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TRANSVERSE HOMOCLINIC ORBIT BIFURCATED FROM A HOMOCLINIC MANIFOLD BY THE HIGHER ORDER MELNIKOV INTEGRALS*

Bin Long[1,†] and Changrong Zhu[2]

Abstract Consider an autonomous ordinary differential equation in $\mathbb{R}^n$ that has a d dimensional homoclinic solution manifold W^H. Suppose the homoclinic manifold can be locally parametrized by $(\alpha, \theta) \in \mathbb{R}^{d-1} \times \mathbb{R}$. We study the bifurcation of the homoclinic solution manifold W^H under periodic perturbations. Using exponential dichotomies and Lyapunov-Schmidt reduction, we obtain the higher order Melnikov function. For a fixed (α_0, θ_0) on W^H, if the Melnikov function have a simple zeros, then the perturbed system can have transverse homoclinic solutions near W^H.

Keywords Homoclinic manifold, Lyapunov-Schmidt reduction, exponential dichotomies, Melnikov integral, chaos.

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1. Introduction

The problem of the bifurcation of homoclinic orbit under small periodic perturbation is very important in dynamic system because they are related to some complex dynamic behaviors, such as chaotic motions. Homoclinic orbit is a special invariant set of a differential equation. Suppose the equation $\dot{x} = f(x)$ has a solution $\gamma(t)$, which asymptotic to the hyperbolic equilibrium $x = 0$ in both forward and backward time direction. The orbit γ corresponding to the solution $\gamma(t)$ in phase space is called homoclinic orbit [11]. Suppose the variational equation of $\dot{x} = f(x)$ along γ has d linearly independent bounded solutions. Let $W^s(0)$, $W^u(0)$ denote the stable and unstable manifolds of the equilibrium 0, respectively. Clearly, the homoclinic orbit γ lies on $W^s(0) \bigcap W^u(0)$. From the number of the linearly independent bounded solutions of the variational equation, we know the dimension of the intersection of the tangent space of $W^s(0)$ and $W^u(0)$ is equal to d. Apparently $\dot{\gamma} \in T_{\gamma(0)} W^s(0) \bigcap T_{\gamma(0)} W^u(0)$, so $d \geqslant 1$. If $d = 1$, the homoclinic orbit γ is called nondegenerate; otherwise it is called degenerate [15].

[†]The corresponding author. Email address:longbin210@126.com(B. Long)
[1]Department of Mathematics, Shaanxi University of Science and Technology, Xi'an, 710021, China
[2]School of Mathematics and Statistics, Chongqing University, Chongqing, 401331, China
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From geometrical viewpoint, in 1963 Melnikov [17] used Poincare map to investigate the persistence of homoclinic orbit in R^2. In 1980, Chow, Hale and Mallet-Parret [3] used functional methods to study the persistence of homoclinic orbit of Duffing's equation under damping and periodic forcing. Later Palmer [18] extended the work in [3] to N-dimensional system. He assumed the unperturbed system has a nondegenerate homoclinic solution. By the Exponential dichotomies and the methods of the Lyapunove-Schmidt reduction, he obtained the bifurcation function which are Melnikov types integrals. The zeros of the function correspond to the persistence of the homolicnic orbit for the perturbed system. Also by the Shadowing Lemma, Palmer proved that the persistent homoclinic orbit is transversal. Hence the periodic map of the perturbed system exhibits chaotic motion.

In 1984, Hale [12] suggested a further extension of the functional method to a more general case where the unperturbed equation has a degenerate homoclinic orbit. In 1992, Gruendler [8] studied the persistence of the homoclinic orbit for an autonomous ordinary differential equation with an autonomous perturbation in $\mathbb{R}^N$. He assumed the autonomous system has a homoclinic solution and the variational equation has d bounded solutions, $d \geqslant 1$. By using functional methods, he obtained the bifurcation function which deponed on d dimension independent parameters except for the perturbation parameters. The low order term of the bifurcation function are also Melnikov types integrals. He expanded the bifurcation function about those parameters to the second derivative by Taylor's Theorem. And by a final application of the Implicit Function Theorem, he proved the bifurcation function has a zero. Therefore the perturbed system has a homoclinic orbit. In 1995, Gruendler [9] generalized this result to the periodic perturbed ordinary differential equation. In 1996, Gruendler [10] showed that the variational equation alone the perturbed homoclinic orbit has no nozero bounded solution by the exponential dichotomies and Lyapunov-Schmidt reduction. Hence the perturbed system exhibits chaos.

In 1990, Palmer [19] considered the bifurcation of the degenerate homoclinic orbit under the periodic perturbation. He assumed unperturbed equation has a family of homoclinic orbits which depend on two parameter family. In 1992, Battelli and Palmer [2] given us one way of degenerate homoclinic orbit. That is the intersection of the stable and unstable manifold have branches which is a two dimensional homoclinic solution manifold. And this can occur in the integrable Hamiltonian system [16]. Moreover in [20], Zhu and Zhang investigated the bifurcations of a degenerate homoclinic loop in $\mathbb{R}^N$. They obtained an invariant manifold of a definite dimension bifurcated from the degenerate homoclinic orbit.

The definition of the degenerate homoclinic orbit is equivalent to require the tangent space of $W^s(0)$ and $W^u(0)$ at least have a two dimension intersection. In certain integrable Hamiltonian system with more than two degrees of freedom, the corresponding stable and unstable manifold can coincide or intersect in a more than two dimension submanifold. In this paper we suppose $W^s(0) \bigcap W^u(0)$ have a branch which is a d dimension homoclinic solution manifold denoted by W^H. Let $(t, \alpha) \in \mathbb{R} \times \mathbb{R}^{d-1}$, $d > 1$, be the local coordinates on W^H. For each $\alpha \in \mathbb{R}^{d-1}$, $d > 1$, let $\gamma(t, \alpha)$ are homoclinic orbits which asymptotic to the hyperbolic equilibrium 0. We will using the higher order melnikov types integrals to study the bifurcation of the homoclinic manifold W^H under the periodic perturbation. Meantime many authors studied the homoclinic orbit bifurcation and limit cycle bifurcations by high order Melnikov method [1, 4–7, 13, 14].

Consider the following system:

$$\dot{x}(t) = f(x(t)) + \mu g(x(t), \mu, t), \quad x \in \mathbb{R}^n,\ \mu \in \mathbb{R}, \tag{1.1}$$

we make the following assumptions:

(H1) $f \in C^3$.

(H2) $f(0) = 0$ and the eigenvalues of $Df(0)$ lie off the imaginary axis.

