Journal of Applied Analysis and Computation
Volume 10, Number 1, February 2020, 378–390
Website:http://jaac.ijournal.cn
DOI:10.11948/20190294

EXACT BOUND ON THE NUMBER OF LIMIT CYCLES ARISING FROM A PERIODIC ANNULUS BOUNDED BY A SYMMETRIC HETEROCLINIC LOOP*

Xianbo Sun[1,2]

Abstract In this paper, the bound on the number of limit cycles by Poincaré bifurcation in a small perturbation of some seventh-degree Hamiltonian system is concerned. The lower and upper bounds on the number of limit cycles have been obtained in two previous works, however, the sharp bound is still unknown. We will employ some new techniques to determine which is the exact bound between 3 and 4. The asymptotic expansions are used to determine the four vertexes of a tetrahedron, and the sharp bound can be reached when the parameters belong to this tetrahedron.

Keywords Limit cycle, Abelian integral, heteroclinic loop, sharp bound.

MSC(2010) 34C07, 34D10, 37G20.

1. Introduction

The classical Liénard system

$$\ddot{x} + f(x)\dot{x} + x = 0, \tag{1.1}$$

and its generalized form

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{1.2}$$

have extensive application in real world. It can describe the oscillation which is evident in many fields of science, not only in electronics, physics and mathematics, but also in civil engineering, chemistry, biology, astronomy, see a very new monograph [2], in which various kinds of nonlinear oscillators and their applications have been studied systematically by methods of numerical analysis and dynamical system. From another point of view, the number of limit cycles of systems (1.1) and (1.2) provides very important information for studying Hilbert's 16th problem [16]. This problem asks for the maximal number of limit cycles for the polynomial differential system $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$ of a given degree n. It is rather difficult and still open even for $n = 2$. Several weaker versions had been proposed, for example,

Email address: xianbo01@126.com(X. Sun)
[1]Department of Applied Mathematics, Guangxi University of Finance and Economics, 100-West-Mingxiu, Nanning, 530003, China
[2]Department of Applied Mathematics, Western University, 1151 Richmond Street London, Ontario, N6A 5B7, Canada
*The author is supported by National Natural Science Foundation of China (11861009), National Science Foundation of Guangxi (2018GXNSFAA138198) and Program for Innovative Team of GUFE(2018-2021).

S. Smale [21] proposed to study the maximal number of limit cycles of the classical Liénard system (1.1), and VI. Arnold confined the problem to studying limit cycles arising from perturbing a family of ovals of Hamiltonian, see [1]. In detail, $H(x,y)$, $p(x,y)$ and $q(x,y)$ are polynomials, and the Hamiltonian system $\dot{x} = H_y(x,y)$, $\dot{y} = -H_x(x,y)$ has at least one family of closed orbits $L_h \subseteq \{(x,y)|H(x,y)=0\}$. After a small polynomial vector perturbation by $(p(x,y), q(x,y))$, only a finite number of closed orbits persist. These persisting closed orbits are limit cycles of the perturbed Hamiltonian system,

$$\dot{x} = H_y(x,y) + \epsilon p(x,y),\ \dot{y} = -H_x(x,y) + \epsilon q(x,y), \tag{1.3}$$

where ϵ is a sufficiently small parameter. The phenomenon is called Poincaré bifurcation firstly studied by Jules Henri Poincaré (1854-1912). The Poincaré Theorem [9] reveals that the number of limit cycles by Poincaré bifurcation of (1.3) can be evaluated by the number of zeros of the following integral,

$$M(h) = \oint_{L_h} q(x,y)dx - p(x,y)dy, \tag{1.4}$$

which is called Abelian integral. However, both of Smale's version and Arnold's version of Hilbert's 16th problem are still rather difficult and only solved for lower degree n, see [3].

A simpler form of (1.3) by choosing $H(x,y) = \frac{y^2}{2} + \int g(x)dx,\ p(x,y) = 0$ and $q(x,y) = f(x)y$ is

$$\dot{x} = y,\ \dot{y} = -g(x) + \epsilon f(x)y, \tag{1.5}$$

which is the planar form of system (1.2) with weak damping term $-\epsilon f(x)\dot{x}$. Therefore, studying the maximal number of zeros of the Abelian integral of system (1.5),

$$I(h) = \oint_{L_h} f(x)ydx, \tag{1.6}$$

can be regarded as an intersection version of Smale's problem and Arnold's problem. There have been many excellent works on this rather simple versions, however, it is not easy at all. Note that system (1.5) of a fixed degree n can be classified into different cases according to the topological portraits of system $(1.5)_{\epsilon=0}$. For convenience, the maximal number of zeros of (1.6) for any concrete system (1.5) of degree n is denoted by $\mathcal{N}(n)$ when it's need for convenience. Dumortier and Li obtained the sharp bounds of $\mathcal{N}(3)$ for 5 different topological portraits of unperturbed system (1.5) in a series paper [4–7]. For system (1.5) of degree 4, the lower and upper bounds on $\mathcal{N}(4)$ for topologically different periodic annulus have been summarized in a new work [22], in which the sharp bounds on $\mathcal{N}(4)$ are obtained when the outer boundary of the period annulus is a heteroclinic (homoclinic) loop connecting a nilpotent singularity. The method and analysis are based on the Chebyshev criterion and combination technique. For $\mathcal{N}(5)$, the results for system (1.5) with symmetry have been only reported, see [23,31] and the references therein. When system (1.5) is nonsymmetric and has a degree that is larger than 4, the Abelian integrals have more than 4 elements. Then, the classical idea based on Picard-Focus equation fails and applying the Chebyshev criterion leads to rather complicated algebraic computation. The symmetry assumption on (1.5) reduces the generating elements of $I(h)$. For example, $I(h)$ has four generating elements for symmetric system (1.5) of degree 7. The symmetric system (1.5) of degree 7 can be normalized as follows:

$$\dot{x} = y,\ \dot{y} = \zeta x(x^2-1)(x^2-\alpha)(x^2-\beta) + \varepsilon f(x)y, \tag{1.7}$$

where $\zeta = \pm 1$, α and β are real number, $f(x) = a_0 + a_1x^2 + a_2x^4 + a_3x^6$. We assume system $(1.7)_{\epsilon=0}$ has a heteroclinic loop without other poly-cycle, then we have eight topological portraits, see Figure 1.

