Journal of Applied Analysis and Computation
Volume 10, Number 5, October 2020, 1962–1979
Website:http://jaac-online.com
DOI:10.11948/20190295

SPATIOTEMPORAL DYNAMICS IN A PREDATOR-PREY MODEL WITH A FUNCTIONAL RESPONSE INCREASING IN BOTH PREDATOR AND PREY DENSITIES*

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Abstract In this paper, we studied a diffusive predator-prey model with a functional response increasing in both predator and prey densities. The Turing instability and local stability are studied by analyzing the eigenvalue spectrum. Delay induced Hopf bifurcation is investigated by using time delay as bifurcation parameter. Some conditions for determining the property of Hopf bifurcation are obtained by utilizing the normal form method and center manifold reduction for partial functional differential equation.

Keywords Predator-prey, delay, Turing instability, Hopf bifurcation.

MSC(2010) 34K18, 35B32.

1. Introduction

Many scholars have established and studied predator-prey models in the form of differential equations, since the predator-prey relationships are widespread in nature [2, 8, 11, 13, 16]. The functional response is essential since it reflects the specific interaction between predator and prey. In [4], the authors propose a functional response increasing in both predator and prey densities as follow

$$\psi(u,v) = \frac{Ce_0uv}{hCe_0uv+1}, \tag{1.1}$$

where u and v denote the densities of prey and predator, respectively. C is the fraction of a prey item killed per predator per encounter. h is the handling time per prey. e_0 is the total encounter coefficient between the predator and the prey. This type functional response reflects a higher hunting rate of predator when the predators' population is large [4].

Based on the functional response (1.1), Kimun et al. proposed the following

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*The authors were supported by Fundamental Research Funds for the Central Universities (2572019BC01), Natural Science Foundation of Heilongjiang (A2018001), Postdoctoral Science Foundation of China (2019M651237) and National Nature Science Foundation of China (11601070).

model [9]

$$\begin{cases} \dot{u} = ru(1 - \frac{u}{K}) - \frac{Ce_0 uv}{hCe_0 uv + 1}v, \\ \dot{v} = \frac{\epsilon Ce_0 uv}{hCe_0 uv + 1}v - \mu v. \end{cases} \tag{1.2}$$

Using the following parameter transformation,

$$rt = \bar{t}, \quad \frac{x}{K} = \bar{x}, \quad hCe_0 Ky = \bar{y}, \quad \frac{1}{Ce_0(hK)^2 r} = \alpha, \quad \frac{\epsilon}{rh} = \beta, \quad \frac{\mu h}{\epsilon} = \gamma,$$

the model (1.2) becomes

$$\begin{cases} \dot{u} = u(1 - u) - \frac{\alpha uv^2}{uv + 1}, \\ \dot{v} = \beta v(\frac{uv}{uv + 1} - \gamma). \end{cases} \tag{1.3}$$

In [9], the authors analyzed the saddle-node, Hopf and Bogdanov-Takens bifurcations.

In predator-prey models, reaction diffusion term and time delay are two important factors [1, 3, 5, 6, 10, 12, 14]. Since the spatial distributions predator and prey population are inhomogeneous and spread around. There is a delay in energy conversion between predator and prey, too. Many scholars investigate delayed diffusive predator-prey models and show some new dynamical phenomena (Turing instability, spatial inhomogeneity bifurcating period solution, spatial pattern, and so on). Motivated by this, we consider the following model.

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = d_1 \Delta u + u(1 - u(t - \tau)) - \frac{\alpha uv^2}{uv + 1}, \\ \frac{\partial v(x,t)}{\partial t} = d_2 \Delta v + \beta v(\frac{uv}{uv + 1} - \gamma), \quad x \in \Omega, \ t > 0, \\ \frac{\partial u(x,t)}{\partial \nu} = \frac{\partial v(x,t)}{\partial \nu} = 0, \quad x \in \partial\Omega, \ t > 0, \\ u(x,t) = u_1(x,t) \geq 0, v(x,t) = v_1(x,t) \geq 0, \quad x \in \Omega, t \in [-\tau, 0]. \end{cases} \tag{1.4}$$

All parameter are positive. d_1 and d_2 are diffusion coefficients of prey and predator, respectively. τ is the resource limitation of the prey logistic equation. The aim of this paper is to study the effect of diffusion and time delay on the model (1.4). Compare with the model (1.3), whether some new dynamical phenomena occurs.

The organization of this paper is as follows. In the section 2, we study the non-delay model, including Turing instability and local stability of positive equilibrium. In the section 3, we analyze the delay model, including delay induced instability and Hopf bifurcation at positive equilibrium, and the property of Hopf bifurcation. In the section 4, we give a brief conclusion.

2. Non-delay model

When $\tau = 0$, the model (1.4) becomes

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1 \Delta u + u(1-u) - \dfrac{\alpha u v^2}{uv+1}, \\ \dfrac{\partial v}{\partial t} = d_2 \Delta v + \beta v(\dfrac{uv}{uv+1} - \gamma). \end{cases} \tag{2.1}$$

2.1. Equilibria

The equilibria of model (1.4) are the roots of the following equations,

$$\begin{cases} u(1-u) - \dfrac{\alpha u v^2}{uv+1} = 0, \\ \beta v(\dfrac{uv}{uv+1} - \gamma) = 0. \end{cases} \tag{2.2}$$

Referring to literature [9], we have the following conclusion about the equilibria of model (1.4).

Lemma 2.1 ([9]). *For the model* (1.4)*, the following statements are true.*

(i) $(0,0)$ *and* $(1,0)$ *are two boundary equilibria.*

(ii) If $\alpha > \frac{4(1-\gamma)}{27\gamma^2}$, *the model* (1.4) *has no coexisting equilibrium.*

(iii) If $\alpha = \frac{4(1-\gamma)}{27\gamma^2}$, *the model* (1.4) *has a unique coexisting equilibrium* $(\frac{2}{3}, \frac{3\gamma}{2(1-\gamma)})$.

(iv) If $\alpha < \frac{4(1-\gamma)}{27\gamma^2}$, *the model* (1.4) *has two coexisting equilibria* $(u_1, \frac{\gamma}{u_1(1-\gamma)})$ *and* $(u_2, \frac{\gamma}{u_2(1-\gamma)})$, *where* $0 < u_1 < \frac{2}{3} < u_2 <$ *are two roots of the following equation.*

$$u^3 - u^2 + \frac{\alpha\gamma^2}{1-\gamma} = 0. \tag{2.3}$$

In this paper, we mainly consider the property of coexisting equilibrium. We just denote the (u_*, v_*) as a coexisting equilibrium of the model (1.4).