(H3) Unperturbed equation $\dot{x}(t) = f(x(t))$ has a homoclinic solution manifold W^H which can be parameterized by $\gamma(t, \alpha)$, where $(t, \alpha) \in \mathbb{R} \times \mathbb{R}^{d-1}$, $d > 1$, $\gamma \in C^3$.

$$\dot{\gamma}(t, \alpha) = f(\gamma(t, \alpha)) \text{ and } \lim_{t \to \pm\infty} \gamma(t, \alpha) = 0,$$

which uniformly with respect to $\alpha \in \mathbb{R}^{d-1}, d > 1$.

(H4) $g \in C^3,\ g(0, \mu, t) = 0$ and $g(x, \mu, t + 2) = g(x, \mu, t)$.

Clearly by $(H3)$, the homoclinic solution $\gamma(t, \alpha)$ lies on $W^s(0) \bigcap W^u(0)$. The variational equation of unperturbed equation of (1.1) along the homoclinic solution $\gamma(t, \alpha)$ is

$$\dot{u}(t) = Df(\gamma(t, \alpha))u(t). \tag{1.2}$$

From $\dot{\gamma}(t, \alpha) = f(\gamma(t, \alpha))$, we can take derivative with respect to variables t and α_i, for $i = 1, ...d - 1$. Then we can get $\dot{\gamma}(t, \alpha)$, $\frac{\partial \gamma}{\partial \alpha_i}(t, \alpha)$ are solutions of system (1.2), $i = 1, ..., d - 1$. For simplicity, let

$$\dot{\gamma}(t, \alpha) = u_1(t, \alpha), \frac{\partial \gamma}{\partial \alpha_i}(t, \alpha) = u_{i+1}(t, \alpha),$$

$i = 1, ..., d - 1$. We assume u_i satisfies

(H5) $u_1(t, \alpha), u_2(t, \alpha), ..., u_d(t, \alpha)$ are linearly independent and uniformly bounded about $(t, \alpha) \in \mathbb{R} \times \mathbb{R}^{d-1}$, $d > 1$.

We will investigate the homoclinic bifurcations of (1.1) near the homoclinic manifold $\gamma(t, \alpha)$ by the higher order melnikov integrals. This paper is organized as following. In section 2, we list some properties of fundamental solutions of the variational equation along the homoclinic manifold $\gamma(t, \alpha)$. The main result is presented. In section 3, we prove the main result. By using the functional analytic method, we obtained the bifurcation function. And expanded it about d dimensional parameters to the third order derivatives by Taylor's Theorem. We obtained the lower order term of the bifurcation function which denote by M. Take a fixed point (α_0, θ_0) on the homoclinic manifold W^H, if M has a simple zero, then the system (1.1) has a homoclinic solutions near $\gamma(t, \alpha_0)$. In section 4, we prove the homoclinic orbit bifurcated from the homoclinic manifold W^H is transversal. Hence the periodic map of the system (1.1) can have chaotic motion.

2. Preliminaries and Main result

For each $\alpha \in \mathbb{R}^{d-1}$, $d > 1$, since $\gamma(t, \alpha)$ is a homoclinic orbit which asymptotic to the hyperbolic equilibrium 0. Hence, from [9, 18], variational equation (1.2) has exponential dichotomies on $J = \mathbb{R}^{\pm}$ respectively. Let $U(t,\alpha)$ be the fundamental solution matrix of the system (1.2). In particular, there exist projections to the stable and unstable subspaces, $P_s + P_u = I$, and constants $m > 0$, $K_0 \geq 1$ which are uniformly with respect to $\alpha \in \mathbb{R}^{d-1}$, such that

$$\begin{aligned} (i)\ \ & |U(t,\alpha)P_s U^{-1}(s,\alpha)| \leq K_0 e^{2m(s-t)}, \quad \text{for } s \leqslant t \text{ on } J, \\ (ii)\ \ & |U(t,\alpha)P_u U^{-1}(s,\alpha)| \leq K_0 e^{2m(t-s)}, \quad \text{for } t \leqslant s \text{ on } J. \end{aligned} \tag{2.1}$$

By $H(3)$, we know $u_1, u_2, ..., u_d$ are linearly independent bounded solutions of system (1.2). Hence $d = dim(T_{\gamma(0,\alpha)}W^s(0) \bigcap T_{\gamma(0,\alpha)}W^u(0))$.

Take the same m in (2.1), define the Banach space

$$\mathcal{Z} = \{z \in C^1(\mathbb{R}, \mathbb{R}^n) : \sup_{t\in\mathbb{R}} |z(t)|e^{m|t|} < \infty\},$$

with the norm $\|z\| = \sup_{t\in\mathbb{R}} |z(t)|e^{m|t|}$. The linear variational system

$$L_\alpha(u) := \dot{u} - Df(\gamma(t,\alpha))u = h \tag{2.2}$$

will be considered in $\mathcal{Z}$. The adjoint operator for L_α is

$$L^*_\alpha(\psi) := \dot{\psi} + (Df(\gamma(t,\alpha)))^*\psi. \tag{2.3}$$

The domains of (2.2) and (2.3) are the dense subset of $\mathcal{Z}$, defined as

$$D(L_\alpha) := \{u(t,\alpha) : u, u_t, u_\alpha \in \mathcal{Z}\}, \quad D(L^*_\alpha) := \{\psi(t,\alpha) : \psi, \psi_t, \psi_\alpha \in \mathcal{Z}\}.$$

From the theory of homoclinic bifurcation [18], $\gamma(t,\alpha)$ asymptotic to the hyperbolic equilibrium $x = 0$, so $L_\alpha : \mathcal{Z} \to \mathcal{Z}$ be Fredholm operators with index 0, for $\alpha \in \mathbb{R}^{d-1}$. And we have

$$h \in R(L_\alpha) \text{ iff } \int_{-\infty}^{\infty} \langle \psi, h\rangle dt = 0, \text{ for all } \psi \in N(L^*_\alpha). \tag{2.4}$$

From $H(3)$, we know $N(L_\alpha)$ is d dimensional. Let $(u_1(t,\alpha), ..., u_d(t,\alpha))$ be an orthonormal unit basis of $N(L_\alpha)$ and $(\varphi_1(t,\alpha), ..., \varphi_d(t,\alpha))$ be an orthonormal unit basis of $N(L^*_\alpha)$.

