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EQUIVALENCE OF INITIALIZED RIEMANN-LIOUVILLE AND CAPUTO DERIVATIVES

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Abstract Initialization of fractional differential equations remains an ongoing problem. The initialization function approach and the infinite state approach provide two effective ways of dealing with this issue. The purpose of this paper is to prove the equivalence of the initialized Riemann-Liouville derivative and the initialized Caputo derivative with arbitrary order. By synthesizing the above two initialization theories, diffusive representations of the two initialized derivatives with arbitrary order are derived. The Laplace transforms of the two initialized derivatives are shown to be identical. Therefore, the two most commonly used derivatives are proved to be equivalent as long as initial conditions are properly imposed.

Keywords Fractional calculus, initialized fractional derivatives, diffusive representation, equivalence of fractional derivatives.

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1. Introduction

Fractional calculus provides a powerful tool of modeling real-world phenomena exhibiting memory and hereditary properties [10]. For fractional-order dynamical systems, initial conditions are required to characterize the historical effects. Therefore, initial conditions of fractional differential equations should be imposed in a different way from that of integer-order differential equations. However, proper initialization of fractional-order systems remains an ongoing problem [2,9,12,17,19,20]. This issue dates back as far as Riemann's complementary function theory, in which many mathematicians were made confused, including Liouville, Peacoch, Cayley, and Riemann himself [4].

In recent years, the initialization function approach and the infinite state approach provide two effective ways of imposing physically coherent initial conditions to fractional systems. In the initialization function theory [5], initialization functions are proposed to initialize fractional differential equations. The initialization function is a time-varying function. It can be viewed as generalization of the constant of integration required for the order-one integral. By virtue of using the initialization function, the Riemann-Liouville derivative and the Caputo derivative

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can be properly initialized. In the infinite state theory, the Riemann-Liouville fractional integral is viewed as a linear system which is characterized by the impulse function and excited by the integrand function. The linear system is termed as the fractional integrator. It can be equivalently converted into an infinite dimensional frequency distributed differential system. As a result, initial conditions of fractional systems can be represented by the distributed initial conditions [13, 14]. The equivalence and compatibility of the above two initialization theories have been proved in [3, 18].

The Riemann-Liouville derivative and the Caputo derivative are the two most commonly used derivatives in the real-world modeling of factional systems. However, definitions of the two derivatives are different. Thus, which derivative to choose is a trial and error process. On the other hand, the two derivatives are expected to be equivalent in many practical applications, as long as the initial conditions are properly taken into account [1, 11]. As a result, from mathematical point of view, it is a primary task to prove equivalence of the two definitions. In [6], the two derivatives are shown to be identical in the special case where the history function is a constant and the fractional order lies in between 0 and 1. In [15], the Laplace transforms of Riemann-Liouville derivative and Caputo derivative are calculated based on the infinite state approach. However, relationships between these Laplace transforms are not mentioned by the authors. Motivated by the above work, we go further in this paper to prove the equivalence and compatibility of the two initialized derivatives with arbitrary orders and history functions. By synthesizing the initialization function approach and the infinite state approach, the diffusive models of the initialized Riemann-Liouville derivatives and the initialized Caputo derivatives are derived. The Laplace transforms of the two initialized derivatives are shown to be identical. Consequently, the Riemann-Liouville derivative and the Caputo derivative are proved to equivalent as long as initial conditions are properly imposed.

The rest of this paper is organized as follows. Section 2 revisits the diffusive model for the initialized Riemann-Liouville fractional integral. Section 3 presents the equivalence of the initialized Riemann-Liouville derivatives and the initialized Caputo derivatives with order between 0 and 1. Section 4 shows the equivalence of the two initialized derivatives with order between 1 and 2. Section 5 proves the equivalence of the two initialized derivatives with arbitrary orders. Two examples of elementary functions are presented in Section 6. Finally, the paper is concluded in Section 7.

2. Preliminaries

2.1. Diffusive model of the fractional integrator

The Riemann-Liouville fractional integral of a function $f(t)$ with order $0 < \alpha < 1$ is defined as

$$ {}_{t_0}I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} f(\tau)\, d\tau \tag{2.1}$$

where α is an non-integer order of the factional integral, the subscripts t_0 and t are lower and upper terminals respectively.

On the other hand, Eq.(2.1) can be viewed as a convolution of the function $f(t)$

with the the impulse response $h_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, namely,

$$ {}_{t_0}I_t^{\alpha} f(t) = h_\alpha(t) * f(t) . \tag{2.2} $$

From this viewpoint, the fractional integral can be obtained as the output of a linear system. It is characterized by the impulse response $h_\alpha(t)$ and excited by $f(t)$, namely,

$$ x(t) = h_\alpha(t) * f(t) . \tag{2.3} $$

This linear system is terms as the fractional integrator.

Note that

$$ h_\alpha(t) = \int_0^{\infty} \mu_\alpha(\omega) e^{-\omega t} d\omega \tag{2.4} $$

where the elementary frequency ω is ranging from 0 to ∞, and

$$ \mu_\alpha(\omega) = \frac{sin\alpha\pi}{\pi} \omega^{-\alpha}. $$

Eq.(2.3) becomes

$$ x(t) = \int_0^{+\infty} \mu_\alpha(\omega) z(\omega, t) d\omega \tag{2.5} $$

where $z(\omega, t)$ is the frequency distributed state and it verifies the following ordinary differential equation:

$$ \frac{\partial z(\omega, t)}{\partial t} = -\omega z(\omega, t) + f(t) . \tag{2.6} $$

The relations Eq.(2.5) and Eq.(2.6) are termed as frequency distributed model or diffusive model of fractional integrator [16].

