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STATISTICAL ANALYSIS OF TWO-PARAMETER GENERALIZED BIRNBAUM-SAUNDERS CAUCHY DISTRIBUTION*

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Abstract The image features of density function and failure rate function are studied in detail for two-parameter generalized Birnbaum-Saunders Cauchy fatigue life distribution. The logarithmic moment estimation and other two point estimations of parameters are proposed under full sample, and the precisions of point estimations are investigated by Monte-Carlo simulations. The approximate interval estimations of parameters are given by using Taylor expansion, and the precisions of approximate interval estimations are investigated by Monte-Carlo simulations. Finally, several examples show the feasibility of the methods.

Keywords Two-parameter generalized Birnbaum-Saunders Cauchy fatigue life distribution, shape parameter, scale parameter, point estimation, approximate interval estimation.

MSC(2010) 62N05.

1. Introduction

Birnbaum-Saunders model is an important failure distribution model in the probabilistic physical method, which is deduced by Birnbaum and Saunders in 1969 when they studied on the material failure process caused by dominant crack propagation. It is widely applied in the study of mechanical production reliability, mainly used in the study of fatigue failure. Besides, it has the important application in the failure analysis of electronic products performance degradation.

Suppose that T follows two-parameter Birnbaum-Saunders fatigue life distribution $BS(\alpha,\beta)$, its distribution function and density function are respectively

$$F(t) = \Phi\left[\frac{1}{\alpha}\left(\sqrt{\frac{t}{\beta}} - \sqrt{\frac{\beta}{t}}\right)\right],$$

$$f(t) = \frac{1}{2\alpha\sqrt{\beta}}\left(\frac{1}{\sqrt{t}} + \frac{\beta}{t\sqrt{t}}\right)\varphi\left[\frac{1}{\alpha}\left(\sqrt{\frac{t}{\beta}} - \sqrt{\frac{\beta}{t}}\right)\right], t > 0$$

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where $\alpha > 0$ is a shape parameter, $\beta > 0$ is a scale parameter and $\varphi(x), \Phi(x)$ are respectively density function and distribution function of standard normal distribution, that is, $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \Phi(x) = \int_{-\infty}^{x} \varphi(y)dy$.

Birnbaum-Saunders fatigue life distribution is derived from basic characteristics of fatigue process, so it is more suitable for describing the life regularity of several fatigue failure products than common life distributions such as Weibull distribution and log-normal distribution. Besides, it has become one of common distributions in reliability statistical analysis.

Due to the relations between the two-parameters BS distribution $BS(\alpha, \beta)$ and standard normal distribution $N(0,1)$ which is a symmetric distribution, if $N(0,1)$ is replaced by other symmetric distribution, then the obtained distribution is called generalized BS distribution. For example, if $N(0,1)$ is replaced by standard Laplace distribution, that is, $\frac{1}{\alpha}\left(\sqrt{\frac{T}{\beta}} - \sqrt{\frac{\beta}{T}}\right) \sim L(0,1)$, then it is called $T \sim \mathrm{GBS} - \mathrm{Laplace}(\alpha, \beta)$; if $N(0,1)$ is replaced by standard Cauchy distribution, that is, $\frac{1}{\alpha}\left(\sqrt{\frac{T}{\beta}} - \sqrt{\frac{\beta}{T}}\right) \sim t(1)$, then it is called $T \sim \mathrm{GBS} - \mathrm{Cauchy}(\alpha, \beta)$ or $T \sim \mathrm{GBS} - \mathrm{Student}(\alpha, \beta)$; if $N(0,1)$ is replaced by standard Logistic distribution, that is, $\frac{1}{\alpha}\left(\sqrt{\frac{T}{\beta}} - \sqrt{\frac{\beta}{T}}\right) \sim \mathrm{Logistic}(1)$, then it is called $T \sim \mathrm{GBS} - \mathrm{Logistic}(\alpha, \beta)$. The researches of this kind of generalized BS distributions see reference [1,3–8,12]. They mostly refer to image features of density function and failure rate function, numerical characteristics and the discussion on MLE of parameter. Xiaojun Zhu and Balakrishnan [12] made a further analysis on $\mathrm{GBS} - \mathrm{Laplace}(\alpha, \beta)$ in 2015, and proved that MLE is existent and unique, but the process of proof is not perfect. Ronghua Wang [10] proposed the test statistics of fitting test and two new approximate interval estimation methods of environmental factor for two-parameter BS fatigue life distribution.

The image features of density function and failure rate function are studied in detail for two-parameter generalized Birnbaum-Saunders Cauchy fatigue life distribution $\mathrm{GBS} - \mathrm{Cauchy}(\alpha, \beta)$ in this paper. The logarithmic moment estimation and other two point estimations of parameters are proposed under full sample, and the precisions of point estimations are investigated by Monte-Carlo simulations. The approximate interval estimations of parameters are given by using Taylor expansion, and the precisions of approximate interval estimations are investigated by Monte-Carlo simulations. Finally, several examples show the feasibility of the methods.

2. Image Features of Density Function and Failure Rate Function for Two-parameter $\mathrm{GBS} - \mathrm{Cauchy}$ (α, β) Distribution

Suppose non-negative continuous random variable X follows two-parameter Birnbaum-Saunders Cauchy fatigue life distribution $\mathrm{GBS} - \mathrm{Cauchy}(\alpha, \beta)$, its distribution function $F_X(x)$ and density function $f_X(x)$ are respectively

$$F_X(x)=\Phi\left[\frac{1}{\alpha}\left(\sqrt{\frac{x}{\beta}}-\sqrt{\frac{\beta}{x}}\right)\right], f_X(x)=\frac{1}{2\alpha x}\left(\sqrt{\frac{x}{\beta}}+\sqrt{\frac{\beta}{x}}\right)\varphi\left[\frac{1}{\alpha}\left(\sqrt{\frac{x}{\beta}}-\sqrt{\frac{\beta}{x}}\right)\right],$$
$$x>0,\alpha>0,\beta>0$$

where $\varphi(t)=\frac{1}{\pi(1+t^2)}, \Phi(t)=\int_{-\infty}^{t}\varphi(y)dy=\frac{1}{2}+\frac{1}{\pi}\arctan t, -\infty<t<+\infty$.

2.1. Image Feature of Density Function $f_X(x)$

Theorem 2.1. *Suppose non-negative continuous random variable X follows two-parameter Birnbaum-Saunders Cauchy fatigue life distribution* $\mathrm{GBS}-\mathrm{Cauchy}(\alpha,\beta)$, *$f(x)$ is firstly strictly monotonic decreasing, then strictly monotonic increasing, and finally strictly monotonic decreasing again for $\alpha\le 0.910721$; $f(x)$ is strictly monotonic decreasing for $\alpha>0.910721$.*

Proof. The density function is

$$f_X(x)=\frac{1}{2\alpha x}\frac{\sqrt{\frac{x}{\beta}}+\sqrt{\frac{\beta}{x}}}{\pi\left[1+\frac{1}{\alpha^2}\left(\sqrt{\frac{x}{\beta}}-\sqrt{\frac{\beta}{x}}\right)^2\right]}=\frac{\alpha\sqrt{\beta}}{2\pi}\frac{x+\beta}{\sqrt{x}\,[x^2+(\alpha^2-2)\beta x+\beta^2]}.$$

It is obvious that $\lim\limits_{x\to 0}f_X(x)=+\infty$, $\lim\limits_{x\to+\infty}f_X(x)=0$.

