\begin{aligned} I(u) &= \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}\,dx - \int_{\Omega_2} \frac{Q(x)}{r(x)}|u|^{r(x)}\,dx \\ &\le \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}\,dx - \frac{Q_2}{r^+}\int_{\Omega_2} (|u|^{r_2}-1)\,dx \\ &\le \frac{1}{p^-}\max\{\|u\|^{p^-}, \|u\|^{p^+}\} - \frac{Q_2 \tilde{a}_k^{r_2}}{r^+}\|u\|^{r_2} + \frac{Q_2}{r^+}|\Omega_2|. \end{aligned}$$

Since $r_2 > p^+$, (4.5) holds for large R_k. □

Define the minimax value

$$c_k = \inf_{h\in G_k} \max_{u\in D_k} I(h(u)),$$

where $D_k = \overline{B}_{R_k} \cap Y^k$ and $G_k = \{h \in C(D_k, X)\,|\, h \text{ is odd and } h = id \text{ on } \partial B_{R_k} \cap Y^k\}$.

Remark 4.3. Using the arguments in the proof of [15, Lemma 4.9], we see that the minimax value c_k is independent of the choice of R_k satisfying (4.5). Therefore, we can replace R_k by a larger number such that $\{R_k\}$ is strictly increasing and $\lim_{k\to\infty} R_k = +\infty$.

Lemma 4.4. *$c_k \to +\infty$ as $k \to +\infty$.*

We postpone the proof of Lemma 4.4 for a moment and prove Theorem 1.2 in the following.

Proof of Theorem 1.2. It follows from Lemma 4.4 that there exists $k_0 \in \mathbb{N}$ and $\alpha > 0$ such that

$$c_k \ge \alpha > 0, \quad \text{for any } k \ge k_0.$$

We claim that, for $k \ge k_0$, the minimax value c_k is a critical value of I. If this is false, then, by Lemma 4.1, there would exist $\varepsilon \in (0, \alpha)$ and $\eta \in C([0,1]\times X, X)$ such that

- $\eta(0,u) = u$ for all $u \in X$;
- $\eta(1, I^{c_k+\varepsilon}) \subset I^{c_k-\varepsilon}$, where $I^d = \{u \in X \,|\, I(u) \le d\}$;
- If $I(u) \notin [c_k-\varepsilon, c_k+\varepsilon]$, then $\eta(t,u) = u$ for all $t \in [0,1]$;
- $\eta(t,u)$ is odd in u.

Choose $h \in G_k$ such that $\max_{u\in D_k} I(h(u)) < c_k + \varepsilon$. Then $\eta(1, h(\cdot)) \in G_k$ and

$$I(\eta(1,h(u))) \le c_k - \varepsilon, \quad \text{for all } u \in D_k.$$

This contradicts the definition of c_k.

For $k \ge k_0$, let v_k be a critical point corresponding to c_k. Then we have

$$\int_\Omega |\nabla v_k|^{p(x)}\,dx = \int_\Omega Q(x)|v_k|^{r(x)}dx,$$

which combined with $I(v_k) = c_k$ leads to

$$c_k = \int_\Omega \frac{|\nabla v_k|^{p(x)}}{p(x)}\,dx - \int_\Omega \frac{Q(x)}{r(x)}|v_k|^{r(x)}dx$$

$$\leq \frac{1}{p^-}\int_\Omega |\nabla v_k|^{p(x)}\,dx - \frac{1}{r^+}\int_\Omega Q(x)|v_k|^{r(x)}dx$$
$$= \left(\frac{1}{p^-}-\frac{1}{r^+}\right)\int_\Omega |\nabla v_k|^{p(x)}\,dx$$
$$\leq \left(\frac{1}{p^-}-\frac{1}{r^+}\right)\max\{\|v_k\|^{p^-}, \|v_k\|^{p^+}\}.$$

By Lemma 4.4 and $r^+ > p^-$, we have $\|v_k\| \to \infty$ as $k \to \infty$. The proof is complete.

To reach the conclusion, it only remains to prove Lemma 4.4. For this purpose, we recall the definition and properties of genus which is due to Krasnoselski.

Definition 4.5. Let E be a Banach space. A subset A of E is said to be symmetric if $u \in A$ implies $-u \in A$. Let $\mathcal{A}$ denote the family of closed symmetric subsets A of $E \setminus \{0\}$. For $A \in \mathcal{A}$, we define the genus $\gamma(A)$ of A by the smallest integer m such that there exists an odd continuous map from A to $\mathbb{R}^m \setminus \{0\}$. If there does not exist a finite such m, we define $\gamma(A) = \infty$. Moreover, we set $\gamma(\emptyset) = 0$.

Lemma 4.6. *Let $A, B \in \mathcal{A}$. Then we have*
(i) If $A \subset B$, then $\gamma(A) \leq \gamma(B)$.
(ii) If there exists an odd continuous map $f \in C(A, B)$, then $\gamma(A) \leq \gamma(B)$.
(iii) If V is a bounded symmetric neighborhood of 0 in $\mathbb{R}^N$, then $\gamma(\partial V) = N$.
(iv) If Y is a subspace of E such that codim$\,Y = m$ and $\gamma(A) > m$, then $A \cap Y \neq \emptyset$.

