GLOBAL ANALYSIS OF THE SHADOW GIERER-MEINHARDT SYSTEM WITH GENERAL LINEAR BOUNDARY CONDITIONS IN A RANDOM ENVIRONMENT

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Abstract The global analysis of the shadow Gierer-Meinhardt system with multiplicative white noise and general linear boundary conditions is investigated in this paper. For this reaction-diffusion system, we employ a fixed point argument to prove local existence and uniqueness. Our results on global existence are based on *a priori* estimates of solutions.

Keywords Shadow Gierer-Meinhardt system, multiplicative noise, global existence, linear Robin-Neumann boundary conditions.

MSC(2010) 60H05, 60H15, 60H30.

1. Introduction

In 1744, Trembley's discovery in developmental biology pointed out that fragments of the small, fresh water animal called hydra can regenerate into a complete animal [22]. Based on Turing's (1952) idea of "diffusion-driven instability" [23], Gierer and Meinhardt [8] in 1972 proposed a theory of biological pattern formation that placed special emphasis on certain striking features on developmental biology, in particular, they proposed a system to model the head formation in the hydra. Mathematical modeling of biological spatial pattern formation has become one of the most popular areas of investigation in applied mathematics in recent times. Many models involved in these biological phenomena are of the general reaction-diffusion type considered in [17,23]. Several researchers have been able to provide great insights into the underlying mechanisms of biological processes realized by the

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Gierer-Meinhardt system of the following form.

$$
\begin{cases}
A_t = \epsilon^2 \Delta A - A + \dfrac{A^p}{H^q + b} & \text{in } D \times (0, T), \\[2mm]
\tau H_t = d \Delta H - H + \dfrac{A^r}{H^s} & \text{in } D \times (0, T), \\[2mm]
\epsilon \dfrac{\partial A}{\partial \nu} + a A = 0 = \dfrac{\partial H}{\partial \nu} & \text{on } \partial D \times (0, T), \\[2mm]
H(x,0) = H_0(x) > 0, \quad A(x,0) = A_0(x) \geq 0 & \text{in } \overline{D},
\end{cases}
\tag{1.1}
$$

where $\epsilon > 0$, $d > 0$, $\tau > 0$, $a \geq 0$, $b > 0$ and $D \subset \mathbb{R}^N$ ($N \geq 1$) is a bounded domain with a smooth boundary ∂D, and A and H are activator and inhibitor, respectively; Δ is the Laplace or diffusion operator in $\mathbb{R}^N$ acting on A and H; $\nu(x)$ is the unit outer normal vector at $x \in \partial D$, $\partial / \partial \nu := \nabla \cdot \nu$ is the directional derivative in the direction of the vector ν. The reaction exponents p, q, r, and s are positive, and satisfy $(p-1)(s+1) < qr$. The constants ϵ and d are the diffusion coefficients for the activator and inhibitor respectively. The constant b provides additional support to the inhibitor and may be thought of as a measure of the effectiveness of the inhibitor in suppressing the production of the activator. The time relaxation constant τ plays a significant role on the stability of the system. The two chemical substances A and H, representing the concentrations of certain biochemicals, are initially produced by an outside source. Then they interact as represented by the coupled nonlinear terms in the system (see e.g. [16] and references therein).

There are several results for equation (1.1) with homogeneous linear Neumann boundary conditions (i.e., $a = 0$ and $b = 0$) in [5, 10, 16, 21, 25] and references therein. Chen et al. [4] studied the generalized (singular) Gierer-Meinhardt system with Dirichlet boundary conditions. Recently, Antwi-Fordjour and Nkashama [2] studied the global existence of (1.1). It is well known that it is quite challenging to study the solvability of the equation (1.1) since it does not have a standard variational structure. In [1], we observe another interesting problem that attracts attention to explain the influence of the Robin boundary conditions to the Gierer-Meinhardt system.

One way to initiate the study of (1.1) is to first examine the shadow system suggested by Keener [11]. Shadow systems are mostly employed to approximate the reaction-diffusion systems when one of the diffusion coefficients is large. Indeed, when the diffusion coefficient of the second equation in (1.1) is sufficiently large; that is, $d \to \infty$, and $\gamma(t)$ is the formal limit of $H(x, t)$, then the system (1.1) can be reduced to the shadow Gierer-Meinhardt system:

$$
\begin{cases}
A_t = \epsilon^2 \Delta A - A + \dfrac{A^p}{\gamma^q + b} & \text{in } D \times (0, T), \\[2mm]
\tau \gamma' = -\gamma + \dfrac{\overline{A^r}}{\gamma^s} & \text{in } (0, T), \\[2mm]
\epsilon \dfrac{\partial A}{\partial \nu} + a A = 0 & \text{on } \partial D \times (0, T), \\[2mm]
\gamma(0) = \gamma_0 > 0 \text{ in } \mathbb{R}, \qquad A(x,0) = A_0(x) \geq 0 & \text{in } \overline{D},
\end{cases}
\tag{1.2}
$$

where we define

$$
\gamma(t) := \frac{1}{|D|} \int_D H(x,t)\,dx, \qquad \overline{A^r}(t) := \frac{1}{|D|} \int_D A^r(x,t)\,dx,
$$

and $|D|$ is the (Lebesgue) measure of D. It is important to note here that, the second equation is a nonlocal ordinary differential equation.

Global existence and finite-time blow-up for equation (1.2) have been investigated by Li and Ni [13] when $a = b = 0$, provided we have $\frac{p-1}{r} < \frac{2}{N+2}$. Phan [19] studied the global existence of solutions for $a = b = 0$ in (1.2) provided $\frac{p-1}{r} = \frac{2}{N+1}$. Maini et al. [15] studied the stability of spikes for (1.2) with $b = 0$.

Physical and biological systems are inevitably affected by random fluctuations from the environment. It is therefore important to incorporate the random effects from the environment into (1.2). In stochastic modeling, these random effects are conceived as stochastic fluctuations.