Since β is a scale parameter, we choose $\beta=1$ without loss of generality. Then the density function is $f_X(x)=\frac{\alpha}{2\pi}\frac{x+1}{\sqrt{x}[x^2+(\alpha^2-2)x+1]}$, and its derivative is

$$\begin{aligned}f'_X(x)=&\frac{\alpha}{2\pi}\frac{-3x^3-(\alpha^2+3)x^2-(3\alpha^2-7)x-1}{2x\sqrt{x}[x^2+(\alpha^2-2)x+1]^2}\\=&-\frac{\alpha}{4\pi}\frac{3x^3+(\alpha^2+3)x^2+(3\alpha^2-7)x+1}{x\sqrt{x}[x^2+(\alpha^2-2)x+1]^2}.\end{aligned}$$

Let the function be $g(x)=3x^3+(\alpha^2+3)x^2+(3\alpha^2-7)x+1, x>0$, and we know

$$\lim_{x\to 0}g(x)=1, g(1)=4\alpha^2, \lim_{x\to+\infty}g(x)=+\infty, g'(x)=9x^2+2(\alpha^2+3)x+3\alpha^2-7.$$

Then let the function be $g_1(x)=9x^2+2(\alpha^2+3)x+3\alpha^2-7, x>0$, and we know

$$\begin{aligned}&\lim_{x\to 0}g_1(x)=3\alpha^2-7, g_1(1)=5\alpha^2+8, \lim_{x\to+\infty}g_1(x)=+\infty,\\ \Delta=&4(\alpha^2+3)^2-36(3\alpha^2-7)=4(\alpha^4-21\alpha^2+72)\\=&4\left(\alpha^2-\frac{21-\sqrt{153}}{2}\right)\left(\alpha^2-\frac{21+\sqrt{153}}{2}\right)\\=&4\left(\alpha+\sqrt{\frac{21-\sqrt{153}}{2}}\right)\left(\alpha+\sqrt{\frac{21+\sqrt{153}}{2}}\right)\\&\times\left(\alpha-\sqrt{\frac{21-\sqrt{153}}{2}}\right)\left(\alpha-\sqrt{\frac{21+\sqrt{153}}{2}}\right).\end{aligned}$$

Since $\sqrt{\frac{21-\sqrt{153}}{2}} = 2.07734, \sqrt{\frac{21+\sqrt{153}}{2}} = 4.08469$, we have $\Delta > 0$ for $\alpha < \sqrt{\frac{21-\sqrt{153}}{2}}$; $\Delta \leq 0$ for $\sqrt{\frac{21-\sqrt{153}}{2}} \leq \alpha \leq \sqrt{\frac{21+\sqrt{153}}{2}}$; and $\Delta > 0$ for $\alpha > \sqrt{\frac{21+\sqrt{153}}{2}}$.

1. When $\alpha < \sqrt{\frac{21-\sqrt{153}}{2}}$, we have $\Delta > 0$.

(1) If $\alpha \leq \sqrt{7/3} = 1.52753$, then the equation $g_1(x) = 0$ has unique positive root

$$x_1 = \frac{-(\alpha^2+3) + \sqrt{\alpha^4 - 21\alpha^2 + 72}}{9}.$$

We have $g_1(x) < 0, g'(x) < 0$ for $x < x_1$; and $g_1(x) > 0, g'(x) > 0$ for $x > x_1$. Hence $g(x)$ has the minimum value at the point $x = x_1$, and the minimum value is

$$\begin{aligned} g(x_1) =& \frac{1}{243}\Big[2\alpha^6 + 42\alpha^2\sqrt{\alpha^4 - 21\alpha^2 + 72} - 144\left(-6 + \sqrt{\alpha^4 - 21\alpha^2 + 72}\right) \\ & -\alpha^4\left(63 + 2\sqrt{\alpha^4 - 21\alpha^2 + 72}\right)\Big]. \end{aligned}$$

Let the function be $h(\alpha) = g(x_1), \alpha \leq \sqrt{7/3}$, then the image of $h(\alpha)$ in the interval $(0, \sqrt{7/3}]$ is shown as Figure 1. The root of the equation $h(\alpha) = 0$ is 0.910721. Then we have $h(\alpha) \leq 0$ for $\alpha \leq 0.910721$, and $h(\alpha) > 0$ for $0.910721 < \alpha \leq \sqrt{7/3}$.

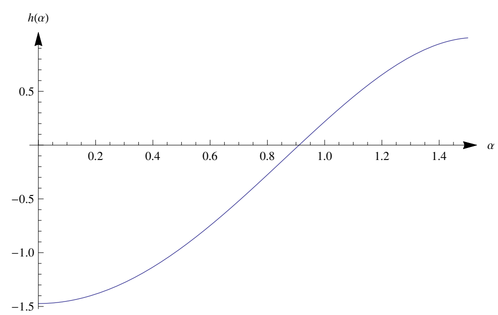

Figure 1. Image of $h(\alpha)$ in the interval $(0, \sqrt{7/3}]$.

(i) When $\alpha \leq 0.910721$, we have $h(\alpha) \leq 0$, and there are $x_{01}, x_{02}, x_{02} > x_{01} > 0$ that satisfy $g(x_{01}) = g(x_{02}) = 0$. If $x < x_{01}, g(x) > 0, f'(x) < 0$; if $x_{01} < x < x_{02}, g(x) < 0, f'(x) > 0$; if $x > x_{02}, g(x) > 0, f'(x) < 0$. Then $f(x)$ is strictly monotonic decreasing, then strictly monotonic increasing, and finally strictly monotonic decreasing again.

(ii)When $0.910721 < \alpha \leq \sqrt{7/3}$, we have $h(\alpha) > 0, g(x) > 0, f'(x) < 0$. Then $f(x)$ is strictly monotonic decreasing.

(2) If $\alpha > \sqrt{7/3}$, that is, $\sqrt{7/3} < \alpha \leq \sqrt{\frac{21-\sqrt{153}}{2}}$,then the equation $g_1(x) = 0$ has no positive root. We have $g_1(x) > 0, g'(x) > 0, g(x) > 0, f'(x) < 0$, and then $f(x)$ is strictly monotonic decreasing.