By the defination of $u_j(t,\alpha)$, we take derivatives about t and α on both side of (1.2). So we have

$$\begin{aligned} &\ddot{u}_j(t,\alpha) = Df(\gamma(t,\alpha))\dot{u}_j(t,\alpha) + D_{11}f(\gamma(t,\alpha))u_j(t,\alpha)u_1(t,\alpha), \\ &\frac{d}{dt}(\frac{\partial u_j}{\partial \alpha_{k-1}}(t,\alpha)) = Df(\gamma(t,\alpha))\frac{\partial u_j}{\partial \alpha_{k-1}}(t,\alpha) + D_{11}f(\gamma(t,\alpha))u_j(t,\alpha)u_k(t,\alpha), \end{aligned} \tag{2.5}$$

where $j = 1, ..., d, k = 2, ..., d$. Let

$$v_j(t,\alpha) = \dot{u}_j(t,\alpha) \text{ or } \frac{\partial u_j}{\partial \alpha_{k-1}}(t,\alpha) \tag{2.6}$$

So from (2.5) , we have

$$L_\alpha(v_j(t,\alpha)) = D_{11}f(\gamma(t,\alpha))u_j(t,\alpha)u_k(t,\alpha), \tag{2.7}$$

$j, k = 1, ..., d$.

We define some Melnikov types of integrals that will be used in the future. For integers i, j, k, l from the set $\{1, ..., d\}$, let

$$a^{(i)}(\alpha, \theta) = \int_{-\infty}^{+\infty} \langle \psi_i(t, \alpha), g(\gamma(t, \alpha), 0, t+\theta) \rangle dt; \tag{2.8}$$

$$\begin{aligned} c^{(i)}_{jkl}(\alpha) = & \int_{-\infty}^{+\infty} \langle \psi_i(t, \alpha), D_{11} f(\gamma(t, \alpha)) u_j(t, \alpha) v_k(t, \alpha)) \rangle dt \\ & + \int_{-\infty}^{+\infty} \langle \psi_i(t, \alpha), D_{11} f(\gamma(t, \alpha)) u_k(t, \alpha) v_l(t, \alpha)) \rangle dt \\ & + \int_{-\infty}^{+\infty} \langle \psi_i(t, \alpha), D_{11} f(\gamma(t, \alpha)) u_l(t, \alpha) v_j(t, \alpha)) \rangle dt \\ & + \int_{-\infty}^{+\infty} \langle \psi_i(t, \alpha), D_{111} f(\gamma(t, \alpha)) u_j(t, \alpha) u_k(t, \alpha) u_l(t, \alpha) \rangle dt, \end{aligned} \tag{2.9}$$

where $v_j(t, \alpha)$ be as in (2.6).

Define $M : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^{d-1} \times \mathbb{R} \to \mathbb{R}^d$ be given by

$$M(\boldsymbol{\beta}, \mu, \alpha, \theta) = (M_1(\boldsymbol{\beta}, \mu, \alpha, \theta), ..., M_d(\boldsymbol{\beta}, \mu, \alpha, \theta)),$$

where

$$M_i(\boldsymbol{\beta}, \mu, \alpha, \theta) = a^{(i)}(\alpha, \theta)\mu + \frac{1}{6} \sum_{j=1}^{d} \sum_{k=1}^{d} \sum_{l=1}^{d} c^{(i)}_{jkl}(\alpha) \beta_j \beta_k \beta_l.$$

Theorem 2.1. *Assume that* $(H1)$–$(H5)$ *hold. If there are some fixed* $(\boldsymbol{\beta}_0, \mu_0, \alpha_0, \theta_0)$ *such that* $M(\boldsymbol{\beta}_0, \mu_0, \alpha_0, \theta_0) = 0$ *and* $D_1 M(\boldsymbol{\beta}_0, \mu_0, \alpha_0, \theta_0)$ *is a nonsingular* $d \times d$ *matrix. Then there exits an open interval* $I \in \mathbb{R}$ *which contain zeros, such that* (1.1) *with* $\mu = s^3 u_0$ *has a homoclinic solutions* γ_s *near* $\gamma(t, \alpha_0)$ *for* $s \in I$.

Moreover γ_s *is transverse for* $s \in I \setminus \{0\}$.

3. The proof of the main result

By (H3), system (1.1) with $\mu = 0$ has a homoclinic solution manifold W^H which can be parameterized by $\gamma(t, \alpha)$. In this section, we will find conditions such that (1.1), with small $\mu \neq 0$, has homoclinic solution γ_μ near W^H for some α_0.

Let $D_i h$, $D_{ij} h$ or $D_{ijk} h$ denote the derivatives of a multivariate function h with respect to its i-th , i, j-th or the i, j, k-th variables. And suppose $d > 1$ in the rest of the paper. With the change of variable

$$x(t+\theta) = \gamma(t, \alpha) + z(t),$$

then (1.1) is transformed to

$$\dot{z} = Df(\gamma(t, \alpha))z + \widetilde{g}(z, \mu, \alpha, \theta), \tag{3.1}$$

where

$$\begin{aligned} \widetilde{g}(z, \mu, \alpha, \theta)(t) = & f(\gamma(t, \alpha) + z(t)) - f(\gamma(t, \alpha)) - Df(\gamma(t, \alpha))z \\ & + \mu g(\gamma(t, \alpha) + z(t), \mu, t+\theta). \end{aligned} \tag{3.2}$$

As in [9], one can prove that $\widetilde{g}$ satisfies the following properties:

Lemma 3.1. *The function* $\widetilde{g}(\cdot,\mu,\alpha,\theta):\mathcal{Z}\mapsto\mathcal{Z}$ *satisfies the following properties:*

(1) $\widetilde{g}(0,0,\alpha,\theta)=0$,
(2) $D_1\widetilde{g}(0,0,\alpha,\theta)=0, \quad D_2\widetilde{g}(0,0,\alpha,\theta)=g(\gamma(t,\alpha),0,t+\theta)$,
(3) $D_{11}\widetilde{g}(0,0,\alpha,\theta)=D_{11}f(\gamma(t,\alpha))$,
(4) $D_{111}\widetilde{g}(0,0,\alpha,\theta)=D_{111}f(\gamma(t,\alpha))$.

For any $\alpha\in\mathbb{R}^{d-1}, i=1,...,d$, we define the subspace of $\mathcal{Z}$ which consists the range of L_α by the method in [9,18]. Let

$$\widetilde{\mathcal{Z}}=\{h\in\mathcal{Z}:\int_{-\infty}^{\infty}\langle\psi_i(s,\alpha),h(s)\rangle ds=0, i=1,...,d\}.$$

Consider a nonhomogeneous equation

$$\dot{z}-Df(\gamma(t,\alpha))z=h. \tag{3.3}$$

If $h\in\widetilde{\mathcal{Z}}$, using the variation of constants, with some phase condition, there exists an operator $K:\widetilde{\mathcal{Z}}\to\mathcal{Z}$ such that Kh is a solution of (3.3). Clearly, the general bounded solution of (3.3) is

$$z(t)=\sum_{i=1}^{d}\beta_i u_i(t,\alpha)+(Kh)(t),$$

where $\beta_i\in\mathbb{R}$.

