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POSITIVE PERIODIC SOLUTION FOR SECOND-ORDER NONLINEAR DIFFERENTIAL EQUATION WITH SINGULARITY OF ATTRACTIVE TYPE*

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Abstract This paper is devoted to investigate the following second-order nonlinear differential equation with singularity of attractive type

$$x'' - a(t)x = f(t,x) + e(t),$$

where the nonlinear term f has a singularity at the origin. By using the Green's function of the linear differential equation with constant coefficient and Schauder's fixed point theorem, we establish some existence results of positive periodic solutions.

Keywords Positive periodic solutions, singularity of attractive type, attractive-repulsive singularities.

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1. Introduction

In this paper, we discuss the existence of positive periodic solutions of the following nonlinear differential equation with singularity

$$x'' - a(t)x = f(t,x) + e(t), \tag{1.1}$$

where $a(t) \in C(\mathbb{R}, \mathbb{R}^+)$ and $e(t) \in L^1(\mathbb{R})$ are ω-periodic functions, $f(t,x) \in C(\mathbb{R} \times \mathbb{R}^+, \mathbb{R})$ is an ω-periodic function on t. The nonlinear term f of equation (1.1) can be with a singularity at the origin, i.e.,

$$\lim_{x\to 0^+} f(t,x) = -\infty, \qquad (\text{or } \lim_{x\to 0^+} f(t,x) = +\infty), \qquad \text{uniformly in } t.$$

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It is said that equation (1.1) is of attractive type (resp. repulsive type) if $f(t,x) \to -\infty$ (resp. $f(t,x) \to +\infty$) as $x \to 0^+$.

Since 1980s, there have been published many works in which singularity of differential equations is discussed. More concretely, in 1987, Lazer and Solimini [10] investigated the model equations with singularity

$$x'' = -\frac{\nu}{x^\lambda} + f(t) \tag{1.2}$$

and

$$x'' = \frac{\mu}{x^\lambda} + f(t), \tag{1.3}$$

where $\lambda,\ \nu,\ \mu$ are positive constants and f is a continuous periodic functions with period ω. It is said that equation (1.2) has an attractive singularity, whereas equation (1.3) has a repulsive singularity. The authors provided the necessary and sufficient conditions for the existence of periodic solutions of equations (1.2) and (1.3). One of the common conditions to guarantee the existence of positive periodic solution is a so-called strong force condition (corresponds to the case $\lambda \geq 1$ in equation (1.2)), see [1,6,7,17,18,20,23] and references therein. On the other hand, the existence of positive periodic solution of the singular differential equations has been established with a weak force condition (corresponds to the case $0 < \lambda < 1$ in equation (1.2)), see [2,5,11,14].

During the last two decades, the study of the existence of positive periodic solutions for second-order differential equations with singularity of repulsive type has attracted the attention of many researchers (see [1,2,4–9,11–19,23]). For example, Torres [14] in 2007 investigated a kind of second order non-autonomous singular differential equation

$$x'' + a(t)x = f(t,x) + e(t). \tag{1.4}$$

Applying Schauder's fixed point theorem, the author showed that the additional assumption of a weak singularity enabled the obtention of new criteria for the existence of positive periodic solutions. In 2010, Wang [17] discussed the existence and multiplicity of positive periodic solutions of singular systems (1.4) with superlinearity or sublinearity assumptions at infinity for an appropriately chosen parameter $e(t)$. The proofs of their results are based on the Krasnoselskii fixed point theorem in cones. In 2014, Ma, Chen and He [11] improved above results and presented a new assumption which was weaker than the singular condition in [14].

All the aforementioned results are related to second-order differential equations with singularity of repulsive type. Naturally, a new question arises: how second-order differential equation works on singularity of attractive type? In this paper, we try to establish the existence of a positive periodic solution of equation (1.1) by using the Green's function of the linear differential equation with constant coefficient and Schauder's fixed point theorem. This trick has been used to investigate a third-order singular differential equation with variable coefficients in [21].

Remark 1.1. As far as we know, the calculation of the Green's function of the second-order linear differential equation with variable coefficient

$$x'' - a(t)x = h(t), \quad h \in C(\mathbb{R}, \mathbb{R}^-) \text{ is an } \omega\text{-periodic function},$$

is very complicated. In this paper, we will discuss the Green's function $G(t,s)$ of the second-order linear differential equation with constant coefficient

$$x'' - Mx = h(t), \tag{1.5}$$

where $M = \max_{t\in[0,\omega]} a(t)$.

Remark 1.2. It is worth noting that the nonlinear term f of equation (1.1) satisfies singularity of attractive type, i.e., $\lim_{x\to 0^+} f(t,x) = -\infty$, uniformly in t. Obviously, attractive condition and repulsive condition are contradiction. Therefore, the above conditions in [11, 14, 17] are no long applicable to the proof of existence of positive periodic solution for equation (1.1) with singularity of attractive type. In this paper, we need find another conditions to overcome this problems.