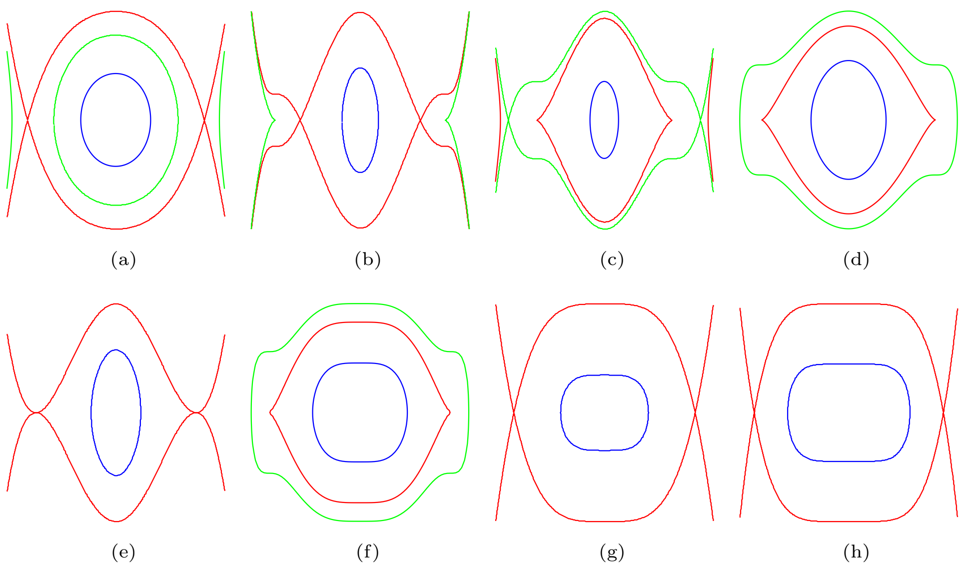

Figure 1. Periodic annului bounded by heteroclinic loops (red curves) for system $(1.7)_{\varepsilon=0}$. (a) $\zeta = 1,\ \alpha < \beta < 0$; (b) $\zeta = 1,\ \alpha = \beta > 1$; (c) $\zeta = 1,\ \alpha = 1,\ \beta > 1$; (d) $\zeta = -1,\ \alpha < 0, \beta = 1$; (e) $\zeta = 1,\ \alpha = \beta = 1$; (f) $\zeta = -1,\ \alpha = 0, \beta = 1$; (g) $\zeta = 1,\ \alpha < 0, \beta = 0$; (h) $\zeta = 1,\ \alpha = \beta = 0$. For cases a and b, the heteroclinic loop connects two hyperbolic saddles and the inner boundary is an elementary center. For cases c, d and e, the heteroclinic loop connects two nilpotent saddles (cusps) and the inner boundary is an elementary center. For case f, the heteroclinic loop connects two nilpotent cusps and the inner boundary is a nilpotent center. For cases g and h, the heteroclinic loop connects two hyperbolic saddles and the inner boundary is a nilpotent center.

For cases a and b, it was proved that 4 is the sharp bound of $\mathcal{N}(7)$ in [24] and [30] by different researchers, while in the latter work the same result was obtained for case c only on the inner part of the periodic annulus. Kazemi and Zangeneh [17] studied the perturbation of period annulus belonging to case e, and proved 3 and 4 are the lower and upper bounds on the corresponding $\mathcal{N}(7)$. An equivalent system was studied in [25] and almost same results were obtained with different methods on asymptotic expansion of $I(h)$. Sun and Zhao [26] obtained that 3 and 4 are the lower and upper bounds on the corresponding $\mathcal{N}(7)$ by perturbing the period annulus of case f. There are no results reported on cases d, g and h up to now, and it is still unknown that which is the exact bound on the corresponding $\mathcal{N}(7)$ between 3 and 4 for cases e and f. The sharp bounds are only obtained for system (1.7) belonging to cases a, b, it is interesting that there exist a heteroclinic loop connecting two hyperbolic saddles and an elementary center. In a very recent work [32], an equivalent system that belongs to case a was investigated by choosing different α and β from those in [24]. They obtained that 3 and 4 are the lower bound and upper bound on the corresponding $\mathcal{N}(7)$, respectively. The authors also claimed that the

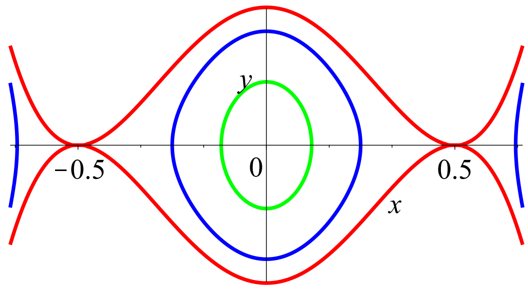

Figure 2. The level set of $\mathcal{H}(x, y)$

non-existence of nilpotent singularity on the boundary of period annulus may not be the key condition for obtaining the sharp bound 4. An additional technique based on combination of two elements in $I(h)$ has been used in [24, 26]. It plays an key role in obtaining the exact bound and the upper bound. The sharp bound in [24] could not be obtained without the combination technique and only a larger upper bound 5 can be derived by directly applying the Chebyshev criterion, see the final remark in [24]. Chebyshev criteria [8, 19] failed to bound $\mathcal{N}(7)$ on case f. However, the combination produces a parameter in the integral systems and the computational analysis becomes more difficult.