Recall that L_α be a family Fredholm operators with index 0 which independent of α. And we know $\dim N(L_\alpha)=\dim(\mathcal{Z}/R(L_\alpha))=d$. So we suppose $\varphi_1(t,\alpha),...,\varphi_d(t,\alpha)$ be an orthonormal unit basis of $\mathcal{Z}/R(L_\alpha)$. Define a map $P:\mathcal{Z}\to\mathcal{Z}$ by

$$(Pz)(t)=\sum_{i=1}^{d}\varphi_i(t,\alpha)\int_{-\infty}^{\infty}\langle\psi_i(s,\alpha),z(s)\rangle ds,$$

which $\varphi_i(s,\alpha)$ satisfying $\langle\psi_i(s,\alpha),\varphi_j(s,\alpha)\rangle=\delta_{ij}$, for $\alpha\in\mathbb{R}^{d-1}$, δ_{ij} be the Kronecker delta.

As in [18], one can prove that

Lemma 3.2. *The map* P *satisfies the following properties:*
(1) P *and* $I-P$ *are projections;*
(2) $R(P)\oplus R(L_\alpha)=\mathcal{Z}$*;*
(3) $R(I-P)=N(P)=R(L_\alpha)=\widetilde{\mathcal{Z}}$.

We now use the Lyapunov-Schmidt reduction to solve (3.1). Applying P and $(I-P)$ on (3.1), we find that (3.1) is equivalent to the following system

$$\dot{z}=Df(\gamma(t,\alpha))z-(I-P)\widetilde{g}(z,\mu,\alpha,\theta), \tag{3.4}$$
$$P\widetilde{g}(z,\mu,\alpha,\theta)=0. \tag{3.5}$$

First, we solve (3.4) for $z\in\mathcal{Z}$. Then the bifurcation equations are obtained by substituting the solution z into (3.5).

Lemma 3.3. *There exist open balls* $B_1(\delta_0) \subset \mathbb{R}^d$, $B_2(\delta_0) \subset \mathbb{R}$ *with radius* $\delta_0 > 0$ *centered at the origins and a* C^2 *map* $\phi : B_1(\delta_0) \times B_2(\delta_0) \times \mathbb{R}^{d-1} \times \mathbb{R} \to \mathcal{Z}$, *denoted by* $\phi(\boldsymbol{\beta}, \mu, \alpha, \theta)$, *such that* $z = \phi(\boldsymbol{\beta}, \mu, \alpha, \theta)$ *is a solution of equation* (3.4). *Moreover for any* $(\alpha, \theta) \in \mathbb{R}^{d-1} \times \mathbb{R}$, $\phi(\boldsymbol{\beta}, \mu, \alpha, \theta)$ *satisfies:*

(1) $\phi(0, 0, \alpha, \theta) = 0$;

(2) $\frac{\partial \phi}{\partial \beta_i}|_{(0,0,\alpha,\theta)} = u_i(t, \alpha),\ i = 1, ..., d;$

(3) $\frac{\partial^2 \phi}{\partial \beta_i \partial \beta_j}|_{(0,0,\alpha,\theta)} = K(I - P)D_{11}f(\gamma(t, \alpha))u_i(t, \alpha)u_j(t, \alpha), i, j = 1, ..., d.$

Proof. Since $R(I - P) = \widetilde{\mathcal{Z}}$ and $K : \widetilde{\mathcal{Z}} \to \mathcal{Z}$, we define a C^2 map: $F : \mathcal{Z} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^{d-1} \times \mathbb{R} \to \mathcal{Z}$ by

$$F(z, \boldsymbol{\beta}, \mu, \alpha, \theta)(t) = \sum_{i=1}^{d} \beta_i u_i(t, \alpha) + K(I - P)\widetilde{g}(z, \mu, \alpha, \theta)(t), \tag{3.6}$$

where $\boldsymbol{\beta} = (\beta_1, ..., \beta_d) \in \mathbb{R}^d$. Clearly, the fixed point z of (3.6) is a solution of (3.4) in $\mathcal{Z}$.

For any $(\alpha, \theta) \in \mathbb{R}^{d-1} \times \mathbb{R}$, from (1) of Lemma 3.1 and the definition of $\widetilde{g}$ in (3.2) , we have

$$F(0, 0, 0, \alpha, \theta) = 0, \quad D_1F(0, 0, 0, \alpha, \theta) = 0. \tag{3.7}$$

By the smoothness of F, given any $\delta > 0$, there exists $c > 0$ such that

$$\|D_2F\| < c, \quad \|D_3F\| < c, \quad \|D_{11}F\| < c, \quad \|D_{12}F\| < c, \quad \|D_{13}F\| < c,$$

for $(z, \boldsymbol{\beta}, \mu, \alpha, \theta) \in \bar{B}(\delta) \times \bar{B}_1(\delta) \times \bar{B}_2(\delta) \times \mathbb{R}^{d-1} \times \mathbb{R}$, where $\bar{B}(\delta) \subset \mathcal{Z}$, $\bar{B}_1(\delta) \subset \mathbb{R}^d$, $\bar{B}_2(\delta) \subset \mathbb{R}$ are closed balls of radius δ. Let

$$\delta_1 = min\{\delta, \frac{1}{4c}\}, \quad \delta_2 = min\{\delta, \delta_1, \frac{\delta_1}{8c}\}.$$

For any $(z, \boldsymbol{\beta}, \mu, \alpha, \theta) \in \bar{B}(\delta_1) \times \bar{B}_1(\delta_2) \times \bar{B}_2(\delta_2) \times \mathbb{R}^{d-1} \times \mathbb{R}$, define a map $\varphi_1 : [0, 1] \to \mathcal{L}(\mathcal{Z}, \mathcal{Z})$ by $\varphi_1(s) = D_1F(sz, s\boldsymbol{\beta}, s\mu, \alpha, \theta)$. By the smoothness of F, we see $\varphi_1 \in C^1$. By (3.7) we know $\varphi_1(0) = 0$. Then there exists $s_1 \in (0, 1)$, such that

$$\begin{aligned}\|D_1F(z, \boldsymbol{\beta}, \mu, \alpha, \theta)\| &= \|\varphi_1(1) - \varphi_1(0)\| = \|\varphi_1'(s_1)\| \\ &\leqslant \|D_{11}F(s_1z, s_1\boldsymbol{\beta}, s_1\mu, \alpha, \theta)\| \cdot \|z\| \\ &\quad + \|D_{12}F(s_1z, s_1\boldsymbol{\beta}, s_1\mu, \alpha, \theta)\| \cdot \|\boldsymbol{\beta}\| \\ &\quad + \|D_{13}F(s_1z, s_1\boldsymbol{\beta}, s_1\mu, \alpha, \theta)\| \cdot \|\mu\| \\ &\leqslant c \cdot \frac{1}{4c} + c \cdot \frac{1}{4c} + c \cdot \frac{1}{4c} = \frac{3}{4}.\end{aligned} \tag{3.8}$$