From now on, we denote the essential supremum and infimum of the external force $e(t) \in C[0,\omega]$ by e^* and e_*. We define the function $\gamma : \mathbb{R} \to \mathbb{R}$ by

$$\gamma(t) = \int_0^\omega G(t,s)e(s)ds,$$

which is the unique ω-periodic solution of equation

$$x''(t) - Mx(t) = e(t). \tag{1.6}$$

Obviously, $\gamma(t)$ is closely depending on the external force $e(t)$. We would like to emphasize that the value of $\gamma(t)$ can influence the existence of a positive periodic solution of equation (1.1) with strong singularity or weak singularity. Specifically, the existence of a positive periodic solution of equation (1.1) with weak and strong singularities of attractive type if the following conditions satisfies:

$$\gamma_* > 0.$$

The existence of a positive periodic solution to equation (1.1) with weak singularity of attractive type if the following conditions satisfies:

$$\gamma_* = 0 \quad \text{or} \quad \gamma^* \leq 0.$$

Moreover, we consider the existence of a positive periodic solution for equation (1.1) with attractive-repulsive singularities.

The paper is organized as follows: In Section2, the Green's function for constant coefficients differential equation (1.5) will be given. Some useful properties for the Green's function are shown also. In Section 3, we consider the positive periodic solution of (1.1) with attractive singularities in three cases: $\gamma_* > 0$, $\gamma_* = 0$ and $\gamma_* < 0$. Moreover, we also proved the existence of a positive periodic solution when (1.1) has an attractive-repulsive singularity. To conclude this introduction, we write $d(t) \prec 0$ if $d(t) \leq 0$ for a.e. $t \in [0,\omega]$ and it is negative in a set of negative measure.

2. Preparation

2.1. Constant coefficients differential equation

In this section, we discuss the Green's function of the differential equation with constant coefficients

$$\begin{cases} x'' - Mx = h(t), \\ x(0) = x(\omega), x'(0) = x'(\omega). \end{cases} \tag{2.1}$$

Let $M := \varrho^2$, then equation (2.1) is transformed into

$$\begin{cases} y'(t) + \varrho y(t) = h(t), \\ y(0) = y(\omega), \end{cases} \tag{2.2}$$

and

$$\begin{cases} x'(t) - \varrho x(t) = y(t), \\ x(0) = x(\omega). \end{cases} \tag{2.3}$$

Solution of equation (2.2) is written as

$$y(t) = \int_0^\omega G_1(t,s)h(s)ds, \tag{2.4}$$

where

$$G_1(t,s) = \begin{cases} \dfrac{e^{-\varrho(t-s)}}{1-e^{-\omega\varrho}}, & 0 \le s \le t \le \omega, \\ \dfrac{e^{-\varrho(\omega+t-s)}}{1-e^{-\omega\varrho}}, & 0 \le t < s \le \omega. \end{cases}$$

Solution of equation (2.3) is written as

$$x(t) = \int_0^\omega G_2(t,s)y(s)ds, \tag{2.5}$$

where

$$G_2(t,s) = \begin{cases} \dfrac{e^{\varrho(t-s)}}{1-e^{\omega\varrho}}, & 0 \le s \le t \le \omega, \\ \dfrac{e^{\varrho(\omega+t-s)}}{1-e^{\omega\varrho}}, & 0 \le t < s \le \omega. \end{cases}$$

Therefore, we know that the solution of equation (2.1) is written as

$$x(t) = \int_0^\omega G_2(t,\tau)\int_0^\omega G_1(\tau,s)h(s)dsd\tau = \int_0^\omega \left[\int_0^\omega G_2(t,\tau)G_1(\tau,s)d\tau\right] h(s)ds.$$

Denote

$$G(t,s) = \int_0^\omega G_2(t,\tau)G_1(\tau,s)d\tau, \tag{2.6}$$

then we can get

$$x(t) = \int_0^\omega G(t,s)h(s)ds. \tag{2.7}$$

Lemma 2.1. *The Green function $G(t,s) < 0$ for all $(t,s) \in [0,\omega] \times [0,\omega]$.*

Proof. Since $G_1(t,s) > 0$ and $G_2(t,s) < 0$ for all $(t,s) \in [0,\omega] \times [0,\omega]$, then we know $G(t,s) < 0$ for all $(t,s) \in [0,\omega] \times [0,\omega]$ from (2.6). $\square$

2.2. Variable coefficients differential equation

In this section, we consider variable coefficients differential equations

$$\begin{cases} x'' - a(t)x = h(t), \\ x(0) = x(\omega), \ x'(0) = x'(\omega), \end{cases} \tag{2.8}$$

where $h(t) \in C(\mathbb{R}, \mathbb{R}^-)$ is an ω-periodic function. Obviously, the calculation of the Green's function of equation (2.8) is very complicated. To overcome this difficuly, we will make a shift on the linear term.

