Journal of Applied Analysis and Computation
Volume 10, Number 5, October 2020, 1995–2007
Website:http://jaac-online.com
DOI:10.11948/20190309

GENERALIZED P(X)-ELLIPTIC SYSTEM WITH NONLINEAR PHYSICAL DATA

Elhoussine Azroul[1] and Farah Balaadich[1,†]

Abstract This paper considers the following Dirichlet problem of the form

$$-\mathrm{div}\,\big(\Phi(Du-\Theta(u)\big)=v(x)+f(x,u)+\mathrm{div}\,\big(g(x,u)\big),$$

which corresponds to a diffusion problem with a source v in moving and dissolving substance, the motion is described by g and the dissolution by f. By the theory of Young measure we will prove the existence result in variable exponent Sobolev spaces $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$.

Keywords p(x)-Laplacian systems, variable exponents, weak solutions, young measures.

MSC(2010) 35Jxx, 46E30, 35D30, 28Axx.

1. Introduction and main result

Let Ω be a bounded open domain in $\mathbb{R}^n$, $n\geq 2$. Throughout this paper, we denote by $\mathbb{M}^{m\times n}$ the set of real m by n matrices equipped with the inner product $\xi:\eta=\sum_{i,j}\xi_{ij}\eta_{ij}$. For $\xi\in\mathbb{M}^{m\times n}$, $|\xi|$ is the norm of ξ when regarded as a vector of $\mathbb{R}^{mn}$. In [6], the following quasilinear elliptic system was considered

$$\begin{cases} -\mathrm{div}\big(\Phi(Du-\Theta(u))\big)=v \quad \text{in } \Omega, \\ u=0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Du is the symmetric part of the gradient of $u:\Omega\to\mathbb{R}^m$, v belongs to $W^{-1,p'}(\Omega;\mathbb{R}^m)$, $\frac{1}{p}+\frac{1}{p'}=1$, $\Phi(\xi)=|\xi|^{p-2}\xi$ for all $\xi\in\mathbb{M}^{m\times n}$ and $1<p<\infty$ (a constant). And $\Theta:\mathbb{R}^m\to\mathbb{M}^{m\times n}$ is a continuous function satisfying

$$\Theta(0)=0 \quad \text{and} \quad |\Theta(a)-\Theta(b)|\leq c|a-b|, \quad \forall a,b\in\mathbb{R}^m, \tag{1.2}$$

where the positive constant c satisfies

$$c<\frac{1}{\mathrm{diam}(\Omega)}\Big(\frac{1}{2}\Big)^{\frac{1}{p}}. \tag{1.3}$$

Here diam(Ω) is the diameter of Ω. The authors used the theory of Young measure and Galerkin method to prove that (1.1) had a weak solution $u\in W_0^{1,p}(\Omega;\mathbb{R}^m)$ under the conditions (1.2) and (1.3). See also [7,9] for related topics.

†The corresponding author. Email: balaadich.edp@gmail.com(F. Balaadich)
1Department of Mathematics, Faculty of Sciences Dhar El Mehraz, B.P. 1796 Atlas, Fez-Morocco

When the exponent p is not constant, but depends on x, i.e. $p \equiv p(x)$, Azroul and Balaadich [8] established the existence result for (1.1), where v belongs to $W^{-1,p'(x)}(\Omega;\mathbb{R}^m)$ and the constant c in (1.2) is assumed to satisfy

$$c < \frac{1}{\operatorname{diam}(\Omega)}\Big(\frac{1}{2}\Big)^{\frac{1}{p^+}}.$$

Here p^+ is the essential sup of the variable exponent $p(x)$ for $x \in \Omega$. They used also the tool of Young measure in establishing their result.

The operator $-\operatorname{div}\big(|Du|^{p(x)-2}Du\big)$ is said to be the $p(x)$-Laplacian and becomes p-Laplacian when $p(x) \equiv p$. When $p(x) > 2$ (resp. $1 < p(x) < 2$), then the problem (1.1) is nonlinear degenerate (resp. singular) elliptic systems. Partial differential equations with nonlinearities involving nonconstant exponents have attracted an increasing amont of attention in recent years. The impulse for this maybe comes from the sound physical applications in play, or perhaps it is just the thrill of developping a mathematical theory where PDEs again meet functional analysis in a truly two-way street.

As we know, the $p(x)$-Laplacian is inhomogeneous. This implies that it posesses more complicated nonlinearities than the case of p constant. Problems with variable exponent appear in several domains. For example; in the mathematical modeling of stationary thermorheological viscous of the process filtration of an ideal barotropic gas through a porous medium (cf. [2,3]). In image processing [18], to outline the borders of a true image and to elliminate possible noise, the variable nonlinearity find its applications. For the case of calculus of variations, the reader can see [1,14] and references therein.

The authors in [10] considered the following $p(x)$-curl systems

$$\begin{cases} \nabla\times\big(|\nabla\times u|^{p(x)-2}\nabla\times u\big) = \lambda g(x,u) - \mu f(x,u), & \nabla.u = 0 \quad \text{in } \Omega \\ |\nabla\times u|^{p(x)-2}\nabla\times u\times\mathbf{n} \quad = 0, & u.\mathbf{n} = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Here $\nabla\times u$ is the curl of $u = (u_1, u_2, u_3)$. They studied the existence and nonexistence of solutions. Note that the above system is arising in electromagnetism.

E. Azroul et al. [4] investigated a class of nonlinear $p(x)$-Laplacian problems, in the scalar case, of the form

$$\begin{cases} -\operatorname{div}\Phi(\nabla u - \Theta(u)) + |u|^{p(x)-2}u + \alpha(u) = f \quad \text{in } \Omega, \\ \Phi(\nabla u - \Theta(u)).\eta + \gamma(u) = g \quad \text{on } \partial\Omega, \end{cases}$$

where the source term f was assumed to belong to $L^1(\Omega)$. They used the techniques of entropy solutions to prove the existence of a solution. See also [5,13].