2. When $\sqrt{\frac{21-\sqrt{153}}{2}} \leq \alpha \leq \sqrt{\frac{21+\sqrt{153}}{2}}$, we have $\Delta \leq 0$. Then we know $g_1(x) \geq 0, g'(x) \geq 0, g(x) \geq 0, f'(x) < 0$, and $f(x)$ is strictly monotonic decreasing.

3. When $\alpha > \sqrt{\frac{21+\sqrt{153}}{2}}$, we have $\Delta > 0$, and the equation $g_1(x) = 0$ has no positive root. Then we know $g_1(x) > 0, g'(x) > 0, g(x) > 0, f'(x) < 0$, and $f(x)$ is strictly monotonic decreasing. $\square$

For a given scale parameter $\beta = 1$, we choose the shape parameter α respectively as 0.25, 0.5, 0.75, 0.8, 0.85, 0.9, 0.95, 1, 1.5, 2, and the images of the corresponding density function $f(x)$ are shown from Figure 2 to Figure 11.

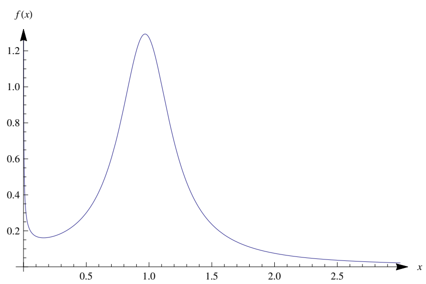

Figure 2. $\alpha = 0.25$

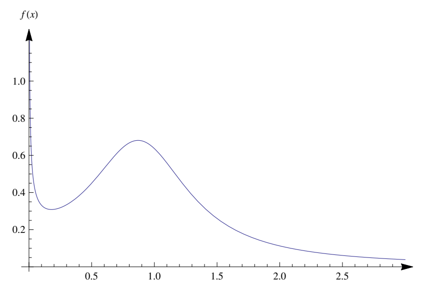

Figure 3. $\alpha = 0.5$

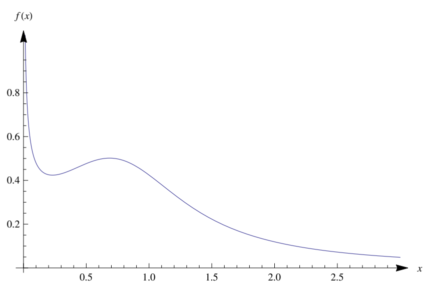

Figure 4. $\alpha = 0.75$

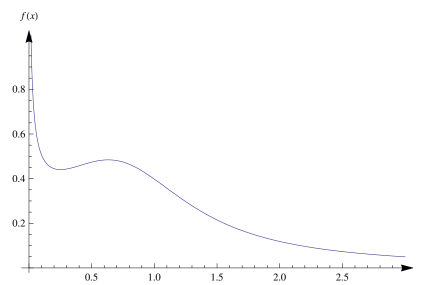

Figure 5. $\alpha = 0.8$

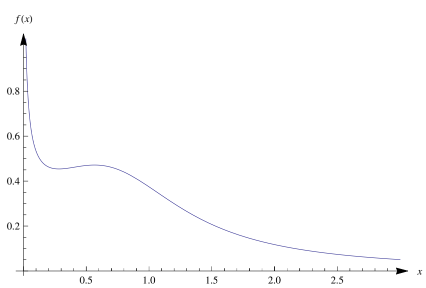

Figure 6. $\alpha = 0.85$

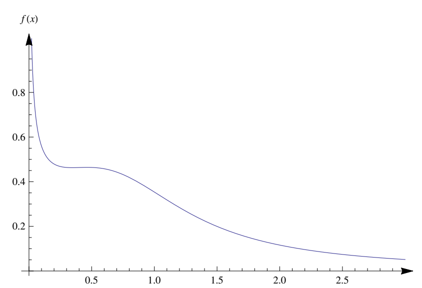

Figure 7. $\alpha = 0.9$

2.2. Discussion on images of failure rate functions

Lemma 2.1 ([2]). *Suppose T is a non-negative continuous random variable, its second order derivation of density function $f(t)$ exists. Let $\eta(t) = -\frac{f'(t)}{f(t)}$, and we have the following conclusions:*

(i) If $\eta'(t) > 0$, that is, $\eta(t)$ is a strictly monotonic increasing function, then $\lambda(t)$ is strictly monotonic increasing;

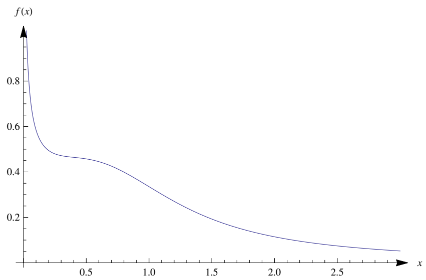

Figure 8. $\alpha = 0.95$

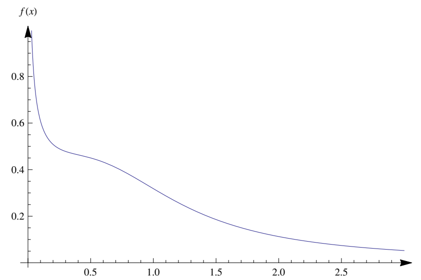

Figure 9. $\alpha = 1$

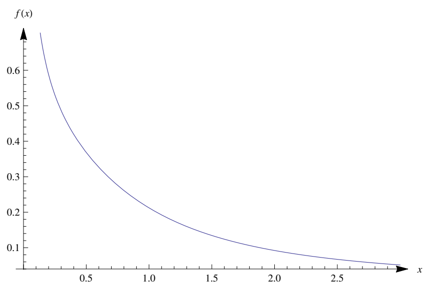

Figure 10. $\alpha = 1.5$

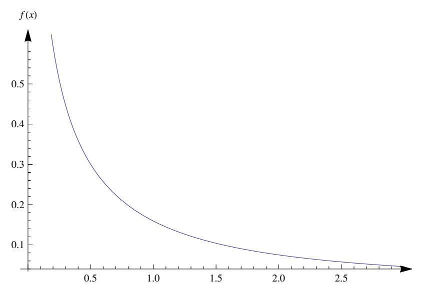

Figure 11. $\alpha = 2$

(ii) *If $\eta'(t) < 0$, that is, $\eta(t)$ is a strictly monotonic decreasing function, then $\lambda(t)$ is strictly monotonic decreasing;*

(iii) *If there is $t_0, t_0 > 0$ that satisfies $\eta'(t_0) = 0$, and $\eta(t)$ is strictly monotonic increasing and then strictly monotonic decreasing, that is, inverse-bathtub shape, then $\lambda(t)$ may be inverse-bathtub shaped, or strictly monotonic decreasing;*

(iv) *If there is $t_0, t_0 > 0$ that satisfies $\eta'(t_0) = 0$, and $\eta(t)$ is strictly monotonic decreasing and then strictly monotonic increasing, that is, bathtub shape, then $\lambda(t)$ may be bathtub shaped, or strictly monotonic increasing.*