For $(z, \boldsymbol{\beta}, \mu, \alpha, \theta) \in \bar{B}(\delta_1) \times \bar{B}_1(\delta_2) \times \bar{B}_2(\delta_2) \times \mathbb{R}^{d-1} \times \mathbb{R}$, define a map $\varphi_2 : [0, 1] \to \mathcal{Z}$ by $\varphi_2(s) = F(sz, s\boldsymbol{\beta}, s\mu, \alpha, \theta)$. Clearly $\varphi_2 \in C^1$ and $\varphi_2(0) = 0$. Then there exists $s_2 \in (0, 1)$ such that

$$\|F(z, \boldsymbol{\beta}, \mu, \alpha, \theta)\| = \|\varphi_2(1) - \varphi_2(0)\| = \|\varphi_2'(s_2)\|$$

$$\begin{aligned}&\leqslant \|D_1F(s_2z, s_2\boldsymbol{\beta}, s_2\mu, \alpha, \theta)\| \cdot \|z\| \\ &\quad + \|D_2F(s_2z, s_2\boldsymbol{\beta}, s_2\mu, \alpha, \theta)\| \cdot \|\boldsymbol{\beta}\| \\ &\quad + \|D_3F(s_2z, s_2\boldsymbol{\beta}, s_2\mu, \alpha, \theta)\| \cdot \|\mu\| \\ &\leqslant \frac{3}{4}\delta_1 + c \cdot \frac{\delta_1}{8c} + c \cdot \frac{\delta_1}{8c} = \delta_1,\end{aligned}$$

which implies that $F(\cdot, \boldsymbol{\beta}, \mu, \alpha, \theta)$ maps $\bar{B}(\delta_1)$ into itself.

For $z_1, z_2 \in \bar{B}(\delta_1)$, $(\boldsymbol{\beta}, \mu, \alpha, \theta) \in \bar{B}_1(\delta_2) \times \bar{B}_2(\delta_2) \times \mathbb{R}^{d-1} \times \mathbb{R}$, define a map $\varphi_3 : [0,1] \to \mathcal{Z}$ by $\varphi_3(s) = F(sz_1 + (1-s)z_2, \boldsymbol{\beta}, \mu, \alpha, \theta)$. Then $\varphi_3 \in C^2$, there exists $s_3 \in (0,1)$, such that

$$\begin{aligned}&\|F(z_1, \boldsymbol{\beta}, \mu, \alpha, \theta) - F(z_2, \boldsymbol{\beta}, \mu, \alpha, \theta)\| \\ =&\|\varphi_3(1) - \varphi_3(0)\| = \|\varphi_3'(s_3)\| \\ \leqslant&\|D_1F(s_3z_1 + (1-s_3)z_2^{(k)}, \boldsymbol{\beta}, \mu, \alpha, \theta)\| \cdot \|z_1 - z_2\| \\ \leqslant&\frac{3}{4}\|z_1 - z_2\|.\end{aligned}$$

Therefore F is a uniform contraction in $\bar{B}(\delta_1)$. By the contraction mapping principle, there are $\delta_{21}, \delta_{22} > 0$ and a C^2 map $\phi : B_1(\delta_{21}) \times B_2(\delta_{22}) \times \mathbb{R}^{d-1} \times \mathbb{R} \to B(\delta_1)$ such that $\phi(0,0,\alpha,\theta) = 0$, $(\alpha, \theta) \in \mathbb{R}^{d-1} \times \mathbb{R}, d > 1$ and

$$\phi(\boldsymbol{\beta}, \mu, \alpha, \theta) = F(\phi(\boldsymbol{\beta}, \mu, \alpha, \theta), \boldsymbol{\beta}, \mu, \alpha, \theta).$$

Let $\delta_0 = \min\{\delta_2, \delta_{21}, \delta_{22}\}$. From (3.6), we have

$$\phi(\boldsymbol{\beta}, \mu, \alpha, \theta)(t) = \sum_{i=1}^{d} \beta_i u_i(t, \alpha) + K(I-P)\widetilde{g}(\phi(\boldsymbol{\beta}, \mu, \alpha, \theta), \mu, \alpha, \theta). \tag{3.9}$$

Differentiating (3.9) with respect to β_i and the second derivative about β_i, β_j, we can get

$$\begin{aligned}\frac{\partial \phi}{\partial \beta_i} =&u_i(t,\alpha) + K(I-p)D_1\widetilde{g}(\phi, \mu, \alpha, \theta)\frac{\partial \phi}{\partial \beta_i}, \\ \frac{\partial^2 \phi}{\partial \beta_i \beta_j} =&K(I-p)D_{11}\widetilde{g}(\phi, \mu, \alpha, \theta)\frac{\partial \phi}{\partial \beta_i}\frac{\partial \phi}{\partial \beta_j} \\ &+ K(I-p)D_1\widetilde{g}(\phi, \mu, \alpha, \theta)\frac{\partial^2 \phi}{\partial \beta_i \partial \beta_j},\end{aligned}$$

where $i, j = 1, ..., d$. By Lemma 3, evaluating above formulas at $(0,0,\alpha,\theta)$, we get

$$\begin{aligned}&\left.\frac{\partial \phi}{\partial \beta_i}\right|_{(0,0,\alpha,\theta)}(t) = u_i(t,\alpha), \\ &\left.\frac{\partial^2 \phi}{\partial \beta_i \beta_j}\right|_{(0,0,\alpha,\theta)}(t) = K(I-p)D_{11}f(\gamma(t,\alpha)u_i(t,\alpha)u_j(t,\alpha),\end{aligned}$$

where $i, j = 1, ..., d$. The proof has been completed. □

By Lemma 3.3, (3.4) has a solution $\phi(\boldsymbol{\beta}, \mu, \alpha, \theta)$. Substituting $\phi(\boldsymbol{\beta}, \mu, \alpha, \theta)$ into (3.5), we have the bifurcation equation

$$0 = P\widetilde{g}(\phi(\boldsymbol{\beta}, \mu, \alpha, \theta), \mu, \alpha, \theta)$$

$$= \sum_{i=1}^{d} \varphi_i(t, \alpha) \int_{-\infty}^{+\infty} \langle \psi_i(s, \alpha), \widetilde{g}(\phi(\boldsymbol{\beta}, \mu, \alpha, \theta), \mu, \alpha, \theta)(s) \rangle ds, \tag{3.10}$$

where the definition of projection P is used. By the linear independence of $\psi_1, ..., \psi_d$, (3.10) is equivalent to

$$H_i(\boldsymbol{\beta}, \mu, \alpha, \theta) := \int_{-\infty}^{+\infty} \langle \psi_i(s, \theta), \widetilde{g}(\phi(\boldsymbol{\beta}, \mu, \alpha, \theta), \mu, \alpha, \theta)(s) \rangle ds = 0, \ i = 1, ..., d.$$

If there are some parameter values $(\boldsymbol{\beta}, \mu, \alpha, \theta) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^{d-1} \times \mathbb{R}$ such that

$$H_i(\boldsymbol{\beta}, \mu, \alpha, \theta) = 0, \quad i = 1, ..., d,$$

then $z = \phi$ is a solution of (3.1) and hence the perturbed system (1.1) has a homoclinic orbit $x = \gamma + \phi$, where ϕ is given in (3.9). Let

$$H(\boldsymbol{\beta}, \mu, \alpha, \theta) = (H_1(\boldsymbol{\beta}, \mu, \alpha, \theta), ..., H_d(\boldsymbol{\beta}, \mu, \alpha, \theta)).$$

We have the following Lemma.