Let $X := \{\phi \in C(\mathbb{R}, \mathbb{R}) : \phi(t + \omega) = \phi(t)\}$ with the maximum norm $\|\phi\| = \max\limits_{t\in[0,\omega]} |\phi(t)|$. Obviously, X is a Banach space. Denote

$$M := \max_{t\in[0,\omega]} a(t) \quad \text{and} \quad m := \min_{t\in[0,\omega]} a(t),$$

then equation (2.8) can be rewritten as

$$x'' - Mx = (a(t) - M)x + h(t). \tag{2.9}$$

Define operators $T,\ H : X \to X$ by

$$(Th)(t) := \int_0^\omega G(t, s)h(s)ds \quad \text{and} \quad (Hx)(t) := (a(t) - M)x(t). \tag{2.10}$$

Since $h(t) \in (\mathbb{R}, \mathbb{R}^-)$, we have $(Th)(t) > 0$ for any $t \in \mathbb{R}$ by Lemma 2.1. At the same time, we know $\|H\| \leq M - m$. By equation (2.7), the solution of equation (2.9) can be written in the form

$$x(t) = (Th)(t) + (THx)(t).$$

Since

$$\|TH\| \leq \|T\|\|H\| \leq \frac{M - m}{M} = 1 - \frac{m}{M} < 1,$$

where we used the fact $\int_0^\omega G(t, s)ds = -\frac{1}{M}$ (see Lemma 2.2 in [3]), hence we have

$$x(t) = (I - TH)^{-1}(Th)(t).$$

Define an operator $P : X \to X$ by

$$(Ph)(t) = (I - TH)^{-1}(Th)(t), \tag{2.11}$$

it is obvious that $x(t) = (Ph)(t)$ is the unique periodic solution of equation (2.8) for any $h(t)$. Moveover, we arrive at

Lemma 2.2. *P satisfies*

$$(Th)(t) \leq (Ph)(t), \quad \forall\ t \in \mathbb{R} \quad \textit{and} \quad \|Ph\| \leq \frac{M}{m}\|Th\|. \tag{2.12}$$

Proof. By the Neumann expansion of P, we have

$$\begin{aligned} Ph =& (I - TH)^{-1}Th \\ =& (I + TH + (TH)^2 + \cdots + (TH)^n + \cdots)Th. \end{aligned} \tag{2.13}$$

Since $Th(t) > 0$ for any t, we get

$$(Th)(t) \leq (Ph)(t).$$

Noting that $\|TH\| < 1$, we get

$$\|Ph\| \leq \frac{M}{m}\|Th\|.$$

□

3. Singularity of attractive type

In this section, we establish the existence of a positive periodic solution of equation (1.1) by applications of Schauder's fixed point theorem. Define an operator $Q : X \to X$ by

$$(Qx)(t) = P(f(t, x(t)) + e(t)). \tag{3.1}$$

3.1. Case (I) $\gamma_* > 0$.

Theorem 3.1. *Assume that the following conditions hold:*

(H_1) There exist continuous, non-negative functions $g(x)$, $p(x)$ and continuous, non-positive $\zeta(t)$ such that

$$\zeta(t)(g(x) + p(x)) \leq f(t, x) \leq 0 \quad \text{for all } (t, x) \in [0, \omega] \times (0, \infty),$$

and $g(x) > 0$ is non-increasing and $p(x)$ is non-decreasing in $x \in (0, \infty)$.

(H_2) There exists a constant $R > 0$ such that

$$\frac{M}{m}\left(g(\gamma_*)\left(1 + \frac{p(R)}{g(R)}\right)\Lambda^* + \gamma^*\right) \leq R,$$

where $\Lambda(t) = \int_0^\omega G(t, s)\zeta(s)ds$ and $\Lambda^* = \sup\limits_{t\in[0,\omega]} \Lambda(t)$.

If $\gamma_* > 0$, then equation (1.1) has at least one positive periodic solution.

Proof. Obviously, an ω-periodic solution of equation (1.1) is just a fixed point of operator equation

$$(Qx)(t) = P(f(t, x) + e(t)). \tag{3.2}$$

Let R be the positive constant and $r := \gamma_*$, then we have $R > r > 0$, since $R > \gamma^*$. Define

$$\Omega = \{x \in X : \ r \leq x(t) \leq R \ \text{ for all } t\}, \tag{3.3}$$

then, Ω is a closed convex set. For any $x \in \Omega$, $t \in \mathbb{R}$, from equation (3.1), we deduce

$$(Qx)(t + \omega) = P(f(t + \omega, x(t + \omega) + e(t + \omega)) = P(f(t, x(t) + e(t)) = (Qx)(t),$$

which show that $(Qx)(t)$ is ω-periodic.