Motivated by the work of Kelkel and Surulescu [12] and Winter et al. [24], we consider the following stochastic shadow Gierer-Meinhardt system:

$$
\begin{cases}
A_t = \epsilon^2 \Delta A - A + \dfrac{A^p}{\gamma^q + b} & \text{in } D \times (0, T), \\[2mm]
\tau d\gamma = -\gamma dt + \dfrac{\overline{A^r}}{\gamma^s} dt + \sqrt{\eta}\gamma dB_t & \text{in } (0, T), \\[2mm]
\epsilon \dfrac{\partial A}{\partial \nu} + aA = 0 & \text{on } \partial D \times (0, T), \\[2mm]
\gamma(0) = \gamma_0 > 0 \text{ in } \mathbb{R}, \qquad A(x, 0) = A_0(x) \geq 0 & \text{in } \overline{D},
\end{cases}
\tag{1.3}
$$

where $\eta > 0$ is small and represents the noise intensity, and B_t is a white noise (or statistically Brownian motion at time t).

Analytical results for the equation (1.3) were obtained with Neumann boundary conditions ($a = 0$) but there is a lack of theoretical considerations for the problem with general linear boundary conditions (see e.g. [14, 24]). Thus, investigating the equation (1.3) with general linear boundary conditions of Robin-Neumann type plays an important role in understanding various kinds of biological phenomena.

To the best of our knowledge, this appears to be the first paper on stochastic shadow Gierer-Meinhardt system with general linear boundary conditions of Robin-Neumann type. In this paper, motivated by [24] and the above considerations, we shall prove the following main result on the global existence of strong positive solutions for the problem with general linear boundary conditions of Robin-Neumann type.

Theorem 1.1. *Suppose that $D \subset \mathbb{R}^N$ is a bounded domain with a smooth boundary ∂D, and assume that the exponents satisfy the inequality*

$$
\frac{p-1}{r} < \min\left\{\frac{2}{N+2}, \frac{q}{s+1}\right\}.
$$

Let $A_0 \in W^{2,l}(D)$ where $l > \max\{N, 2\}$, and $\gamma_0 \in \mathbb{R}$ with $\gamma_0 > 0$. Then, with probability 1, there is a unique solution $(A(x,t), \gamma(t))$ of the stochastic shadow equation (1.3) which exists globally. Moreover, the component γ satisfies the estimate

$$
\gamma(t) \geq \left(\frac{\eta}{\tau}\right)^{\frac{1}{s+1}} e^{-\frac{3}{2\eta}t - \frac{1}{\sqrt{\eta}}|B_t|} \gamma_0.
\tag{1.4}
$$

The paper is organized as follows. In Section 2 we show the unique local existence of solutions. In Section 3 we prove global existence of positive solutions.

2. Unique Local Existence

In this section, we use several concepts from probability theory and semigroup of linear operators theory (see e.g. [3, 6, 7, 9, 18, 20]) along with estimates obtained herein and a fixed point argument to prove the unique local existence of positive solutions. Let us consider the (standard) probability space $(\Omega, \mathbb{F}, \mathbb{P})$ where Ω is the sample space, $\mathbb{F}$ is the σ-algebra, $\mathbb{P}$ is the probability measure and define

$$B_t^* := \sup_{0 \leq s \leq t} |B_s|, \quad \forall t > 0, \text{ and } \tau_K(\omega) := \inf\{t > 0 : |B_t(\omega)| \geq K\}, \ \omega \in \Omega. \quad (2.1)$$

Note that $\tau_K(\omega)$ denotes an optional stopping time (see e.g. [6] for background). It is easy to see that

$$E^c := \{\omega \in \Omega : \tau_K(\omega) \leq t\} = \{\omega \in \Omega : B_t^*(\omega) \geq K\}. \quad (2.2)$$

Since the distribution of B_t^* is a normal distribution function, we can ascertain that for sufficiently large $K > 0$, we have that

$$\mathbb{P}(E^c) < \frac{C}{K^2} \ll 1, \quad C > 0;$$

which means that we can think of the complement E^c as a negligible set. Next, we define the following operators;

$$S(t) := e^{-(-\epsilon^2 \Delta + I)t} \text{ and } R(t, B_t) := e^{-\frac{3}{2\eta}t + \frac{1}{\sqrt{\eta}}B_t}. \quad (2.3)$$

Notice that here $S(t)$ denotes the semigroup associated with the Laplace operator subject to homogeneous Robin-Neumann boundary conditions where $(-\epsilon^2 \Delta + I)$ is a strongly elliptic operator.

Consider the function space

$$C(\overline{D}, \mathbb{R}) = \{f : \overline{D} \to \mathbb{R} |\ f \text{ is a continuous function}\}$$

endowed with the sup-norm

$$\|f\|_C = \sup_{x \in \overline{D}} |f(x)|. \quad (2.4)$$

It follows that

$$\|S(t)f\|_C \leq \|f\|_C, \quad \||f|^p\|_C \leq \|f\|_C^p, \quad p \geq 1, \ f \in C(\overline{D}, \mathbb{R}). \quad (2.5)$$

We also consider the following operator norm (on the appropriate space):

$$\|(A, \gamma)\|_{C([0,T];C \times \mathbb{R})} := \|A\|_{C([0,T];C)} + \|\gamma\|_{C([0,T];\mathbb{R})}. \quad (2.6)$$

Finally, we define

$$x \wedge y := \min\{x, y\} \text{ and } x \vee y := \max\{x, y\} \quad \text{for } x, y \in \mathbb{R}.$$

Based on the aforementioned preliminaries, we shall prove the following result on local existence and uniqueness of solutions to equation (1.3).