2.2. The initialized fractional integral

In the time-varying initialization theory [7,8], the fractional integration is assumed to take place for $t > -a$ rather than $t > 0$, thus the integrand $v(t)$ is required to be zero for all $t < -a$. The time period between $t = -a$ and $t = 0$ represents the "history" of the fractional integral. Accordingly, the integrand $v(t)$ is described as

$$ v(t) = \begin{cases} 0, \ t < -a \\ f_{in}(t), \ \ -a \le t \le 0 \\ f(t), \ t > 0 \end{cases} $$

where $t = -a$ is the starting time of integral, $t = 0$ is the initial time, $f_{in}(t)$ is the history function describing the behavior during the initialization period $[-a, 0]$, $f(t)$ is the function of primary interest after the initial time $t = 0$.

The initialized Riemann-Liouville fractional integral of order α is defined as

$$ {}_0D_t^{-\alpha} f(t) = {}_0d_t^{-\alpha} f(t) + \psi(t) , \ t > 0. \tag{2.7} $$

In Eq.(2.7), ${}_0D_t^{-\alpha} f(t)$ is called the initialized fractional integral. ${}_0d_t^{-\alpha} f(t)$ is called the uninitialized fractional integral and defined as

$$ {}_0d_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau. \tag{2.8} $$

Eq.(2.8) is actually the standard definition of fractional Riemann-Liouville integral. $\psi(t)$ is defined as

$$\psi(t) = \frac{1}{\Gamma(\alpha)} \int_{-a}^{0} (t-\tau)^{\alpha-1} f_{in}(\tau)\, d\tau. \tag{2.9}$$

$\psi(t)$ is termed as the initialization function, as it describes the hereditary effect of the past.

2.3. Diffusive model of the initialized Riemann-Liouville fractional integral [13]

Lemma 2.1. *The uninitialized Riemann-Liouville fractional integral* ${}_0d_t^{-\alpha} f(t)$ *with order* $0 < \alpha < 1$ *is equivalent to the diffusive model with zero initial condition:*

$$\begin{cases} \dfrac{\partial z(\omega,t)}{\partial t} = -\omega z(\omega,t) + f(t) \\ z(\omega,t_0) = 0 \end{cases}$$

and

$${}_0d_t^{-\alpha} f(t) = \int_0^{+\infty} \mu_\alpha(\omega)\, z(\omega,t) d\omega.$$

Lemma 2.2. *The initialized Riemann-Liouville fractional integral* ${}_0D_t^{-\alpha} f(t)$ *with order* $0 < \alpha < 1$ *is equivalent to the diffusive model with distributed initial condition:*

$$\begin{cases} \dfrac{\partial z(\omega,t)}{\partial t} = -\omega z(\omega,t) + f(t) \\ z(\omega,0) = \displaystyle\int_{-a}^{0} e^{\omega\tau} f_{in}(\tau) d\tau \end{cases}$$

and

$${}_0D_t^{-\alpha} f(t) = \int_0^{+\infty} \mu_\alpha(\omega) z(\omega,t) d\omega.$$

3. Equivalence of the two Derivatives with order between 0 and 1

3.1. The Laplace transform of initialized Riemann-Liouville derivative

The initialized Riemann-Liouville fractional derivative with order $0 < \alpha < 1$ is defined as

$${}_0^{RL}D_t^\alpha f(t) = \frac{d}{dt} {}_0D_t^{\alpha-1} f(t)$$

where ${}_0^{RL}D_t^\alpha$ represents the initialized fractional derivative in the Riemann-Liouville sense.

By virtue of Lemma 2.2, the diffusive representation of the initialized Riemann-Liouville fractional integral ${}_0D_t^{\alpha-1} f(t)$ is

$${}_0D_t^{\alpha-1} f(t) = \int_0^{\infty} \mu_{1\text{-}\alpha}(\omega) z_{RL}(\omega,t)\, d\omega$$

where $z_{RL}(\omega, t)$ satisfies

$$\begin{cases} \frac{\partial z_{RL}(\omega,t)}{\partial t} = -\omega z_{RL}(\omega,t) + f(t) \\ z_{RL}(\omega,0) = \int_{-a}^{0} e^{\omega\tau} f_{in}(\tau) d\tau \end{cases} \tag{3.1}$$

Therefore, the diffusive model of the initialized Riemann-Liouville derivative is

$${}_{0}^{RL}D_t^{\alpha} f(t) = \frac{d}{dt} \int_0^{\infty} \mu_{1\text{-}\alpha}(\omega) z_{RL}(\omega, t)\, d\omega. \tag{3.2}$$

Taking the Laplace transform of the first equation of Eq.(3.1), yields

$$Z_{RL}(\omega, t) = \frac{z_{RL}(\omega, 0) + F(s)}{s + \omega}. \tag{3.3}$$

Taking the Laplace transform of Eq.(3.2), we have

$$\begin{aligned} L\left\{{}_{0}^{RL}D_t^{\alpha} f(t)\right\} &= L\left\{\frac{d}{dt}\int_0^{\infty} \mu_{1\text{-}\alpha}(\omega) z_{RL}(\omega, t)\, d\omega\right\} \\ &= s\int_0^{\infty} \mu_{1\text{-}\alpha}(\omega) Z_{RL}(\omega, t)\, d\omega - \int_0^{\infty} \mu_{1\text{-}\alpha}(\omega) z_{RL}(\omega, 0)\, d\omega. \end{aligned} \tag{3.4}$$