2.2. Local stability

Linearize system (1.4) at (u_*, v_*) is as follows:

$$\begin{pmatrix} \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial t} \end{pmatrix} = d\Delta \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} + L_1 \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} + L_2 \begin{pmatrix} u(t-\tau) \\ v(t-\tau) \end{pmatrix}, \tag{2.4}$$

where

$$L_1 = \begin{pmatrix} a_1 & -a_2 \\ \beta b_1 & \beta b_2 \end{pmatrix}, \qquad L_2 = \begin{pmatrix} -u_* & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$a_1 = \frac{\alpha\gamma^3}{(1-\gamma)u_*^2} > 0, \quad a_2 = \alpha\gamma(2-\gamma) > 0, \quad b_1 = \frac{\gamma^2}{u_*^2} > 0, \quad b_2 = (1-\gamma)\gamma > 0. \tag{2.5}$$

The characteristic equation of (2.4) is

$$det(\lambda I - M_n - L_1 - L_2 e^{-\lambda\tau}) = 0 \tag{2.6}$$

where $I = \text{diag}\{1,1\}$ and $M_n = -n^2/l^2 \text{diag}\{d_1, d_2\}$, $n \in \mathbb{N}_0$. Then, we have

$$\lambda^2 + \lambda A_n + B_n + (C_n + \lambda u_*)e^{-\lambda\tau} = 0, \quad n \in \mathbb{N}_0 \triangleq \{0\} \cup \mathbb{N}, \tag{2.7}$$

where

$$\begin{aligned} A_n &= (d_1 + d_2)\frac{n^2}{l^2} - (a_1 + \beta b_2), \\ B_n &= d_1 d_2 \frac{n^4}{l^4} - (a_1 d_2 + d_1 \beta b_2)\frac{n^2}{l^2} + \beta(a_2 b_1 + a_1 b_2), \\ C_n &= d_2 u_* \frac{n^2}{l^2} - \beta b_2 u_*. \end{aligned}$$

When $\tau = 0$, the characteristic Eq. (1.4) reduces to the following equation.

$$\lambda^2 - tr_n \lambda + \Delta_n(r) = 0, \quad n \in \mathbb{N}_0, \tag{2.8}$$

where

$$\begin{cases} tr_n = a_1 - u_* + \beta b_2 - \frac{n^2}{l^2}(d_1 + d_2), \\ \Delta_n = \beta[a_2 b_1 + b_2(a_1 - u_*)] - \frac{n^2}{l^2}[d_2(a_1 - u_*) + \beta b_2 d_1] + d_1 d_2 \frac{n^4}{l^4}, \end{cases} \tag{2.9}$$

and the eigenvalues are given by

$$\lambda_{1,2}^{(n)}(r) = \frac{tr_n \pm \sqrt{tr_n^2 - 4\Delta_n}}{2}, \quad n \in \mathbb{N}_0. \tag{2.10}$$

We make the following hypothesis

$$\begin{aligned} &(\mathbf{H_1}) \qquad 3\alpha\gamma^3 - (1-\gamma)\gamma u_*^2 > 0, \\ &(\mathbf{H_2}) \qquad \beta < \beta_c \triangleq \frac{u_0 - a_1}{b_2} = \frac{(1-\gamma)u_*^3 - \alpha\gamma^3}{(1-\gamma)^2\gamma u_*^2}. \end{aligned} \tag{2.11}$$

By direct calculation, we can get the following conclusion.

Proposition 2.1. *If hypothesis* $(\mathbf{H_1})$ *holds, then* $a_2 b_1 + b_2(a_1 - u_*) > 0$. *If hypothesis* $(\mathbf{H_2})$ *holds, then* $a_1 - u_* + \beta b_2 < 0$.

Theorem 2.1. *If* $d_1 = d_2 = 0$, $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$ *hold. Then the equilibrium* (u_*, v_*) *is locally asymptotically stable.*

Proof. By the Proposition 2.1, we know that all eigenvalues have negative real parts. Then the equilibrium (u_*, v_*) is locally asymptotically stable. □

Define some critical value

$$\begin{aligned} \beta_\pm &= \frac{d_2}{b_2^2 d_1}[b_2(a_1 - u_*) + 2a_2 b_1 \pm 2\sqrt{a_2 b_1[b_2(a_1 - u_*) + a_2 b_1]}], \\ \varrho_\pm &= \frac{1}{2d_1 d_2}[d_2(a_1 - u_*) + \beta b_2 d_1 \pm \sqrt{(d_2(a_1 - u_*) + \beta b_2 d_1)^2 - 4\beta d_1 d_2(b_2(a_1 - u_*) + a_2 b_1)}]. \end{aligned} \tag{2.12}$$

Theorem 2.2. *If $d_1 > 0$, $d_2 > 0$, $(\mathbf{H_1})$ and $(\mathbf{H_2})$ hold. For the model (2.1), the following statements are true.*

(i) The equilibrium (u_*, v_*) is locally asymptotically stable for $0 < \beta \leq \frac{d_2}{d_1}\beta_c$.

(ii) If $\beta_c \leq \beta_+$, then the equilibrium (u_*, v_*) is locally asymptotically stable for $\frac{d_2}{d_1}\beta_c < \beta < \beta_c$.

(iii) If $\beta_c > \beta_+$, then the equilibrium (u_*, v_*) is locally asymptotically stable for $\frac{d_2}{d_1}\beta_c < \beta < \beta_+$.

(iv) If $\beta_c > \beta_+$ and there is no $k \in \mathbb{N}$ such that $\frac{k^2}{l^2} \in (\varrho_-, \varrho_+)$ then the equilibrium (u_*, v_*) is locally asymptotically stable for $\beta_+ < \beta < \beta_c$.

(v) If $\beta_c > \beta_+$ and there is a $k \in \mathbb{N}$ such that $\frac{k^2}{l^2} \in (\varrho_-, \varrho_+)$ then the equilibrium (u_*, v_*) is Turing unstable for $\beta_+ < \beta < \beta_c$.

Proof. By direct calculation, we have $\beta_- < \frac{d_2}{d_1}\beta_c \leq \beta_+$. If $\beta \leq \frac{d_2}{d_1}\beta_c$, we can obtain $tr_n < 0$ and $\Delta_n > 0$. This means that all roots of Eq. (2.8) have negative real parts. Then the equilibrium (u_*, v_*) is locally asymptotically stable (statement (1) is ture). Similarly, statements $(1) - (4)$ are also true. If conditions in statement (5) hold, then Eq. (2.8) have at least one root with positive real part. Then the equilibrium (u_*, v_*) is Turing unstable. □

2.3. Example

Fix the following parameters

$$\alpha = 1.5, \quad \gamma = 0.25, \quad d_1 = 1. \tag{2.13}$$

It is easy to obtain that (0.8090, 0.4120) and (0.5000, 0.6667) are two positive equilibria. For the equilibrium (0.8090, 0.4120), $(\mathbf{H_1})$ doesn't hold implying that this equilibrium is unstable. So we mainly investigate the other equilibrium. We choose $(u_*, v_*) \approx (0.5000, 0.6667)$, and $(\mathbf{H_1})$ is satisfied. The bifurcation diagram for d_2 and β is given in Fig. 1. Now, we fix $d_2 = 0.1$ and $l = 4$, then we have

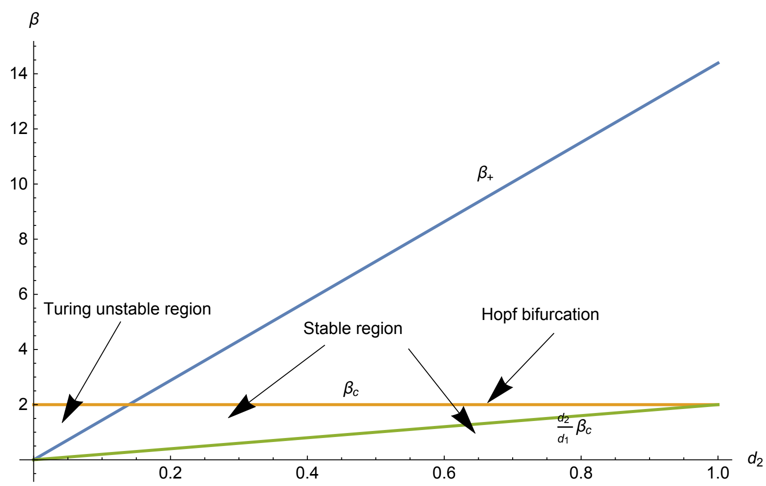

Figure 1. Bifurcation diagram for d_2 and β.

$\beta_+ \approx 1.4388$, and $\beta_c \approx 2.0000$. If we choose $\beta = 1.8$, then (u_*, v_*) is Turing unstable (shown in Fig. 2). If we choose $\beta = 1$, then (u_*, v_*) is locally asymptotically stable (shown in Fig. 3).