Next we will prove $Q(\Omega) \subset \Omega$. In fact, for each $x \in \Omega$ and for all $t \in [0, \omega]$, from Lemma 2.1 and condition (H_1), we know that non-positive sign of the Green's function $G(t, s)$ and the nonlinear term $f(t, x)$ for all $(t, s) \in [0, \omega] \times [0, \omega]$ and $(t, x) \in [0, \omega] \times (0, \infty)$. Together Lemma 2.2, we arrive at

$$(Qx)(t) = P(f(t, x(t) + e(t))$$

$$
\begin{aligned}
&\geq T(f(t,x(t))+e(t))\\
&=\int_0^\omega G(t,s)f(s,x(s))ds+\gamma(t)\\
&=\int_0^\omega G(t,s)f^-(s,x(s))ds+\gamma(t)\\
&\geq \gamma_* := r>0,
\end{aligned}
$$

where $f^-(t,x) := \min\{0, f(t,x)\}$. On the other hand, by Lemma 2.2, we see that

$$
\begin{aligned}
(Qx)(t) &= P(f(t,x(t))+e(t))\\
&\leq \frac{M}{m}\max_{t\in[0,\omega]}\left|\int_0^\omega G(t,s)f(s,x(s))ds+\gamma(t)\right|\\
&\leq \frac{M}{m}\left\{\max_{t\in[0,\omega]}\left|\int_0^\omega G(t,s)f(s,x(s))ds\right|+\gamma^*\right\},
\end{aligned}
$$

since $\gamma_*>0$, we know $\gamma(t)>0$, then $\|\gamma\|=\gamma^*$. Therefore, by conditions (H_1) and (H_2), we get

$$
\begin{aligned}
(Qx)(t) &\leq \frac{M}{m}\left\{\max_{t\in[0,\omega]}\left|\int_0^\omega G(t,s)f^-(s,x(s))ds\right|+\gamma^*\right\}\\
&\leq \frac{M}{m}\left\{\max_{t\in[0,\omega]}\left|\int_0^\omega G(t,s)\zeta(s)(g(x(s))+p(x(s)))ds\right|+\gamma^*\right\}\\
&\leq \frac{M}{m}\left(g(r)\left(1+\frac{p(R)}{g(R)}\right)\Lambda^*+\gamma^*\right)\leq R.
\end{aligned}
$$

In conclusion, we see that $Q(\Omega)\subset\Omega$.

Next, we show that Q is completely continuous. According to equations (2.11), (2.13) and (3.1), we shall prove that T is completely continuous and H is a continuous bounded operator.

Firstly, we show that T is completely continuous. Let $\{h_k\}\in\Omega$ be a convergent sequence of functions, such that $h_k(t)\to h(t)$ as $k\to\infty$. Since Ω is closed, for $h\in\Omega$ and $t\in[0,\omega]$, it is clear that

$$
\begin{aligned}
|(Th_k)(t)-(Th)(t)| &= \left|\int_0^\omega G(t,s)h_k(s)ds-\int_0^\omega G(t,s)h(s)ds\right|\\
&\leq \int_0^\omega |G(t,s)||h_k(s)-h(s)|ds.
\end{aligned}
$$

Since $|h_k(t)-h(t)|\to 0$ as $k\to\infty$, we obtain

$$\lim_{k\to\infty}\|(Th_k)(t)-(Th)(t)\|=0. \tag{3.4}$$

Therefore, T is continuous. On the other hand, we deduce

$$|(Th)(t)|=\left|\int_0^\omega G(t,s)h(s)ds\right|\leq \|h\|\int_0^\omega |G(t,s)|ds\leq\frac{\|h\|}{M},$$

where $\|h\| := \max_{t\in[0,\omega]}|h(t)|$. From Lemma 2.1, it is easy to see that

$$|(T'h)(t)|=\left|\int_0^\omega \frac{\partial G(t,s)}{\partial t}h(s)ds\right|\leq\|h\|\int_0^\omega\left|\frac{\partial G(t,s)}{\partial t}\right|ds\leq\|h\|G_0\omega,$$

where $G_0 := \max_{s,t\in[0,\omega]} \left|\frac{\partial G(t,s)}{\partial t}\right|$. By above two inequalities, we conclude that $\{Th : h \in \Omega\}$ is uniformly bounded and equi-continuous on $t \in [0, \omega]$. Therefore, $T(\Omega)$ is relatively compact. T is compact operator. In conclusion, T is completely continuous.

Secondly, we show that H is a continuous bounded operator. By using similar argument, it is clearly that H is continuous. From equation (2.10), we obtain

$$|(Hy)(t)| = |(a(t) - M)y(t)| \le (M - m)\|y\|,$$

where $\|y\| := \max_{t\in[0,\omega]} |y(t)|$. Therefore, H is a bounded operator.