Substituting Eq.(3.3) into Eq.(3.4), we obtain

$$\begin{aligned} L\left\{{}_{0}^{RL}D_t^{\alpha} f(t)\right\} = sF(s)\int_0^{\infty} \frac{\mu_{1\text{-}\alpha}(\omega)}{s+\omega} d\omega + s\int_0^{\infty} \frac{\mu_{1-\alpha}(\omega) z_{RL}(\omega, 0)}{s+\omega} d\omega \\ - \int_0^{\infty} \mu_{1-\alpha}(\omega) z_{RL}(\omega, 0)\, d\omega. \end{aligned} \tag{3.5}$$

Taking the Laplace transform of Eq.(2.4), yields

$$\int_0^{\infty} \frac{\mu_{1\text{-}\alpha}(\omega)}{s+\omega} d\omega = s^{\alpha-1}.$$

Therefore, Eq.(3.5) becomes

$$\begin{aligned} L\left\{{}_{0}^{RL}D_t^{\alpha} f(t)\right\} &= s^{\alpha} F(s) + s\int_0^{\infty} \frac{\mu_{1\text{-}\alpha}(\omega) z_{RL}(\omega, 0)}{s+\omega} d\omega \\ &\quad - \int_0^{\infty} \mu_{1\text{-}\alpha}(\omega) z_{RL}(\omega, 0)\, d\omega \\ &= s^{\alpha} F(s) + \int_0^{\infty} \left(\frac{s}{s+\omega} - 1\right) \mu_{1\text{-}\alpha}(\omega) z_{RL}(\omega, 0)\, d\omega \\ &= s^{\alpha} F(s) - \int_0^{\infty} \frac{\omega \mu_{1\text{-}\alpha}(\omega) z_{RL}(\omega, 0)}{s+\omega} d\omega. \end{aligned} \tag{3.6}$$

Eq.(3.6) is the Laplace transform of the initialized Riemann-Liouville derivative with order $0 < \alpha < 1$.

3.2. The Laplace transform of initialized Caputo derivative

The initialized Caputo fractional derivative with order $0 < \alpha < 1$ is defined as

$${}_{0}^{C}D_t^{\alpha} f(t) = {}_{0}D_t^{\alpha-1} f'(t)$$

where ${}_0^C D_t^\alpha$ represents the initialized fractional derivative in the Caputo sense.
In terms of Lemma 2.2, the diffusive representation of the initialized Caputo fractional derivative is

$$ {}_0^C D_t^\alpha f(t) = \int_0^\infty \mu_{1\text{-}\alpha}(\omega) z_C(\omega, t)\, d\omega \tag{3.7} $$

where $z_C(\omega, t)$ satisfies

$$ \begin{cases} \frac{\partial z_C(\omega,t)}{\partial t} = -\omega z_C(\omega, t) + f'(t) \\ z_C(\omega, 0) = \int_{-a}^0 e^{\omega\tau} {f_{in}}'(\tau) d\tau \end{cases} \tag{3.8} $$

Taking the Laplace transform of Eq.(3.8), we have

$$ Z_C(\omega, t) = \frac{sF(s) - f(0) + z_C(\omega, 0)}{s + \omega}. \tag{3.9} $$

Taking the Laplace transform of Eq.(3.7), yields

$$ \begin{aligned} L\left\{ {}_0^C D_t^\alpha f(t) \right\} &= L\left\{ \int_0^\infty \mu_{1\text{-}\alpha}(\omega) z_C(\omega, t)\, d\omega \right\} \\ &= \int_0^\infty \mu_{1\text{-}\alpha}(\omega) Z_C(\omega, t)\, d\omega. \end{aligned} \tag{3.10} $$

Substituting Eq.(3.9) into Eq.(3.10), leads to

$$ \begin{aligned} L\left\{ {}_0^C D_t^\alpha f(t) \right\} &= sF(s) \int_0^\infty \frac{\mu_{1\text{-}\alpha}(\omega)}{s+\omega} d\omega + \int_0^\infty \frac{\mu_{1\text{-}\alpha}(\omega) z_C(\omega, 0)}{s+\omega} d\omega \\ &\quad - f(0) \int_0^\infty \frac{\mu_{1\text{-}\alpha}(\omega)}{s+\omega} d\omega \\ &= s^\alpha F(s) + \int_0^\infty \frac{\mu_{1\text{-}\alpha}(\omega) z_C(\omega, 0)}{s+\omega} d\omega - s^{\alpha-1} f(0). \end{aligned} \tag{3.11} $$

In terms of the formula of integration by parts, the initial condition in Eq.(3.8) becomes

$$ z_C(\omega, 0) = \int_{-a}^0 e^{\omega\tau} {f_{in}}'(\tau) d\tau = e^{\omega\tau} f_{in}(\tau)|_{\tau=-a}^{\tau=0} - \omega \int_{-a}^0 e^{\omega\tau} f_{in}(\tau) d\tau. $$

In terms of Eq.(3.1), we have

$$ z_C(\omega, 0) = f(0) - \omega z_{RL}(\omega, 0). \tag{3.12} $$

Then

$$ \begin{aligned} \int_0^\infty \frac{\mu_{1\text{-}\alpha}(\omega) z_C(\omega, 0)}{s+\omega} d\omega &= f(0) \int_0^\infty \frac{\mu_{1-\alpha}(\omega)}{s+\omega} d\omega - \int_0^\infty \frac{\omega \mu_{1\text{-}\alpha}(\omega) z_{RL}(\omega, 0)}{s+\omega} d\omega \\ &= s^{\alpha-1} f(0) - \int_0^\infty \frac{\mu_{1\text{-}\alpha}(\omega) z_{RL}(\omega, 0)}{s+\omega} d\omega. \end{aligned} \tag{3.13} $$