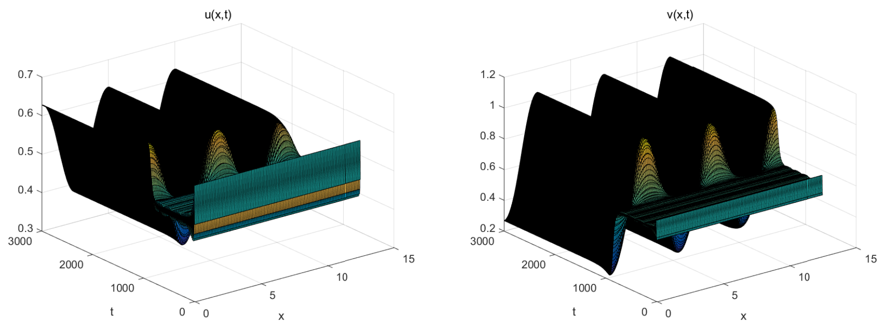

Figure 2. Numerical simulations of system (2.1) for $\beta = 1.8$.

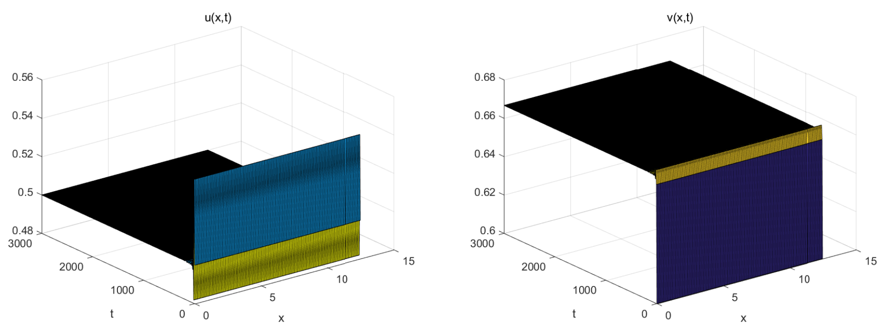

Figure 3. Numerical simulations of system (2.1) for $\beta = 1$.

3. Delay model

3.1. Existence of Hopf bifurcation

To study the stability of $E_*(u_*, v_*)$ when $\tau > 0$, we suppose $(\mathbf{H_1})$, $(\mathbf{H_2})$ and one of conditions $(1) - (4)$ in Theorem 2.2 always hold. Let $\mathrm{i}\omega$ $(\omega > 0)$ be a solution of Eq. (2.7), we have

$$-\omega^2 + \mathrm{i}\omega A_n + B_n + (C_n + \mathrm{i}\omega u_*)(\cos\omega\tau - \mathrm{i}\sin\omega\tau) = 0.$$

Then we have

$$\begin{cases} -\omega^2 + B_n + C_n\cos\omega\tau + \omega u_*\sin\omega\tau = 0, \\ A_n\omega - C_n\sin\omega\tau + \omega u_*\cos\omega\tau = 0. \end{cases}$$

It leads to

$$\omega^4 + (A_n^2 - 2B_n - u_*^2)\omega^2 + B_n^2 - C_n^2 = 0. \tag{3.1}$$

Denote $z=\omega^2$, then (3.1) can be changed into

$$z^2+(A_n^2-2B_n-u_*^2)z+B_n^2-C_n^2=0, \tag{3.2}$$

and the roots of (3.2) are

$$z^{\pm}=\frac{1}{2}[-(A_n^2-2B_n-u_*^2)\pm\sqrt{(A_n^2-2B_n-u_*^2)^2-4(B_n^2-C_n^2)}].$$

By direct computation,

$$\begin{aligned}&A_n^2-2B_n-u_*^2=\left(d_1^2+d_2^2\right)\frac{n^4}{l^4}-2(a_1d_1+\beta b_2d_2)\frac{n^2}{l^2}+b_2^2\beta^2-2a_2b_1\beta-u_*^2+a_1^2,\\&B_n-C_n=d_1d_2\frac{n^4}{l^4}-\left[(d_2(a_1+u_*))+\beta b_2d_1\right]\frac{n^2}{l^2}+\beta[a_2b_1+b_2(a_1+u_*)],\\&B_n+C_n=\Delta_n>0.\end{aligned}$$

Fix parameters $\alpha,\ \beta,\ \gamma$, define

$$\mathcal{D}=\{k\in\mathbb{N}_0\mid \text{Eq. (3.2) has positive roots with } n=k.\} \tag{3.3}$$

For $n\in\mathcal{D}$, if $z^+>0$, then Eq. (2.7) has a pair of purely imaginary roots $\pm\mathrm{i}\omega_n^+$ at $\tau_n^{j,+}$, $j\in\mathbb{N}_0$; if $z^->0$, then Eq. (2.7) has a pair of purely imaginary roots $\pm\mathrm{i}\omega_n^-$ at $\tau_n^{j,-}$, $j\in\mathbb{N}_0$, where

$$\begin{aligned}&\omega_n^{\pm}=\sqrt{z_n^{\pm}},\qquad \tau_n^{j,\pm}=\tau_n^{0,\pm}+\frac{2j\pi}{\omega_n^{\pm}},\ (j=0,1,2,\cdots),\\&\tau_n^{0,\pm}=\frac{1}{\omega_n^{\pm}}\arccos\frac{(C_n-u_*A_n)(\omega_n^{\pm})^2-B_nC_n}{C_n^2+u_*^2(\omega_n^{\pm})^2}.\end{aligned} \tag{3.4}$$

From (3.4), we have $\tau_n^{0,\pm}<\tau_n^{j,\pm}$ $(j\in\mathbb{N})$. For $k\in\mathcal{D}$, define the smallest τ so that the stability will change

$$\tau_*=\min\{\tau_k^{0,\pm}\text{ or }\tau_k^{0,+}\mid k\in\mathcal{D}\}. \tag{3.5}$$

Lemma 3.1. *Suppose* $(\mathbf{H_1})$ *(or* $(\mathbf{H_2})$*) holds. If* $(A_n^2-2B_n-u_*^2)^2-4(B_n^2-C_n^2)>0$, *then* $\mathrm{Re}(\frac{d\lambda}{d\tau})|_{\tau=\tau_n^{j,+}}>0$, $\mathrm{Re}(\frac{d\lambda}{d\tau})|_{\tau=\tau_n^{j,-}}<0$ *for* $\tau\in\mathcal{D}$ *and* $j\in\mathbb{N}_0$.

Proof. Differentiating two sides of (2.7) with respect τ, we have

$$(\frac{d\lambda}{d\tau})^{-1}=\frac{2\lambda+A_n+u_*e^{-\lambda\tau}}{(C_n+\lambda u_*)e^{-\lambda\tau}}-\frac{\tau}{\lambda}.$$

Then

$$\begin{aligned}[\mathrm{Re}(\frac{d\lambda}{d\tau})^{-1}]_{\tau=\tau_n^{j,\pm}}&=\mathrm{Re}[\frac{2\lambda+A_n+u_*e^{-\lambda\tau}}{(C_n+\lambda u_*)e^{-\lambda\tau}}-\frac{\tau}{\lambda}]_{\tau=\tau_n^{j,\pm}}\\&=[\frac{1}{\Lambda}\omega^2(2\omega^2+A_n^2-2B_n-u_*^2)]_{\tau=\tau_n^{j,\pm}}\\&=\pm[\frac{1}{\Lambda}\omega^2\sqrt{(A_n^2-2B_n-u_*^2)^2-4(B_n^2-C_n^2)}]_{\tau=\tau_n^{j,\pm}},\end{aligned}$$

where $\Lambda=\omega^4b_2^2+C_n^2\omega^2>0$. Therefore $\mathrm{Re}(\frac{d\lambda}{d\tau})|_{\tau=\tau_n^{j,+}}>0$, $\mathrm{Re}(\frac{d\lambda}{d\tau})|_{\tau=\tau_n^{j,-}}<0$. □

Theorem 3.1. *Suppose* $(\mathbf{H_1})$, $(\mathbf{H_2})$ *and one of conditions* (1) − (4) *in Theorem 2.2 hold. For system* (1.4), *the following statements are true.*

(i) $E_*(u_*, v_*)$ *is locally asymptotically stable for all* $\tau \geq 0$ *when* $\mathcal{D} = \emptyset$.