In this paper we shall prove the existence of weak solutions for the following problem which is motivated by physics or geometry:

$$-\operatorname{div}\big(\Phi(Du - \Theta(u))\big) = v(x) + f(x,u) + \operatorname{div}\big(g(x,u)\big) \quad \text{in } \Omega, \tag{1.4}$$

supplemented with the Dirichlet boundary condition $u = 0$ on $\partial\Omega$. Here v belongs to $W^{-1,p'(x)}(\Omega;\mathbb{R}^m)$, $\Phi : \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ is given in a simple form $\Phi(\xi) = |\xi|^{p(x)-2}\xi$ for all $\xi \in \mathbb{M}^{m\times n}$ and $\Theta : \mathbb{R}^m \to \mathbb{M}^{m\times n}$ is a continuous function such that

$$\Theta(0) = 0 \quad \text{and} \quad |\Theta(a) - \Theta(b)| \le c|a-b|, \quad \forall a,b \in \mathbb{R}^m, \tag{1.5}$$

where c is a positive constant that satisfies $c < \frac{1}{\operatorname{diam}(\Omega)}\left(\frac{1}{2}\right)^{\frac{1}{p^+}}$. Moreover, f and g satisfy the following continuity and growth conditions:

(F0)(Continuity) $f : \Omega \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$ is a Carathéodory function, i.e. $x \mapsto f(x,u)$ is measurable for every $u \in \mathbb{R}^m$ and $u \mapsto f(x,u)$ is continuous for a.e. $x \in \Omega$.
(F1)(Growth) There exist $b_1 \in L^{p'(x)}(\Omega)$ and $0 < \gamma(x) < p(x) - 1$ such that

$$|f(x,u)| \leq b_1(x) + |u|^{\gamma(x)}.$$

(G0)(Continuity) $g : \Omega \times \mathbb{R}^m \longrightarrow \mathbb{M}^{m\times n}$ is a Carathéodory function in the sense of (F0).
(G1)(Growth) There exist $b_2 \in L^{p'(x)}$ and $0 < q(x) < p(x) - 1$ such that

$$|g(x,u)| \leq b_2(x) + |u|^{q(x)}.$$

Remark 1.1. 1) The strict bound $p(x) - 1$ for $\gamma(x)$ and $q(x)$ in the growth conditions (F1) and (G1) ensures the coercivity of the operator T introduced in Section 3.

2) The function f may depend even on the Jacobien matrix $Du := \frac{1}{2}\big(\nabla u + (\nabla u)^t\big)$ and linear with respect to its variable $\xi \in \mathbb{M}^{m\times n}$, see Appendix.

Let $u : \Omega \to \mathbb{R}^m$ be a vector-valued function.

Definition 1.1. A measurable function $u \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$ is called a weak solution to problem (1.4) if

$$\int_\Omega \Phi\big(Du - \Theta(u)\big) : D\varphi dx = \langle v, \varphi\rangle + \int_\Omega f(x,u).\varphi dx - \int_\Omega g(x,u) : D\varphi dx$$

holds for all $\varphi \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$. Here $\langle .,.\rangle$ is the duality pairing of $W^{-1,p'(x)}(\Omega;\mathbb{R}^m)$ and $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$.

We shall prove the following existence theorem:

Theorem 1.1. *Assume that* (1.5)*, (F0), (F1), (G0) and (G1) hold true. Then there exists at least one weak solution to* (1.4) *in the sense of Definition 1.1.*

The outline of the present paper is as follows: In section 2, we introduce the functional space and its properties, and a brief review on the theory of Young measure. Section 3 is devoted to construct the approximating solutions by the Galerkin method and we give the proof of the main result. This paper ends with an appendix.

2. Preliminaries

2.1. Variable exponent Lebesgue and Sobolev spaces

In order to discuss the solutions to the Dirichlet problem (1.4), we need some theories and basic properties of variable exponent Lebesgue and Sobolev spaces $L^{p(x)}(\Omega;\mathbb{R}^m)$, $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$ respectively (see [16,18,21] and other references therein).

Let Ω be a bounded open domain of $\mathbb{R}^n$. We denote $C_+(\overline{\Omega})$ the following set

$$C_+(\overline{\Omega}) = \big\{p \in C(\overline{\Omega}) : p(x) > 1 \quad \text{for all } x \in \overline{\Omega}\big\}.$$

Throughout this paper,

$$p^- := \operatorname{ess}\inf_{x\in\Omega} p(x) \quad \text{and} \quad p^+ := \operatorname{ess}\sup_{x\in\Omega} p(x)$$

for every $p \in C_+(\overline{\Omega})$. We define the modular of a measurable function $u : \Omega \to \mathbb{R}^m$ by

$$\rho_{p(.)}(u) := \int_\Omega |u(x)|^{p(x)} dx.$$

The variable exponent Lebesgue space $L^{p(x)}(\Omega;\mathbb{R}^m)$ is a Banach space that is the set of all measurable functions $u : \Omega \to \mathbb{R}^m$ such that its modular

$$\rho_{p(.)}(u) < +\infty$$

is finite, equipped with the Luxemburg norm

$$\|u\|_{L^{p(x)}(\Omega;\mathbb{R}^m)} := \|u\|_{p(x)} = \inf\big\{\beta > 0 : \ \rho_{p(.)}\big(\frac{u}{\beta}\big) \leq 1\big\}.$$