Proposition 2.1 (Itô's Lemma)**.** *Suppose that* $f = f(t, B_t) \in C^2$, *i.e., it has continuous partial derivatives up to order two. Then with probability 1, for all* $t > 0$ *and* $x = B_t$,

$$df(t, x) = \left(\frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \right) dt + \frac{\partial f}{\partial x} dB_t. \tag{2.7}$$

Proposition 2.2. *For every* $K > 0$ *there exists* $T = T(K) > 0$ *such that for all* ω *in* $E \subset \Omega$ *as defined in* (2.2), *equation* (1.3) *has a unique solution* $(A, \gamma) \in C([0, T \wedge \tau_K]; C(D, \mathbb{R}) \times \mathbb{R})$ *such that for all* $t \in [0, T \wedge \tau_K]$, *with* $\overline{\gamma}(t)$ *defined by* $\overline{\gamma}(t) := \frac{\tau}{\eta} \gamma(t)$,

$$A(t) = S(t) A_0 + \int_0^t S(t - u) \left(\frac{A^p(u)}{\gamma^q(u) + b} \right) du, \tag{2.8}$$

$$\overline{\gamma}(t) = R(t, B_t) \gamma_0 + \frac{1}{\eta} \int_0^t R(t - u, B_t - B_u) \left(\frac{\overline{A^r}(u)}{\gamma^s(u)} \right) du. \tag{2.9}$$

Proof. Without loss of generality, we assume in what follows that the constants $\tau = \eta = 1$; which implies that $\overline{\gamma}(t)$ reads $\gamma(t)$. Via application of the product rule, it is easy to see from (2.3) that for $x = B_t$,

$$\begin{aligned}
\frac{\partial \gamma}{\partial t} &= -\frac{3}{2} R(t, B_t) \gamma_0 + \frac{\partial}{\partial t} \int_0^t R(t - u, B_t - B_u) \left(\frac{\overline{A^r}(u)}{\gamma^s(u)} \right) du \\
&= -\frac{3}{2} R(t, B_t) \gamma_0 - \frac{3}{2} \int_0^t R(t - u, B_t - B_u) \left(\frac{\overline{A^r}(u)}{\gamma^s(u)} \right) du + \frac{\overline{A^r}(t)}{\gamma^s(t)} \\
&= -\frac{3}{2} \gamma + \frac{\overline{A^r}(t)}{\gamma^s(t)},
\end{aligned} \tag{2.10}$$

and

$$\frac{\partial \gamma}{\partial x} = \gamma, \quad \frac{\partial^2 \gamma}{\partial x^2} = \gamma. \tag{2.11}$$

It follows from Itô's derivative (see (2.7)) that

$$\begin{aligned}
d\gamma &= \left(\frac{\partial \gamma}{\partial t} + \frac{1}{2} \frac{\partial^2 \gamma}{\partial x^2} \right) dt + \frac{\partial \gamma}{\partial x} dB_t \\
&= \left(-\frac{3}{2} \gamma + \frac{\overline{A^r}(t)}{\gamma^s(t)} + \frac{1}{2} \gamma \right) dt + \gamma dB_t \\
&= -\gamma dt + \frac{\overline{A^r}(t)}{\gamma^s(t)} dt + \gamma dB_t.
\end{aligned} \tag{2.12}$$

Thus, (2.9) implies (1.3). Next, for every $\omega \in E \subset \Omega$, we first define the space

$$\begin{aligned}
\mathbb{D}(T, K, L, \omega) := \Big\{ &(A(\omega), \gamma(\omega) \in C\big([0, T \wedge \tau_K(\omega)], C(D, \mathbb{R}) \times (0, \infty)\big) : \\
&\gamma(\omega, t) \geq e^{-\frac{3}{2} - K} \gamma_0, \ A(0) = A_0, \ \gamma(0) = \gamma_0, \\
&\|(A, \gamma)(\omega)\|_{C([0, T \wedge \tau_K(\omega), C \times (0, \infty)])} \leq L \Big\},
\end{aligned} \tag{2.13}$$

where $T \in (0,1]$ depends on K, L, A_0, γ_0 with

$$L > 2 + \|A_0\|_C + e^K \gamma_0. \tag{2.14}$$

We simply denote $\mathbb{D}(T, K, L, \omega)$ by $\mathbb{D}$ (and drop all ω). Next, we define the distance between $(A_1, \gamma_1), (A_2, \gamma_2) \in C\big([0, T \wedge \tau_K(\omega)], C(D, \mathbb{R}) \times \mathbb{R}\big)$ by

$$d\Big((A_1, \gamma_1), (A_2, \gamma_2)\Big) := \big\|(A_1 - A_2, \gamma_1 - \gamma_2)\big\|_{C\big([0, T \wedge \tau_K(\omega)], C \times \mathbb{R}\big)}. \tag{2.15}$$

It is clear that the $\mathbb{D}$ is a closed metric space with the metric d; that is, $\mathbb{D}$ is a complete metric space. Now, consider

$$F_1(A, \gamma)(t) := S(t)A_0 + \int_0^t S(t-u)\Big(\frac{A^p(u)}{\gamma^q(u) + b}\Big) du, \tag{2.16}$$

$$F_2(A, \gamma)(t) := R(t, B_t)\gamma_0 + \int_0^t R(t-u, B_t - B_u)\Big(\frac{\overline{A^r}(u)}{\gamma^s(u)}\Big) du, \tag{2.17}$$

and

$$F(A, \gamma)(t) := \big(F_1(A, \gamma)(t), F_2(A, \gamma)(t)\big). \tag{2.18}$$

In order to use the Banach fixed point theorem (i.e., the contraction mapping theorem) which guarantees the existence of a local unique pair of solutions (i.e., a fixed-point) to (2.8) and (2.9), we shall prove the following:

1. There exists $T := T(K, L, \|A_0\|_C, \gamma_0) > 0$ such that

$$F(A, \gamma)(t) \in \mathbb{D} \text{ whenever } (A, \gamma) \in \mathbb{D}. \tag{2.19}$$