In this paper, we concern about part of the remaindering problems on system (1.7) with a unique heteroclinic loop bounding a periodic annulus. We employ the combination technique to determine which is the exact bound between 3 and 4 for $\mathcal{N}(7)$ on case e. The main idea to prove our result is as follows: we bound the parameter space in $\mathbb{R}^3$ to obtain a cubic set, which might yield maximal 4 zeros of the corresponding Abelian integrals. Furthermore, we exclude existence of 4 zeros on this cubic set, and thus the sharp bound is 3. We adopt the system given below rather than the system $(1.7)_{\alpha=\beta=1}$ investigated in [17], and part of the analysis is based on [24],

$$\dot{x} = y,\ \ \dot{y} = x(x^2 - \frac{1}{4})^3 + \epsilon(a_0 + a_1x^2 + a_2x^4 + a_3x^6)y, \tag{1.8}$$

where ϵ is sufficiently small and a_is $(i = 0, 1, 2, 3)$ are bounded parameters. System (1.8) can be transformed into system $(1.7)_{\alpha=\beta=1}$ by scaling $x = \frac{\widetilde{x}}{2}$, $y = \frac{\widetilde{y}}{16}$, $t = 8\widetilde{t}$ and $a_i = 2^{2i+1}\widetilde{a}_i$.

The unperturbed system ($\epsilon = 0$) is an integrable system with the Hamiltonian,

$$\mathcal{H}(x, y) = \frac{y^2}{2} + \frac{x^2}{128} - \frac{3}{64}x^4 + \frac{x^6}{8} - \frac{x^8}{8}. \tag{1.9}$$

The bounded level sets are sketched in Figure 2. $\mathcal{H}(x, y) = h$ defines a family of closed curves γ_h (orbits of system $(1.8)_\epsilon = 0$) forms a periodic annulus $\mathcal{PA}$ for $0 < h < \frac{1}{2048}$. The inner boundary is an elementary center at the origin $(0, 0)$ and the outer one is a heteroclinic loop γ_l connecting two nilpotent saddles $(\pm\frac{1}{2}, 0)$ with $\mathcal{H}(\gamma_l) = \frac{1}{2048}$.

Corresponding to the discussion above, the first order approximation of the return map constructed on $\mathcal{PA}$ is given as below:

$$\mathcal{I}(h,\delta) = \oint_{\gamma_h} (a_0 + a_1x^2 + a_2x^4 + a_3x^6)ydx \equiv a_0\mathcal{I}_0(h) + a_1\mathcal{I}_1(h) + a_2\mathcal{I}_2(h) + a_3\mathcal{I}_3(h) \tag{1.10}$$

where

$$\mathcal{I}_i(h) = \oint_{\gamma_h} x^{2i}ydx$$

for $i = 0, 1, 2, 3$, $\delta = (a_0, a_1, a_2, a_3)$. The following results have been obtained.

Theorem 1.1 ([25]). *For all possible $(a_0, a_1, a_2, a_3) \in \mathbb{R}^4$, $\mathcal{I}(h,\delta)$ has at most 4 zeros, and there exist some $(a_0, a_1, a_2, a_3) \in \mathbb{R}^4$ such that $\mathcal{I}(h,\delta)$ has at least 3 zeros.*

As we mentioned before the same result was given in [17] for the related Abelian integral. However, the exact bound had not been obtained. Our new result is given as belows:

Theorem 1.2. *The exact bound on the number of zeros of $\mathcal{I}(h,\delta)$ is 3.*

As we have shown that $(1.7)_{\alpha=\beta=1}$ is equivalent to system (1.8). Therefore, Theorem 1.2 holds for system $(1.7)_{\alpha=\beta=1}$. The rest of the paper is organized as follows: in section 2, we give some primary results of the elements $\mathcal{I}_i(h)$ near the endpoints of $(0, \frac{1}{2048})$ and give a short description of the main tool that is used in this paper. In section 3, we prove our main result and give four vertexes of a tetrahedron $\mathbb{T}_4$. $\mathcal{I}(h)$ has exact 3 zeros when the parameter belongs to $\mathbb{T}_4$. Finally, we end this paper with a discussion and a summary on the bound of $\mathcal{N}(7)$ associated with the eight periodic annuli bounded by a heteroclinic loop.

2. The ratios of some $\frac{\mathcal{I}_i(h)}{\mathcal{I}_0(h)}$ at the endpoints and Chebyshev criterion

In this section, we analyze the ratios of $\frac{\mathcal{I}_i(h)}{\mathcal{I}_0(h)}$ for $i = 1, 2, 3$ near the center and heteroclinic loop and give a brief introduction of the Chebyshev criterion. We denote $\mathcal{A}(x) = \mathcal{H}(x,y) - \frac{y^2}{2}$. First, $\mathcal{I}_0(h) = \oint_{\gamma_h} ydx = \iint_{\mathcal{R}} dxdy > 0$, where $\mathcal{R}$ is the region bounded by γ_h. Then, the ratio $\frac{\mathcal{I}_i(h)}{\mathcal{I}_0(h)}$ $(i = 1, 2, 3)$ is well defined. Further, we have the following result.

Lemma 2.1. $\lim_{h\to 0} \frac{\mathcal{I}_i(h)}{\mathcal{I}_0(h)} = 0$ *for* $i = 1, 2, 3$, $\lim_{h\to\frac{1}{2048}} \frac{\mathcal{I}_1(h)}{\mathcal{I}_0(h)} = \frac{1}{28}$, $\lim_{h\to\frac{1}{2048}} \frac{\mathcal{I}_2(h)}{\mathcal{I}_0(h)} = \frac{1}{336}$ and $\lim_{h\to\frac{1}{2048}} \frac{\mathcal{I}_3(h)}{\mathcal{I}_0(h)} = \frac{5}{14784}$.