Theorem 2.2. *Suppose non-negative continuous random variable X follows two-parameter Birnbaum-Saunders Cauchy fatigue life distribution $GBS-Cauchy(\alpha, \beta)$, the failure rate function $\lambda(x)$ is strictly monotonic decreasing for $\alpha \geq 1.59643$.*

Proof. The failure rate function is

$$\begin{aligned}\lambda(x) =& \frac{f(x)}{1-F(x)} \\ =& \alpha\sqrt{\beta}\frac{x+\beta}{\sqrt{x}\left[x^2+(\alpha^2-2)\beta x+\beta^2\right]}\left\{\pi - 2\arctan\left[\frac{1}{\alpha}\left(\sqrt{\frac{x}{\beta}}-\sqrt{\frac{\beta}{x}}\right)\right]\right\}^{-1}\end{aligned}$$

and we have $\lambda(0) = +\infty, \lambda(+\infty) = \lim\limits_{x\to+\infty} \frac{f(x)}{1-F(x)} = \lim\limits_{x\to+\infty} \frac{f'(x)}{-f(x)} = 0$.

Since β is the scale parameter, we choose $\beta = 1$ without loss of generality. Then we have

$$\eta(x) = -\frac{f'(x)}{f(x)} = \frac{\alpha}{4\pi}\frac{3x^3+(\alpha^2+3)x^2+(3\alpha^2-7)x+1}{x\sqrt{x}[x^2+(\alpha^2-2)x+1]^2}\frac{2\pi}{\alpha}\frac{\sqrt{x}\left[x^2+(\alpha^2-2)x+1\right]}{x+1}$$

$$=\frac{1}{2}\frac{3x^3+(\alpha^2+3)x^2+(3\alpha^2-7)x+1}{x(x+1)\left[x^2+(\alpha^2-2)x+1\right]},$$

and $\eta(0)=+\infty, \eta(+\infty)=0, \eta(1)=1, \eta'(x)=\frac{1}{2}\frac{g(x)}{x^2(x+1)^2[x^2+(\alpha^2-2)x+1]^2}$, where

$$\begin{aligned}g(x)=&-[3x^6+2(\alpha^2+3)x^5+(\alpha^4+8\alpha^2-21)x^4+2(3\alpha^4-10\alpha^2+6)x^3\\&+(3\alpha^4-8\alpha^2+1)x^2+2(\alpha^2-1)x+1].\end{aligned}$$

Let the function be

$$\begin{aligned}h(x)=&3x^6+2(\alpha^2+3)x^5+(\alpha^4+8\alpha^2-21)x^4+2(3\alpha^4-10\alpha^2+6)x^3\\&+(3\alpha^4-8\alpha^2+1)x^2+2(\alpha^2-1)x+1, x>0,\end{aligned}$$

and we have $h(0)=1, h(1)=2\alpha^2(5\alpha^2-8), h(+\infty)=+\infty$,

$$\begin{aligned}h'(x)=&18x^5+10(\alpha^2+3)x^4+4(\alpha^4+8\alpha^2-21)x^3+6(3\alpha^4-10\alpha^2+6)x^2\\&+2(3\alpha^4-8\alpha^2+1)x+2(\alpha^2-1)\\=&2\left[9x^5+5(\alpha^2+3)x^4+2(\alpha^4+8\alpha^2-21)x^3+3(3\alpha^4-10\alpha^2+6)x^2\right.\\&\left.+(3\alpha^4-8\alpha^2+1)x+(\alpha^2-1)\right].\end{aligned}$$

Let the function be

$$\begin{aligned}h_1(x)=&9x^5+5(\alpha^2+3)x^4+2(\alpha^4+8\alpha^2-21)x^3+3(3\alpha^4-10\alpha^2+6)x^2\\&+(3\alpha^4-8\alpha^2+1)x+(\alpha^2-1), x>0,\end{aligned}$$

and we have $h_1(0)=\alpha^2-1, h_1(0)\le 0$ for $\alpha\le 1$; $h_1(0)>0$ for $\alpha>1$,

$$\begin{aligned}&h_1(+\infty)=+\infty,\quad h_1(1)=2\alpha^2(7\alpha^2-8),\\&h_1'(x)=45x^4+20(\alpha^2+3)x^3+6(\alpha^4+8\alpha^2-21)x^2\\&\qquad+6(3\alpha^4-10\alpha^2+6)x+(3\alpha^4-8\alpha^2+1).\end{aligned}$$

Let the function be

$$\begin{aligned}h_2(x)=&45x^4+20(\alpha^2+3)x^3+6(\alpha^4+8\alpha^2-21)x^2\\&+6(3\alpha^4-10\alpha^2+6)x+(3\alpha^4-8\alpha^2+1), x>0\end{aligned}$$

and we have

$$\begin{aligned}h_2(0)&=3\alpha^4-8\alpha^2+1=3\left(\alpha^2-\frac{4-\sqrt{13}}{3}\right)\left(\alpha^2-\frac{4+\sqrt{13}}{3}\right)\\&=3(\alpha+0.362606)(\alpha+1.59223)(\alpha-0.362606)(\alpha-1.59223),\end{aligned}$$

where $\sqrt{\frac{4-\sqrt{13}}{3}}=0.362606, \sqrt{\frac{4+\sqrt{13}}{3}}=1.59223$.

Then we have $h_2(0)\ge 0$ for $\alpha\le 0.362606$; $h_2(0)<0$ for $0.362606<\alpha<1.59223$; $h_2(0)\ge 0$ for $\alpha\ge 1.59223$, and $h_2(+\infty)=+\infty, h_2(1)=27\alpha^4+16$,

$$\begin{aligned}h_2'(x)=&45\cdot 4x^3+60(\alpha^2+3)x^2+12(\alpha^4+8\alpha^2-21)x+6(3\alpha^4-10\alpha^2+6)\\=&6\left[30x^3+10(\alpha^2+3)x^2+2(\alpha^4+8\alpha^2-21)x+(3\alpha^4-10\alpha^2+6)\right].\end{aligned}$$

Let the function be

$$h_3(x) = 30x^3 + 10(\alpha^2+3)x^2 + 2(\alpha^4+8\alpha^2-21)x + (3\alpha^4-10\alpha^2+6), x>0$$

and we have

$$\begin{aligned} h_3(0) =& 3\alpha^4 - 10\alpha^2 + 6 = 3\left(\alpha^2 - \frac{5-\sqrt{7}}{3}\right)\left(\alpha^2 - \frac{5+\sqrt{7}}{3}\right) \\ =& 3(\alpha+0.885861)(\alpha+1.59643)(\alpha-0.885861)(\alpha-1.59643) \end{aligned}$$

where $\sqrt{\frac{5-\sqrt{7}}{3}} = 0.885861, \sqrt{\frac{5+\sqrt{7}}{3}} = 1.59643$.