2. There exists $T := T(K, L, \|A_0\|_C, \gamma_0) > 0$ such that

$$d\Big(F(A_1, \gamma_1), F(A_2, \gamma_2)\Big) \leq \frac{1}{2} d\Big((A_1, \gamma_1), (A_2, \gamma_2)\Big), (A_i, \gamma_i) \in \mathbb{D}, \ i = 1, 2. \tag{2.20}$$

We first show (1). It is clear that

$$F(A, \gamma)(0) = (A_0, \gamma_0). \tag{2.21}$$

Now, let $(A, \gamma) \in \mathbb{D}$ be given. By using (2.4) and (2.16), we get

$$\|F_1(A, \gamma)\|_{C([0,t];C)} \leq \|A_0\|_C + b^{-1} \int_0^t \|A(u)\|_C^p du$$
$$\leq \|A_0\|_C + b^{-1} L^p t \tag{2.22}$$

and by (2.17), we obtain

$$\|F_2(A, \gamma)\|_{C([0,t];\mathbb{R})} \leq e^{-\frac{3}{2}t + B_t}\gamma_0 + \gamma_0^{-s} \int_0^t e^{-\frac{3}{2}(t-u) + B_t - B_u}\|A(u)\|_C^p du$$
$$\leq e^K \gamma_0 + e^{\frac{3}{2}s + Ks + 2K} L^p t. \tag{2.23}$$

Setting
$$T_1 := bL^{-p}, \ T_2 := e^{-\frac{3}{2}s - Ks - 2K} L^{-p} \text{ and } \hat{T} := \min\{T_1, T_2\},$$
it follows from (2.14), (2.22) and (2.23) that
$$\|F(A,\gamma)\|_{C([0,T\wedge\tau_K,C\times(0,\infty)])} \le 2 + \|A_0\|_C + e^K \gamma_0 \le L,$$
which implies immediately that $F(A,\gamma) \in \mathbb{D}$.

Next, let us show (2). Indeed, for all $(A_1,\gamma_1),(A_2,\gamma_2) \in \mathbb{D}$,

$$\|F_1(A_1,\gamma_1) - F_1(A_2,\gamma_2)\|_{C([0,t];C)}$$
$$\le \int_0^t \left\| \frac{A_1^p(u)}{\gamma_1^q(u) + b} - \frac{A_2^p(u)}{\gamma_2^q(u) + b} \right\|_C du$$
$$\le \int_0^t \frac{\|A_1^p(u) - A_2^p(u)\|_C}{\gamma_1^q(u) + b} du + \int_0^t \|A_2^p(u)\|_C \left| \frac{1}{\gamma_1^q(u) + b} - \frac{1}{\gamma_2^q(u) + b} \right| du. \quad (2.24)$$

Now, let us estimate the first term. Considering the convex combination
$$A_\lambda(t) := \lambda A_1(t) + (1-\lambda) A_2(t), \quad \lambda \in [0,1],$$
we have by (2.5) that

$$\|A_1^p(u) - A_2^p(u)\|_C \le p \int_0^1 \|A_\lambda(u)\|_C^{p-1} \|A_1(u) - A_2(u)\|_C \, d\lambda$$
$$\le pL^{p-1} \|A_1(u) - A_2(u)\|_C; \quad (2.25)$$

which implies that

$$\int_0^t \frac{\|A_1^p(u) - A_2^p(u)\|_C}{\gamma_1^q(u) + b} du \le tpb^{-1} L^{p-1} \|A_1 - A_2\|_{C([0,1],C)}. \quad (2.26)$$

Similarly, considering the convex combination
$$\gamma_\lambda(t) := \lambda \gamma_1(t) + (1-\lambda)\gamma_2(t), \quad \lambda \in [0,1],$$
we have that

$$\int_0^t \|A_2^p(u)\|_C \left| \frac{1}{\gamma_1^q(u) + b} - \frac{1}{\gamma_2^q(u) + b} \right| du$$
$$\le pL^p b^{-2} \int_0^t \int_0^1 |\gamma_\lambda(u)|^{p-1} |\gamma_1(u) - \gamma_2(u)| d\lambda du$$
$$\le tpL^{2p-1} b^{-2} \|\gamma_1 - \gamma_2\|_{C([0,t],\mathbb{R})}. \quad (2.27)$$

Combining (2.26) and (2.27), we get that

$$\|F_1(A_1,\gamma_1) - F_1(A_2,\gamma_2)\|_{C([0,t],C)}$$
$$\le tpL^{p-1} b^{-1}(1 + L^p b^{-1}) \|(A_1,\gamma_1) - (A_2,\gamma_2)\|_{C([0,t],C\times\mathbb{R})}. \quad (2.28)$$

By a similar argument as above, we ascertain that

$$\|F_2(A_1,\gamma_1) - F_2(A_2,\gamma_2)\|_{C([0,t],\mathbb{R})}$$
$$\le te^{2K + (\frac{3}{2}+K)s} \gamma_0^{-s} L^{p-1}(1 + e^{\frac{3}{2}+K} L\gamma_0^{-1}) \|(A_1,\gamma_1) - (A_2,\gamma_2)\|_{C([0,t],C\times\mathbb{R})}. \quad (2.29)$$

It now follows from (2.28) and (2.29) that there exists $\tilde{T} = \tilde{T}(K,L,\gamma_0) > 0$ such that the inequality (2.20) holds. The proof is complete. $\qquad\square$

3. Global Existence

In this section, we shall establish existence and uniqueness of global positive solutions. To prove the global existence and uniqueness result; i.e., Theorem 1.1, we assume that $(A(t), \gamma(t))_{0 \leq t \leq \tilde{t}}$ is a solution of (1.3) such that for all $\omega \in E$,

$$A(\omega) \in C([0, \tilde{t}]; C(\overline{D}, \mathbb{R})), \quad \gamma(\omega) \in C([0, \tilde{t}]; \mathbb{R}),$$

and then we prove an *a priori* estimate for $(A(t), \gamma(t))$ almost surely.