Substituting Eq.(3.13) into Eq.(3.11), yields

$$ L\left\{ {}_0^C D_t^\alpha f(t) \right\} = s^\alpha F(s) - \int_0^\infty \frac{\omega \mu_{1\text{-}\alpha}(\omega) z_{RL}(\omega, 0)}{s+\omega} d\omega. \tag{3.14} $$

Eq.(3.14) is the Laplace transform of the initialized Caputo derivative with order $0 < \alpha < 1$.

By comparing Eq.(3.6) with Eq.(3.14), one finds that the Laplace transforms of the two initialized derivatives are identical, i.e.,

$$L\left\{{}_0^{RL}D_t^{\alpha} f\left(t\right)\right\} = L\left\{{}_0^{C}D_t^{\alpha} f\left(t\right)\right\}. \tag{3.15}$$

Therefore,

$${}_0^{RL}D_t^{\alpha} f\left(t\right) = {}_0^{C}D_t^{\alpha} f\left(t\right). \tag{3.16}$$

Eq.(3.16) shows the equivalence of the initialized Riemann-Liouville derivative and Caputo derivative with order lying in between 0 and 1.

4. Equivalence of the two derivatives with order between 1 and 2

4.1. The Laplace transform of initialized Riemann-Liouville derivative

The initialized Riemann-Liouville fractional derivative ${}_0D_t^{\alpha} f(t)$ with order $1 < \alpha < 2$ is defined as

$${}_0^{RL}D_t^{\alpha} f\left(t\right) = \frac{d^2}{dt^2}{}_0D_t^{\alpha-2} f\left(t\right).$$

In terms of of Lemma 2.2, the diffusive representation of the initialized Riemann-Liouville fractional integral ${}_0D_t^{\alpha-2} f(t)$ is

$${}_0D_t^{\alpha-2} f\left(t\right) = \int_0^{\infty} \mu_{2-\alpha}(\omega) z_{RL}\left(\omega, t\right) d\omega$$

where $z_{RL}\left(\omega, t\right)$ satisfies Eq.(3.1).

Thus, the diffusive model of the initialized Riemann-Liouville derivative is

$${}_0^{RL}D_t^{\alpha} f\left(t\right) = \frac{d^2}{dt^2} \int_0^{\infty} \mu_{1\text{-}\alpha}(\omega) z_{RL}\left(\omega, t\right) d\omega. \tag{4.1}$$

Taking the Laplace transform of Eq.(4.1), we have

$$\begin{aligned} L\left\{{}_0^{RL}D_t^{\alpha} f\left(t\right)\right\} &= L\left\{\frac{d^2}{dt^2} \int_0^{\infty} \mu_{1-\alpha}(\omega) z_{RL}\left(\omega, t\right) d\omega\right\} \\ &= s^2 \int_0^{\infty} \mu_{2-\alpha}(\omega) Z_{RL}\left(\omega, t\right) d\omega - s \int_0^{\infty} \mu_{2\text{-}\alpha}(\omega) z_{RL}\left(\omega, 0\right) d\omega \\ &\quad - \int_0^{\infty} \mu_{2\text{-}\alpha}(\omega) \left.\frac{\partial z_{RL}(\omega, t)}{\partial t}\right|_{t=0} d\omega. \end{aligned} \tag{4.2}$$

Substituting Eq.(3.3) into Eq.(4.2), yields

$$\begin{aligned} L\left\{{}_0^{RL}D_t^{\alpha} f\left(t\right)\right\} &= sF\left(s\right) \int_0^{\infty} \frac{\mu_{2-\alpha}(\omega)}{s+\omega} d\omega + s^2 \int_0^{\infty} \frac{\mu_{2-\alpha}(\omega) z_{RL}\left(\omega, 0\right)}{s+\omega} d\omega \\ &\quad - s \int_0^{\infty} \mu_{2-\alpha}(\omega) z_{RL}\left(\omega, 0\right) d\omega - \int_0^{\infty} \mu_{2-\alpha}(\omega) \left.\frac{\partial z_{RL}(\omega, t)}{\partial t}\right|_{t=0} d\omega. \end{aligned} \tag{4.3}$$

Note that

$$\int_0^\infty \frac{\mu_{2-\alpha}(\omega)}{s+\omega} d\omega = s^{\alpha-2}.$$

Substituting the first equation of Eq.(3.1) into Eq.(4.3), we get

$$\begin{aligned} L\left\{{}_0^{RL}D_t^\alpha f(t)\right\} &= s^\alpha F(s) + \int_0^\infty \left(\frac{s^2}{s+\omega} - s + \omega\right) \mu_{2\text{-}\alpha}(\omega) z_{RL}(\omega, 0)\, d\omega \\ &\quad - f(0) \int_0^\infty \mu_{2-\alpha}(\omega) d\omega \\ &= s^\alpha F(s) + \int_0^\infty \frac{\omega^2 \mu_{2\text{-}\alpha}(\omega) z_{RL}(\omega, 0)}{s+\omega} d\omega \\ &\quad - f(0) \int_0^\infty \mu_{2\text{-}\alpha}(\omega) d\omega. \end{aligned} \tag{4.4}$$

Eq.(4.4) is the Laplace transform of the initialized Riemann-Liouville derivative with order $1 < \alpha < 2$.