(ii) $E_*(u_*, v_*)$ *is locally asymptotically stable for* $\tau \in [0, \tau_*)$, *and unstable for* $\tau \in [\tau_*, \tau_* + \epsilon)$ *with some* ϵ *when* $\mathcal{D} \neq \emptyset$, *where* τ_* *is defined in* (3.5).

(iii) System (1.4) *undergoes a Hopf bifurcation at the equilibrium* $E_*(u_*, v_*)$ *when* $\tau = \tau_n^{j,+}$ *(or* $\tau = \tau_n^{j,-}$*),* $j \in \mathbb{N}_0$, $n \in \mathcal{D}$ *when* $\mathcal{D} \neq \emptyset$. *The bifurcating periodic solutions are spatially homogeneous (inhomogeneous) when* $n = 0$ $(n > 0)$.

3.2. Properties of Hopf bifurcation

Now, we will study the property of Hopf bifurcation by the method [7, 15]. For a critical value $\tau_n^{j,+}$ (or $\tau_n^{j,-}$), we denote it as $\tilde{\tau}$. Let $\tilde{u}(x,t) = u(x, \tau t) - u_*$ and $\tilde{v}(x,t) = v(x, \tau t) - v_*$, then the system (1.4) is (drop the tilde)

$$\begin{cases} \dfrac{\partial u}{\partial t} = \tau[d_1 \Delta u + (u + u_*)\left(1 - u(t-1) - u_* - \dfrac{\alpha\,(v + v_*)^2}{1 + (u + u_*)\,(v + v_*)}\right)], \\ \dfrac{\partial v}{\partial t} = \tau[d_2 \Delta v + \beta\,(v + v_*)\left(\dfrac{(u + u_*)\,(v + v_*)}{1 + (u + u_*)\,(v + v_*)} - \gamma\right)]. \end{cases} \tag{3.6}$$

Denote $\tau = \tilde{\tau} + \varepsilon$, and $U = (u(x,t), v(x,t))^T$. In the phase space $\mathbb{C}_1 := C([-1, 0], X)$, (3.6) can be rewritten as

$$\frac{\mathrm{d}U(t)}{\mathrm{d}t} = \tilde{\tau} D \Delta U(t) + L_{\tilde{\tau}}(U_t) + F(U_t, \varepsilon), \tag{3.7}$$

where $L_\varepsilon(\varphi)$ and $F(\varphi, \varepsilon)$ are

$$L_\varepsilon(\phi) = \varepsilon \begin{pmatrix} a_1\phi_1(0) - a_2\phi_2(0) - u_*\phi_1(-1) \\ \beta b_1 \phi_1(0) + \beta b_2 \phi_2(0) \end{pmatrix} \tag{3.8}$$

$$F(\phi, \varepsilon) = \varepsilon D \Delta \phi + L_\varepsilon(\phi) + f(\phi, \varepsilon), \tag{3.9}$$

with

$$\begin{aligned} f(\phi, \varepsilon) &= (\tilde{\tau} + \varepsilon)(F_1(\phi, \varepsilon), F_2(\phi, \varepsilon))^T, \\ F_1(\phi, \varepsilon) &= (\phi_1(0) + u_*)\left(1 - \phi_1(-1) - u_* - \frac{\alpha\,(\phi_2(0) + v_*)^2}{1 + (\phi_1(0) + u_*)\,(\phi_2(0) + v_*)}\right) \\ &\quad - a_1\phi_1(0) + a_2\phi_2(0) + u_*\phi_1(-1), \\ F_2(\phi, \varepsilon) &= \beta\,(\phi_2(0) + v_*)\left(\frac{(\phi_1(0) + u_*)\,(\phi_2(0) + v_*)}{1 + (\phi_1(0) + u_*)\,(\phi_2(0) + v_*)} - \gamma\right) - \beta b_1 \phi_1(0) - \beta b_2 \phi_2(0), \end{aligned}$$

respectively, for $\phi = (\phi_1, \phi_2)^T \in \mathbb{C}_1$.

Consider the linear equation

$$\frac{dU(t)}{dt} = \tilde{\tau} D \Delta U(t) + L_{\tilde{\tau}}(U_t). \tag{3.10}$$

We know that $\Lambda_n := \{\mathrm{i}\omega_n\tilde{\tau}, -\mathrm{i}\omega_n\tilde{\tau}\}$ are characteristic roots of

$$\frac{dz(t)}{dt} = -\tilde{\tau}D\frac{n^2}{l^2}z(t) + L_{\tilde{\tau}}(z_t). \tag{3.11}$$

Choose

$$\eta^n(\sigma,\tau) = \begin{cases} \tau E & \sigma = 0, \\ 0 & \sigma \in (-1,0), \\ -\tau F & \sigma = -1, \end{cases} \tag{3.12}$$

where

$$E = \begin{pmatrix} a_1 - d_1\frac{n^2}{l^2} & -a_2 \\ \beta b_1 & \beta b_2 - d_2\frac{n^2}{l^2} \end{pmatrix}, \qquad F = \begin{pmatrix} -u_* & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.13}$$

Then

$$-\tilde{\tau}D\frac{n^2}{l^2}\phi(0) + L_{\tilde{\tau}}(\phi) = \int_{-1}^{0} d\eta^n(\sigma,\tau)\phi(\sigma)$$

for $\phi \in C([-1,0],\mathbb{R}^2)$.

Define the bilinear paring

$$\begin{aligned}(\psi,\varphi) &= \psi(0)\varphi(0) - \int_{-1}^{0}\int_{\xi=0}^{\sigma}\psi(\xi-\sigma)d\eta^n(\sigma,\tilde{\tau})\varphi(\xi)d\xi \\ &= \psi(0)\varphi(0) + \tilde{\tau}\int_{-1}^{0}\psi(\xi+1)F\varphi(\xi)d\xi,\end{aligned} \tag{3.14}$$

for $\varphi \in C([-1,0],\mathbb{R}^2)$, $\psi \in C([0,1],\mathbb{R}^2)$. $A(\tilde{\tau})$ has a pair of simple purely imaginary eigenvalues $\pm\mathrm{i}\omega_n\tilde{\tau}$, and they are also eigenvalues of A^*.