First, we need the following results.

Lemma 3.1. *For the function $\gamma(t)$, we have the following estimates:*

$$\gamma(t) \geq \left(\frac{\eta}{\tau}\right)^{\frac{1}{s+1}} e^{-\frac{3}{2\eta}t + \frac{1}{\sqrt{\eta}}B_t}\gamma_0 \quad t > 0, \tag{3.1}$$

$$\inf_{0 \leq s \leq t} \gamma(s) \geq \left(\frac{\eta}{\tau}\right)^{\frac{1}{s+1}} e^{-\frac{3}{2\eta}t - \frac{1}{\sqrt{\eta}}B_t^*}\gamma_0 \quad t > 0. \tag{3.2}$$

Proof. Using Itô's Lemma and the identity (2.10), we have that, for $x = B_t$,

$$\frac{\partial}{\partial t}\gamma^{s+1}(t)dt = (s+1)\gamma^s(t)\frac{\partial\gamma(t)}{\partial t}dt$$

$$= (s+1)\gamma^s(t)\left(-\frac{3}{2\tau}\gamma(t)dt + \frac{1}{\tau}\frac{\overline{A^r}(t)}{\gamma^s(t)}dt\right) \tag{3.3}$$

$$= -\frac{3}{2\tau}(s+1)\gamma^{s+1}(t)dt + \frac{1}{\tau}(s+1)\overline{A^r}(t)dt,$$

$$\frac{\partial}{\partial x}\gamma^{s+1}(t)dB_t = (s+1)\gamma^s(t)\frac{\partial\gamma(t)}{\partial x}dB_t = \frac{\sqrt{\eta}}{\tau}(s+1)\gamma^{s+1}(t)dB_t, \tag{3.4}$$

$$\frac{\partial^2}{\partial x^2}(\gamma(t))^{s+1}dt = \frac{\partial}{\partial x}\left(\frac{\sqrt{\eta}}{\tau}(s+1)(\gamma(t))^{s+1}\right)dt = \frac{1}{\tau}(s+1)^2\gamma^{s+1}(t)dt. \tag{3.5}$$

It follows from (3.3) – (3.5) that

$$\tau d\gamma^{s+1}(t) = \tau\left(\frac{\partial}{\partial t}\gamma^{s+1}(t) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\gamma^{s+1}(t)\right)dt + \tau\frac{\partial}{\partial x}\gamma^{s+1}(t)dB_t$$

$$= \frac{1}{2}(s+1)(s-2)\gamma^{s+1}(t)dt + \sqrt{\eta}(s+1)\gamma^{s+1}(t)dB_t + (s+1)\overline{A^r}(t)dt; \tag{3.6}$$

which implies that

$$\tau\gamma^{s+1}(t) = \eta e^{-\frac{3}{2\eta}(s+1)t + \frac{1}{\sqrt{\eta}}(s+1)B_t}\gamma_0^{1+s}$$

$$+ \eta(s+1)\int_0^{\frac{t}{\eta}} e^{-\frac{3}{2\eta}(s+1)(t-u) + \frac{1}{\sqrt{\eta}}(s+1)(B_t - B_u)}\overline{A^r}(u)du$$

$$\geq \eta e^{-\frac{3}{2\eta}(s+1)t + \frac{1}{\sqrt{\eta}}(s+1)B_t}\gamma_0^{1+s}; \tag{3.7}$$

from which we derive the estimates (3.1) and (3.2). The proof is complete. $\qquad\square$

Lemma 3.2. *For every constant $\delta > 0$, define the function*

$$h_\delta(x, t) := \frac{A^r(x, t)}{\gamma^{s+1+\delta}(t)}, \quad (x, t) \in D \times [0, T). \tag{3.8}$$

Then, $h_\delta \in L^1\left(D \times [0,T)\right)$ almost surely, and one has that

$$\int_0^{\tilde{t}} \int_D h_\delta(x,t)dxdt \leq \frac{\tau}{\delta\gamma_0^\delta} + \frac{\delta-3}{2\gamma_0^\delta}\tilde{t}\left(\frac{\eta}{\tau}\right)^{-\frac{\delta}{s+1}} e^{\frac{3\delta}{2\eta}+\frac{\delta}{\eta}K} + \sqrt{\eta} \sup_{0 \leq t \leq \tilde{t}}\left|\int_0^t \frac{1}{\gamma^\delta(s)}dB_s\right|.$$

$$(3.9)$$

Proof.   By using a similar argument as in Lemma 3.1, we have that

$$\tau\frac{\partial}{\partial t}\gamma^{-\delta}(t)dt = -\frac{3\delta}{2}\gamma^{-\delta}(t)dt - \delta\frac{\overline{A^r}(t)}{\gamma^{s+1+\delta}(t)}dt, \tag{3.10}$$

$$\tau\frac{\partial}{\partial x}\gamma^{-\delta}(t)dB_t = -\delta\sqrt{\eta}\gamma^{-\delta}(t)dB_t, \tag{3.11}$$

$$\tau\frac{\partial^2}{\partial x^2}\gamma^{-\delta}(t)dt = \delta^2\gamma^{-\delta}(t)dt. \tag{3.12}$$

It follows from (3.10) – (3.12) that

$$\tau d\gamma^{-\delta}(t) = \frac{1}{2}\delta(\delta-3)\gamma^{-\delta}(t)dt - \delta\sqrt{\eta}\gamma^{-\delta}(t)dB_t - \delta\frac{\overline{A^r}(t)}{\gamma^{s+1+\delta}(t)}dt. \tag{3.13}$$