Similar to [21, Proposition 9.23], we have

Lemma 4.7. *If Y is a closed subspace of X with codim$\,Y < k$, then*

$$h(D_k) \cap \partial B_R \cap Y \neq \emptyset, \quad \text{for all } h \in G_k \text{ and } 0 < R < R_k,$$

where $G_k = \{h \in C(D_k, X) \,|\, h \text{ is odd and } h = id \text{ on } \partial B_{R_k} \cap Y^k\}$ and R_k is from Lemma 4.2.

Proof. Set $V := \{u \in Y^k \,|\, \|u\| < R_k \text{ and } h(u) \in B_R\} \subset D_k$. Then $0 \in V$ and V is bounded and symmetric in Y^k. By Lemma 4.6, we have $\gamma(h(\partial V)) \geq \gamma(\partial V) = k$. Next we claim that

$$\|u\| < R_k, \quad \text{for } u \in \partial V.$$

Suppose by contradiction that $\|w_k\| = R_k$ for some $w_k \in \partial V$. Then, since $h = id$ on $\partial B_{R_k} \cap Y^k$, we have $h(w_k) = w_k$. Hence $R_k = \|w_k\| = \|h(w_k)\| \leq R$, which contradicts $R < R_k$. Consequently, $\|u\| < R_k$ for $u \in \partial V$. From the definition of V we see that

$$h(u) \in \partial B_R, \text{ for } u \in \partial V. \tag{4.8}$$

Since codim$\,Y < k \leq \gamma(h(\partial V))$, using Lemma 4.4 yields that $Y \cap h(\partial V) \neq \emptyset$. Then there is a point $w_0 \in \partial V$ such that $h(w_0) \in Y$. By (4.8), we have $h(w_0) \in Y \cap \partial B_R$. The proof is complete. □

It is known that there exist $\{e_n\} \subset X$ and $\{f_n\} \subset X^*$ such that

$$X = \overline{\text{span}\{e_n \,|\, n = 1, 2, \cdots\}}, \quad X^* = \overline{\text{span}\{f_n \,|\, n = 1, 2, \cdots\}},$$

and

$$f_n(e_m) = \begin{cases} 1, \text{ if } m = n, \\ 0, \text{ if } m \neq n. \end{cases}$$

For $k \in \mathbb{N}$, we set

$$Y_k = \overline{\operatorname{span}\{e_n \mid n = k, k+1, \cdots\}}, \quad Z_k = \operatorname{span}\{e_n \mid n = 1, 2, \cdots, k-1\}.$$

Then $X = Y_k + Z_k$ and $\operatorname{codim} Y_k = k - 1$. Similar to [24, Lemma 4.1], we have the following lemma.

Lemma 4.8. *There exists a sequence $\{\delta_k\}$ of positive numbers such that $\lim_{k\to\infty} \delta_k = 0$ and*

$$\|u\|_{r^+} \le \delta_k \|u\|, \quad \textit{for all } u \in Y_k.$$

Now we are ready to prove Lemma 4.4.

Proof of Lemma 4.4. Since $\operatorname{codim} Y_k = k - 1$, it follows from Lemma 4.7 that

$$h(D_k) \cap \partial B_R \cap Y_k \ne \emptyset, \quad \text{for all } h \in G_k \text{ and } 0 < R < R_k.$$

Then

$$\max_{u \in D_k} I(h(u)) \ge \inf_{\partial B_R \cap Y_k} I(u), \quad \text{for all } h \in G_k \text{ and } 0 < R < R_k,$$

which implies that

$$c_k \ge \inf_{\partial B_R \cap Y_k} I(u), \quad \text{for all } 0 < R < R_k. \tag{4.9}$$

By Lemma 4.8, we have

$$\begin{aligned}
I(u) &= \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}\, dx - \int_\Omega \frac{Q(x)}{r(x)} |u|^{r(x)}\, dx \\
&\ge \frac{1}{p^+} \min\{\|u_n\|^{p^-}, \|u_n\|^{p^+}\} - C_1 \int_\Omega (|u|^{r^+} + 1)\, dx \\
&\ge \frac{1}{p^+} \min\{\|u_n\|^{p^-}, \|u_n\|^{p^+}\} - C_2 \|u\|_{r^+}^{r^+} - C_3 \\
&\ge \frac{1}{p^+} \|u_n\|^{p^-} - C_4 \delta_k^{r^+} \|u\|^{r^+} - C_5,
\end{aligned}$$

for all $u \in Y_k$. Combining this with (4.9) leads to

$$c_k \ge \frac{1}{p^+} R^{p^-} - C_4 \delta_k^{r^+} R^{r^+} - C_5, \quad \text{for all } \ 0 < R < R_k.$$

Set $\xi_k = \left(\frac{p^-}{C_4 p^+ r^+ \delta_k^{r^+}}\right)^{\frac{1}{r^+ - p^-}}$ and, by Remark 4.3, we may assume that $R_k \ge \xi_k$. Then we have

$$c_k \ge \frac{1}{p^+} \xi_k^{p^-} - C_4 \delta_k^{r^+} \xi_k^{r^+} - C_5 = \frac{r^+ - p^-}{p^+ r^+} \left(\frac{p^-}{C_4 p^+ r^+ \delta_k^{r^+}}\right)^{\frac{p^-}{r^+ - p^-}} - C_5.$$

The desired conclusion follows easily from $\lim_{k\to\infty} \delta_k = 0$ and $r^+ > p^-$.

Acknowledgements

Thanks to Professor Zhi-Qiang Wang at Utah State University for his great help and valuable advice in this paper.

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