Lemma 3.4. *For $i, j, k, l \in \{1, ..., d\}$, the function $H_i(\boldsymbol{\beta}, \mu, \alpha, \theta)$ has the following properties:*

$$(i) \quad H_i(0, 0, \alpha, \theta) = 0;$$
$$(ii) \quad \frac{\partial H_i}{\partial \beta_j}(0, 0, \alpha, \theta) = 0, \qquad \frac{\partial H_i}{\partial \mu}(0, 0, \alpha, \theta) = a^{(i)}(\alpha, \theta);$$
$$(iii) \quad \frac{\partial^2 H_i}{\partial \beta_j \partial \beta_k}(0, 0, \alpha, \theta) = 0;$$
$$(iv) \quad \frac{\partial^3 H_i}{\partial \beta_j \partial \beta_k \partial \beta_l}(0, 0, \alpha, \theta) = c^{(i)}_{jkl}(\alpha),$$

where $a^{(i)}(\alpha, \theta)$ and $c^{(i)}_{jkl}(\alpha)$ be as in (2.8) and (2.9).

Proof. To prove those properties of H_i ,we should take derivatives with respect to μ and β in (3.2) by the formula of H_i, where z is equal to ϕ which is in (3.9). From Lemma 3.1,Lemma 3.2, so it is easy to check (i), (ii) through direct calculations. Next we prove (iii) and (iv).

By the definition of $u_1, ..., u_d$, we have

$$\dot{u}_j(t, \alpha) = Df(\gamma(t, \alpha))u_j(t, \alpha), \tag{3.11}$$

where $j = 1, ..., d$. From Eq.(2.5), we have

$$\ddot{u}_j(t, \alpha) = D_{11}f(\gamma(t, \alpha))u_j(t, \alpha)u_1(t, \alpha) + Df(\gamma(t, \alpha))\dot{u}_j(t, \alpha),$$
$$\frac{d}{dt}(\frac{\partial u_j}{\partial \alpha_{k-1}}(t, \alpha)) = D_{11}f(\gamma(t, \alpha))u_j(t, \alpha)u_k(t, \alpha) + Df(\gamma(t, \alpha))\frac{\partial u_j}{\partial \alpha_{k-1}}(t, \alpha),$$

where $k = 2, ..., d$. So from the above formula, $D_{11}f(\gamma(t, \alpha))u_j(t, \alpha)u_k(t, \alpha)$ is equal to

$$\ddot{u}_j(t, \alpha) - Df(\gamma(t, \alpha))\dot{u}_j(t, \alpha)$$

or

$$\frac{d}{dt}(\frac{\partial u_j}{\partial \alpha_{k-1}}(t,\alpha)) - Df(\gamma(t,\alpha))\frac{\partial u_j}{\partial \alpha_{k-1}}(t,\alpha),$$

where $j, k = 2, ..., d$. Hence

$$D_{11}f(\gamma(t,\alpha))u_j(t,\alpha)u_k(t,\alpha) \in R(L_\alpha) = R(I-P). \tag{3.12}$$

When $k = 1$, we have

$$\begin{aligned}
\frac{\partial^2 H_i}{\partial \beta_j \partial \beta_k}(0,0,\alpha,\theta) &= \int_{-\infty}^{+\infty} \langle \psi_i, D_{11}f(\gamma)u_j u_k\rangle dt \\
&= \int_{-\infty}^{+\infty} \langle \psi_i, \ddot{u}_j - Df(\gamma)\dot{u}_j\rangle dt \\
&= \int_{-\infty}^{+\infty} \langle \psi_i, \ddot{u}_j\rangle dt - \int_{-\infty}^{+\infty} \langle \psi_i, Df(\gamma)\dot{u}_j\rangle dt \\
&= \int_{-\infty}^{+\infty} \langle \psi_i, \ddot{u}_j\rangle dt - \int_{-\infty}^{+\infty} \langle (Df(\gamma))^*\psi_i, \dot{u}_j\rangle dt \\
&= \int_{-\infty}^{+\infty} \langle \psi_i, \ddot{u}_j\rangle dt + \int_{-\infty}^{+\infty} \langle \dot{\psi}_i, \dot{u}_j\rangle dt \\
&= \int_{-\infty}^{+\infty} d\langle \psi_i, \ddot{u}_j\rangle \\
&= \langle \psi_i, \dot{u}_j\rangle|_{-\infty}^{+\infty} \\
&= 0.
\end{aligned}$$

When $k = 2, ..., d$, we can prove it by the similar process. Hence (iii) is proved.

By the definition of K, $K(h)$ can be regard as a particular solution of (3.3) in the formula of the variation of constants. So from (2.5), without loss of generality we can take

$$K(I-P)D_{11}f(\gamma(t,\alpha))u_j(t,\alpha)u_k(t,\alpha) = v_j(t,\alpha),$$

where

$$v_j(t,\alpha) = \dot{u}_j(t,\alpha) \ or \ \frac{\partial u_j}{\partial \alpha_{k-1}},$$

$j = 1, ..., d$, $k = 2, ..., d$. Through direct calculations, we have

$$\begin{aligned}
\frac{\partial^3 H_i}{\partial \beta_j \partial \beta_k \partial \beta_l}(0,0,\alpha,\theta) &= \int_{-\infty}^{+\infty} \langle \psi_i, D_{111}f(\gamma)u_j u_k u_l\rangle dt \\
&\quad + \int_{-\infty}^{+\infty} \langle \psi_i, D_{11}f(\gamma)u_j(K(I-P)D_{11}f(\gamma)u_k u_l)\rangle dt \\
&\quad + \int_{-\infty}^{+\infty} \langle \psi_i, D_{11}f(\gamma(t,\alpha))u_k(K(I-P)D_{11}f(\gamma)u_j u_l)\rangle dt \\
&\quad + \int_{-\infty}^{+\infty} \langle \psi_i, D_{11}f(\gamma)u_l(K(I-P)D_{11}f(\gamma)u_j u_k)\rangle dt \\
&= \int_{-\infty}^{+\infty} \langle \psi_i, D_{11}f(\gamma)u_j v_k)\rangle dt + \int_{-\infty}^{+\infty} \langle \psi_i, D_{11}f(\gamma)u_k v_j)\rangle dt
\end{aligned}$$

$$+\int_{-\infty}^{+\infty}\langle\psi_i, D_{11}f(\gamma)u_l v_j\rangle\rangle dt+\int_{-\infty}^{+\infty}\langle\psi_i, D_{111}f(\gamma)u_j u_k u_l\rangle dt$$
$$=c_{jkl}^{(i)}(\alpha).$$