Define $p_1(\sigma) = (1,\xi)^T e^{\mathrm{i}\omega_n\tilde{\tau}\sigma}$ $(\sigma \in [-1,0])$, $q_1(r) = (1,\eta)e^{-\mathrm{i}\omega_n\tilde{\tau}r}$ $(r \in [0,1])$, where

$$\xi = \frac{\beta b_1}{-\beta b_2 + d_2 n^2/l^2 + \mathrm{i}\omega}, \qquad \eta = \frac{a_2}{\beta b_2 - d_2 n^2/l^2 + \mathrm{i}\omega}.$$

Let $\Phi = (\Phi_1, \Phi_2)$ and $\Psi^* = (\Psi_1^*, \Psi_2^*)^T$ with

$$\Phi_1(\sigma) = \frac{p_1(\sigma)+p_2(\sigma)}{2} = \begin{pmatrix} \mathrm{Re}\left(e^{\mathrm{i}\omega_n\tilde{\tau}\sigma}\right) \\ \mathrm{Re}\left(\xi e^{\mathrm{i}\omega_n\tilde{\tau}\sigma}\right) \end{pmatrix}, \quad \Phi_2(\sigma) = \frac{p_1(\sigma)-p_2(\sigma)}{2i} = \begin{pmatrix} \mathrm{Im}\left(e^{\mathrm{i}\omega_n\tilde{\tau}\sigma}\right) \\ \mathrm{Im}\left(\xi e^{\mathrm{i}\omega_n\tilde{\tau}\sigma}\right) \end{pmatrix}$$

for $\theta \in [-1,0]$, and

$$\Psi_1^*(r) = \frac{q_1(r)+q_2(r)}{2} = \begin{pmatrix} \mathrm{Re}\left(e^{-\mathrm{i}\omega_n\tilde{\tau}\mathrm{r}}\right) \\ \mathrm{Re}\left(\eta e^{-\mathrm{i}\omega_n\tilde{\tau}\mathrm{r}}\right) \end{pmatrix}, \quad \Psi_2^*(r) = \frac{q_1(r)-q_2(r)}{2i} = \begin{pmatrix} \mathrm{Im}\left(e^{-\mathrm{i}\omega_n\tilde{\tau}\mathrm{r}}\right) \\ \mathrm{Im}\left(\eta e^{-\mathrm{i}\omega_n\tilde{\tau}\mathrm{r}}\right) \end{pmatrix}$$

for $r \in [0,1]$. Then we can compute by (3.14)

$$D_1^* := (\Psi_1^*, \Phi_1),\ D_2^* := (\Psi_1^*, \Phi_2),\ D_3^* := (\Psi_2^*, \Phi_1),\ D_4^* := (\Psi_2^*, \Phi_2).$$

Define $(\Psi^*, \Phi) = (\Psi_j^*, \Phi_k) = \begin{pmatrix} D_1^* & D_2^* \\ D_3^* & D_4^* \end{pmatrix}$ and $\Psi = (\Psi_1, \Psi_2)^T = (\Psi^*, \Phi)^{-1}\Psi^*$. Then $(\Psi, \Phi) = I_2$. In addition, define $f_n := (\beta_n^1, \beta_n^2)$, where

$$\beta_n^1 = \begin{pmatrix} \cos \frac{n}{l}x \\ 0 \end{pmatrix}, \quad \beta_n^2 = \begin{pmatrix} 0 \\ \cos \frac{n}{l}x \end{pmatrix}.$$

We also define

$$c \cdot f_n = c_1\beta_n^1 + c_2\beta_n^2, \quad \text{for} \quad c = (c_1, c_2)^T \in \mathbb{C}_1.$$

and

$$< u, v >:= \frac{1}{l\pi}\int_0^{l\pi} u_1\overline{v_1}dx + \frac{1}{l\pi}\int_0^{l\pi} u_2\overline{v_2}dx$$

for $u = (u_1, u_2)$, $v = (v_1, v_2)$, $u, v \in X$ and $< \varphi, f_0 >= (< \varphi, f_0^1 >, < \varphi, f_0^2 >)^T$.

Rewrite Eq. (3.6) as

$$\frac{dU(t)}{dt} = A_{\tilde{\tau}}U_t + R(U_t, \varepsilon), \tag{3.15}$$

where

$$R(U_t, \varepsilon) = \begin{cases} 0, & \theta \in [-1, 0); \\ F(U_t, \varepsilon), & \theta = 0. \end{cases} \tag{3.16}$$

The solution is

$$U_t = \Phi \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} f_n + h(x_1, x_2, \varepsilon), \tag{3.17}$$

where

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (\Psi, < U_t, f_n >),$$

and

$$h(x_1, x_2, \varepsilon) \in P_S\mathbb{C}_1, \quad h(0, 0, 0) = 0, \quad Dh(0, 0, 0) = 0.$$

Then

$$U_t = \Phi \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} f_n + h(x_1, x_2, 0). \tag{3.18}$$

Let $z = x_1 - \mathrm{i}x_2$, then

$$\Phi \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} f_n = (\Phi_1, \Phi_2) \begin{pmatrix} \frac{z+\overline{z}}{2} \\ \frac{i(z-\overline{z})}{2} \end{pmatrix} f_n = \frac{1}{2}(p_1 z + \overline{p_1 z})f_n,$$

and

$$h(x_1, x_2, 0) = h(\frac{z+\overline{z}}{2}, \frac{\mathrm{i}(z-\overline{z})}{2}, 0).$$

Eq. (3.18) is

$$\begin{aligned} U_t &= \frac{1}{2}(p_1 z + \overline{p_1 z}) f_n + h(\frac{z+\overline{z}}{2}, \frac{\mathrm{i}(z-\overline{z})}{2}, 0) \\ &= \frac{1}{2}(p_1 z + \overline{p_1 z}) f_n + W(z, \overline{z}), \end{aligned} \tag{3.19}$$

where

$$W(z, \overline{z}) = h(\frac{z+\overline{z}}{2}, \frac{\mathrm{i}(z-\overline{z})}{2}, 0).$$

From [15], z satisfies

$$\dot{z} = \mathrm{i}\omega_n \tilde{\tau} z + g(z, \overline{z}), \tag{3.20}$$

where

$$g(z, \overline{z}) = (\Psi_1(0) - \mathrm{i}\Psi_2(0)) < F(U_t, 0), f_n > . \tag{3.21}$$

Let

$$W(z, \overline{z}) = W_{20}\frac{z^2}{2} + W_{11} z\overline{z} + W_{02}\frac{\overline{z}^2}{2} + \cdots , \tag{3.22}$$

$$g(z, \overline{z}) = g_{20}\frac{z^2}{2} + g_{11} z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + \cdots , \tag{3.23}$$

then

$$\begin{aligned} u_t(0) &= \frac{1}{2}(z+\overline{z})\cos\left(\frac{nx}{l}\right) + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0) z\overline{z} + W_{02}^{(1)}(0)\frac{\overline{z}^2}{2} + \cdots , \\ v_t(0) &= \frac{1}{2}(\xi + \overline{\xi}\overline{z})\cos\left(\frac{nx}{l}\right) + W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0) z\overline{z} + W_{02}^{(2)}(0)\frac{\overline{z}^2}{2} + \cdots , \\ u_t(-1) &= \frac{1}{2}(z e^{-\mathrm{i}\omega_n\tilde{\tau}} + \overline{z} e^{\mathrm{i}\omega_n\tilde{\tau}})\cos(\frac{nx}{l}) + W_{20}^{(1)}(-1)\frac{z^2}{2} + W_{11}^{(1)}(-1) z\overline{z} + W_{02}^{(1)}(-1)\frac{\overline{z}^2}{2} + \cdots , \end{aligned}$$

and

$$\begin{aligned} \overline{F}_1(U_t, 0) = \frac{1}{\tilde{\tau}} F_1 = &- u_t(0) u_t(-1) + \alpha_1 u_t^2(0) + \alpha_2 u_t(0) v_t(0) + \alpha_3 v_t^2(0) + \alpha_4 u_t^3(0) \\ &+ \alpha_5 u_t^2(0) v_t(0) + \alpha_6 u_t(0) v_t^2(0) + \alpha_7 v_t^3(0) + O(4), \end{aligned} \tag{3.24}$$

$$\begin{aligned} \overline{F}_2(U_t, 0) = \frac{1}{\tilde{\tau}} F_2 = &\beta_1 u_t^2(0) + \beta_2 u_t(0) v_t(0) + \beta_3 v_t^2(0) + \beta_4 u_t^3(0) + \beta_5 u_t^2(0) v_t(0) \\ &+ \beta_6 u_t(0) v_t^2(0) + \beta_7 v_t^3(0) + O(4), \end{aligned} \tag{3.25}$$