Note that the generalized Lebesgue space $L^{p(x)}(\Omega;\mathbb{R}^m)$ is a kind of Musielak-Orlicz space. If

$$1 \leq p^- \leq p^+ < \infty,$$

$L^{p(x)}(\Omega;\mathbb{R}^m)$ is separable and, in particular, $C_0^\infty(\Omega;\mathbb{R}^m)$ is dense in $L^{p(x)}(\Omega;\mathbb{R}^m)$. If we restrict $p(.)$ to satisfy

$$1 < p^- \leq p^+ < \infty,$$

then $L^{p(x)}(\Omega;\mathbb{R}^m)$ becomes reflexive, and its dual is given for $p'(x) = p(x)/(p(x)-1)$ by $L^{p'(x)}(\Omega;\mathbb{R}^m)$, where $p'(x)$ is the conjugate of $p(x)$. In these spaces, a version of Hölder's inequality

$$\int_\Omega uv dx \leq \Big(\frac{1}{p^-} + \frac{1}{p^+}\Big)\|u\|_{p(x)}\|v\|_{p'(x)} \leq 2\|u\|_{p(x)}\|v\|_{p'(x)}$$

is valid for $u \in L^{p(x)}(\Omega;\mathbb{R}^m)$ and $v \in L^{p'(x)}(\Omega;\mathbb{R}^m)$. For the relation between the modular $\rho_{p(.)}(.)$ and the norm $\|.\|_{p(x)}$, we recall the following properties: if $u_k, u \in L^{p(x)}(\Omega;\mathbb{R}^m)$ and $1 < p^- \leq p^+ < \infty$, then:

$$\begin{aligned}
&\text{if} \quad \|u\|_{p(x)} > 1 \Rightarrow \|u\|_{p(x)}^{p^-} \leq \rho_{p(x)}(u) \leq \|u\|_{p(x)}^{p^+};\\
&\text{if} \quad \|u\|_{p(x)} < 1 \Rightarrow \|u\|_{p(x)}^{p^+} \leq \rho_{p(x)}(u) \leq \|u\|_{p(x)}^{p^-};\\
&\|u_k\|_{p(x)} \to 0 \quad (\text{resp.} +\infty) \Leftrightarrow \rho_{p(x)}(u_k) \to 0 \quad (\text{resp.} +\infty).
\end{aligned}$$

We define the generalized Lebesgue-Sobolev space $W^{1,p(x)}(\Omega;\mathbb{R}^m)$ as the set of all $u \in L^{p(x)}(\Omega;\mathbb{R}^m)$ such that $Du \in L^{p(x)}(\Omega;\mathbb{M}^{m\times n})$, which is a Banach space for the norm

$$\|u\|_{1,p(x)} = \|u\|_{p(x)} + \|Du\|_{p(x)}.$$

This space is again a special case of Sobolev-Orlicz spaces and inherits many of the properties of $L^{p(x)}(\Omega;\mathbb{R}^m)$. In particular, $W^{1,p(x)}(\Omega;\mathbb{R}^m)$ is separable if $1 \leq p^- \leq p^+ < \infty$, and is reflexive if $1 < p^- \leq p^+ < \infty$. Further, $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$ is the closure of $C_0^\infty(\Omega;\mathbb{R}^m)$ in the norm of $W^{1,p(x)}(\Omega;\mathbb{R}^m)$. If $p(.) \in C_+(\overline{\Omega})$, then an equivalent norm in $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$ is $\|Du\|_{p(x)}$. The dual space of $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$ can be identified with $W^{-1,p'(x)}(\Omega;\mathbb{R}^m)$ for $\frac{1}{p(x)}+\frac{1}{p'(x)}=1$. As in [19], the following Poincaré's inequality: there exists a positive constant $\alpha=\text{diam}(\Omega)$ such that

$$\|u\|_{p(x)} \leq \frac{\alpha}{2}\|Du\|_{p(x)}, \quad \forall u \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m),$$

together with Hölder's inequality, are central in this paper.

Let us summarize the above properties in the following proposition:

Proposition 2.1 ([18]). *1) $W^{1,p(x)}(\Omega;\mathbb{R}^m)$ and $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$ are Banach spaces which are separable if $p(.)\in L^\infty(\Omega)$ and reflexive if $1<p^-\leq p^+<\infty$.*
2) If $q\in C_+(\overline{\Omega})$ with $q(x)<p^*(x):=\frac{np(x)}{n-p(x)}$ for all $p(x)<n$, then the following compact embedding $W^{1,p(x)} \hookrightarrow L^{q(x)}(\Omega;\mathbb{R}^m)$ holds true. In particular, since $p(x)<p^*(x)$ for all $x\in\Omega$ then

$$W^{1,p(x)} \hookrightarrow\hookrightarrow L^{p(x)}(\Omega;\mathbb{R}^m).$$

3) There exists a constant $C>0$ with $\|u\|_{p(x)}\leq C\|Du\|_{p(x)}$ for all $u\in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$, hence $\|Du\|_{p(x)}$ and $\|u\|_{1,p(x)}$ are two equivalent norms on $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$.

2.2. Young measures

The weak convergence is a basic tool of modern nonlinear analysis, because it has the same compactness properties as the convergence in finite dimensional spaces as discussed in the paper of Evans [15]. However this convergence, sometimes, does not behave as we desire with respect to nonlinear functionals and operators. To solve this difficulties, we can use the tool of Young measure which we will use in this paper to prove the needed result.

For convenience of the reader not familiar with this concept, we recall some basic notions and properties (see eg. [11, 15, 17, 20] and references therein).