Then we have $h_3(0) \geq 0$ for $\alpha \leq 0.885861$; $h_3(0) < 0$ for $0.885861 < \alpha < 1.59643$; $h_3(0) \geq 0$ for $\alpha \geq 1.59643$, and $h_3(+\infty) = +\infty, h_3(1) = 5\alpha^4 + 16\alpha^2 + 24$,

$$\begin{aligned} h_3'(x) =& 90x^2 + 20(\alpha^2+3)x + 2(\alpha^4+8\alpha^2-21) \\ =& 2\left[45x^2 + 10(\alpha^2+3)x + (\alpha^4+8\alpha^2-21)\right]. \end{aligned}$$

Let the function be $h_4(x) = 45x^2 + 10(\alpha^2+3)x + (\alpha^4+8\alpha^2-21), x>0$, and we have

$$\begin{aligned} h_4(0) =& \alpha^4 + 8\alpha^2 - 21 = \left[\alpha^2 + (4+\sqrt{37})\right]\left[\alpha^2 - (-4+\sqrt{37})\right] \\ =& \left[\alpha^2 + (4+\sqrt{37})\right](\alpha+1.44318)(\alpha-1.44318), \end{aligned}$$

where $\sqrt{-4+\sqrt{37}} = 1.44318$.

Then we have $h_4(0) \leq 0$ for $\alpha \leq 1.44318$; $h_4(0) > 0$ for $\alpha > 1.44318$, and

$$h_4(+\infty) = +\infty, h_4(1) = \alpha^4 + 18\alpha^2 + 54.$$

For $h_4(x)$, we know

$$\begin{aligned} \Delta =& 100(\alpha^2+3)^2 - 4\cdot 45(\alpha^4+8\alpha^2-21) = -40(2\alpha^4+21\alpha^2-117) \\ =& -80\left[\alpha^2 + \frac{3}{4}(7+3\sqrt{17})\right]\left[\alpha^2 - \frac{3}{4}(-7+3\sqrt{17})\right] \\ =& -80\left[\alpha^2 + \frac{3}{4}(7+3\sqrt{7})\right](\alpha+2.00674)(\alpha-2.00674), \end{aligned}$$

where $\sqrt{\frac{3}{4}(-7+3\sqrt{17})} = 2.00674$.

Then we have $\Delta \geq 0$ for $\alpha \leq 2.00674$; $\Delta < 0$ for $\alpha > 2.00674$. The value of α is divided into eight situations, which is shown in Table 1. Then we only discuss Situation 7 and Situation 8.

Table 1. Eight Situations for the Value of α

Situation 1	Situation 2	Situation 3	Situation 4	Situation 5	Situation 6	Situation 7	Situation 8
0∼ 0.362606	0.362606∼ 0.885861	0.885861∼ 1	1∼ 1.44318	1.44318∼ 1.59223	1.59223∼ 1.59643	1.59643∼ 2.00674	2.00674∼ $+\infty$
$h_1(0) < 0$	$h_1(0) < 0$	$h_1(0) \leq 0$	$h_1(0) > 0$	$h_1(0) > 0$	$h_1(0) > 0$	$h_1(0) > 0$	$h_1(0) > 0$
$h_2(0) \geq 0$	$h_2(0) < 0$	$h_2(0) < 0$	$h_2(0) < 0$	$h_2(0) < 0$	$h_2(0) \geq 0$	$h_2(0) > 0$	$h_2(0) > 0$
$h_3(0) > 0$	$h_3(0) \geq 0$	$h_3(0) < 0$	$h_3(0) < 0$	$h_3(0) < 0$	$h_3(0) < 0$	$h_3(0) \geq 0$	$h_3(0) > 0$
$h_4(0) < 0$	$h_4(0) < 0$	$h_4(0) < 0$	$h_4(0) \leq 0$	$h_4(0) > 0$	$h_4(0) > 0$	$h_4(0) > 0$	$h_4(0) > 0$
$\Delta > 0$	$\Delta > 0$	$\Delta > 0$	$\Delta > 0$	$\Delta > 0$	$\Delta > 0$	$\Delta \geq 0$	$\Delta < 0$

(1) When $\alpha > 2.00674$, we have $h_1(0) > 0, h_2(0) > 0, h_3(0) > 0, h_4(0) > 0, \Delta < 0$. It is obvious that $h_4(x) > 0, h_3'(x) > 0, h_3(x) > 0, h_2'(x) > 0, h_2(x) > 0, h_1'(x) > 0, h_1(x) > 0, h'(x) > 0, h(x) > 0, g(x) < 0, \eta'(x) < 0$. Then the image of $\lambda(x)$ is strictly monotonic decreasing.

(2) When $1.59643 \leq \alpha \leq 2.00674$, we have $h_1(0) > 0, h_2(0) > 0, h_3(0) \geq 0, h_4(0) > 0, \Delta \geq 0$. It is obvious that $h_4(x) > 0, h_3'(x) > 0, h_3(x) > 0, h_2'(x) > 0, h_2(x) > 0, h_1'(x) > 0, h_1(x) > 0, h'(x) > 0, h(x) > 0, g(x) < 0, \eta'(x) < 0$. Then the image of $\lambda(x)$ is strictly monotonic decreasing. $\square$

Remark 2.1. By drawing the images of $\lambda(x)$, it can be concluded that the image of $\lambda(x)$ is firstly strictly monotonic decreasing and then strictly monotonic increasing and finally strictly monotonic decreasing again for $\alpha \leq 1.1$; while the image of $\lambda(x)$ is gradually strictly monotonic decreasing for $\alpha > 1.1$ .That is, with the increase of shape parameter α , the image of $\lambda(x)$ gradually changes from firstly strictly monotonic decreasing, then strictly monotonic increasing and finally strictly monotonic decreasing again to strictly monotonic decreasing.

Let scale parameter be $\beta = 1$ and shape parameter α be $0.1(0.1)2, 1.15, 2.5, 3, 5$, then the images of failure rate function $\lambda(x)$ are shown from Figure 12 to Figure 35.