with

$$\alpha_1=\frac{\alpha v_*^3}{(u_*v_*+1)^3},\quad \alpha_2=-\frac{2\alpha v_*}{(u_*v_*+1)^3},\quad \alpha_3=-\frac{\alpha u_*}{(u_*v_*+1)^3},\quad \alpha_4=-\frac{\alpha v_*^4}{(u_*v_*+1)^4},$$
$$\alpha_5=\frac{3\alpha v_*^2}{(u_*v_*+1)^4},\quad \alpha_6=\frac{\alpha(2u_*v_*-1)}{(u_*v_*+1)^4},\quad \alpha_7=\frac{\alpha u_*^2}{(u_*v_*+1)^4},\quad \beta_1=-\frac{\beta v_*^3}{(u_*v_*+1)^3},$$
$$\beta_2=\frac{2\beta v_*}{(u_*v_*+1)^3},\quad \beta_3=\frac{\beta u_*}{(u_*v_*+1)^3},\quad \beta_4=\frac{\beta v_*^4}{(u_*v_*+1)^4},\quad \beta_5=-\frac{3\beta v_*^2}{(u_*v_*+1)^4},$$
$$\beta_6=\frac{\beta(1-2u_*v_*)}{(u_*v_*+1)^4},\quad \beta_7=-\frac{6\beta u_*^2}{(u_*v_*+1)^4}. \tag{3.26}$$

Hence,

$$\overline{F}_1(U_t,0)=\cos^2(\frac{nx}{l})(\frac{z^2}{2}\chi_{20}+z\overline{z}\chi_{11}+\frac{\overline{z}^2}{2}\overline{\chi}_{20})+\frac{z^2\overline{z}}{2}(\chi_1\cos\frac{nx}{l}+\chi_2\cos^3\frac{nx}{l})+\cdots,$$
$$\overline{F}_2(U_t,0)=\cos^2(\frac{nx}{l})(\frac{z^2}{2}\varsigma_{20}+z\overline{z}\varsigma_{11}+\frac{\overline{z}^2}{2}\overline{\varsigma}_{20})+\frac{z^2\overline{z}}{2}(\varsigma_1\cos\frac{nx}{l}+\varsigma_2\cos^3\frac{nx}{l})+\cdots, \tag{3.27}$$

$$\begin{aligned}< F(U_t,0),f_n > &=\tilde{\tau}(\overline{F}_1(U_t,0)f_n^1+\overline{F}_2(U_t,0)f_n^2)\\ &=\frac{z^2}{2}\tilde{\tau}\begin{pmatrix}\chi_{20}\\ \varsigma_{20}\end{pmatrix}\Gamma+z\overline{z}\tilde{\tau}\begin{pmatrix}\chi_{11}\\ \varsigma_{11}\end{pmatrix}\Gamma+\frac{\overline{z}^2}{2}\tilde{\tau}\begin{pmatrix}\overline{\chi}_{20}\\ \overline{\varsigma}_{20}\end{pmatrix}\Gamma+\frac{z^2\overline{z}}{2}\tilde{\tau}\begin{pmatrix}\kappa_1\\ \kappa_2\end{pmatrix}+\cdots.\end{aligned} \tag{3.28}$$

with

$$\begin{aligned}\Gamma&=\frac{1}{l\pi}\int_0^{l\pi}\cos^3(\frac{nx}{l})dx,\\ \kappa_1&=\frac{\chi_1}{l\pi}\int_0^{l\pi}\cos^2(\frac{nx}{l})dx+\frac{\chi_2}{l\pi}\int_0^{l\pi}\cos^4(\frac{nx}{l})dx,\\ \kappa_2&=\frac{\varsigma_1}{l\pi}\int_0^{l\pi}\cos^2(\frac{nx}{l})dx+\frac{\varsigma_2}{l\pi}\int_0^{l\pi}\cos^4(\frac{nx}{l})dx\end{aligned}$$

and

$$\begin{aligned}\chi_{20}=&\frac{1}{2}\left(\alpha_1+\xi(\alpha_2+\alpha_3\xi)-e^{-\mathrm{i}\tilde{\tau}\omega_n}\right),\\ \chi_{11}=&-\frac{1}{4}\left(-(2\alpha_1+\alpha_2(W_{11}^{(1)}(0)+\xi)+2\alpha_3W_{11}^{(1)}(0)\xi)+e^{-\mathrm{i}\tilde{\tau}\omega_n}+e^{\mathrm{i}\tilde{\tau}\omega_n}\right),\\ \chi_1=&W_{11}^{(1)}(0)(2\alpha_1+\alpha_2\xi-e^{-\mathrm{i}\tilde{\tau}\omega_n})+W_{11}^{(2)}(0)(2\alpha_2+2\alpha_3\xi)-W_{11}^{(1)}(-1)-\frac{1}{2}W_{20}^{(1)}(-1)\\ &+W_{20}^{(1)}(0)(-\frac{1}{2}\left(-2\alpha_1-\alpha_2W_{11}^{(1)}(0)+e^{\mathrm{i}\tilde{\tau}\omega_n}\right))\\ &+W_{20}^{(2)}(0)(\frac{1}{2}(\alpha_2+2\alpha_3W_{11}^{(1)}(0))),\\ \chi_2=&\frac{1}{4}(3\alpha_4+\alpha_5(W_{11}^{(1)}(0)+2\xi)+\xi(2\alpha_6W_{11}^{(1)}(0)+\alpha_6\xi+3\alpha_7W_{11}^{(1)}(0)\xi)),\end{aligned}$$

$$
\begin{aligned}
\varsigma_{20} =& \frac{1}{2}(\beta_1 + \xi(\beta_2 + \beta_3\xi)),\\
\varsigma_{11} =& \frac{1}{4}(2\beta_1 + \beta_2(W_{11}^{(1)}(0) + \xi) + 2\beta_3 W_{11}^{(1)}(0)\xi),\\
\varsigma_1 =& W_{11}^{(1)}(0)\left(\beta_1 + \frac{\beta_2\xi}{2}\right) + \frac{1}{2}W_{11}^{(2)}(0)(\beta_2 + 2\beta_3\xi) + \frac{1}{4}W_{20}^{(1)}(0)(2\beta_1 + \beta_2 W_{11}^{(1)}(0))\\
&+ \frac{1}{4}W_{20}^{(2)}(0)(\beta_2 + 2\beta_3 W_{11}^{(1)}(0)),\\
\varsigma_2 =& \frac{1}{8}(3\beta_4 + \beta_5(W_{11}^{(1)}(0) + 2\xi) + \xi(2\beta_6\overline{\xi} + \beta_6\xi + 3\beta_7 W_{11}^{(1)}(0)\xi)).
\end{aligned}
$$

Denote

$$
\Psi_1(0) - \mathrm{i}\Psi_2(0) := (\gamma_1 \quad \gamma_2).
$$

Notice that

$$
\frac{1}{l\pi}\int_0^{l\pi} \cos^3\frac{nx}{l}dx = 0, \quad n = 1, 2, 3, \cdots,
$$

and we have

$$
\begin{aligned}
&(\Psi_1(0) - \mathrm{i}\Psi_2(0)) < F(U_t, 0), f_n >\\
=& \frac{z^2}{2}(\gamma_1\chi_{20} + \gamma_2\varsigma_{20})\Gamma\tilde{\tau} + z\overline{z}(\gamma_1\chi_{11} + \gamma_2\varsigma_{11})\Gamma\tilde{\tau} + \frac{\overline{z}^2}{2}(\gamma_1\overline{\chi}_{20} + \gamma_2\overline{\varsigma}_{20})\Gamma\tilde{\tau}\\
&+ \frac{z^2\overline{z}}{2}\tilde{\tau}[\gamma_1\kappa_1 + \gamma_2\kappa_2] + \cdots.
\end{aligned} \tag{3.29}
$$

Then by (3.21), (3.23) and (3.29), we have $g_{20} = g_{11} = g_{02} = 0$, for $n = 1, 2, 3, \cdots$. If $n = 0$, we have:

$$
g_{20} = \gamma_1\tilde{\tau}\chi_{20} + \gamma_2\tilde{\tau}\varsigma_{20}, \qquad g_{11} = \gamma_1\tilde{\tau}\chi_{11} + \gamma_2\tilde{\tau}\varsigma_{11}, \qquad g_{02} = \gamma_1\tilde{\tau}\overline{\chi}_{20} + \gamma_2\tilde{\tau}\overline{\varsigma}_{20}.
$$

And for $n \in \mathbb{N}_0$, $g_{21} = \tilde{\tau}(\gamma_1\kappa_1 + \gamma_2\kappa_2)$.