By $C_0(\mathbb{R}^m)$ we denote the set of functions $g\in C(\mathbb{R}^m)$ satisfying $\lim_{|\lambda|\to\infty} g(\lambda)=0$. Its dual is the well known space of signed Radon measures with finite mass and denoted by $\mathcal{M}(\mathbb{R}^m)$. The duality pairing of these spaces is defined for $\nu:\Omega\longrightarrow\mathcal{M}(\mathbb{R}^m)$ as

$$\langle \nu, g\rangle = \int_{\mathbb{R}^m} g(\lambda)d\nu(\lambda).$$

Lemma 2.1 ([15]). *Assume that the sequence $\{w_k\}_{k\geq 1}$ is bounded in $L^\infty(\Omega;\mathbb{R}^m)$. Then there exist a subsequence (still denoted by $\{w_k\}$) and a Borel probability measure ν_x on $\mathbb{R}^m$ for a.e. $x\in\Omega$, such that for each $g\in C(\mathbb{R}^m)$ we have $g(w_k)\rightharpoonup^* \overline{g}$ weakly in $L^\infty(\Omega;\mathbb{R}^m)$ where*

$$\overline{g}(x) = \langle \nu_x, g\rangle = \int_{\mathbb{R}^m} g(\lambda)d\nu_x(\lambda) \quad \textit{for a.e. } x\in\Omega.$$

We call $\nu=\{\nu_x\}_{x\in\Omega}$ the family of Young measure associated with the subsequence $\{w_k\}$. In [11] it is shown that if for all $R>0$

$$\lim_{L\to\infty}\sup_{k\in\mathbb{N}}\left|\Big\{x\in\Omega\cap B_R(0):\ |w_k(x)|\geq L\Big\}\right| = 0,$$

then for any measurable $\Omega' \subset \Omega$

$$h(.,w_k) \rightharpoonup \langle \nu_x, h(x,.)\rangle = \int_{\mathbb{R}^m} h(x,\lambda)d\nu_x(\lambda)$$

weakly in $L^1(\Omega')$, for any Carathéodory function $h : \Omega' \times \mathbb{R}^m \to \mathbb{R}$ provided that the sequence $\{h(.,w_k)\}$ is weakly precompact in $L^1(\Omega')$.

Moreover, if $|\Omega| < \infty$,

$$w_k \longrightarrow w \quad \text{in measure} \Longleftrightarrow \nu_x = \delta_{w(x)}. \tag{2.1}$$

The Young measure associated to the sequence (y_k, w_k) is given by

$$\delta_{y(x)} \otimes \nu_x \tag{2.2}$$

if $y_k \to y$ in measure and if ν_x is the Young measure associated to w_k.

Lemma 2.2 ([8]). *Let (w_k) be the sequence defined as above. Then the Young measure ν_x generated by Dw_k in $L^{p(x)}(\Omega;\mathbb{M}^{m\times n})$ has the following properties:*

(i) ν_x is a probability measure, i.e. $\|\nu_x\|_{\mathcal{M}(\mathbb{M}^{m\times n})} = 1$ for almost every $x \in \Omega$.

(ii) The weak L^1-limit of Dw_k is given by $\langle \nu_x, id\rangle = \int_{\mathbb{M}^{m\times n}} \lambda d\nu_x(\lambda)$.

(iii) ν_x satisfies $\langle \nu_x, id\rangle = Dw(x)$ for almost every $x \in \Omega$.

3. Approximating solutions

To construct the approximating solutions, we will use the Galerkin method. To this purpose, we consider the following map $T : W_0^{1,p(x)}(\Omega;\mathbb{R}^m) \longrightarrow W^{-1,p'(x)}(\Omega;\mathbb{R}^m)$ defined for arbitrary $u \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$, by

$$\langle T(u),\varphi\rangle = \int_\Omega \Phi\big(Du-\Theta(u)\big) : D\varphi dx - \langle v,\varphi\rangle - \int_\Omega f(x,u)\varphi dx + \int_\Omega g(x,u) : D\varphi dx$$

for all $\varphi \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$. As a consequence, our problem (1.4) is then equivalent to find $u \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$ such that

$$\langle T(u),\varphi\rangle = 0 \quad \text{for all} \quad \varphi \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m).$$

Lemma 3.1. *The mapping $T(u)$ is well defined, linear and bounded.*

Proof. For arbitrary $u \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$, $T(u)$ is linear. For all $\varphi \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$,

$$\begin{aligned}|\langle T(u),\varphi\rangle| &\leq \Big|\int_\Omega \Phi\big(Du-\Theta(u)\big) : D\varphi dx - \langle v,\varphi\rangle - \int_\Omega f(x,u)\varphi dx + \int_\Omega g(x,u) : D\varphi dx\Big| \\ &\leq \int_\Omega \big|Du-\Theta(u)\big|^{p(x)-1}|D\varphi|dx + \big|\langle v,\varphi\rangle\big| \\ &\quad + \int_\Omega |f(x,u)||\varphi|dx + \int_\Omega |g(x,u)||D\varphi|dx.\end{aligned}$$

Note that since

$$|a+b|^r \leq 2^{r-1}\big(|a|^r + |b|^r\big) \quad \text{for } r > 1, \tag{3.1}$$

it follows by Hölder's inequality that

$$\begin{aligned} I_1 &:= \int_\Omega \big|Du - \Theta(u)\big|^{p(x)-1} |D\varphi| dx \\ &\leq \Big(\int_\Omega \big|Du - \Theta(u)\big|^{p(x)} dx \Big)^{\frac{1}{p'(x)}} \|D\varphi\|_{p(x)} \\ &\leq \Big(\int_\Omega 2^{p(x)-1} \big(|Du|^{p(x)} + |\Theta(u)|^{p(x)}\big) dx \Big)^{\frac{p(x)-1}{p(x)}} \|D\varphi\|_{p(x)} \\ &\leq 2^{\frac{(p^+-1)^2}{p^-}} \Big(\|Du\|_{p(x)}^{p(x)} + \|\Theta(u)\|_{p(x)}^{p(x)} \Big)^{\frac{p(x)-1}{p(x)}} \|D\varphi\|_{p(x)}. \end{aligned}$$