The proof is completed. $\square$

Let $M:\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^{d-1}\times\mathbb{R}\to\mathbb{R}^d$ be given by

$$M(\boldsymbol{\beta},\mu,\alpha,\theta)=(M_1(\boldsymbol{\beta},\mu,\alpha,\theta),...,M_d(\boldsymbol{\beta},\mu,\alpha,\theta)),$$

where

$$M_i(\boldsymbol{\beta},\mu,\alpha,\theta)=a^{(i)}(\alpha,\theta)\mu+\frac{1}{6}\sum_{j=1}^{d}\sum_{k=1}^{d}\sum_{l=1}^{d}c_{jkl}^{(i)}(\alpha)\beta_j\beta_k\beta_l.$$

So

$$H(\boldsymbol{\beta},\mu,\alpha,\theta)=M(\boldsymbol{\beta},\mu,\alpha,\theta)+H.O.T.$$

Lemma 3.5. *If there are some fixed $(\boldsymbol{\beta}_0,\mu_0,\alpha_0,\theta_0)$ such that $M(\boldsymbol{\beta}_0,\mu_0,\alpha_0,\theta_0)=0$ and $D_1M(\boldsymbol{\beta}_0,\mu_0,\alpha_0,\theta_0)$ is a nonsingular $d\times d$ matrix. Then there exits an open interval $I\in\mathbb{R}$ which contain zeros and a differentiable functions $\delta:I\to\mathbb{R}^d$ with $\delta(0)=0$, such that*

$$H(s(\boldsymbol{\beta}_0+\delta(s)),s^3\mu_0,\alpha_0,\theta_0)=0,$$

for $s\in I$.

Proof. Define a C^2 function $W:\mathbb{R}^d\times\mathbb{R}\mapsto\mathbb{R}^2$ by

$$W(x,s)=\begin{cases}\dfrac{1}{s^3}H(s(\boldsymbol{\beta}_0+x),s^3\mu_0,\alpha_0,\theta_0), & \text{for } s\neq 0,\\ M(\boldsymbol{\beta}_0+x,\mu_0,\alpha_0,\theta_0), & \text{for } s=0.\end{cases}$$

Clearly, $H=0$ if and only if $W=0$ for $s\neq 0$. Through direct calculations, we have $W(0,0)=0$ and $D_xW(0,0)=D_{\boldsymbol{\beta}}M(\boldsymbol{\beta}_0,\mu_0,\alpha_0,\theta_0)$ is a nonsingular matrix. By the implicit function theorem there exist an open region $I\subset\mathbb{R}$ containing zero and a differentiable functions, $\delta:I\to\mathbb{R}^d$ with $\delta(0)=0$ such that $W(\delta(s),s)=0$ for $s\in I$. Hence we have

$$H(s(\boldsymbol{\beta}_0+\delta(s)),s^3\mu_0,\alpha_0,\theta_0)=0 \quad\text{for}\quad s\neq 0.$$

The proof has been completed. $\square$

By Lemma 3.5, the bifurcation function H vanishes at $\boldsymbol{\beta}=s(\boldsymbol{\beta}_0+\delta(s))$ and $\mu=s^3\mu_0$. Then system (3.1) has the solution $\phi(s(\boldsymbol{\beta}_0+\delta(s)),s^3\mu_0,\alpha_0,\theta_0)$. Hence system (1.1) have homoclinic solutions given by

$$\begin{aligned}\gamma_s(t)=&\gamma(t,\alpha_0)+\sum_{i=1}^{d}s(\beta_{0i}+\delta_i(s))u_i(t,\alpha_0)\\&+K(I-P)\widetilde{g}(\phi,s^3\mu_0,\alpha_0,\theta_0)(t),\end{aligned}\tag{3.13}$$

for $s\in I$. Clearly, $\lim_{s\to 0}\gamma_s(t,\alpha_0)=\gamma(t,\alpha_0)$.

4. The Transversality

In this section we will prove that the homoclinic solutions $\gamma_s(t)$ are transverse for $0 \neq s \in I$. From Section three, the solution $\gamma_s(t)$ satisfying

$$\dot{\gamma}_s(t) = f(\gamma_s(t, \alpha_0)) + s^3\mu_0 g(\gamma_s(t), s^3\mu_0, t + \theta_0). \tag{4.1}$$

From (3.13), through calculations, we have

$$\frac{\partial \gamma_s}{\partial s}|_{s=0} = \sum_{j=1}^{d} \beta_{0j} u_j(t, \alpha_0)$$

and

$$\begin{aligned}\frac{\partial^2 \gamma_s}{\partial s^2}|_{s=0} = K(I-P)D_{11}f(\gamma(t,\alpha_0)\sum_{j=1}^{d}\sum_{k=1}^{d}\beta_{0j}\beta_{0k}u_j(t,\alpha_0)u_k(t,\alpha_0)\\ + 2\sum_{j=1}^{d} D\delta_j(0)u_j(t,\alpha_0).\end{aligned} \tag{4.2}$$

Differentiating with respect to t in (4.1), we have

$$\begin{aligned}\ddot{\gamma}_s(t) =&(Df(\gamma_s(t)) + s^3\mu_0 D_1 g(\gamma_s(t), s^3\mu_0, t+\theta_0))\dot{\gamma}_s(t)\\ &+ s^3\mu_0 D_3 g(\gamma_s(t), s^3\mu_0, t+\theta_0).\end{aligned}$$

Due to the extra term $s^3\mu_0 D_3 g(\gamma_s(t), s^3\mu_0, t+\theta_0)$, $\dot{\gamma}_s(t)$ is not a solution of the variational equation along $\gamma_s(t)$:

$$\dot{u}(t) = (Df(\gamma(t,\alpha_0) + S(s)(t))u(t), \tag{4.3}$$

where

$$S(s)(t) = Df(\gamma_s(t)) + s^3\mu_0 D_1 g(\gamma_s(t), s^3\mu_0, t+\theta_0) - Df(\gamma(t,\alpha_0)).$$

Note that $S(0) = 0$, $DS(0) = D_{11}f(\gamma(t,\alpha_0))\sum_{j=1}^{d}\beta_{0j}u_j(t,\alpha_0)$ and

$$\begin{aligned}D_{11}S(0) =&D_{11}f(\gamma(t,\alpha_0))(2\sum_{j=1}^{d} D\delta_j(0)u_j(t,\alpha_0)\\ &+ K(I-p)D_{11}f(\gamma(t,\alpha_0))\sum_{j=1}^{d}\beta_{0j}u_j(t,\alpha_0)\sum_{k=1}^{d}\beta_{0k}u_k(t,\alpha_0))\\ &+ D_{111}f(\gamma(t,\alpha_0))\sum_{j=1}^{d}\beta_{0j}u_j(t,\alpha_0)\sum_{k=1}^{d}\beta_{0k}u_k(t,\alpha_0),\end{aligned} \tag{4.4}$$

where (4.2) be used.