This implies by (3.2) that

$$\int_0^{\tilde{t}} \frac{\overline{A^r}(t)}{\gamma^{s+1+\delta}(t)}dt \leq \frac{\tau}{\delta}\gamma_0^{-\delta} + \frac{1}{2}(\delta-3)\int_0^{\tilde{t}} \gamma^{-\delta}(t)dt + \sqrt{\eta}\sup_{0 \leq t \leq \tilde{t}}\left|\int_0^t \frac{1}{\gamma^\delta(s)}dB_s\right|$$

$$\leq \frac{\tau}{\delta\gamma_0^\delta} + \frac{\delta-3}{2}\tilde{t}\left(\frac{\eta}{\tau}\right)^{-\frac{\delta}{s+1}}e^{\frac{3\delta}{2}+\frac{\delta}{\eta}K}\gamma_0^{-\delta} + \sqrt{\eta}\sup_{0 \leq t \leq \tilde{t}}\left|\int_0^t \frac{1}{\gamma^\delta(s)}dB_s\right|.$$

Now, it suffices to show that

$$\sup_{0 \leq t \leq \tilde{t}}\left|\int_0^t \frac{1}{\gamma^\delta(s)}dB_s\right| < \infty \quad \text{almost surely.}$$

Indeed, using Hölder's inequality, martingale inequality and Itô's isometry, we have by (3.2) that

$$\mathbb{E}\sup_{0 \leq t \leq \tilde{t}}\left|\int_0^t \frac{1}{\gamma^\delta(s)}dB_s\right| \leq \left(\mathbb{E}\sup_{0 \leq t \leq \tilde{t}}\left|\int_0^t \frac{1}{\gamma^\delta(s)}dB_s\right|^2\right)^{1/2}$$

$$\leq \sqrt{2}\left(\mathbb{E}\left|\int_0^t \frac{1}{\gamma^\delta(s)}dB_s\right|^2\right)^{1/2}$$

$$\leq \sqrt{2}\left(\mathbb{E}\int_0^t \frac{1}{\gamma^{2\delta}(s)}dB_s\right)^{1/2}$$

$$\leq \sqrt{2}\gamma_0^{-\delta}\left(\frac{\eta}{\tau}\right)^{-\frac{\delta}{s+1}}\left(\int_0^{\tilde{t}} \mathbb{E}e^{\frac{3}{\eta}t-\frac{2}{\sqrt{\eta}}B_t^*}dt\right)^{1/2} < \infty.$$

The proof is complete.   □

Lemma 3.3. *For any two constants $\alpha > 1, \beta \geq 0$, define the function*

$$h_{\alpha,\beta}(t, B_t) := \int_D \frac{A^\alpha(x, t)}{\gamma^\beta(t, B_t)} dx, \quad 0 \leq t < T.$$

$$\frac{p-1}{r} < \min\left\{\frac{2}{N+2}, \frac{q}{s+1}\right\},$$

it follows that for all $\omega \in E$ defined in (2.2) up to negligible set,

$$dh_{\alpha,\beta} \leq \left(\frac{1}{2\tau}(3\beta + \beta^2) - \alpha\right)h_{\alpha,\beta}dt - \left(\frac{\sqrt{\eta}\beta}{\tau} + \beta\right)h_{\alpha,\beta}dB_t + v(t)h_{\alpha,\beta}, \quad (3.14)$$

where $v(t)$ is an integrable function on $(0, T)$, almost surely.

Proof.   Let $\alpha > 1$ and $\beta \geq 0$. Using Itô's Lemma, we have that for $s = B_t$,

$$dh_{\alpha,\beta}(t, s) = \left(\frac{\partial h_{\alpha,\beta}}{\partial t} + \frac{1}{2}\frac{\partial^2 h_{\alpha,\beta}}{\partial s^2}\right)dt + \frac{\partial h_{\alpha,\beta}}{\partial s}dB_t.$$

By using similar arguments as in $(3.4) - (3.5)$ and (3.13), we get

$$\frac{\partial h_{\alpha,\beta}}{\partial t}dt$$
$$= \int_D \left[\alpha\frac{A^{\alpha-1}}{\gamma^\beta}A_t dt - \beta\frac{A^\alpha}{\gamma^{\beta+1}}\frac{\partial\gamma}{\partial t}dt\right]dx$$
$$= \int_D \left[\alpha\frac{A^{\alpha-1}}{\gamma^\beta}\left(\epsilon^2\Delta A - A + \frac{A^p}{\gamma^q + b}\right)dt - \beta\frac{A^\alpha}{\gamma^{\beta+1}}\left(-\frac{3}{2\tau}\gamma dt + \frac{1}{\tau}\frac{\overline{A^r}(t)}{\gamma^s}dt\right)\right]dx$$
$$= \left(\frac{3\beta}{2\tau} - \alpha\right)h_{\alpha,\beta}dt + \alpha\epsilon^2\int_D \frac{A^{\alpha-1}}{\gamma^\beta}\Delta A dx dt + \alpha\int_D \frac{A^{\alpha+p-1}}{\gamma^\beta(\gamma^q + b)}dx dt$$
$$- \frac{\beta}{\tau}\int_D \frac{A^\alpha\overline{A^r}}{\gamma^{\beta+s+1}}dx dt,$$

$$(3.15)$$

$$\frac{\partial^2 h_{\alpha,\beta}}{\partial s^2}dt = \frac{\beta^2}{\tau}\int_D \frac{A^\alpha}{\gamma^\beta}dx dt = \frac{\beta^2}{\tau}h_{\alpha,\beta}, \quad (3.16)$$

and

$$\frac{\partial h_{\alpha,\beta}}{\partial s}dB_t = -\frac{\sqrt{\eta}\beta}{\tau}\int_D \frac{A^\alpha}{\gamma^\beta}dx dB_t = -\frac{\sqrt{\eta}\beta}{\tau}h_{\alpha,\beta}dB_t. \quad (3.17)$$