Proof. From section 3 of [25], $\mathcal{I}(h,\delta)$ has the following asymptotic expansion for h psotive and near $h = 0$,

$$\mathcal{I}(h,\delta) = b_0(\delta)h + b_1(\delta)h^2 + b_2(\delta)h^3 + O(h^4), \tag{2.1}$$

where $b_0(\delta) = 16\pi a_0$, $b_1(\delta) = 4608\pi a_0 + 512\pi a_1$, $b_2(\delta) = 3850240\pi a_0 + 491520\pi a_1 + 32768\pi a_2$. The first assertion holds obviously by (2.1). For the rest parts, we only prove $\lim\limits_{h\to\frac{1}{2048}} \frac{\mathcal{I}_1(h)}{\mathcal{I}_0(h)} = \frac{1}{28}$, others are similar proved. Direct computation gives

$$\lim_{h\to\frac{1}{2048}} \frac{\mathcal{I}_1(h)}{\mathcal{I}_0(h)} = \frac{\lim\limits_{h\to\frac{1}{2048}} \mathcal{I}_1(h)}{\lim\limits_{h\to\frac{1}{2048}} \mathcal{I}_0(h)} = \frac{\oint_{\gamma_l} x^2 y dx}{\oint_{\gamma_l} y dx} = \frac{2\int_{-\frac{1}{2}}^{\frac{1}{2}} x^2(\frac{1}{32} - \frac{x^2}{4} + \frac{x^4}{2})dx}{2\int_{-\frac{1}{2}}^{\frac{1}{2}} (\frac{1}{32} - \frac{x^2}{4} + \frac{x^4}{2})dx} = \frac{1}{28}.$$

□

The following result is taken from [25] which was obtained based on Lemma 4.1 of [8].

Lemma 2.2 ([25]). *For $i \in \mathbb{N}$, we have*

$$8h^3 \mathcal{I}_i(h) = 8h^3 \oint\limits_{\gamma_h} x^{2i} y dx = \oint\limits_{\gamma_h} f_i(x) y^7 dx \equiv \widetilde{I}_i(h),$$

where $f_i(x) = \frac{8x^{2i}\left(\sum\limits_{j=0}^{12} g_j(i) x^{2j}\right)}{105(2x-1)^{12}(2x+1)^{12}}$ *with $g_j(i)s$ $(i = 0, 1, \cdots, 12)$ are given in Appendix A.*

The technique to bound the number of zeros of the Abelian integral used in [17, 25] is the Chebyshev criterion. Generally speaking, we call a set of integrals, taking $\{\mathcal{I}_0(h), \mathcal{I}_1(h), \mathcal{I}_2(h), \cdots, \mathcal{I}_{n-1}(h)\}$ for example, forms a Chebyshev system if any linear combination $\alpha_0\mathcal{I}_0(h) + \alpha_1\mathcal{I}_1(h) + \alpha_2\mathcal{I}_2(h) + \cdots + \alpha_{n-1}\mathcal{I}_{n-1}(h)$ has at most $n-1$ zeros for $n \in \mathbb{N}^+$. A clever criterion to determine the Chebyshev system for a special integral set was proposed in [8, 19], which reduces the problem into determining a set of analytic functions forming a Chebyshev system. The reduced problem is algebraic and one only check if the related Wronskians vanishes or not. For completeness and readability, we give the exact criterion of the main criterion [19] based on the integrals $\{\oint\limits_{\gamma_h} f_i(x) y^7 dx\}$ for system (1.8) which is defined on the periodic annulus $\mathcal{PA}$, see [8, 17, 19, 25] for more details.

Lemma 2.3. *Considering*

$$\widetilde{I}_i(h) = \int\limits_{\gamma_h} f_i(x) y^{2s-1} dx, \quad i = 0, 1, \cdots, n-1,$$

where $s = 4$, each $f_i(x)$ is given in Lemma 2.2 and analytic function on $(-\frac{1}{2}, \frac{1}{2})$. Let

$$l_i(x) := \frac{f_i(x)}{\mathcal{A}'(x)} - \frac{f_i(z(x))}{\mathcal{A}'(z(x))}.$$

If the following condition are verified:

(a) $\mathcal{W}[l_0, ..., l_i]$ is non vanishing on $(0, \frac{1}{2})$ for $i = 0, 1, \cdots, n-2$,

(b) $\mathcal{W}[l_0, ..., l_{n-1}]$ has k zeros on $(0, \frac{1}{2})$ with multiplicities, and

(c) $s > n + k - 2$.

Then any nontrivial linear combination of $\{\mathcal{I}_0, \mathcal{I}_1, \cdots, \mathcal{I}_{n-1}\}$ has at most $n+k-1$ zeros on $(0, \frac{1}{2048})$ counted with multiplicities, and $\{\mathcal{I}_0, \mathcal{I}_1, \cdots, \mathcal{I}_{n-1}\}$ is usually called a Chebyshev system with accuracy k on $(0, \frac{1}{2048})$.

As we have mentioned above, the criterion has been successfully used to system (1.8) in [17, 25], however, only a larger upper bound was derived. It is still unknown that whether 4 or 3 is the exact bound. We will combine the algebraic criterion with a new skill and then answer the remaining question.

3. Proof of Main result

In this section, we prove the main result. The main idea is to intercept the parameter space to obtain a cubic set $\mathcal{S}$, only on which $\mathcal{I}(h,\delta)$ may have maximal 4 zeros. Then, the maximum of $\mathcal{I}(h,\delta)$ on $(0,\frac{1}{2048})$ for $(a_0,a_1,a_2)\in\mathcal{S}$ contradicts existence of zeros of $\mathcal{I}(h,\delta)$. Hence, the sharp bound is 3.

The following lemma is obtained by direct computation, and part of them had been proved in [25]. The proof is omitted.