From above analysis, we conclude that TH is completely continuous. From equation (3.1), we have Q is completely continuous. Therefore, the proof is finished by Schauder's fixed point theorem. □

Corollary 3.1. *Assume the following condition holds:*

(F_1) *There exist a continuous function $d(t) \prec 0$ and a constant $\rho > 0$ such that satisfy*

$$\frac{d(t)}{x^\rho} \le f(t, x) \le 0, \quad \textit{for all } x > 0 \textit{ and a.e. } t.$$

If $\gamma_* > 0$, then equation (1.1) *has at least one positive periodic solution.*

Proof. Take

$$\zeta(t) = d(t), \quad g(x) = \frac{1}{x^\rho} \text{ and } \ p(x) = 0.$$

Then condition (H_1) is satisfied and the existence condition (H_2) is also satisfied if we take $R > 0$ with

$$\frac{M}{m}\left(\frac{\Psi^*}{(\gamma_*)^\rho} + \gamma^*\right) \le R,$$

where $\Psi(t) = \int_0^\omega G(t, s)d(s)dt$ and $\Psi^* = \sup_{t\in[0,\omega]} \Psi(t)$. □

Corollary 3.2. *Assume the following condition holds:*

(F_2) *There exist a continuous function $d(t) \prec 0$ and constants $\rho > 0,\ 0 \le \eta < 1$ such that*

$$\frac{d(t)}{x^\rho} + d(t)x^\eta \le f(t, x) \le 0, \quad \textit{for all } x > 0, \quad \textit{and a.e. } t.$$

If $\gamma_* > 0$, then equation (1.1) *has at least one positive periodic solution.*

Proof. Take

$$\zeta(t) = d(t), \quad g(x) = \frac{1}{x^\rho} \text{ and } \ p(x) = x^\eta.$$

Then condition (H_1) is satisfied and the existence condition (H_2) is also satisfied if we take $R > 0$ with

$$\frac{M}{m}\left(\Psi^*\left(\frac{1}{(\gamma_*)^\rho} + (R)^\eta\right) + \gamma^*\right) \le R.$$

□

In the following, we investigate equation (1.1) with attractive-repulsive singularities.

Corollary 3.3. *Assume the following condition holds:*

(F_3) *There exists continuous function* $d(t) \prec 0$, $\alpha > \beta > 0$ *and* $\mu > 0$ *such that*

$$\frac{d(t)}{x^\alpha} - \frac{\mu d(t)}{x^\beta} \leq f(t,x), \qquad \text{for all } x > 0 \quad \text{and a.e. } t.$$

If $\gamma_* > 0$, *then there exists a positive constant* μ_1 *such that equation* (1.1) *has at least one positive periodic solution for each* $0 \leq \mu \leq \mu_1$.

Proof. Take

$$g(x) = \frac{1}{x^\alpha} \quad \text{and} \quad \zeta(t) = d(t),$$

and choose $R > 0$ with

$$\frac{\Psi^*}{(\gamma_*)^\alpha} + \gamma^* < R,$$

then the condition (H_2) is satisfied. Next, we consider condition (H_1). In fact, $f(t,x) \leq 0$ if and only if $\mu \leq x^{\beta-\alpha}$. In view of $\beta < \alpha$, then we have $\mu < R^{\beta-\alpha}$. As a consequence, the result holds for

$$\mu_1 := \left(\frac{\Psi^*}{(\gamma_*)^\rho} + \gamma^*\right)^{\beta-\alpha}.$$

$\square$

3.2. Case (II) $\gamma_* = 0$

Theorem 3.2. *Assume that the condition* (H_1) *holds. Furthermore, suppose that the following conditions hold:*

(H_3) *For each* $L > 0$, *there exists a continuous function* $\phi_L \prec 0$ *such that* $f(t,x) \leq \phi_L(t)$ *for all* $(t,x) \in [0,\omega] \times (0,L]$.

(H_4) *There exists a positive constant* $R > 0$ *such that* $R > (\Phi_R)_*$ *and*

$$\frac{M}{m}\left(\left(g((\Phi_R)_*)\right)\left(1 + \frac{p(R)}{g(R)}\right)\right)\Lambda^* + \|\gamma\|\right) \leq R,$$

where $\Phi_R(t) = \int_0^\omega G(t,s)\phi_R(s)ds$ *and* $\|\gamma\| = \max_{t\in[0,\omega]} |\gamma(t)|$.

If $\gamma_* = 0$, *then equation* (1.1) *has at least one positive periodic solution.*

Proof. We follow the same strategy and notation as in the proof of Theorem 3.1. Let R be the positive constant satisfying condition (H_4) and let $r = (\Phi_{R_*})$, then $R > r > 0$ since $R > (\Phi_{R_*})$.

Next we prove that $Q(\Omega) \subset \Omega$. For each $x \in \Omega$ and for all $t \in [0,\omega]$, by the non-positive sign of the Green's function $G(t,s)$ and the nonlinear term $f(t,x)$ we have, from condition (H_3),

$$\begin{aligned}
(Qx)(t) &= P(f(t,x(t)) + e(t)) \\
&\geq T(f(t,x(t)) + e(t)) \\
&= \int_0^\omega G(t,s)f(s,x(s))ds + \gamma(t) \\
&= \int_0^\omega G(t,s)f^-(s,x(s))ds + \gamma(t)
\end{aligned}$$

$$
\begin{aligned}
&\geq \int_0^\omega G(t,s)\phi_R(s)ds + \gamma(t)\\
&\geq (\Phi_R)_* := r > 0.
\end{aligned}
$$