The generalized Hölder inequality implies that

$$I_2 := \big|\langle v, \varphi\rangle\big| \leq \|v\|_{-1,p'(x)} \|\varphi\|_{1,p(x)}.$$

On the other hand, it follows from the growth condition (F1) (without loss of generality, we can assume that $\gamma(x) = p(x) - 1$), that

$$I_3 := \int_\Omega |f(x,u)||\varphi| dx \leq \|b_1\|_{p'(x)} \|\varphi\|_{p(x)} + \|u\|_{p(x)}^{p(x)-1} \|\varphi\|_{p(x)}.$$

Finally, the growth condition (G1) (without loss of generality, we may assume that $q(x) = p(x) - 1$) allows to estimate (by application of the Hölder inequality)

$$I_4 := \int_\Omega |g(x,u)||D\varphi| dx \leq \|b_2\|_{p'(x)} \|D\varphi\|_{p(x)} + \|u\|_{p(x)}^{p(x)-1} \|D\varphi\|_{p(x)}.$$

By virtual of the Poincaré inequality, the I_i for $i = 1, .., 4$ are finite, then $T(u)$ is well defined. Moreover, for all $\varphi \in W_0^{1,p(x)}(\Omega; \mathbb{R}^m)$

$$\big|\langle T(u), \varphi\rangle\big| \leq \sum_{i=1}^{4} I_i \leq C \|D\varphi\|_{p(x)},$$

and this implies that $T(u)$ is bounded. □

Lemma 3.2. *The restriction of T to a finite dimensional linear subspace V of $W_0^{1,p(x)}(\Omega; \mathbb{R}^m)$ is continuous.*

Proof. Let r be the dimension of V and $(e_i)_{i=1}^r$ a basis of V. Let $(u_j = a_j^i e_i)$ be a sequence in V which converges to $u = a^i e_i$ in V (with conventional summation). Then $u_j \to u$ and $Du_j \to Du$ almost everywhere. The continuity of Θ, f and g implies that

$$\begin{aligned} &\Phi(Du_j - \Theta(u_j)) : D\varphi \to \Phi(Du - \Theta(u)) : D\varphi, \\ &f(x, u_j)\varphi \to f(x,u)\varphi \quad \text{and} \quad g(x, u_j) : D\varphi \to g(x,u) : D\varphi \end{aligned}$$

almost everywhere. Since $u_j \to u$ strongly in V,

$$\int_\Omega |u_j - u|^{p(x)} dx \longrightarrow 0 \quad \text{and} \quad \int_\Omega |Du_j - Du|^{p(x)} \longrightarrow 0.$$

According to [12] (Chapter IV, Section 3, Theorem 3) there exist a subsequence of $\{u_j\}$ still denoted by $\{u_j\}$ and $h_1, h_2 \in L^1(\Omega)$ such that $|u_j - u|^{p(x)} \leq h_1$, $|Du_j - Du|^{p(x)} \leq h_2$. By vertue of the Eq (3.1), we can write

$$\begin{aligned} |u_j|^{p(x)} = |u_j - u + u|^{p(x)} &\leq 2^{p^+-1}\big(|u_j - u|^{p(x)} + |u|^{p(x)}\big) \\ &\leq 2^{p^+-1}\big(h_1 + |u|^{p(x)}\big), \end{aligned}$$

from which (similarly) we get $|Du_j|^{p(x)} \leq 2^{p^+-1}\big(h_2 + |Du|^{p(x)}\big)$. Consequently, the sequences $\|u_j\|_{p(x)}$ and $\|Du_j\|_{p(x)}$ are bounded by a constant denoted C. Now, if $\Omega' \subset \Omega$ is a measurable subset and $\varphi \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$, then by Poincaré's inequality

$$\begin{aligned} &\int_{\Omega'} \big|\Phi(Du_j - \Theta(u_j)) : D\varphi\big| dx \\ \leq & 2^{\frac{(p^+-1)^2}{p^-}} \Big(\|Du_j\|_{p(x)}^{p(x)} + \|\Theta(u_j)\|_{p(x)}^{p(x)}\Big)^{\frac{p(x)-1}{p(x)}} \Big(\int_{\Omega'} |D\varphi|^{p(x)}\Big)^{\frac{1}{p(x)}} \\ \leq & 2^{\frac{(p^+-1)^2}{p^-}} \Big(\underbrace{\|Du_j\|_{p(x)}^{p(x)}}_{\leq C} + c^{p^+} \underbrace{\|u_j\|_{p(x)}^{p(x)}}_{\leq C}\Big)^{\frac{p(x)-1}{p(x)}} \Big(\int_{\Omega'} |D\varphi|^{p(x)}\Big)^{\frac{1}{p(x)}}, \end{aligned}$$

where the small c is the constant in (1.5), and (without loss of generality, we can assume that $\gamma(x) = p(x) - 1$ and $q(x) = p(x) - 1$)

$$\int_{\Omega'} |f(x, u_j)\varphi| dx \leq C\Big(\|b_1\|_{p'(x)} + \underbrace{\|u_j\|_{p(x)}^{p(x)-1}}_{\leq C}\Big)\Big(\int_{\Omega'} |D\varphi|^{p(x)} dx\Big)^{\frac{1}{p(x)}},$$

and

$$\int_{\Omega'} |g(x, u_j) : D\varphi| dx \leq \Big(\|b_2\|_{p'(x)} + \underbrace{\|u_j\|_{p(x)}^{p(x)-1}}_{\leq C}\Big)\Big(\int_{\Omega'} |D\varphi|^{p(x)} dx\Big)^{\frac{1}{p(x)}}.$$