From [15], we have

$$
\begin{aligned}
\dot{W}(z,\overline{z}) &= W_{20}z\dot{z} + W_{11}\dot{z}\overline{z} + W_{11}z\dot{\overline{z}} + W_{02}\overline{z}\dot{\overline{z}} + \cdots,\\
A_{\tilde{\tau}}W(z,\overline{z}) &= A_{\tilde{\tau}}W_{20}\frac{z^2}{2} + A_{\tilde{\tau}}W_{11}z\overline{z} + A_{\tilde{\tau}}W_{02}\frac{\overline{z}^2}{2} + \cdots,
\end{aligned}
$$

and

$$
\dot{W}(z,\overline{z}) = A_{\tilde{\tau}}W + H(z,\overline{z}),
$$

where

$$
\begin{aligned}
H(z,\overline{z}) &= H_{20}\frac{z^2}{2} + W_{11}z\overline{z} + H_{02}\frac{\overline{z}^2}{2} + \cdots\\
&= F(U_t, 0) - \Phi(\Psi, < F(U_t, 0), f_n > \cdot f_n).
\end{aligned} \tag{3.30}
$$

Hence, we have

$$
(2\mathrm{i}\omega_n\tilde{\tau} - A_{\tilde{\tau}})W_{20} = H_{20}, \quad -A_{\tilde{\tau}}W_{11} = H_{11}, \quad (-2\mathrm{i}\omega_n\tilde{\tau} - A_{\tilde{\tau}})W_{02} = H_{02}, \tag{3.31}
$$

that is

$$
W_{20} = (2\mathrm{i}\omega_n\tilde{\tau} - A_{\tilde{\tau}})^{-1}H_{20}, \quad W_{11} = -A_{\tilde{\tau}}^{-1}H_{11}, \quad W_{02} = (-2\mathrm{i}\omega_n\tilde{\tau} - A_{\tilde{\tau}})^{-1}H_{02}. \tag{3.32}
$$

Then

$$\begin{aligned}
H(z,\overline{z}) &= -\Phi(\theta)\Psi(\theta) < F(U_t,\theta), f_n > \cdot f_n \\
&= -(\frac{p_1(\theta)+p_2(\theta)}{2}, \frac{p_1(\theta)-p_2(\theta)}{2\mathrm{i}}) \begin{pmatrix} \Phi_1(\theta) \\ \Phi_2(\theta) \end{pmatrix} < F(U_t,\theta), f_n > \cdot f_n \\
&= -\frac{1}{2}[p_1(\theta)(\Phi_1(\theta) - \mathrm{i}\Phi_2(\theta)) + p_2(\theta)(\Phi_1(\theta) + \mathrm{i}\Phi_2(\theta))] < F(U_t,\theta), f_n > \cdot f_n \\
&= -\frac{1}{2}[(p_1(\theta)g_{20} + p_2(\theta)\overline{g}_{02})\frac{z^2}{2} + (p_1(\theta)g_{11} + p_2(\theta)\overline{g}_{11})z\overline{z} \\
&\quad + (p_1(\theta)g_{02} + p_2(\theta)\overline{g}_{20})\frac{\overline{z}^2}{2}] + \cdots .
\end{aligned}$$

Therefore,

$$H_{20}(\theta) = \begin{cases} 0 & n \in \mathbb{N}, \\ -\frac{1}{2}(p_1(\theta)g_{20} + p_2(\theta)\overline{g}_{02}) \cdot f_0 & n = 0, \end{cases}$$

$$H_{11}(\theta) = \begin{cases} 0 & n \in \mathbb{N}, \\ -\frac{1}{2}(p_1(\theta)g_{11} + p_2(\theta)\overline{g}_{11}) \cdot f_0 & n = 0, \end{cases}$$

$$H_{02}(\theta) = \begin{cases} 0 & n \in \mathbb{N}, \\ -\frac{1}{2}(p_1(\theta)g_{02} + p_2(\theta)\overline{g}_{20}) \cdot f_0 & n = 0, \end{cases}$$

and

$$H(z,\overline{z})(0) = F(U_t,0) - \Phi(\Psi, < F(U_t,0), f_n >) \cdot f_n,$$

where

$$H_{20}(0) = \begin{cases} \tilde{\tau} \begin{pmatrix} \chi_{20} \\ \varsigma_{20} \end{pmatrix} \cos^2(\frac{nx}{l}), & n \in \mathbb{N}, \\ \tilde{\tau} \begin{pmatrix} \chi_{20} \\ \varsigma_{20} \end{pmatrix} - \frac{1}{2}(p_1(0)g_{20} + p_2(0)\overline{g}_{02}) \cdot f_0, & n = 0. \end{cases} \tag{3.33}$$

$$H_{11}(0) = \begin{cases} \tilde{\tau} \begin{pmatrix} \chi_{11} \\ \varsigma_{11} \end{pmatrix} \cos^2(\frac{nx}{l}), & n \in \mathbb{N}, \\ \tilde{\tau} \begin{pmatrix} \chi_{11} \\ \varsigma_{11} \end{pmatrix} - \frac{1}{2}(p_1(0)g_{11} + p_2(0)\overline{g}_{11}) \cdot f_0, & n = 0. \end{cases}$$

By the definition of $A_{\tilde{\tau}}$ and (3.31), we have

$$\dot{W}_{20} = A_{\tilde{\tau}} W_{20} = 2\mathrm{i}\omega_n \tilde{\tau} W_{20} + \frac{1}{2}(p_1(\theta)g_{20} + p_2(\theta)\overline{g}_{02}) \cdot f_n, \quad -1 \le \theta < 0.$$

That is

$$W_{20}(\theta) = \frac{\mathrm{i}}{2\mathrm{i}\omega_n\tilde{\tau}}(g_{20}p_1(\theta) + \frac{\overline{g}_{02}}{3}p_2(\theta)) \cdot f_n + E_1 e^{2\mathrm{i}\omega_n\tilde{\tau}\theta},$$

where

$$E_1 = \begin{cases} W_{20}(0) & n = 1, 2, 3, \cdots, \\ W_{20}(0) - \frac{\mathrm{i}}{2\mathrm{i}\omega_n\tilde{\tau}}(g_{20}p_1(\theta) + \frac{\overline{g}_{02}}{3}p_2(\theta)) \cdot f_0 & n = 0. \end{cases}$$