Therefore, $(3.15) - (3.17)$ imply that

$$dh_{\alpha,\beta}(t, s) = \left(\frac{1}{2\tau}(3\beta + \beta^2) - \alpha\right)h_{\alpha,\beta}(t, s)dt - \frac{\sqrt{\eta}\beta}{\tau}h_{\alpha,\beta}(t, s)dB_t$$
$$+ \left(\alpha\epsilon^2\int_D \frac{A^{\alpha-1}}{\gamma^\beta}\Delta A dx + \alpha\int_D \frac{A^{\alpha+p-1}}{\gamma^\beta(\gamma^q + b)}dx - \frac{\beta}{\tau}\int_D \frac{A^\alpha\overline{A^r}}{\gamma^{\beta+s+1}}dx\right)dt.$$

Since

$$\alpha\epsilon^2\int_D \frac{A^{\alpha-1}}{\gamma^\beta}\Delta A dx = \alpha\epsilon^2\int_{\partial D} \frac{A^{\alpha-1}}{\gamma^\beta}\nabla A \cdot \nu dS - \alpha\epsilon^2(\alpha - 1)\int_D \frac{A^{\alpha-2}}{\gamma^\beta}|\nabla A|^2 dx$$

$$= \alpha\epsilon^2 \int_{\partial D} \frac{A^{\alpha-1}}{\gamma^\beta} \left(\frac{-aA}{\epsilon}\right) dS - \alpha\epsilon^2(\alpha-1) \int_D \frac{A^{\alpha-2}}{\gamma^\beta} |\nabla A|^2 dx$$

$$= -\alpha\epsilon a \int_{\partial D} \frac{A^\alpha}{\gamma^\beta} dS - \alpha\epsilon^2(\alpha-1) \int_D \frac{A^{\alpha-2}}{\gamma^\beta} |\nabla A|^2 dx$$

$$\leq -\alpha\epsilon^2(\alpha-1) \int_D \frac{A^{\alpha-2}}{\gamma^\beta} |\nabla A|^2 dx,$$

we obtain the following inequality,

$$dh_{\alpha,\beta}(t,s) \leq \left(\frac{1}{2\tau}(3\beta + \beta^2) - \alpha\right) h_{\alpha,\beta}(t,s)dt - \frac{\sqrt{\eta}\beta}{\tau} h_{\alpha,\beta}(t,s)dB_t + E_1 + E_2, \tag{3.18}$$

where

$$E_1 = -\alpha\epsilon^2(\alpha-1) \int_D \frac{A^{\alpha-2}}{\gamma^\beta} |\nabla A|^2 \, dxdt, \tag{3.19}$$

and

$$E_2 = \alpha \int_D \frac{A^{\alpha+p-1}}{\gamma^\beta (\gamma^q + b)} \, dxdt. \tag{3.20}$$

Now, we concentrate on estimates of E_1 and E_2. To do so, let us define the number $0 < \kappa < 1$ by

$$\kappa =: \frac{p-1}{r} = \frac{q}{s+1+\delta} \text{ for some } \delta > 0 \text{ and } (p-1) < \kappa r, \ q = \kappa(s+1+\delta).$$

Then we obtain

$$\frac{A^{\alpha+p-1}}{\gamma^q} = \left(\frac{A^r}{\gamma^{s+1+\delta}}\right)^\kappa A^\alpha = (h_\delta)^\kappa z^2, \tag{3.21}$$

where $h_\delta = \dfrac{A^r}{\gamma^{s+1+\delta}}$ is defined in the statement of Lemma 3.2 and

$$z := A^{\alpha/2}. \tag{3.22}$$

Notice that (3.22) implies that

$$|\nabla z|^2 = \frac{\alpha^2}{4} A^{\alpha-2} |\nabla A|^2. \tag{3.23}$$

By Hölder's inequality, it follows from (3.21) – (3.22) that

$$\int_D \frac{A^{\alpha+p-1}}{\gamma^q} dx = \int_D (h_\delta)^\kappa z^2 \leq \|h_\delta\|_{L^1(D)}^\kappa \|z\|_{L^{\frac{2}{1-\kappa}}(D)}^2. \tag{3.24}$$

Since $0 < \kappa \leq \frac{2}{N+2} < \frac{2}{N}$, it follows from Gagliardo-Nirenberg inequality (see e.g. [18]) that there is a constant $C = C(D, N, \kappa) > 0$ such that for $\theta := \frac{N\kappa}{2} \in (0,1)$,

$$\|z\|_{L^{\frac{2}{1-\kappa}}(D)}^2 \leq C\left[\|\nabla z\|_{L^2(D)}^\theta \|z\|_{L^2(D)}^{1-\theta} + \|z\|_{L^2(D)}\right]^2$$

$$\leq 4C\Big[\|\nabla z\|_{L^2(D)}^{2\theta}\|z\|_{L^2(D)}^{2(1-\theta)} + \|z\|_{L^2(D)}^2\Big]. \tag{3.25}$$

It follows from (3.24) and (3.25) that

$$\int_D \frac{A^{\alpha+p-1}}{\gamma^q}dx \leq 4C\|h_\delta\|_{L^1(D)}^{\kappa}\Big[\|\nabla z\|_{L^2(D)}^{2\theta}\|z\|_{L^2(D)}^{2(1-\theta)} + \|z\|_{L^2(D)}^2\Big]. \tag{3.26}$$