Therefore, the sequences $\big(\Phi(Du_j - \Theta(u_j)) : D\varphi\big)$, $\big(f(x, u_j)\varphi\big)$ and $\big(g(x, u_j) : D\varphi\big)$ are equiintegrable, since $\int_{\Omega'} |D\varphi|^{p(x)} dx$ is arbirary small if the measure of Ω' is chosen small enough. Applying the Vitali Theorem, it follows for all $\varphi \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$ that $\lim_{j\to\infty}\langle T(u_j), \varphi\rangle = \langle T(u), \varphi\rangle$ as we desire. □

Lemma 3.3. *The mapping T is coercive.*

Proof. Taking $\varphi = u$ in the definition of T, then

$$\langle T(u), u\rangle = \int_\Omega \Phi(Du - \Theta(u)) : Du dx - \langle v, u\rangle - \int_\Omega f(x, u) u dx + \int_\Omega g(x, u) : Du dx. \tag{3.2}$$

We know that

$$|\xi|^{r-2}\xi : (\xi - \eta) \geq \frac{1}{r}|\xi|^r - \frac{1}{r}|\eta|^r,$$

then for $\xi = Du - \Theta(u)$ and $\eta = -\Theta(u)$, we get

$$J_1 := \int_\Omega |Du - \Theta(u)|^{p(x)-2}(Du - \Theta(u)) : (Du - \Theta(u) + \Theta(u)) dx$$

$$\geq \int_\Omega \frac{1}{p(x)}|Du-\Theta(u)|^{p(x)}dx - \int_\Omega \frac{1}{p(x)}|\Theta(u)|^{p(x)}dx.$$

On the other hand, by (3.1)

$$\begin{aligned}|Du|^{p(x)} &= |Du-\Theta(u)+\Theta(u)|^{p(x)}\\ &\leq 2^{p^+-1}\Big(|Du-\Theta(u)|^{p(x)}+|\Theta(u)|^{p(x)}\Big),\end{aligned}$$

thus

$$\begin{aligned}J_1 &\geq \frac{1}{p^+}\frac{1}{2^{p^+-1}}\int_\Omega |Du|^{p(x)}dx - \frac{2}{p^+}\int_\Omega |\Theta(u)|^{p(x)}dx\\ &\geq \Big(\frac{1}{p^+}\frac{1}{2^{p^+-1}} - \frac{2}{p^+}\frac{1}{2\alpha^{p^+}}\big(\frac{\alpha}{2}\big)^{p^+}\Big)\int_\Omega |Du|^{p(x)}dx\\ &= \frac{1}{p^+}\frac{1}{2^{p^+}}\int_\Omega |Du|^{p(x)}dx,\end{aligned}$$

by definition of the function Θ and Poincaré's inequality. Next the Hölder inequality implies that

$$|J_2| := |\langle v,u\rangle| \leq \|v\|_{-1,p'(x)}\|u\|_{1,p(x)}.$$

Finally, it follows from the growth conditions (F1) and (G1) that

$$\begin{aligned}J_3 := \int_\Omega f(x,u)udx &\leq \|b_1\|_{p'(x)}\|u\|_{p(x)} + \|u\|_{p(x)}^{\gamma(x)+1}\\ &\leq C\|b_1\|_{p'(x)}\|Du\|_{p(x)} + C^{\gamma^++1}\|Du\|_{p(x)}^{\gamma(x)+1}\end{aligned}$$

and

$$|J_4| := \Big|\int_\Omega g(x,u):Dudx\Big| \leq \|b_2\|_{p'(x)}\|Du\|_{p(x)} + C^{q^+}\|Du\|_{p(x)}^{q(x)+1}.$$

From these estimations it follows that

$$\langle T(u),u\rangle = J_1 - J_2 - J_3 + J_4 \longrightarrow +\infty \quad \text{as} \quad \|u\|_{1,p(x)} \to +\infty$$

since $p^+ > \max\{1,\gamma^++1,q^++1\}$. Hence T is coercive. $\square$

Now, let $V_1 \subset V_2 \subset ... \subset W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$ be a sequence of finite dimensional subspaces with the property that $\cup_{k\in\mathbb{N}}V_k$ is dense in $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$. Notice that such a sequence (V_k) exists since $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$ is separable. Let dim $V_k = r$ and $e_1,..,e_r$ is a basis of V_k for a fixed k. To construct the approximating solution, we define the map

$$S:\mathbb{R}^r \to \mathbb{R}^r, \quad \begin{pmatrix} a^1\\ a^2\\ .\\ .\\ a^r\end{pmatrix} \mapsto \begin{pmatrix}\langle T(a^ie_i),e_1\rangle\\ \langle T(a^ie_i),e_2\rangle\\ .\\ .\\ \langle T(a^ie_i),e_r\rangle\end{pmatrix}.$$

Lemma 3.4. *1) The map S is continuous.*
2) For all $k \in \mathbb{N}$ there exists $u_k \in V_k$ such that

$$\langle T(u_k), \varphi\rangle = 0 \quad \textit{for all} \quad \varphi \in V_k. \tag{3.3}$$

3) The sequence constructed in 2) is uniformly bounded in $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$, i.e. there exists a constant $R > 0$ such that

$$\|u_k\|_{1,p(x)} \leq R \quad \textit{for all} \quad k \in \mathbb{N}. \tag{3.4}$$

Proof. 1) Since T restricted to V_k is continuous (see Lemma 3.2), then S is continuous.
2) Let $a \in \mathbb{R}^r$ and $u = a^i e_i \in V_k$. Then $S(a).a = \langle T(u), u\rangle$. Notice that $\|a\|_{\mathbb{R}^r} \to +\infty$ is equivalent to $\|u\|_{1,p(x)} \to +\infty$. It follows by Lemma 3.3 that

$$S(a).a \longrightarrow +\infty \quad \text{as} \quad \|a\|_{\mathbb{R}^r} \to +\infty.$$