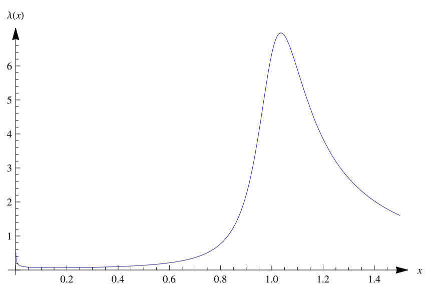

Figure 12. $\alpha = 0.1$

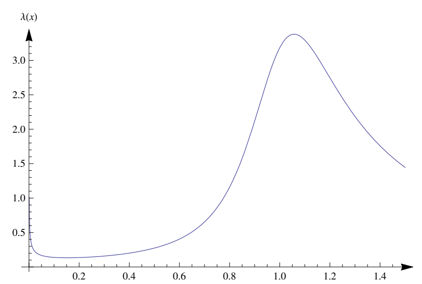

Figure 13. $\alpha = 0.2$

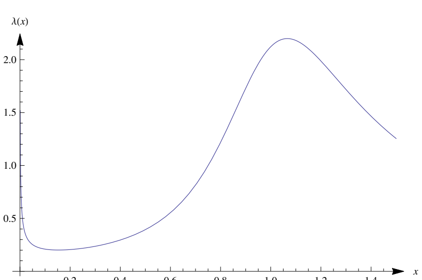

Figure 14. $\alpha = 0.3$

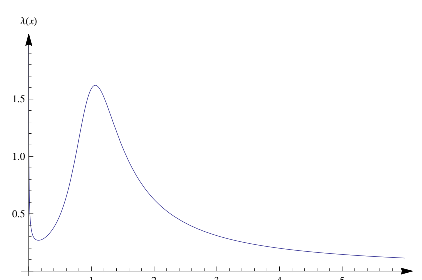

Figure 15. $\alpha = 0.4$

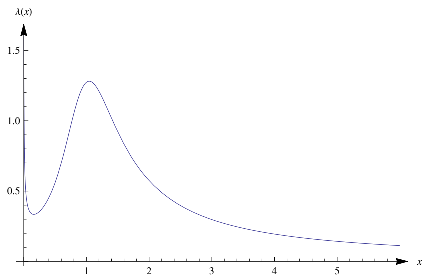

Figure 16. $\alpha = 0.5$

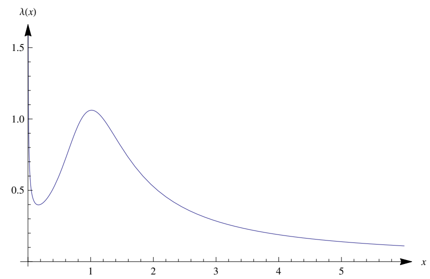

Figure 17. $\alpha = 0.6$

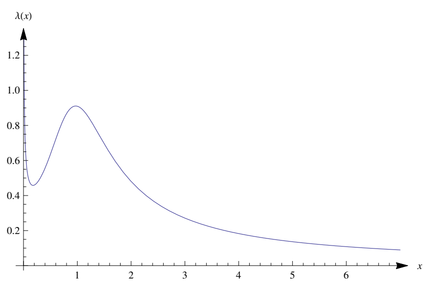

Figure 18. $\alpha = 0.7$

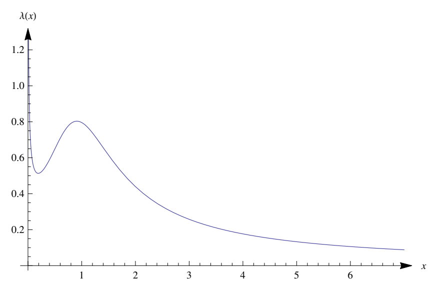

Figure 19. $\alpha = 0.8$

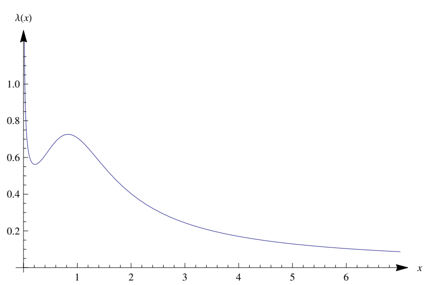

Figure 20. $\alpha = 0.9$

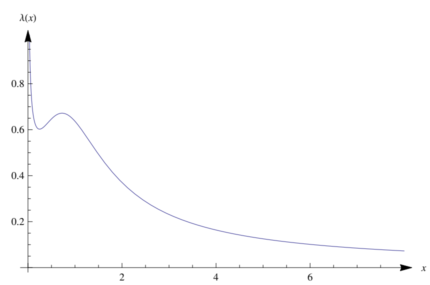

Figure 21. $\alpha = 1$

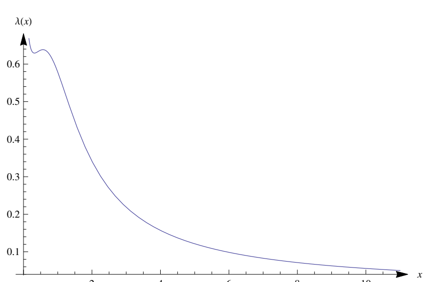

Figure 22. $\alpha = 1.1$

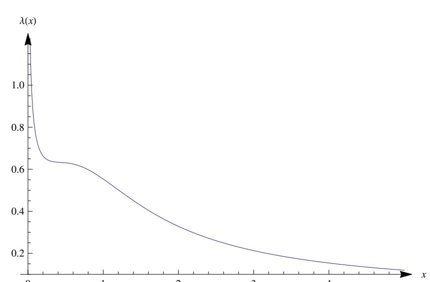

Figure 23. $\alpha = 1.15$

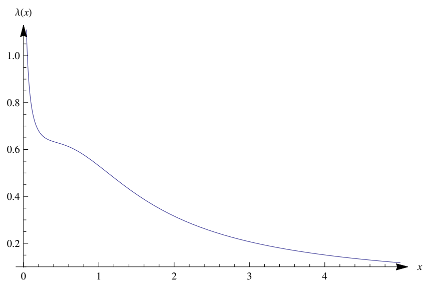

Figure 24. $\alpha = 1.2$

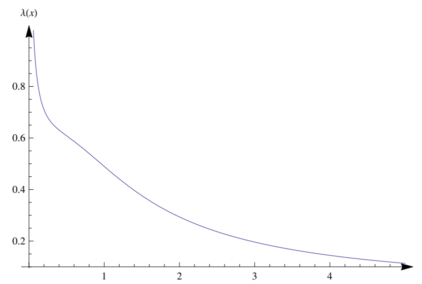

Figure 25. $\alpha = 1.3$

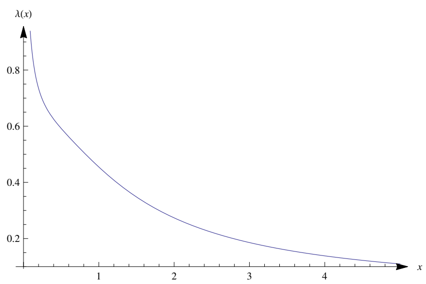

Figure 26. $\alpha = 1.4$

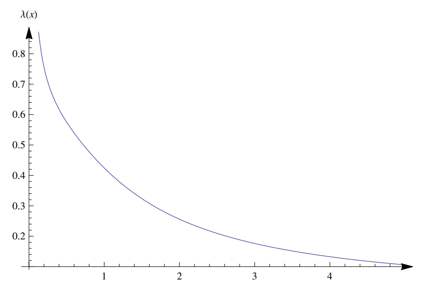

Figure 27. $\alpha = 1.5$

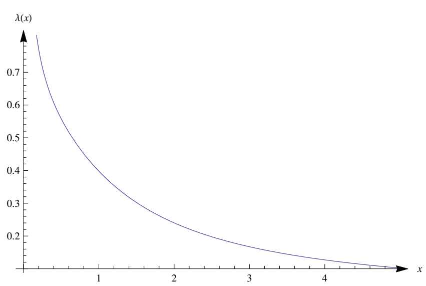

Figure 28. $\alpha = 1.6$

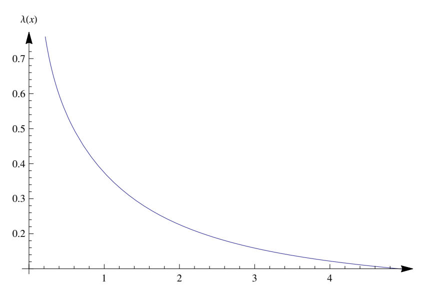

Figure 29. $\alpha = 1.7$

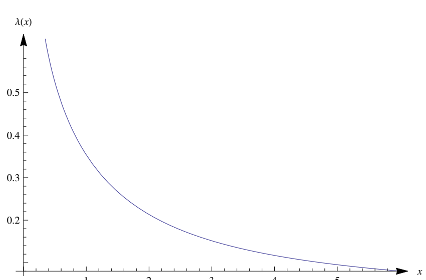

Figure 30. $\alpha = 1.8$

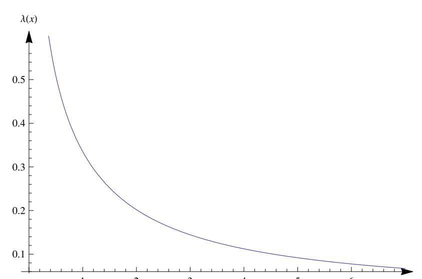

Figure 31. $\alpha = 1.9$

3. Point Estimations of Parameters for Two-parameter Distribution $GBS - Cauchy(\alpha, \beta)$

Suppose that $X_1, X_2, \cdots, X_n$ is a simple random sample from two-parameter Birnbaum-Saunders Cauchy fatigue life distribution $GBS - Cauchy(\alpha, \beta)$ with sample size

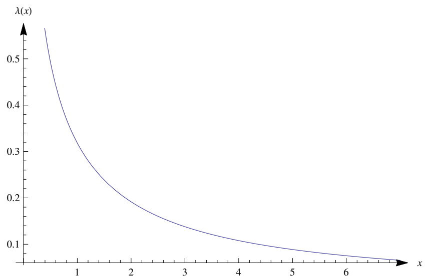

Figure 32. $\alpha = 2$

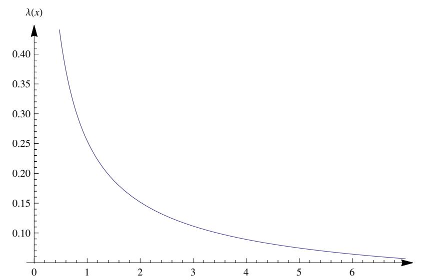

Figure 33. $\alpha = 2.5$

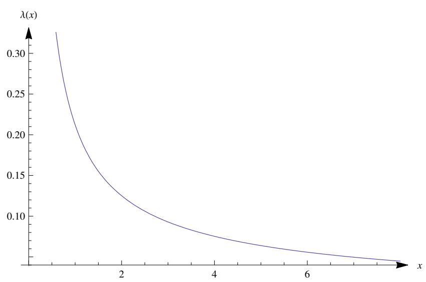

Figure 34. $\alpha = 3$

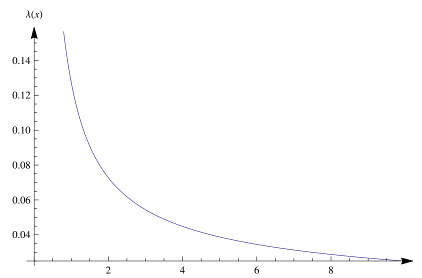

Figure 35. $\alpha = 5$

n , the sample observations are denoted by $x_1, x_2, \cdots, x_n$. The order staistics are denoted by $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$, and the order observations are denoted by $x_{(1)}, x_{(2)}, \cdots, x_{(n)}$.