Lemma 3.1. *The Wronskians $\mathcal{W}(l_0(x))$, $\mathcal{W}(l_1(x))$, $\mathcal{W}(l_0(x),l_1(x))$, $\mathcal{W}(l_0(x),l_2(x))$, $\mathcal{W}(l_0(x),l_3(x))$, $\mathcal{W}(l_1(x),l_2(x))$ and $\mathcal{W}(l_0(x),l_1(x),l_2(x))$ do not vanish on for $x\in(0,\frac{1}{2})$.*

We have the following two propositions by applying Lemma 2.3 and Lemma 3.1.

Proposition 3.1. *The linear combinations $a_0\mathcal{I}_0(h)+a_i\mathcal{I}_{\mathcal{I}}(h)$ $(i=1,2,3)$ has at most one zero on $(0,\frac{1}{2048})$, in other words, $\frac{\mathcal{I}_i(h)}{\mathcal{I}_0(h)}$ $(i=1,2,3)$ is monotonic on $(0,\frac{1}{2048})$.*

Proposition 3.2. *The linear combinations $a_0\mathcal{I}_0(h)+a_1\mathcal{I}_1(h)+a_2\mathcal{I}_2(h)$ has at most two zeros on $(0,\frac{1}{2048})$, in other words, $\mathcal{I}(h,\delta)$ has at most two zeros $(0,\frac{1}{2048})$ provided that $a_3=0$.*

It is without loss of generality to assume $a_3=1$ and denote $\delta=(a_0,a_1,a_2)$ if $a_3\neq 0$. Introduce

$$\begin{aligned}\mathcal{I}_{23}(h)&=\oint\limits_{\gamma_h}\Big(\alpha_2x^4+x^6\Big)ydx,\\ \mathcal{I}_{13}(h)&=\oint\limits_{\gamma_h}\Big(\alpha_1x^2+x^6\Big)ydx,\\ \mathcal{I}_{03}(h)&=\oint\limits_{\gamma_h}\Big(\alpha_0+x^6\Big)ydx.\end{aligned}\tag{3.1}$$

Then, we have

$$\begin{aligned}\mathcal{I}(h,\delta)&=\alpha_0\mathcal{I}_0(h)+\alpha_1\mathcal{I}_1(h)+\mathcal{I}_{23}(h)\\ &=\alpha_0\mathcal{I}_0(h)+\alpha_2\mathcal{I}_2(h)+\mathcal{I}_{13}(h)\\ &=\alpha_1\mathcal{I}_1(h)+\alpha_2\mathcal{I}_2(h)+\mathcal{I}_{03}(h).\end{aligned}$$

The following equations inherit from Lemma 2.2.

Lemma 3.2.

$$
\begin{aligned}
8h^3\mathcal{I}_{23}(h) &= \oint_{\gamma_h} (\alpha_2 f_2(x) + f_3(x)) y^7 dx \overset{\triangle}{=} \widetilde{\mathcal{I}}_{23}(h), \\
8h^3\mathcal{I}_{13}(h) &= \oint_{\gamma_h} (\alpha_1 f_1(x) + f_3(x)) y^7 dx \overset{\triangle}{=} \widetilde{\mathcal{I}}_{13}(h), \\
8h^3\mathcal{I}_{03}(h) &= \oint_{\gamma_h} (\alpha_0 f_0(x) + f_3(x)) y^7 dx \overset{\triangle}{=} \widetilde{\mathcal{I}}_{03}(h).
\end{aligned}
$$

Further, introduce

$$
\mathcal{L}_{i3}(x) = \Big(\frac{\alpha_i f_i + f_3}{\mathcal{A}'}\Big)(x) - \Big(\frac{\alpha_i f_i + f_3}{\mathcal{A}'}\Big)(-x).
$$

Direct computations yield

$$
\begin{aligned}
\mathcal{W}[l_0(x), l_1(x), \mathcal{L}_{23}] &= \tfrac{17179869184(a_2 p_2(x) - p_1(x))}{8575(2x-1)^{45}(2x+1)^{45}}, \\
\mathcal{W}[l_0(x), l_2(x), \mathcal{L}_{13}] &= \tfrac{17179869184(a_1 q_2(x) - q_1(x))}{8575(2x-1)^{45}(2x+1)^{45}}, \\
\mathcal{W}[l_1(x), l_2(x), \mathcal{L}_{03}] &= \tfrac{17179869184(a_0 r_2(x) - r_1(x))}{8575(2x-1)^{45}(2x+1)^{45}},
\end{aligned}
\tag{3.2}
$$

where the polynomials p_1, p_2, q_1, q_2, r_1 and r_2 in x have the degrees 74, 72, 76, 72, 78 and 72, respectively. Applying Sturm's Theorem yields that p_2, q_2 and r_2 do not vanish on $(0, \frac{1}{2})$. Then, the equations $a_2 p_2(x) - p_1(x) = 0$, $a_1 q_2(x) - q_1(x) = 0$ and $a_0 r_2(x) - r_1(x) = 0$ can define three continuous and smooth functions on $(0, \frac{1}{2})$:

$$
a_2(x) = \frac{p_1(x)}{p_2(x)},\ a_1(x) = \frac{q_1(x)}{q_2(x)},\ a_0(x) = \frac{r_1(x)}{r_2(x)}.
$$

We have the following result.