4.2. The Laplace transform of initialized Caputo derivative

The initialized Caputo fractional derivative ${}_0D_t^\alpha f(t)$ with order $1 < \alpha < 2$ is defined as

$${}_0^C D_t^\alpha f(t) = {}_0D_t^{\alpha-2} f''(t).$$

By virtue of Lemma 2.2, the diffusive representation of the initialized Caputo fractional derivative is

$${}_0^C D_t^\alpha f(t) = \int_0^\infty \mu_{2\text{-}\alpha}(\omega) z_C(\omega, t)\, d\omega \tag{4.5}$$

where $z_C(\omega, t)$ satisfies

$$\begin{cases} \frac{\partial z_C(\omega,t)}{\partial t} = -\omega z_C(\omega, t) + f''(t) \\ z_C(\omega, 0) = \int_{-a}^0 e^{\omega\tau} {f_{in}}''(\tau) d\tau \end{cases} \tag{4.6}$$

Taking the Laplace transform of the first equation in Eq.(4.6), we have

$$Z_C(\omega, t) = \frac{s^2 F(s) - s f(0) - f'(0) + z_C(\omega, 0)}{s+\omega}. \tag{4.7}$$

In terms of the formula of integration by parts, the initial condition in Eq.(4.6) becomes

$$\begin{aligned} z_C(\omega, 0) &= \int_{-a}^0 e^{\omega\tau} {f_{in}}''(\tau) d\tau \\ &= e^{\omega\tau} {f_{in}}'(\tau)\big|_{\tau=-a}^{\tau=0} - \omega \int_{-a}^0 e^{\omega\tau} {f_{in}}'(\tau) d\tau \\ &= f'(0) - \omega \int_{-a}^0 e^{\omega\tau} {f_{in}}'(\tau) d\tau \\ &= f'(0) - \omega f(0) + \omega^2 z_R(\omega, 0). \end{aligned} \tag{4.8}$$

Substituting Eq.(4.8) into Eq.(4.7), leads to

$$Z_C(\omega,t) = \frac{s^2F(s) - (s+\omega) f(0) + \omega^2 z_{RL}(\omega,0)}{s+\omega}. \tag{4.9}$$

Taking the Laplace transform of Eq.(4.5)and substituting Eq.(4.9) into it, we obtain

$$\begin{aligned} L\left\{{}_0^C D_t^\alpha f(t)\right\} &= L\left\{\int_0^\infty \mu_{2\text{-}\alpha}(\omega) z_C(\omega,t)\, d\omega\right\} \\ &= \int_0^\infty \mu_{2\text{-}\alpha}(\omega) Z_C(\omega,t)\, d\omega \\ &= s^2F(s)\int_0^\infty \frac{\mu_{2-\alpha}(\omega)}{s+\omega} d\omega + \int_0^\infty \frac{\omega^2\mu_{2\text{-}\alpha}(\omega) z_{RL}(\omega,0)}{s+\omega} d\omega \\ &\quad - f(0)\int_0^\infty \mu_{2\text{-}\alpha}(\omega) d\omega \\ &= s^\alpha F(s) + \int_0^\infty \frac{\omega^2\mu_{2\text{-}\alpha}(\omega) z_{RL}(\omega,0)}{s+\omega} d\omega \\ &\quad - f(0)\int_0^\infty \mu_{2\text{-}\alpha}(\omega) d\omega. \end{aligned} \tag{4.10}$$

Eq.(4.10) is the Laplace transform of initialized Caputo derivative with order $1 < \alpha < 2$.

From Eq.(4.4) and Eq.(4.10), one easily shows that

$$L\left\{{}_0^{RL} D_t^\alpha f(t)\right\} = L\left\{{}_0^C D_t^\alpha f(t)\right\}. \tag{4.11}$$

Thus,

$${}_0^{RL} D_t^\alpha f(t) = {}_0^C D_t^\alpha f(t). \tag{4.12}$$

Eq.(4.12) shows the equivalence of the initialized Riemann-Liouville derivative and Caputo derivative with order $1 < \alpha < 2$.

5. Equivalence of the two derivatives with arbitrary orders

5.1. The Laplace transform of initialized Riemann-Liouville derivative

The initialized Riemann-Liouville fractional derivative ${}_0D_t^\alpha f(t)$ with order $n-1 < \alpha < n$ is defined as

$${}_0^{RL} D_t^\alpha f(t) = \frac{d^n}{dt^n} {}_0D_t^{\alpha-n} f(t).$$

By virtue of Lemma 2, the diffusive representation of the initialized Riemann-Liouville fractional integral ${}_0D_t^{\alpha-n} f(t)$ is

$${}_0D_t^{\alpha-n} f(t) = \int_0^\infty \mu_{n-\alpha}(\omega) z_{RL}(\omega,t)\, d\omega$$

where $z_{RL}(\omega,t)$ satisfies Eq.(4).