3.1. Quantile estimations and maximum likelihood estimations of parameters

Since $F(\beta) = \Phi(0) = 0.5$, the point estimation $\hat{\beta}_1$ of scale parameter β can be the sample median, that is,

$$\hat{\beta}_1 = \begin{cases} \frac{1}{2}\left(X_{(n/2)} + X_{(n/2+1)}\right), & \text{when } n \text{ is an even number,} \\ X_{((n+1)/2)}, & \text{when } n \text{ is an odd number.} \end{cases}$$

Since $f(x) = \frac{1}{2\alpha x}\left(\sqrt{\frac{x}{\beta}} + \sqrt{\frac{\beta}{x}}\right)\varphi\left[\frac{1}{\alpha}\left(\sqrt{\frac{x}{\beta}} - \sqrt{\frac{\beta}{x}}\right)\right] = \frac{\alpha}{2\pi x}\frac{\sqrt{\frac{x}{\beta}}+\sqrt{\frac{\beta}{x}}}{\alpha^2+\left(\sqrt{\frac{x}{\beta}}-\sqrt{\frac{\beta}{x}}\right)^2}$, the likelihood function is

$$\begin{aligned} L(\alpha, \beta) =& \prod_{i=1}^{n} \frac{\alpha}{2\pi x_i} \frac{\sqrt{\frac{x_i}{\beta}} + \sqrt{\frac{\beta}{x_i}}}{\alpha^2 + \left(\sqrt{\frac{x_i}{\beta}} - \sqrt{\frac{\beta}{x_i}}\right)^2} \\ =& (2\pi)^{-n}\left(\prod_{i=1}^{n} x_i\right)^{-1} \alpha^n \prod_{i=1}^{n}\left(\sqrt{\frac{x_i}{\beta}}+\sqrt{\frac{\beta}{x_i}}\right)\prod_{i=1}^{n}\left[\alpha^2+\left(\sqrt{\frac{x_i}{\beta}}-\sqrt{\frac{\beta}{x_i}}\right)^2\right]^{-1}, \end{aligned}$$

$$\ln L(\alpha,\beta) = -n\ln(2\pi) - \sum_{i=1}^{n}\ln x_i + n\ln\alpha + \sum_{i=1}^{n}\ln\left(\sqrt{\frac{x_i}{\beta}} + \sqrt{\frac{\beta}{x_i}}\right)$$
$$- \sum_{i=1}^{n}\ln\left[\alpha^2 + \left(\sqrt{\frac{x_i}{\beta}} - \sqrt{\frac{\beta}{x_i}}\right)^2\right],$$

$$\frac{\partial \ln L(\alpha,\beta)}{\partial\alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n}\frac{2\alpha}{\alpha^2 + \left(\sqrt{\frac{x_i}{\beta}} - \sqrt{\frac{\beta}{x_i}}\right)^2}.$$

Let $\frac{\partial \ln L(\alpha,\beta)}{\partial\alpha} = 0$, and we get the function $\frac{n}{\alpha} - \sum_{i=1}^{n}\frac{2\alpha}{\alpha^2 + \left(\sqrt{\frac{x_i}{\beta}} - \sqrt{\frac{\beta}{x_i}}\right)^2} = 0$.