Lemma 3.3. (i) *$a_2(x)$ is decreasing from $(0,0)$ to a minimum (x^*, a_2^*) and then increasing to $(\frac{1}{2}, -\frac{3}{4})$;*

(ii) *$a_1(x)$ is increasing from $(0,0)$ to a maximum $(x^\dagger, a_1^*)$ and then decreasing to $(\frac{1}{2}, \frac{3}{16})$;*

(iii) *$a_0(x)$ is decreasing from $(0,0)$ to a minimum $(x^\ddagger, a_0^*)$ and then increasing to $(\frac{1}{2}, -\frac{1}{64})$, where*

$$
x^*, x^\dagger, x^\ddagger \in \underbrace{\Big[\frac{824775837939361}{2251799813685248}, \frac{6598206703529315}{18014398509481984}\Big]}_{1/10^{10}},
$$

and

$$a_2^* \in \underbrace{\Big[-\frac{92850\cdots 55904}{97206\cdots 02491}, -\frac{12932\cdots 31675}{13538\cdots 53024}\Big]}_{1/10^{11}}$$
$$\approx [-0.95518814209, -0.95518814206],$$
$$a_1^* \in \underbrace{\Big[\frac{28544\cdots 33125}{1023\cdots 67552}, \frac{27104\cdots 41696}{97206\cdots 02491}\Big]}_{1/10^{11}}$$
$$\approx [0.27883338139, 0.27883338140],$$
$$a_0^* \in \underbrace{\Big[-\frac{31136\cdots 75957}{12033\cdots 41984}, -\frac{85958\cdots 78125}{33220\cdots 45312}\Big]}_{1/10^{12}}$$
$$\approx [-0.025874713153, -0.025874713152].$$

The critical point (x^*, a_2^*) divides the curve $\{(x, a_2(x)|0 < x < \frac{1}{2}\}$ into two simple segments (curves). The points on the two curves correspond to the simple roots of $a_2 p_2(x) - p_1(x) = 0$, while x^* is a root of multiplicity 2. The portrait of the function $a_2(x)$ reveals that $\mathcal{W}[l_0(x), l_1(x), \mathcal{L}_{23}]$ will have 2, 1 and 0 zero with multiplicities counted when a_2 belongs to the intervals $[a_2^*, -\frac{3}{4})$, $(-\frac{3}{4}, 0)$ and $(0, +\infty) \bigcup (-\infty, a_2^*)$, respectively. Combining Lemma 3.1 and applying Lemma 2.2, we have the following result.

Proposition 3.3. *$\mathcal{I}(h,\delta)$ has at most 4, 3, 2 zeros in $(0, \frac{1}{2048})$ when a_2 is located in the intervals $[a_2^*, -\frac{3}{4})$, $[-\frac{3}{4}, 0)$, and $(-\infty, a_2^*) \bigcup [0, +\infty)$, respectively.*

With the same arguments, we have

Proposition 3.4. *$\mathcal{I}(h,\delta)$ has at most 4, 3, 2 zeros in $(0, \frac{1}{2048})$ when a_1 belongs to the intervals $(\frac{3}{16}, a_1^*]$, $(0, \frac{3}{16}]$, and $(-\infty, 0] \bigcup (a_1^*, +\infty)$, respectively.*

Proposition 3.5. *$\mathcal{I}(h,\delta)$ has at most 4, 3, 2 zeros in $(0, \frac{1}{2048})$ when a_0 is located in the intervals $[a_0^*, -\frac{1}{64})$, $(-\frac{1}{64}, 0]$, and $(-\infty, a_0^*] \bigcup (0, +\infty)$, respectively.*

Define

$$\mathcal{S} = \Big\{(a_0, a_1, a_2)|a_0 \in \Big[a_0^*, -\frac{1}{64}\Big),\ a_1 \in \Big(\frac{3}{16}, a_1^*\Big],\ a_2 \in \Big(-\frac{3}{4}, a_2^*\Big]\Big\}.$$

Then, Propositions 3.3, 3.4 and 3.5 imply that

Proposition 3.6. *$\mathcal{I}(h,\delta)$ may have 4 zeros only if $\delta = (a_0, a_1, a_2) \in \mathcal{S}$.*

Last, we will show $\mathcal{I}(h,\delta)$ cannot have 4 zeros when $\delta \in \mathcal{S}$.

Proposition 3.7. *$\mathcal{I}(h,\delta) < 0$ when $\delta = (a_0, a_1, a_2) \in \mathcal{S}$.*

Proof. Lemma 3.3 gives the exact bound on the value of a_i^* for $i = 1, 2, 3$. The right endpoints of the intervals bounding a_0^* and a_1^* are denoted by a_0^{*+} and a_1^{*+} for convenience, respectively. Lemma 2.1 and Proposition 3.1 provide intervals bounding $\frac{\mathcal{I}_i(h)}{\mathcal{I}_0(h)}$ for $i = 1, 2, 3$,

$$\frac{\mathcal{I}_1(h)}{\mathcal{I}_0(h)} \in \Big(0, \frac{1}{28}\Big),\quad \frac{\mathcal{I}_2(h)}{\mathcal{I}_0(h)} \in \Big(0, \frac{1}{336}\Big),\quad \frac{\mathcal{I}_3(h)}{\mathcal{I}_0(h)} \in \Big(0, \frac{5}{14784}\Big).$$

Then, we have $a_2\frac{\mathcal{I}_2(h)}{\mathcal{I}_0(h)} \leq 0$. Then

$$
\begin{aligned}
a_0 + a_1\frac{\mathcal{I}_1(h)}{\mathcal{I}_0(h)} + a_2\frac{\mathcal{I}_2(h)}{\mathcal{I}_0(h)} + \frac{\mathcal{I}_3(h)}{\mathcal{I}_0(h)} &\leq a_0 + a_1\frac{\mathcal{I}_1(h)}{\mathcal{I}_0(h)} + \frac{\mathcal{I}_3(h)}{\mathcal{I}_0(h)} \\
&< -\frac{1}{64} + a_1^{*+}\frac{1}{336} + \frac{5}{14784} \\
&= -\tfrac{113}{7392} + \tfrac{a_1^{*+}}{28} \\
&= -O(10^{-2}) + O(10^{-3}) < 0.
\end{aligned}
$$

Hence,

$$
\mathcal{I}(h,\delta) = \mathcal{I}_0(h)\Big(a_0 + a_1\frac{\mathcal{I}_1(h)}{\mathcal{I}_0(h)} + a_2\frac{\mathcal{I}_2(h)}{\mathcal{I}_0(h)} + \frac{\mathcal{I}_3(h)}{\mathcal{I}_0(h)}\Big) < 0.
$$

□

Proof of Theorem 1.2. Combining Propositions 3.2, 3.6 and 3.7 proves Theorem 1.2.