As a result, the diffusive model of the initialized Riemann-Liouville derivative is

$$ {}_0^{RL}D_t^{\alpha} f(t) = \frac{d^n}{dt^n} \int_0^{\infty} \mu_{\text{n-}\alpha}(\omega) z_{RL}(\omega, t)\, d\omega. \tag{5.1}$$

Taking the Laplace transform of Eq.(5.1), we have

$$\begin{aligned} L\left\{ {}_0^{RL}D_t^{\alpha} f(t) \right\} &= L\left\{ \frac{d^n}{dt^n} \int_0^{\infty} \mu_{n-\alpha}(\omega) z_{RL}(\omega, t)\, d\omega \right\} \\ &= s^n \int_0^{\infty} \mu_{n-\alpha}(\omega) Z_{RL}(\omega, t)\, d\omega - s^{n-1} \int_0^{\infty} \mu_{n-\alpha}(\omega) z_{RL}(\omega, 0)\, d\omega \\ &- s^{n-2} \int_0^{\infty} \mu_{n-\alpha}(\omega) {z'}_{RL}(\omega, 0) d\omega \\ &- s^{n-3} \int_0^{\infty} \mu_{n-\alpha}(\omega) {z''}_{RL}(\omega, 0) d\omega \\ &- \cdots - \int_0^{\infty} \mu_{n-\alpha}(\omega) {z^{(n-1)}}_{RL}(\omega, 0) d\omega. \end{aligned} \tag{5.2}$$

Substituting Eq.(3.3) into Eq.(5.2), and applying $\int_0^{\infty} \frac{\mu_{2-\alpha}(\omega)}{s+\omega} d\omega = s^{\alpha-2}$, we have

$$\begin{aligned} L\left\{ {}_0^{RL}D_t^{\alpha} f(t) \right\} = s^{\alpha} F(s) &+ s^n \int_0^{\infty} \frac{\mu_{n-\alpha}(\omega) z_{RL}(\omega, 0)}{s+\omega} d\omega \\ &- s^{n-1} \int_0^{\infty} \mu_{n-\alpha}(\omega) z_{RL}(\omega, 0)\, d\omega \\ &- s^{n-2} \int_0^{\infty} \mu_{n-\alpha}(\omega) {z'}_{RL}(\omega, 0)\, d\omega \\ &- s^{n-3} \int_0^{\infty} \mu_{n-\alpha}(\omega) {z''}_{RL}(\omega, 0)\, d\omega \\ &- \cdots - \int_0^{\infty} \mu_{n-\alpha}(\omega) z_{RL}^{(n-1)}(\omega, 0)\, d\omega. \end{aligned} \tag{5.3}$$

From Eq.(3.1), we can calculate high-order derivatives of $z_{RL}(\omega, t)$, namely,

$$\begin{aligned} \frac{\partial^2 z_{RL}(\omega, t)}{\partial t^2} &= \omega^2 z_{RL}(\omega, t) - \omega f(t) + f''(t) \\ \frac{\partial^3 z_{RL}(\omega, t)}{\partial t^3} &= -\omega^3 z_{RL}(\omega, t) + \omega^2 f(t) - \omega f'(t) + f''(t) \\ &\cdots\cdots \\ \frac{\partial^{n-1} z_{RL}(\omega, t)}{\partial t^{n-1}} &= (-\omega)^{n-1} z_{RL}(\omega, t) + (-\omega)^{n-2} f(t) + (-\omega)^{n-3} f'(t) \\ &\quad + \cdots + f^{(n-2)}(t). \end{aligned}$$

Substituting the above derivatives into Eq.(5.3), we obtain

$$
\begin{aligned}
L\left\{{}_0^{RL}D_t^{\alpha} f(t)\right\} &= s^{\alpha}F(s) + s^{n}\int_0^{\infty}\frac{\mu_{n-\alpha}(\omega) z_{RL}(\omega,0)}{s+\omega}d\omega \\
&- s^{n-1}\int_0^{\infty}\mu_{n-\alpha}(\omega) z_{RL}(\omega,0)\,d\omega \\
&- s^{n-2}\int_0^{\infty}\mu_{n-\alpha}(\omega)\left[-\omega z_{RL}(\omega,0)+f(0)\right]d\omega \\
&- s^{n-3}\int_0^{\infty}\mu_{n-\alpha}(\omega)\left[\omega^2 z_{RL}(\omega,0)-\omega f(0)+f''(0)\right]d\omega \\
&- \cdots\cdots \\
&- \int_0^{\infty}\mu_{n-\alpha}(\omega)\Big[(-\omega)^{n-1}z_{RL}(\omega,0)+(-\omega)^{n-2}f(0) \\
&+(-\omega)^{n-3}f'(0)+\cdots+f^{(n-2)}(0)\Big]d\omega \\
&= s^{\alpha}F(s)+\int_0^{\infty}\mu_{n-\alpha}(\omega) z_{RL}(\omega,0)\Big[\frac{s^n}{s+\omega}-s^{n-1}+\omega s^{n-2} \\
&-\omega^2 s^{n-3}+\cdots+(-\omega)^{n-2}s+(-\omega)^{n-1}\Big]d\omega \\
&+\int_0^{\infty}\mu_{n-\alpha}(\omega)\left\{\Delta_1\right\}d\omega \\
&= s^{\alpha}F(s)+\int_0^{\infty}\mu_{n-\alpha}(\omega) z_{RL}(\omega,0)\frac{(-\omega)^n}{s+\omega}d\omega \\
&-\int_0^{\infty}\mu_{n-\alpha}(\omega)\left\{\Delta_1\right\}d\omega
\end{aligned}
\tag{5.4}
$$

where

$$
\begin{aligned}
\Delta_1 &= \left[s^{n-2}+(-\omega)s^{n-3}+(-\omega)^2 s^{n-4}+\cdots(-\omega)^{n-2}\right]f(0) \\
&+\left[s^{n-3}+(-\omega)s^{n-4}+\cdots(-\omega)^{n-3}\right]f'(0)+\cdots+f^{(n-2)}(0)\,.
\end{aligned}
\tag{5.5}
$$

Eq.(5.4) is the Laplace transform of the initialized Riemann-Liouville derivative with order $n-1<\alpha<n$.