On the other hand, from Lemma 2.2, conditions (H_1) and (H_4), we obtain

$$
\begin{aligned}
(Qx)(t) &= P(f(t,x(t)) + e(t))\\
&\leq \frac{M}{m}\max_{t\in[0,\omega]}|T(f(t,x(t)) + e(t))|\\
&= \frac{M}{m}\max_{t\in[0,\omega]}\left|\int_0^\omega G(t,s)(f(s,x(s)) + e(s))ds\right|\\
&= \frac{M}{m}\max_{t\in[0,\omega]}\left|\int_0^\omega G(t,s)f^-(s,x(s))ds + \gamma(t)\right|\\
&\leq \frac{M}{m}\left\{\max_{t\in[0,\omega]}\left|\int_0^\omega G(t,s)f^-(s,x(s))ds\right| + \|\gamma\|\right\}\\
&\leq \frac{M}{m}\max_{t\in[0,\omega]}\left|\int_0^\omega G(t,s)\zeta(t)(g(x(s)) + p(x(s)))ds\right| + \frac{M}{m}\|\gamma\|\\
&\leq \frac{M}{m}\left(g(r)\left(1 + \frac{p(R)}{g(R)}\right)\Lambda^* + \|\gamma\|\right) \leq R.
\end{aligned}
$$

By above two inequalities, we have $Q(\Omega) \subset \Omega$. Therefore, by Schauder's fixed point theorem, our result is proved. □

Corollary 3.4. *Assume the following condition holds:*

(F_4) *There exist continuous functions* $d(t)$, $\hat{d}(t) \prec 0$ *and* $0 < \rho < 1$ *such that satisfy*

$$
\frac{d(t)}{x^\rho} \leq f(t,x) \leq \frac{\hat{d}(t)}{x^\rho} \leq 0, \qquad \textit{for all } x > 0 \ \textit{ and a.e. } t.
$$

If $\gamma_* = 0$, *then equation* (1.1) *has at least one positive periodic solution.*

Proof. Take

$$
\phi_L(t) = \frac{\hat{d}(t)}{L^\rho}, \quad \zeta(t) = d(t), \quad g(x) = \frac{1}{x^\rho} \text{ and } \ p(x) = 0,
$$

then conditions (H_1) and (H_3) are satisfied and the existence condition (H_4) becomes

$$
R > \frac{\hat{\Psi}_*}{R^\rho} = r, \qquad \frac{M}{m}\left(\left(\frac{R^\rho}{\hat{\Psi}_*}\right)^\rho \Psi^* + \|\gamma\|\right) \leq R, \tag{3.5}
$$

where $\hat{\Psi} = \int_0^\omega G(t,s)\hat{d}(t)dt$, for some $R > 0$. Note that $\Psi_* > 0$, since $0 < \rho < 1$, we choose $R > 0$ as large as possible so that equation (3.5) is satisfied and the proof is complete. □

In the following, we investigate equation (1.1) with attractive-repulsive singularities.

Corollary 3.5. *Assume the following condition holds:*

(F_5) *There exist constants* $0 < \beta < \alpha < 1$ *and* $\mu > 0$ *such that* $(\alpha + \beta)\alpha < 1$ *and*

$$
\frac{\mu}{x^\beta} - \frac{1}{x^\alpha} \leq f(t,x), \qquad \textit{for all } x > 0 \ \textit{ and a.e. } t.
$$

If $\gamma_* = 0$, then there exists a positive constant μ_2 such that equation (1.1) *has at least one positive periodic solution for each $0 \le \mu \le \mu_2$.*

Proof. Take
$$\zeta(t) = -1 \text{ and } \; g(x) = \frac{1}{x^\alpha}.$$

Firstly, condition (H_3) is satisfied. Let

$$\Gamma(x) = \frac{\mu}{x^\beta} - \frac{1}{x^\alpha}, \quad x \in (0, +\infty), \quad s_1 = \mu^{-\frac{1}{\alpha-\beta}}, \quad s_2 = \left(\frac{\alpha}{\mu\beta}\right)^{\frac{1}{\alpha-\beta}}.$$

Since $\alpha > \beta$, one can easily verify that $s_1 < s_2$ and

$$\Gamma(s_1) = 0, \quad \Gamma'(s_2) = 0 \text{ and } \; \Gamma'(s) > 0, \quad s \in (0, s_2).$$

Therefore, $\Gamma(s)$ is increasing in $(0, s_1) \subset (0, s_2)$. On the other hand, we can choose $\mu > 0$ small enough such that $R \in (0, s_1)$. Thus,

$$\max_{s\in(0,R)} \Gamma(s) = \Gamma(R) < \Gamma(s_1) = 0.$$

This implies that condition (H_3) is satisfied if we take

$$\phi_R(t) = \Gamma(R).$$

Secondly, the existence condition (H_4) becomes

$$\left(\frac{R^{\alpha+\beta}}{R^\beta - \mu R^\alpha}\right)^\alpha \Upsilon^* + \gamma^* \le R, \tag{3.6}$$

where $\Upsilon(t) = -\int_0^\omega G(t,s)ds$. Since $(\alpha+\beta)\alpha < 1$, we can choose $R > 0$ as large as possible so that equation (3.6) is satisfied.