From the exponential dichotomy theory in [18], if we proved that the variational equation of (4.3) has no nonzero bounded solution, then the equation of (4.3) has an exponential dichotomy on $\mathbb{R}$. Hence the perturbed stable and unstable manifolds intersect transversely. So next we will prove the variational equation of (4.3) has no nonzero bounded solution as the similar methods in [10]. Applying the projections P and $(I-P)$ on equation (4.3), we have

$$\dot{u}(t) = Df(\gamma(t,\alpha_0))u + (I-P)S(s)(t)u(t), \tag{4.5}$$
$$0 = PS(s)(t)u(t). \tag{4.6}$$

The general bounded solution $\hat{u}$ of (4.5) has the following form

$$\hat{u}(t) = \sum_{l=1}^{d} a_l u_l(t,\alpha_0) + K(I-P)S(s)(t)\hat{u}(t),$$

where $a_l \in \mathbb{R}$, $l = 1,...,d$. By the formula of $S(s)$, we have $S(0) = 0$, there exist a small region $\tilde{I}$ contained zero such that $(I - K(I-P)S(s))$ is invertible for $s \in \tilde{I}$. We get

$$\hat{u} = [I - K(I-P)G(s)]^{-1} \sum_{l=1}^{l} a_l u_l \quad \text{for} \quad s \in \tilde{I}.$$

Substituting $u = \hat{u}$ into equation (4.6), we have

$$\begin{aligned} 0 &= PS(s)[I - K(I-P)S(s)]^{-1} \sum_{l=1}^{d} a_l u_l \\ &= \sum_{l=1}^{d} \varphi_i \int_{-\infty}^{+\infty} \langle \psi_i, S(s)[I - K(I-P)S(s)]^{-1} \sum_{l=1}^{d} a_l u_l \rangle ds \\ &= \sum_{i,l=1}^{d} \varphi_i a_l \int_{-\infty}^{+\infty} \langle \psi_i, S(s)[I - K(I-P)S(s)]^{-1} u_l \rangle ds \\ &= (\varphi_1, ..., \varphi_d) V(S(s))(a_1, ..., a_d), \end{aligned}$$

where matrix $V(S(s))$ is given by $V(S(s)) = [v_{il}(s)]_{d\times d}$ and

$$v_{il}(s) = \int_{-\infty}^{+\infty} \langle \psi_i, S(s)[I - K(I-P)S(s)]^{-1} u_l \rangle dt. \tag{4.7}$$

Note that $\varphi_1, ..., \varphi_d$ are linearly independent. If we can prove that $V(G(s))$ is a nonsingular matrix, then $a_l = 0$, $l = 1, ..., d$. Thus the only bounded solution for the linear variational equation along γ_s is $\hat{u} = 0$. The Shadowing Lemma implies that γ_s is a transverse homoclinic solution of (1.1) and its periodic map exhibits chaotic motion.

Since $S(0) = 0$, we have

$$\begin{aligned} &S(s)[I - K(I-P)S(s)]^{-1} \\ =&DS(0)s + \frac{1}{2} D_{11}S(0)s^2 + DS(0)K(I-P)DS(0)s^2 + O(s^3). \end{aligned} \tag{4.8}$$

By the formulas of $DS(0)$ and $D_{11}S(0)$ in (4.4), we have

$$\begin{aligned}v_{il}(s) =& \int_{-\infty}^{+\infty} \langle \psi_i, S(s)[I-K(I-P)S(s)]^{-1}u_l\rangle dt\\ =&s^2(\frac{1}{2}\sum_{j=1}^{d}\sum_{k=1}^{d}\beta_{0j}\beta_{0k}\int_{-\infty}^{+\infty}\langle\psi_i, D_{111}f(\gamma)u_ju_ku_l\rangle dt\\ &+\frac{1}{2}\sum_{j=1}^{d}\sum_{k=1}^{d}\beta_{0j}\beta_{0k}\int_{-\infty}^{+\infty}\langle\psi_i, D_{11}f(\gamma)u_jv_k\rangle dt\\ &+\frac{1}{2}\sum_{j=1}^{d}\sum_{k=1}^{d}\beta_{0j}\beta_{0k}\int_{-\infty}^{+\infty}\langle\psi_i, D_{11}f(\gamma)u_kv_l\rangle dt\\ &+\frac{1}{2}\sum_{j=1}^{d}\sum_{k=1}^{d}\beta_{0j}\beta_{0k}\int_{-\infty}^{+\infty}\langle\psi_i, D_{11}f(\gamma)u_lv_j\rangle dt) + O(s^3),\end{aligned}$$

where $\int_{-\infty}^{+\infty}\langle\psi_i, D_{11}f(\gamma)u_ju_k\rangle dt = 0$ be used. We have the following approximation of $v_{il}(s)$:

$$v_{il}(s) = \frac{1}{2}s^2\sum_{j=1}^{d}\sum_{k=1}^{d}\beta_{0j}\beta_{0k}c_{jkl}^{(i)}(\alpha_0) + O(s^3), \tag{4.9}$$

where $i, l = 1, ..., d$. Therefore

$$\det(V(S(s))) = s^{2d}\det\left(D_1M(\boldsymbol{\beta}_0, \mu_0, \alpha_0, \theta_0)\right) + O(s^{2d+1}).$$

Note that $D_1M(\boldsymbol{\beta}_0, \mu_0, \alpha_0, \theta_0)$ is nonsingular. Then there exists a region $\hat{I}$, $\hat{I} \subset \tilde{I}$ such that $V(S(s))$ is nonsingular when $0 \neq s \in \hat{I}$. And we again take $I = \hat{I}$ Then the variational equation along γ_s has no nonzero bounded solutions. So γ_s is a transverse homoclinic solution of (1.1) and its periodic map exhibits chaotic motion.

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