5.2. The Laplace transform of initialized Caputo derivative

The initialized Caputo fractional derivative ${}_0D_t^{\alpha}f(t)$ with order $n-1<\alpha<n$ is defined as

$$
{}_0^{C}D_t^{\alpha}f(t) = {}_0D_t^{\alpha-n}f^{(n)}(t)\,.
$$

By virtue of Lemma 2.2, the diffusive representation of the initialized Caputo fractional derivative is

$$
{}_0^{C}D_t^{\alpha}f(t) = \int_0^{\infty}\mu_{n-\alpha}(\omega) z_C(\omega,t)\,d\omega \tag{5.6}
$$

where $z_C(\omega,t)$ satisfies

$$
\begin{cases}
\frac{\partial z_C(\omega,t)}{\partial t} = -\omega z_C(\omega,t)+f^{(n)}(t) \\
z_C(\omega,0) = \int_{-a}^{0}e^{\omega\tau}f_{in}^{(n)}(\tau)d\tau
\end{cases}
\tag{5.7}
$$

Taking the Laplace transform of Eq.(5.7), we have

$$Z_C(\omega,s)=\frac{1}{s+\omega}\Big[z_C(\omega,0)+s^nF(s)-s^{n-1}f(0)-s^{n-1}f'(0)-\cdots \\ -sf^{(n-2)}(0)-f^{(n-1)}(0)\Big]. \tag{5.8}$$

In terms of the formula of integration by parts, the initial condition in Eq.(5.7) becomes

$$z_C(\omega,0)=f^{(n-1)}(0)+(-\omega)f^{(n-2)}(0)+(-\omega)^2f^{(n-3)}(0)+(-\omega)^3f^{(n-4)}(0) \\ +\cdots+(-\omega)^{n-2}f'(0)+(-\omega)^{n-1}f(0)+(-\omega)^nz_{RL}(\omega,0). \tag{5.9}$$

Substituting Eq.(5.9) into Eq.(5.8), we have

$$Z_C(\omega,s)=\frac{1}{s+\omega}\Big\{s^nF(s)+\Big[(-\omega)^{n-1}-s^{n-1}\Big]f(0) \\ +\Big[(-\omega)^{n-2}-s^{n-2}\Big]f'(0)+\cdots \\ +[(-\omega)-s]f^{(n-2)}(0)+(-\omega)^nZ_{RL}(\omega,0)\Big\}. \tag{5.10}$$

Taking the Laplace transform of Eq.(5.6) and substituting Eq.(5.10) into it, we obtain

$$\begin{aligned} L\left\{{}_0^CD_t^\alpha f(t)\right\} &= L\left\{\int_0^\infty \mu_{n-\alpha}(\omega)z_C(\omega,t)\,d\omega\right\} \\ &=\int_0^\infty \mu_{n-\alpha}(\omega)Z_C(\omega,t)\,d\omega \\ &=s^\alpha F(s)+\int_0^\infty \frac{(-\omega)^n}{s+\omega}\mu_{n-\alpha}(\omega)z_{RL}(\omega,0)\,d\omega \\ &\quad-\int_0^\infty \mu_{n-\alpha}(\omega)\left\{\Delta_2\right\}d\omega \end{aligned} \tag{5.11}$$

where

$$\Delta_2=\left[\frac{s^{n-1}-(-\omega)^{n-1}}{s+\omega}\right]f(0)+\left[\frac{s^{n-2}-(-\omega)^{n-2}}{s+\omega}\right]f'(0)+\cdots+f^{(n-2)}(0). \tag{5.12}$$

Eq.(5.11) is the Laplace transform of initialized Caputo derivative with order $n-1<\alpha<n$.

By comparing Eq.(5.5) with Eq.(5.12), one easily shows that $\Delta_1=\Delta_2$. Therefore,

$$L\left\{{}_0^{RL}D_t^\alpha f(t)\right\}=L\left\{{}_0^CD_t^\alpha f(t)\right\}. \tag{5.13}$$

Thus,

$${}_0^{RL}D_t^\alpha f(t)={}_0^CD_t^\alpha f(t). \tag{5.14}$$

Eq.(5.14) shows the equivalence of the initialized Riemann-Liouville derivative and Caputo derivative with arbitrary order α.

6. Examples

In this section, examples of two elementary functions are presented to illustrate the equivalence of the two initialized fractional derivatives.