From the discussion above, we know that $\{\mathcal{I}_0(h), \mathcal{I}_1(h), \mathcal{I}_2(h), \mathcal{I}_3(h)\}$ forms a Chebyshev system. Mardešić proved that two Chebyshev systems with same dimension have homeomorphic bifurcation diagrams [20]. Then, there exists a tetrahedron $\mathbb{T}_4$ bounding the parameter space $\mathbb{R}^3$ such that $\mathcal{I}(h,\delta)$ has exact 3 zeros when $\delta \in \mathbb{T}_4$. The four vertexes of $\mathbb{T}_4$ can be obtained by the asymptotic of expansions of $\mathcal{I}(h,\delta)$ near the boundary of $\mathcal{PA}$. It is usually very difficult to obtain the expansions near a homoclinic loop or heteroclinic loop connecting singularities, see a series of papers [10–13,15,27,28,33] for the expansions of the Abelian integrals near various kinds of singular loops. The method has been extended to non-smooth systems [14] and higher dimensional systems [29] The asymptotic expansion of $\mathcal{I}(h,\delta)$ near the heteroclinic loop γ_l was given in [25] based on the relatively new results in [12] at that time,

$$
\mathcal{I}(h,\delta) = e_0 + e_1|h - \frac{1}{2048}|^{\frac{3}{4}} + e_2(h - \frac{1}{2048})\ln|h - \frac{1}{2048}| + e_3(h - \frac{1}{2048}) + O(|h - \frac{1}{2048}|^{\frac{5}{4}}), \tag{3.3}
$$

for $0 < -(h - \frac{1}{2048}) \ll 1$, where

$$
\begin{aligned}
e_0 &= \tfrac{1}{88704} + \tfrac{1}{10080}a_2 + \tfrac{1}{840}a_1 + \tfrac{1}{30}a_0, \\
e_1 &= 8^{\frac{13}{12}}\widetilde{A}_0(a_0 + \tfrac{1}{4}a_1 + \tfrac{1}{16}a_2 + \tfrac{1}{64}), \\
e_2 &= \tfrac{3}{8} + a_2 + 2a_1, \\
e_3|_{c_1=c_2=0} &= \tfrac{7}{3} + 4a_2.
\end{aligned}
$$

The four vertexes of $\mathbb{T}_4$ are determined by the algebraic set formed by the coefficients of expansions in (2.1) and (3.3): $\{(a_0,a_1,a_2)|b_0 = b_1 = b_2 = 0\}$, $\{(a_0,a_1,a_2)|b_0 = b_1 = e_0 = 0\}$, $\{(a_0,a_1,a_2)|b_0 = e_0 = e_1 = 0\}$ and $\{(a_0,a_1,a_2)|e_0 = e_1 = e_2 = 0\}$. Direct computation gives the exact four vertexes of $\mathbb{T}_4$: $(0,0,0)$, $(0,0,-\frac{5}{44})$, $(0,\frac{3}{176},-\frac{7}{22})$ and $(-\frac{1}{704},\frac{13}{176},-\frac{23}{44})$.

4. Conclusion

In this paper, we provide a rigorous proof that the Ablian integral of system (1.8) has at most 3 zeros, and then the sharp bound of the number of zeros is 3. The main tools are Chebyshev criterion and combination technique. The problem on the exact bound of zeros of the Abelian integral for system (1.7) of case e is completely solved. However, there still exist some open problems on the bound of $\mathcal{N}(7)$ for system (1.7) of other cases: (I) it is still unknown that the exact bounds on $\mathcal{N}(7)$ for system (1.7) of cases a, b and c when the parameters α and β in unperturbed system are not fixed; (II) what are the exact bounds on $\mathcal{N}(7)$ for system (1.7) of cases d and g with fixed and unfixed parameter α; (III) what are the exact bound for cases f and h. Those problems are difficult by applying recent methods and techniques. We need improve the algebraic criterion and develop more efficient symbolic computations.

Appendix A

The functions $g_j(i)$ in Lemma 2.2 is presented in this part.

$$
\begin{aligned}
g_0(i) &= (i+4)\,(i+3)\,(i+2)\,,\\
g_1(i) &= -3\,\left(10\,i+23\right)(i+4)\,(i+3)\,,\\
g_2(i) &= 21\,\left(i+4\right)\left(20\,i^2+124\,i+193\right),\\
g_3(i) &= -3640\,i^3-42084\,i^2-160910\,i-203241,\\
g_4(i) &= 4(5448\,i^3+72252\,i^2+313242\,i+445311),\\
g_5(i) &= -8(11880\,i^3+180684\,i^2+887958\,i+1416501),\\
g_6(i) &= 32(9680\,i^3+167796\,i^2+932116\,i+1664031),\\
g_7(i) &= -576(1320\,i^3+25852\,i^2+161470\,i+321661),\\
g_8(i) &= 786(1816\,i^3+39804\,i^2+277554\,i+614269),\\
g_9(i) &= -512(3640i^3+88452i^2+682994i+1668735),\\
g_{10}(i) &= 6144(280\,i^3+7476\,i^2+63386\,i+169683),\\
g_{11}(i) &= -24567(40\,i^3+1164\,i^2+10750\,i+31289),\\
g_{12}(i) &= 32768\,\left(2\,i+13\right)(2\,i+21)\,(2\,i+29)\,.
\end{aligned}
$$

References

[1] V. I. Arnold, *Ten problems*, Adv. Soviet Math. 1990.

[2] L. Cveticanin, *Strong Nonlinear Oscillators*, Springer, Berlin, 2018.