Hence, there exists $R > 0$ such that for all $a \in \partial B_R(0) \subset \mathbb{R}^r$ we have $S(a).a > 0$. The usual topological arguments (see eg. [22, Proposition 2.8]) allows to deduce that $S(x) = 0$ has a solution $x \in B_R(0)$. Therefore, for all $k \in \mathbb{N}$ there exists $u_k \in V_k$ such that

$$\langle T(u_k), \varphi\rangle = 0 \quad \text{for all} \quad \varphi \in V_k.$$

3) We have by Lemma 3.3, that $\langle T(u), u\rangle \to +\infty$ as $\|u\|_{1,p(x)} \to +\infty$. It result then the existence of $R > 0$ with the property, that $\langle T(u), u\rangle > 1$ whenever $\|u\|_{1,p(x)} > R$. Taking this into consideration and the sequence of Galerkin approximations $u_k \in V_k$ which satisfy $\langle T(u_k), u_k\rangle = 0$ (by 2)), it follows that (u_k) is uniformly bounded. □

Before we pass to the limit in the approximating sequences and so to prove Theorem 1.1, notice that since (u_k) is bounded in $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$ (see Lemma 3.4), it follows by Lemma 2.1 the existence of a Young measure ν_x generated by Du_k in $L^{p(x)}(\Omega;\mathbb{M}^{m\times n})$ which satisfies the properties of Lemma 2.2.

Proof of Theorem 1.1. To apply the convergence described in Lemma 2.2 to our approximating problem, we need the convergence in measure of u_k to u. To this purpose, consider $E_{k,\epsilon} = \{x \in \Omega;\ |u_k(x) - u(x)| \geq \epsilon\}$. Since (u_k) is bounded in $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$, then for a subsequence still denoted u_k, $u_k \to u$ in $L^{p(x)}(\Omega;\mathbb{R}^m)$. Therefore

$$\int_\Omega \big|u_k(x) - u(x)\big|^{p(x)} \geq \int_{E_{k,\epsilon}} \big|u_k(x) - u(x)\big|^{p(x)} \geq \epsilon^{p^-}\big|E_{k,\epsilon}\big|,$$

which implies that

$$\big|E_{k,\epsilon}\big| \leq \frac{1}{\epsilon^{p^-}} \int_\Omega \big|u_k(x) - u(x)\big|^{p(x)} \longrightarrow 0 \quad \text{as } k \to \infty.$$

Hence the sequence u_k converges in measure to u on Ω. On the other hand, since $\{Du_k - \Theta(u_k)\}$ is equiintegrable by the condition (1.5) and the boundedness of (u_k), it result that

$$Du_k - \Theta(u_k) \rightharpoonup \int_{\mathbb{M}^{m\times n}} \big(\lambda - \Theta(u)\big) d\nu_x(\lambda)$$

$$= \underbrace{\int_{\mathbb{M}^{m\times n}} \lambda d\nu_x(\lambda)}_{=:Du(x)} - \Theta(u) \underbrace{\int_{\mathbb{M}^{m\times n}} d\nu_x(\lambda)}_{=:1}$$
$$= Du - \Theta(u)$$

weakly in $L^1(\Omega)$, where we have used Lemma 2.2. Further, from the reflexivity of $L^{p'(x)}(\Omega)$ and the boundedness of $\{\Phi(Du_k - \Theta(u_k))\}$, we deduce that $\Phi(Du_k - \Theta(u_k))$ converges in $L^{p'(x)}(\Omega)$ and its weak $L^{p'(x)}$-limit is given by $\Phi(Du - \Theta(u))$. Consequently

$$\lim_{k\to\infty} \int_\Omega \Phi(Du_k - \Theta(u_k)) : D\varphi dx = \int_\Omega \Phi(Du - \Theta(u)) : D\varphi dx \quad \forall \varphi \in \cup_{k\in\mathbb{N}} V_k.$$

Moreover, since $u_k \to u$ in measure for $k \to \infty$, we may infer that, after extraction of a suitable subsequence, if necessary,

$$u_k \longrightarrow u \quad \text{almost everywhere for } k \to \infty.$$

Hence, for arbitrary $\varphi \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$, it follows from the continuity conditions (F0) and (G0), that

$$f(x,u_k)\varphi \to f(x,u)\varphi \quad \text{and} \quad g(x,u_k) : D\varphi \to g(x,u) : D\varphi$$

almost everywhere. As in the proof of Lemma 3.2, we have $f(x,u_k)\varphi$ and $g(x,u_k) : D\varphi$ are equiintegrable, thus

$$f(x,u_k)\varphi \to f(x,u)\varphi \quad \text{and} \quad g(x,u_k) : D\varphi \to g(x,u) : D\varphi$$

in $L^1(\Omega)$ by the Vitali Convergence Theorem. Consequently

$$\lim_{k\to\infty} \int_\Omega f(x,u_k)\varphi dx = \int_\Omega f(x,u)\varphi dx \quad \forall \varphi \in \cup_{k\geq 1} V_k$$

and

$$\lim_{k\to\infty} \int_\Omega g(x,u_k) : D\varphi dx = \int_\Omega g(x,u) : D\varphi dx \quad \forall \varphi \in \cup_{k\geq 1} V_k.$$

Since $\cup_{k\geq 1} V_k$ is dense in $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$, u is then a weak solution of (1.4). $\square$