6.1. Fractional derivatives of the Heaviside function

Consider the Heaviside function:

$$f\left(t\right)=H\left(t\right)=\begin{cases}1, & t\geq 0\\ 0, & t<0\end{cases}$$

Firstly, we calculate the initialized Riemann-Liouville derivative with order $0<\alpha<1$. In terms of Eq.(3.1), we have $z_{RL}(\omega,0)=0$ and

$$z_{RL}(\omega,t)=\int_0^t e^{-\omega(t-\tau)}d\tau=\frac{1}{\omega}\left(1-e^{-\omega t}\right).$$

Therefore,

$$\frac{\partial z_{RL}(\omega,t)}{\partial t}=e^{-\omega t}. \tag{6.1}$$

Substituting Eq.(6.1) into Eq.(3.2), we get

$${}_0^{RL}D_t^{\alpha}f\left(t\right)=\int_0^{\infty}\mu_{1\text{-}\alpha}(\omega)\frac{\partial z_{RL}(\omega,t)}{\partial t}d\omega=\int_0^{\infty}\mu_{1\text{-}\alpha}(\omega)e^{-\omega t}d\omega. \tag{6.2}$$

In terms of Eq.(2.4), we have

$$\int_0^{\infty}\mu_{1\text{-}\alpha}(\omega)e^{-\omega t}d\omega=\frac{t^{-\alpha}}{\Gamma\left(1-\alpha\right)},\ t\geq 0. \tag{6.3}$$

Thus, the initialized Riemann-Liouville derivative is

$${}_0^{RL}D_t^{\alpha}f\left(t\right)=\frac{t^{-\alpha}}{\Gamma\left(1-\alpha\right)},\ t\geq 0. \tag{6.4}$$

Next, we calculate the initialized Caputo derivative with order $0<\alpha<1$.

Note that

$$\frac{df\left(t\right)}{dt}=\delta\left(t\right) \tag{6.5}$$

where $\delta\left(t\right)$ is the Dirac function.

In terms of Eq.(3.8), we have $z_C(\omega,0)=0$ and

$$z_C(\omega,t)=\int_0^t e^{-\omega(t-\tau)}\delta\left(\tau\right)d\tau=e^{-\omega t}. \tag{6.6}$$

In terms of Eq.(3.7), we obtain

$${}_0^{C}D_t^{\alpha}f\left(t\right)=\int_0^{\infty}\mu_{1\text{-}\alpha}(\omega)e^{-\omega t}d\omega=\frac{t^{-\alpha}}{\Gamma\left(1-\alpha\right)},\ t\geq 0. \tag{6.7}$$

By comparing Eq.(6.7) with Eq.(6.4), one can easily get

$${}_0^{RL}D_t^{\alpha}f\left(t\right)={}_0^{C}D_t^{\alpha}f\left(t\right).$$

6.2. Fractional derivatives of the exponential function

Consider the exponential function:

$$f(t) = e^{\lambda t},\ t \in (-\infty, +\infty).$$

Firstly, we evaluate the initialized Riemann-Liouville derivative with order $0 < \alpha < 1$. In terms of Eq.(3.1),

$$z_{RL}(\omega, t) = \int_{-\infty}^{t} e^{-\omega(t-\tau)} e^{\lambda\tau} d\tau = \frac{e^{\lambda t}}{\lambda + \omega}.$$

Therefore,

$$\frac{\partial z_{RL}(\omega, t)}{\partial t} = \frac{\lambda e^{\lambda t}}{\lambda + \omega}. \tag{6.8}$$

Substituting Eq.(6.8) into Eq.(3.2), we get

$${}_{0}^{RL}D_t^{\alpha} f(t) = \int_0^{\infty} \mu_{1-\alpha}(\omega) \frac{\partial z_{RL}(\omega, t)}{\partial t} d\omega = \lambda e^{\lambda t} \int_0^{\infty} \frac{\mu_{1-\alpha}(\omega)}{\lambda + \omega} d\omega. \tag{6.9}$$

Recall that

$$s^{\alpha-1} = \int_0^{\infty} \frac{\mu_{1\text{-}\alpha}(\omega)}{s + \omega} d\omega.$$

Thus, with $s = \lambda$, we have

$$\int_0^{\infty} \frac{\mu_{1\text{-}\alpha}(\omega)}{\lambda + \omega} d\omega = \lambda^{\alpha-1}. \tag{6.10}$$

Substituting Eq.(6.10) into Eq.(6.9), we obtain

$${}_{0}^{RL}D_t^{\alpha} f(t) = \lambda^{\alpha} e^{\lambda t},\ t \geq 0. \tag{6.11}$$

Next, we evaluate the initialized Caputo derivative with order $0 < \alpha < 1$. In terms of Eq.(3.8), we have

$$z_C(\omega, t) = \lambda \int_{-\infty}^{t} e^{-\omega(t-\tau)} e^{\lambda\tau} d\tau = \frac{\lambda e^{\lambda t}}{\lambda + \omega}. \tag{6.12}$$

Substituting Eq.(6.12) into Eq.(3.7), we obtain

$${}_{0}^{C}D_t^{\alpha} f(t) = \lambda e^{\lambda t} \int_0^{\infty} \frac{\mu_{1\text{-}\alpha}(\omega)}{\lambda + \omega} d\omega = \lambda^{\alpha} e^{\lambda t},\ t \geq 0. \tag{6.13}$$

From Eq.(6.13) and Eq.(6.11), one can easily get

$${}_{0}^{RL}D_t^{\alpha} f(t) = {}_{0}^{C}D_t^{\alpha} f(t).$$

7. Conclusions

This paper has proved the equivalence of the initialized Riemann-Liouville derivative and the initialized Caputo derivative with arbitrary order. By synthesizing the initialization function theory and the infinite state theory, the diffusive representations of the two initialized derivatives have been obtained. Laplace transforms of the two initialized derivatives with order $0 < \alpha < 1$, $1 < \alpha < 2$ and arbitrary order have been progressively shown to be identical. As a result, the two most commonly used derivatives have been shown to be equivalent as long as initial conditions are properly imposed. Although definitions of the Riemann-Liouville derivative and the Caputo derivative are different, this result eliminates the distinction of the two derivatives in practical applications. In mathematical modeling and analysis, we need not to dwell on which derivative to choose.

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