[3] F. Chen, C. Li, J. Libre, and Z. Zhang, *A unified proof on the weak Hilbert 16th problem for $n = 2$*, J. Differ. Equat., 2006, 221, 309–342.

[4] F. Dumortier and C. Li, *Perturbations from an elliptic Hamiltonian of degree four: (I) Saddle loop and two saddle cycle*, J. Differ. Equat., 2001, 176, 114–157.

[5] F. Dumortier and C. Li, *Perturbations from an elliptic Hamiltonian of degree four: (II) Cuspidal loop*, J. Differ. Equat., 2001, 175, 209–243.

[6] F. Dumortier and C. Li, *Perturbation from an elliptic Hamiltonian of degree four: III Global center*, J. Differ. Equat., 2003, 188, 473–511.

[7] F. Dumortier and C. Li, *Perturbations from an elliptic Hamiltonian of degree four: (IV) Figure eight-loop*, J. Differ. Equat., 2003, 88, 512–514.

[8] M. Grau, F. Mañosas and J. Villadelprat, *A Chebyshev criterion for Abelian integrals*, Trans. Amer. Math. Soc., 2011, 363, 109–129.

[9] M. Han and P. Yu, *Normal forms, Melnikov functions and bifurcations of limit cycles*, Sringer, 2012.

[10] M. Han, J. Yang, A. Tarta and Y. Gao, *Limit cycles near homoclinic and heteroclinic loops*, J.Dyn Diff Equat., 2008, 20, 923–944.

[11] M. Han, H. Zang and J. Yang, *limit cycle bifurcations by perturbing a suspidal loop in a Hamiltonian system*, J. Differ. Equat., 2009, 246, 129–163.

[12] M. Han, J. Yang and D. Xiao, *Limit cycle bifurcations near a double homoclinic loop with a nilpotent saddle*, Int. J. Bifur. Chaos, 2012, 22, 1250189.

[13] M. Han, *Asymptotic Expansions of Melnikov Functions and Limit Cycle Bifurcations*, Int. J. Bifur. Chaos, 2012, 22, 1250296.

[14] M. Han, L. Sheng, *Bifurcation of limit cycles in piecewise smooth systems via Melnikov function*, J. Appl. Anal. Comput., 2015, 5(4), 809–815.

[15] M. Han, *Bifurcation theory of limit cycles*, Science Press, Beijing, 2016.

[16] D. Hilbert, *Mathematical problems*, Bull. Amer. Math, Soc., 1902, 8, 437–479.

[17] R. Kazemi and H. RZ Zangeneh, *Bifurcation of limit cycles in small perturbations of a hyper-elliptic Hamiltonian system with two nilpotent saddles*, J. Appl. Anal. Comput., 2012, 2, 395–413.

[18] C. Li and C. Rousseau, *A system with three limit cycles appearing in a Hopf bifurcation and dying in a homoclinic bifurcation: the cusp of order 4*, J. Differ. Equat., 1989, 79, 132–167.

[19] F. Mañosas and J. Villadelprat, *Bounding the number of zeros of certain A-belian integrals*, J. Differ. Equat., 2011, 251, 1656–1669.

[20] P. Mardešić, *Chebyshev systems and the versal unfolding of the cusps of order n*, Hermann, Éditeurs des Sciences et des Arts, Paris, 1998.

[21] S. Smale, *Mathematical problems for the next century*, The mathematical intelligencer, 1998, 20, 7–15.

[22] X. Sun and P. Yu, *Exact bound on the number of zeros of Abelian integrals for two hyper-elliptic Hamiltonian systems of degree* 4, J. Differ. Equat., 2019, 267, 7369–7384.

[23] X. Sun and J. Yang, *Sharp bounds of the number of zeros of Abelian integrals with parameters*, E. J. Differ. Equat., 2014, 40, 1–12.

[24] X. Sun, *Perturbation of a period annulus bounded by a heteroclinic loop connecting two hyperbolic saddles*, Qual. Theory Dyn. Syst., 2017, 16, 187–203.

[25] X. Sun, *Bifurcation of limit cycles from a Liénard system with a heteroclinic loop connecting two nilpotent saddles*, Nonlinear Dynam, 2013, 73, 869–880.

[26] X. Sun and L. Zhao, *Perturbations of a class of hyper-elliptic Hamiltonian systems of degree seven with nilpotent singular points*, Appl. Math. Comput., 2016, 289, 194–203.

[27] X. Sun, M. Han and J. Yang, *Bifurcation of limit cycles from a heteroclinic loop with a cusp*, Nonlinear Anal., 2011, 74, 2948–2965.

[28] Y. Tian, M. Han, *Hopf and homoclinic bifurcations for near-Hamiltonian systems*, J. Differ. Equat., 2017, 262(4), 3214–3234.

[29] H. Tian, M. Han, *Bifurcation of periodic orbits by perturbing high-dimensional piecewise smooth integrable systems*, J. Differ. Equat., 2017, 263(11), 7448–7474.

[30] S. Yang, B. Qin, G. Xia and Y. Xia, *Perturbation of a period annulus bounded by a saddle-saddle cycle in a hyperelliptic Hamiltonian systems of degree 7*, Qual. Theory Dyn. Syst., 2019, accepted.

[31] L. Zhao and D. Li, *Bifurcations of limit cycles from a quintic Hamiltonian system with a heteroclinic cycle*, Acta Mathematica Sinica. English Series, 2014, 30(3), 411.

[32] H. Zhu, S. Yang, X. Hu and W. Huang, *Perturbation of a Period Annulus with a Unique Two-Saddle Cycle in Higher Order Hamiltonian*, Complexity, 5813596, 2019.

[33] H. Zang, M. Han and D. Xiao, *On Melnikov functions of a homoclinic loop through a nilpotent saddle for planar near-Hamiltonian system*, J. Differ. Equat., 2008, 245, 1086–1111.