Appendix

Consider the function f depends on $\xi \in \mathbb{M}^{m\times n}$, i.e. $f : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}^m$ and satisfies

$$|f(x,s,\xi| \leq b_1(x) + |s|^{\gamma(x)} + |\xi|^{s(x)}, \tag{3.5}$$

where $b_1 \in L^{p'(x)}(\Omega)$, $0 < \gamma(x) < p(x) - 1$ and $0 < s(x) < p(x) - 1$. By similar arguments as above (since $p^+ > \max\{1, \gamma^+ + 1, q^+ + 1, s^+ + 1\}$), it follows that

$$\lim_{k\to\infty} \int_\Omega f(x,u_k,Du_k)\varphi dx = \int_\Omega f(x,u,Du)\varphi dx \quad \forall \varphi \in \cup_{k\geq 1} V_k$$

for all $\varphi \in \cup_{k\geq 1} V_k$. Now, assume that $\xi \mapsto f(x,u,\xi)$ is linear. We have $f(x,u_k,Du_k)$ is equiinetgrable (by the growth condition (3.5)), this implies

$$\begin{aligned} f(x,u_k,Du_k) &\rightharpoonup \int_{\mathbb{M}^{m\times n}} f(x,u,\lambda)d\nu_x(\lambda) \\ &= f(x,u,.)\underbrace{\int_{\mathbb{M}^{m\times n}} \lambda d\nu_x(\lambda)}_{=:Du(x)} \\ &= f(x,u,Du) \end{aligned}$$

weakly in $L^1(\Omega)$, by linearity of f.

To conclude this paper, let $\varphi \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$, since $\cup_{k\geq 1}V_k$ is dense in $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$, then there exists a sequence $(\varphi_k) \subset \cup_{k\geq} V_k$ such that $\varphi_k \to \varphi$ in $W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$. According to the previous results, we get

$$\lim_{k\to\infty} \langle T(u_k),\varphi_k\rangle = \langle T(u),\varphi\rangle.$$

The equation (3.3) implies that $\langle T(u),\varphi\rangle = 0$ as we desire for all $\varphi \in W_0^{1,p(x)}(\Omega;\mathbb{R}^m)$.

References

[1] E. Acerbi and G. Mingione, *Regularity results for a class of functionals with nonstandard growth*, Arch. Rational Mech. Anal., 2001, 156, 121–140.

[2] S. Antontsev and S. Shmarev, *A model porous medium equation with variable exponent of nonlinearity: existence, uniqueness and localization properties of solutions*, Nonlinear Anal., 2005, 60(3), 515–545.

[3] S. N. Antontsev and J. F. Rodrigues, *On stationary thermorheological viscous flows*, Ann. Univ. Ferrara Sez. VII Sci. Mat., 2006, 52(1), 19–36.

[4] E. Azroul, M. B. Benboubker and S. Ouaro, *Entropy solutions for nonlinear nonhomogeneous Neumann problems involving the generalized p(x)-Laplace operator*, J. Appl. Anal. Comput., 2013, 3(2), 105–121.

[5] E. Azroul, M. B. Benboubker and M. Rhoudaf, *On some p(x)-quasilinear problem with right-hand side measure*, Math. Comput. Simulation., 2013, 102, 117–130.

[6] E. Azroul and F. Balaadich, *Weak solutions for generalized p-Laplacian systems via Young measures*, Moroccan J. of Pure and Appl. Anal., 2018, 4(2), 76–83.

[7] E. Azroul and F. Balaadich, *Quasilinear elliptic systems in perturbed form*, Int. J. Nonlinear Anal. Appl., 2019, 10(2), 255–266.

[8] E. Azroul and F. Balaadich, *Existence of solutions for generalized p(x)-Laplacian systems*, Rend. Circ. Mat. Palermo, 2019, 2. https://doi.org/10.1007/s12215–019–00450–z

[9] E. Azroul and F. Balaadich, *A weak solution to quasilinear elliptic problems with perturbed gradient*, Rend. Circ. Mat. Palermo, 2020, 2. https://doi.org/10.1007/s12215–020–00488–4

[10] A. Bahrouni and D. D. Repovš, *Existence and nonexistence of solutions for p(x)-curl systems arising in electromagnetism*, Complex Var. Elliptic Equ., 2018, 63(2), 292–301.

[11] JM. Ball, *A version of the fundamental theorem for Young measures. In: PDEs and Continuum Models of Phase Transitions (Nice, 1988)*, Lecture Notes in Phys., 1989, 344, 207–215.

[12] N. Bourbaki, *Integration I (S. Berberian, Trans.)*, Springer-Verlag, Berlin, Heidelberg, 2004, (Chapters 1–6).

[13] T. A. Bui, *Global $W^{1,p(\cdot)}$ estimate for renormalized solutions of quasilinear equations with measure data on Reifenberg domains*, Adv. Nonlinear Anal., 2018, 7(4), 517–533.

[14] M. Eleuteri, P. Marcellini and E. Mascolo, *Lipschitz estimates for systems with ellipticity conditions at infnity*, Ann. Math. Pura Appl., 2016, 195, 1575–1603.

[15] LC. Evans, *Weak Convergence Methods for Nonlinear Partial Differential Equations*, Am. Math. Soc., New York, 1990.

[16] X. Fan and D. Zhao, *On the generalized Orlicz-Sobolev space $W^{k,p(x)}(\Omega)$*, J. Gansu Educ., College, 1998, 12(1), 1–6.

[17] N. Hungerƀuhler, *A refinement of Ball's theorem on Young measures*, N.Y. J. Math., 1997, 3, 48–53.

[18] O. Kováčik and J. Ràkosník, *On spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$*, Czechoslovak Math. J., 1991, 41(116), 592–618.

[19] G M. Lieberman, *The natural generalizationj of the natural conditions of ladyzhenskaya and uraltseva for elliptic equations*, Comm. Partial Differential Equations., 1991, 16(2–3), 311–361.

[20] J. Málek, J. Necas, M. Rokyta and M. Ružička, *Weak and measure-valued solutions to evolutionary PDEs*, Chapman & Hall, London, 1996.

[21] V. D. Radulescu and D. D. Repovš, *Partial differential equations with variable exponents. Variational methods and qualitative analysis*, Monogr. Res. Notes Math., 2015.

[22] E. Zeidler, *Nonlinear functional analysis and its application*, volume I. Springer, 1986.