Finally, the condition (H_1) is also satisfied. In fact, $f(t,x) \le 0$ if and only if $\mu < x^{\beta-\alpha}$, condition (H_1) is verified of any $\mu < R^{\beta-\alpha}$ since $\beta < \alpha$. As a consequence, the result holds for $\mu_2 = R^{\beta-\alpha}$. □

3.3. Case (III) $\gamma^* < 0$

Theorem 3.3. *Assume that conditions (H_1) and (H_3) hold. Furthermore, suppose that the following condition holds:*

(H_5) *There exists $R > 0$ such that $R > (\Phi_R)_* + \gamma_* > 0$ and*

$$\frac{M}{m} g((\Phi_R)_* + \gamma_*) \left(1 + \frac{p(R)}{g(R)}\right) \Lambda^* \le R.$$

If $\gamma^* < 0$, then equation (1.1) *has at least one positive periodic solution.*

Proof. Let R be the positive constant satisfying (H_5) and $r = (\Phi_R)_* + \gamma_*$, then $R > r > 0$ since $R > (\Phi_R)_* + \gamma_*$.

Next we prove that $Q(\Omega) \subset \Omega$. For each $x \in \Omega$ and for all $t \in [0, \omega]$, by the non-positive sign of the Green's function $G(t,s)$ and the nonlinear term $f(t,x)$ we have, from conditions (H_3) and (H_5),

$$(Qx)(t) = P(f(t, x(t)) + e(t))$$

$$
\begin{aligned}
&\geq T(f(t,x(t))+e(t)) \\
&= \int_0^\omega G(t,s)f(s,x(s))ds + \gamma(t) \\
&= \int_0^\omega G(t,s)f^-(s,x(s))ds + \gamma(t) \\
&\geq \int_0^\omega G(t,s)\phi_R(s)ds + \gamma(t) \\
&\geq (\Phi_R)_* + \gamma_* = r > 0.
\end{aligned}
$$

On the other hand, from Lemma 2.2, conditions (H_1) and (H_5), we deduce

$$
\begin{aligned}
(Qx)(t) =& P(f(t,x(t))+e(t)) \\
\leq& \frac{M}{m}\max_{t\in[0,\omega]} |T(f(t,x(t))+e(t))| \\
=& \frac{M}{m}\max_{t\in[0,\omega]} \left| \int_0^\omega G(t,s)f(s,x(s))ds + \gamma(t) \right| \\
\leq& \frac{M}{m}\max_{t\in[0,\omega]} \left| \int_0^\omega G(t,s)f(s,x(s))ds \right|,
\end{aligned}
$$

since $\gamma^* \leq 0$, $G(t,s)$ and $f(t,x(t))$ are non-positive, $(\Phi_R)_* + \gamma_* > 0$, then we know

$$
\left| \int_0^\omega G(t,s)f(s,x(s))ds + \gamma(t) \right| \leq \left| \int_0^\omega G(t,s)f(s,x(s))ds \right|.
$$

Therefore, by (H_2) and (H_5), we have

$$
\begin{aligned}
(Qx)(t) &\leq \frac{M}{m}\left\{ \max_{t\in[0,\omega]} \left| \int_0^\omega G(t,s)f^-(s,x(s))ds \right| \right\} \\
&\leq \frac{M}{m}\left\{ \max_{t\in[0,\omega]} \left| \int_0^\omega G(t,s)\zeta(s)(g(x(s))+p(x(s)))ds \right| \right\} \\
&\leq \frac{M}{m} g(r)\left(1+\frac{p(R)}{g(R)}\right)\Lambda^* \\
&\leq R.
\end{aligned}
$$

By above two inequalities, $Q(\Omega) \subset \Omega$. Therefore, by Schauder's fixed point theorem, our result is proven. □

Corollary 3.6. *Assume that condition* (F_4) *holds. If* $\gamma^* \leq 0$ *and*

$$
\gamma_* \geq \left(\frac{\hat{\Psi}_* m^\rho}{(M\Psi^*)^\rho}\rho^2 \right)^{\frac{1}{1-\rho^2}} \left(1-\frac{1}{\rho^2}\right),
$$

then there exists a positive periodic solution of equation (1.1).

Proof. Take

$$
\phi_L(t) = \frac{\hat{d}(t)}{L^\rho}, \quad \zeta(t) = d(t), \quad g(x) = \frac{1}{x^\rho} \text{ and } p(x) = 0.
$$

then conditions (H_1) and (H_3) are satisfied.