By the definition of $A_{\tilde{\tau}}$ and (3.31), we have that for $-1 \le \theta < 0$

$$\begin{aligned} &-(g_{20}p_1(0) + \frac{\overline{g}_{02}}{3}p_2(0)) \cdot f_0 + 2\mathrm{i}\omega_n\tilde{\tau}E_1 - A_{\tilde{\tau}}(\frac{\mathrm{i}}{2\omega_n\tilde{\tau}}(g_{20}p_1(0) + \frac{\overline{g}_{02}}{3}p_2(0)) \cdot f_0) \\ &- A_{\tilde{\tau}}E_1 - L_{\tilde{\tau}}(\frac{\mathrm{i}}{2\omega_n\tilde{\tau}}(g_{20}p_1(0) + \frac{\overline{g}_{02}}{3}p_2(0)) \cdot f_n + E_1 e^{2\mathrm{i}\omega_n\tilde{\tau}\theta}) \\ =&\tilde{\tau}\begin{pmatrix} \chi_{20} \\ \varsigma_{20} \end{pmatrix} - \frac{1}{2}(p_1(0)g_{20} + p_2(0)\overline{g}_{02}) \cdot f_0. \end{aligned}$$

As

$$A_{\tilde{\tau}}p_1(0) + L_{\tilde{\tau}}(p_1 \cdot f_0) = \mathrm{i}\omega_0 p_1(0) \cdot f_0,$$

and

$$A_{\tilde{\tau}}p_2(0) + L_{\tilde{\tau}}(p_2 \cdot f_0) = -\mathrm{i}\omega_0 p_2(0) \cdot f_0,$$

we have

$$2\mathrm{i}\omega_n E_1 - A_{\tilde{\tau}}E_1 - L_{\tilde{\tau}}E_1 e^{2\mathrm{i}\omega_n} = \tilde{\tau}\begin{pmatrix} \chi_{20} \\ \varsigma_{20} \end{pmatrix}\cos^2(\frac{nx}{l}), \quad n \in \mathbb{N}_0.$$

That is

$$E_1 = \tilde{\tau}E\begin{pmatrix} \chi_{20} \\ \varsigma_{20} \end{pmatrix}\cos^2(\frac{nx}{l})$$

where

$$E = \begin{pmatrix} 2\mathrm{i}\omega_n\tilde{\tau} + d_1\frac{n^2}{l^2} - a_1 + u_* e^{-2\mathrm{i}\omega_n\tilde{\tau}} & ra_2 \\ -\beta b_1 & 2\mathrm{i}\omega_n\tilde{\tau} + d_2\frac{n^2}{l^2} - \beta b_2 \end{pmatrix}^{-1}.$$

Similarly, from (3.32), we have

$$-\dot{W}_{11} = \frac{\mathrm{i}}{2\omega_n\tilde{\tau}}(p_1(\theta)g_{11} + p_2(\theta)\overline{g}_{11}) \cdot f_n, \quad -1 \le \theta < 0.$$

That is

$$W_{11}(\theta) = \frac{\mathrm{i}}{2\mathrm{i}\omega_n\tilde{\tau}}(p_1(\theta)\overline{g}_{11} - p_1(\theta)g_{11}) + E_2.$$

Similarly, we have

$$E_2 = \tilde{\tau}E^*\begin{pmatrix} \chi_{11} \\ \varsigma_{11} \end{pmatrix}\cos^2(\frac{nx}{l}),$$

where

$$E^* = \begin{pmatrix} d_1\frac{n^2}{l^2} - a_1 + u_* & a_2 \\ -\beta b_1 & d_2\frac{n^2}{l^2} - \beta b_2 \end{pmatrix}^{-1}.$$

Thus, we have

$$\begin{aligned} c_1(0) &= \frac{\mathrm{i}}{2\omega_n\tilde{\tau}}(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}) + \frac{1}{2}g_{21}, \quad \mu_2 = -\frac{\mathrm{Re}(c_1(0))}{\mathrm{Re}(\lambda'(\tau_n^j))}, \\ T_2 &= -\frac{1}{\omega_n\tilde{\tau}}[\mathrm{Im}(c_1(0)) + \mu_2\mathrm{Im}(\lambda'(\tau_n^j))], \quad \beta_2 = 2\mathrm{Re}(c_1(0)). \end{aligned} \tag{3.34}$$

Theorem 3.2. *For any critical value $\tau_n^{j,\pm}$, the bifurcating periodic solutions exists for $\tau > \tau_n^{j,\pm}$ (or $\tau < \tau_n^{j,\pm}$) when $\mu_2 > 0$ (or $\mu_2 < 0$), and are orbitally asymptotically stable (or unstable) when $\beta_2 < 0$ (or $\beta_2 > 0$).*

3.3. Example

Fix parameters in (2.13), $d_2 = 0.1$, $l = 4$ and $\beta = 1$. By direct computation, we have $\mathcal{D} = \{0, 1, 2, 3\} \neq \emptyset$, $\tau_* = \tau_0^{0,-} \approx 1.7335$. By Theorem 3.1, we know that (u_*, v_*) is locally asymptotically stable when $\tau \in [0, \tau_*)$ (shown in Fig. 4). The Hopf bifurcation occurs when $\tau = \tau_*$. By Theorem 3.2, we have

$$\mu_2 \approx 47.5263 > 0, \quad \beta_2 \approx -1.2381 < 0, \quad \text{and} \quad T_2 \approx -3.9621 < 0.$$

Hence, the locally asymptotically stable homogeneous bifurcating periodic solutions exists for $\tau > 3.4595$, and the period of bifurcating periodic solutions decrease (shown in Fig. 5).

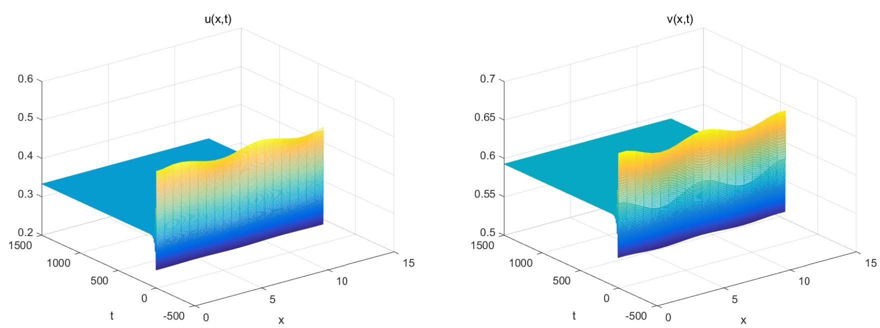

Figure 4. Numerical simulations of system (1.4) for $\tau = 1.5$.

4. Conclusion

In this paper, we consider a delayed diffusive predator-prey system with a functional response increasing in both predator and prey densities. We mainly analyze the Turing instability and Hopf bifurcation of coexisting equilibrium. We also give some parameters that determining the property of Hopf bifurcation: bifurcation direction and the stability of the bifurcating periodic solution. Compare with the model

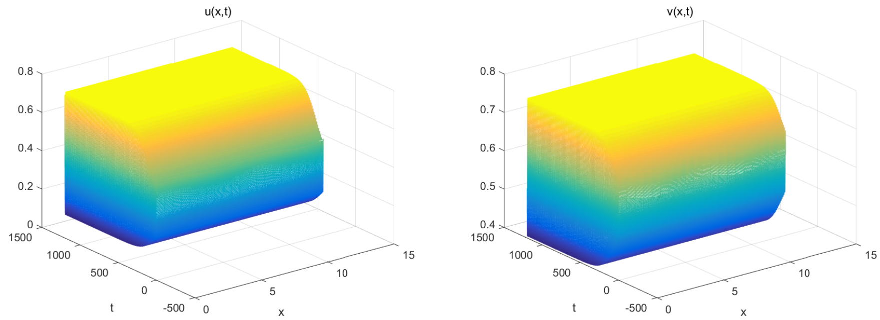

Figure 5. Numerical simulations of system (1.4) for $\tau = 2$.

(1.3), diffusion induced Turing instability and spatial inhomogeneity bifurcating period solution occur. In addition, time delay may induce instability and Hopf bifurcation.

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