Since by Young's inequality one has that, for $\lambda > 0$,

$$\|h_\delta\|_{L^1(D)}^{\kappa}\|\nabla z\|_{L^2(D)}^{2\theta}\|z\|_{L^2(D)}^{2(1-\theta)} \leq \theta\lambda^{1/\theta}\|\nabla z\|_{L^2(D)}^2 + \frac{1-\theta}{\lambda^{\frac{1}{1-\theta}}}\|h_\delta\|_{L^1(D)}^{\frac{\kappa}{1-\theta}}\|z\|_{L^2(D)}^2, \tag{3.27}$$

then by choosing $\lambda > 0$ sufficiently small such that

$$4\alpha\theta C\lambda^{1/\theta} < \frac{\epsilon^2(\alpha-1)}{\alpha},$$

one has that, by using (3.23) and (3.26) – (3.27),

$$\begin{aligned}
E_1 + E_2 &\leq \gamma^{-\beta}\Big(-\alpha^{-1}\epsilon^2(\alpha-1)\|\nabla z\|_{L^2(D)}^2 + 4\alpha C\|h_\delta\|_{L^1(D)}^{\kappa}\|\nabla z\|_{L^2(D)}^{2\theta}\|z\|_{L^2(D)}^{2(1-\theta)} \\
&\quad + 4\alpha C\|h_\delta\|_{L^1(D)}^{\kappa}\|z\|_{L^2(D)}^2\Big)dt \\
&\leq \gamma^{-\beta}\Big(-\alpha^{-1}\epsilon^2(\alpha-1)\|\nabla z\|_{L^2(D)}^2 + 4\alpha\theta C\lambda^{1/\theta}\|\nabla z\|_{L^2(D)}^2 \\
&\quad + \Big[\frac{1-\theta}{\lambda^{\frac{1}{1-\theta}}}\|h_\delta\|_{L^1(D)}^{\frac{\kappa}{1-\theta}} + 4\alpha C\|h_\delta\|_{L^1(D)}^{\kappa}\Big]\|z\|_{L^2(D)}^2\Big)dt \\
&\leq C_1\Big[\|h_\delta\|_{L^1(D)}^{\frac{\kappa}{1-\theta}} + \|h_\delta\|_{L^1(D)}^{\kappa}\Big]h_{\alpha,\beta}dt, \quad C_1 > 0. \tag{3.28}
\end{aligned}$$

Therefore, by using (3.18),

$$dh_{\alpha,\beta} \leq \Big(\frac{1}{2\tau}(3\beta+\beta^2)-\alpha\Big)h_{\alpha,\beta}(t,s)dt - \frac{\sqrt{\kappa}\beta}{\tau}h_{\alpha,\beta}(t,s)dB_t + C_1v(t)h_{\alpha,\beta}dt, \tag{3.29}$$

where

$$v(t) = \|h_\delta(t)\|_{L^1(D)}^{\frac{\kappa}{1-\theta}} + \|h_\delta(t)\|_{L^1(D)}^{\kappa}.$$

Since $\frac{\kappa}{1-\theta} \leq 1$, it follows from Lemma 3.2 that $v(t)$ is integrable on $(0,T)$ almost surely. The proof is complete. $\qquad\square$

Lemma 3.4. *Under the conditions in Lemma 3.3 and the set E defined in (2.2) up to negligible set, there exists a constant $C(T) := C_{\alpha,\beta}(T) \leq \infty$ such that for all $t \in [0,T)$ and $\omega \in E \subset \Omega$,*

$$h_{\alpha,\beta}(t,\omega) \leq C(T). \tag{3.30}$$

Proof. Using Ito's Lemma and inequality (3.14) in Lemma 3.3, it follows that for all $t \in [0,T)$ and $\omega \in E \subset \Omega$,

$$d\left[\exp\Big(-\Big(\frac{1}{2\tau}(3\beta+\beta^2)-\alpha\Big)t + C_1\int_0^t v(s)ds + \frac{\sqrt{\eta}\beta}{\tau}B_t(\omega)\Big)h_{\alpha,\beta}(t,\omega)\right] \leq 0. \tag{3.31}$$

Integrating from 0 to t, we get from (3.31) that

$$h_{\alpha,\beta}(t,\omega) \leq C_2 \exp\left(-C_1 \int_0^T v(s)ds\right) h_{\alpha,\beta}(0,\omega),$$

where

$$C_2 = \exp\left(\left(\frac{1}{2\tau}(3\beta + \beta^2) - \alpha\right)T\right) \exp\left(\frac{\sqrt{\eta}\beta}{\tau} \sup_{0 \leq t \leq T} |B_t|\right).$$

The proof is complete.   □

From Lemma 3.1 and Lemma 3.4, we deduce the Corollary below.

Corollary 3.1. *Let $\ell \geq 1$ and all other assumptions in Theorem 1.1, Lemma 3.3 and Lemma 3.4 hold true. Define*

$$g_1(A,\gamma) = \frac{A^p}{\gamma^q + b},$$

$$g_2(A,\gamma) = \frac{A^r}{\gamma^s},$$

then there exist positive constant $C_\ell(T)$, such that

$$\|g_j(A,\gamma)\|_{L^\ell(\Omega)} \leq C_\ell(T) \qquad j = 1, 2$$

for all $0 \leq t < T$.

Proof.   The proof to this Corollary follows from Lemma 3.4.   □

Proof of Theorem 1.1.   Under the conditions in Lemma 3.3 and the set E defined in (2.2) up to negligible set, using (2.5), (2.8) and Corollary 3.1, we have that for all $0 \leq t \leq T$,

$$\begin{aligned}
\|A(t)\|_{L^2(D)} \leq &\|S(t)A_0\|_{L^2(D)} + \int_0^t \left\|S(t-u)\left(\frac{A^p(u)}{\gamma^q(u) + b}\right)\right\|_{L^2(D)} du \\
\leq &\|A_0\|_{L^2(D)} + T\left\|\frac{A^p(u)}{\gamma^q(u) + b}\right\|_{L^2(D)} \\
\leq &\|A_0\|_{L^2(D)} + TC_2(T) \qquad\qquad\qquad\qquad\qquad (3.32)
\end{aligned}$$

In addition, one is able to obtain the estimate (1.4) from (3.2). With these estimates, the unique local solution obtained in Proposition 2.2 may now be continued indefinitely to obtain a global solution. The proof is complete.   □

Acknowledgements

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