Next, we consider the condition (H_5) is also satisfied. Take $R = \frac{M\Psi^*}{m(r)^\rho}$, then $(\Phi_R)_* + \gamma_* > 0$ holds if r verifies

$$\frac{\hat{\Psi}_* m^\rho}{(M\Psi^*)^\rho}(r)^{\rho^2} + \gamma_* \geq r,$$

or equivalently,

$$\gamma_* \geq f(r) := r - \frac{\hat{\Psi}_* m^\rho}{(M\Psi^*)^\rho}(r)^{\rho^2}.$$

The function $f(r)$ possesses a minimum at

$$r_0 := \left[\frac{\hat{\Psi}_* m^\rho}{(M\Psi^*)^\rho}\rho^2\right]^{\frac{1}{1-\rho^2}}.$$

Let $r = r_0$, then $(\Phi_R)_* + \gamma_* > 0$ holds in condition (H_5) if $\gamma_* \geq f(r_0)$, which is just the condition

$$\gamma_* \geq \left(\frac{\hat{\Psi}_* m^\rho}{(M\Psi^*)^\rho}\rho^2\right)^{\frac{1}{1-\rho^2}}\left(1 - \frac{1}{\rho^2}\right).$$

The condition (H_5) holds directly by the choice of R, and it would remain to prove that $R = \frac{M\Psi^*}{m(r_0)^\rho} > r_0$. This is easily verified through elementary computations. □

In the following, we investigate equation (1.1) with attractive-repulsive singularities.

Corollary 3.7. *Assume the following condition holds:*

(F_6) *There exist continuous functions* $c(t)$, $d(t) \prec 0$, $0 < \beta < \alpha < 1$, $\rho > 0$ *and* $\mu > 0$ *such that* $\rho\alpha < 1$ *and*

$$\frac{d(t)}{x^\alpha} - \frac{\mu d(t)}{x^\beta} \leq f(t,x) \leq \frac{c(t)}{x^\rho}, \quad \text{for all } x > 0, \quad \text{and a.e. } t.$$

If $\gamma^* \leq 0$ *and*

$$\gamma_* \geq \left(\alpha\rho\frac{C_*}{(\Psi^*)^\rho}\right)^{\frac{1}{1-\alpha\rho}}\left(1 - \frac{1}{\alpha\rho}\right),$$

where $C(t) = \int_0^\omega G(t,s)c(s)ds$, *then there exists a positive constant* μ_3 *such that equation* (1.1) *has at least one positive periodic solution for each* $0 \leq \mu \leq \mu_3$.

Proof. Take

$$\phi_R(t) = \frac{c(t)}{R^\rho}, \quad \zeta(t) = d(t) \text{ and } \; g(x) = \frac{1}{x^\alpha},$$

then condition (H_1) is satisfied.

Next, we consider condition (H_5) to be satisfied. Take $R = \frac{\Psi^*}{r^\alpha}$, then $(\Phi_R)_* + \gamma_* > 0$ holds if r verifies

$$\frac{C_*}{(\Psi^*)^\rho}(r)^{\alpha\rho} + \gamma_* \geq r,$$

or equivalently,

$$\gamma_* \geq f(r) := r - \frac{C_*}{(\Psi^*)^\rho}(r)^{\alpha\rho}.$$

The function $f(r)$ possesses a minimum at

$$r_0 := \left(\alpha\rho\frac{C_*}{(\Psi^*)^\rho}\right)^{\frac{1}{1-\alpha\rho}}.$$

Let $r = r_0$, then the $(\Phi_R)_* + \gamma_* > 0$ holds if $\gamma_* \geq f(r_0)$, which is just the condition

$$\gamma_* \geq \left(\alpha\rho\frac{C_*}{(\Psi^*)^\rho}\right)^{\frac{1}{1-\alpha\rho}}\left(1 - \frac{1}{\alpha\rho}\right).$$

Then the condition (H_5) holds directly by the choice of R, and it would remain to prove that $R = \frac{\Psi^*}{(r_0)^\rho} > r_0$. This is easily verified through elementary computations.

Finally, we consider that condition (H_3) is satisfied. In fact,

$$\frac{d(t)}{x^\alpha} - \frac{\mu d(t)}{x^\beta} \leq \frac{c(t)}{R^\rho} \quad \text{if and only if} \quad \mu \leq x^{\beta-\alpha} - \frac{c(t)}{d(t)}\frac{x^\beta}{R^\rho}.$$

Condition (H_3) is verified for any $\mu \leq R^{\beta-\alpha} - \frac{c(t)}{d(t)}R^{\beta-\rho}$ since $\beta < \alpha$. As a consequence, the result holds for

$$\mu_3 := \frac{(\Psi^*)^{\beta-\alpha}}{\left(\alpha\rho\frac{B_*}{(\Psi^*)^\rho}\right)^{\frac{\alpha(\beta-\alpha)}{1-\alpha\rho}}} - \frac{c_*}{d^*}\frac{(\Psi^*)^{\beta-\rho}}{\left(\alpha\rho\frac{B_*}{(\Psi^*)^\rho}\right)^{\frac{\alpha(\beta-\rho)}{1-\alpha\rho}}}.$$

□

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