After simplifying, we have $\frac{n}{2} - \sum_{i=1}^{n}\frac{\alpha^2}{\alpha^2 + \left(\sqrt{\frac{x_i}{\beta}} - \sqrt{\frac{\beta}{x_i}}\right)^2} = 0$,

$$\frac{n}{2} - \sum_{i=1}^{n}\left[1 - \frac{\left(\sqrt{\frac{x_i}{\beta}} - \sqrt{\frac{\beta}{x_i}}\right)^2}{\alpha^2 + \left(\sqrt{\frac{x_i}{\beta}} - \sqrt{\frac{\beta}{x_i}}\right)^2}\right] = 0,\ \sum_{i=1}^{n}\frac{\left(\sqrt{\frac{x_i}{\beta}} - \sqrt{\frac{\beta}{x_i}}\right)^2}{\alpha^2 + \left(\sqrt{\frac{x_i}{\beta}} - \sqrt{\frac{\beta}{x_i}}\right)^2} = \frac{n}{2}.$$

If the point estimation of β is the sample median $\hat{\beta}_1$, then the point estimation $\hat{\alpha}_1$ of shape parameter α is the root of the following equation

$$\sum_{i=1}^{n}\frac{\left(\sqrt{\frac{x_i}{\hat{\beta}_1}} - \sqrt{\frac{\hat{\beta}_1}{x_i}}\right)^2}{\alpha^2 + \left(\sqrt{\frac{x_i}{\hat{\beta}_1}} - \sqrt{\frac{\hat{\beta}_1}{x_i}}\right)^2} = \frac{n}{2}.$$

3.2. Regression estimations of parameters

Let $Q(\beta) = \sum_{i=1}^{n}\left(\sqrt{\frac{X_i}{\beta}} - \sqrt{\frac{\beta}{X_i}}\right)^2$, and we have

$$\frac{dQ(\beta)}{d\beta} = -\frac{1}{\beta}\sum_{i=1}^{n}\left(\sqrt{\frac{X_i}{\beta}} + \sqrt{\frac{\beta}{X_i}}\right)\left(\sqrt{\frac{X_i}{\beta}} - \sqrt{\frac{\beta}{X_i}}\right).$$

Let $\frac{dQ(\beta)}{d\beta} = 0$, and we have the equation $\sum_{i=1}^{n}\left(\sqrt{\frac{X_i}{\beta}} + \sqrt{\frac{\beta}{X_i}}\right)\left(\sqrt{\frac{X_i}{\beta}} - \sqrt{\frac{\beta}{X_i}}\right) = 0$, that is,

$$\frac{1}{\beta}\sum_{i=1}^{n}X_i - \beta\sum_{i=1}^{n}\frac{1}{X_i} = 0.$$

Then we have $\hat{\beta}_2 = \sqrt{\frac{\sum_{i=1}^{n}X_i}{\sum_{i=1}^{n}X_i^{-1}}}$, and the point estimation $\hat{\alpha}_2$, of shape parameter α is the root of the following equation

$$\sum_{i=1}^{n}\frac{\left(\sqrt{\frac{X_i}{\hat{\beta}_2}} - \sqrt{\frac{\hat{\beta}_2}{X_i}}\right)^2}{\alpha^2 + \left(\sqrt{\frac{X_i}{\hat{\beta}_2}} - \sqrt{\frac{\hat{\beta}_2}{X_i}}\right)^2} = \frac{n}{2}.$$

3.3. Logarithmic moment estimations of parameters

Let $Y = \ln X, Y_i = \ln X_i, i = 1, 2, \cdots, n$, and $\mu = \ln \beta$, then we have

$$\begin{aligned} F_Y(y) =& P(Y \le y) = P(\ln X \le y) = P(X \le e^y) = \Phi\left[\frac{1}{\alpha}\left(\sqrt{\frac{e^y}{e^\mu}} - \sqrt{\frac{e^\mu}{e^y}}\right)\right] \\ =& \Phi\left\{\frac{1}{\alpha}\left[\exp\left(\frac{y-\mu}{2}\right) - \exp\left(-\frac{y-\mu}{2}\right)\right]\right\}. \end{aligned}$$

Let $Z = \frac{Y-\mu}{2}$, and we have $F_Z(z) = \Phi\left[\frac{1}{\alpha}(e^z - e^{-z})\right] = \int_{-\infty}^{\frac{1}{\alpha}(e^z-e^{-z})} \frac{1}{\pi(1+t^2)}\mathrm{d}t$,

$$f_Z(z) = \frac{1}{\alpha}(e^z + e^{-z})\varphi\left[\frac{1}{\alpha}(e^z - e^{-z})\right] = \frac{1}{\alpha}(e^z + e^{-z})\frac{1}{\pi\left[1 + \frac{1}{\alpha^2}(e^z - e^{-z})^2\right]}.$$

Since $f_Z(-z) = \frac{1}{\alpha}(e^{-z} + e^z)\frac{1}{\pi\left[1+\frac{1}{\alpha^2}(e^{-z}-e^z)^2\right]} = f_Z(z)$, $f_Z(z)$ is an even function. Therefore when k is an odd number, we know $E(Z^k) = 0$.

When k is an even number, we know

$$\begin{aligned} E(Z^k) =& \int_{-\infty}^{+\infty} z^k \frac{1}{\alpha}(e^z + e^{-z})\varphi\left[\frac{1}{\alpha}(e^z - e^{-z})\right]\mathrm{d}z \\ =& 2\int_0^{+\infty}\left(\ln\frac{\alpha t + \sqrt{\alpha^2 t^2 + 4}}{2}\right)^k \varphi(t)\mathrm{d}t \\ =& \frac{2}{\pi}\int_0^{+\infty}\left(\ln\frac{\alpha t + \sqrt{\alpha^2 t^2 + 4}}{2}\right)^k \frac{1}{1+t^2}\mathrm{d}t. \end{aligned}$$

Since $Y = \mu + 2Z, E(Y) = \mu$, the point estimation of parameter μ can be $\hat{\mu} = \bar{Y}$. Then the point estimation of scale parameter β can be $\hat{\beta}_3 = (\prod\limits_{i=1}^{n} X_i)^{1/n}$.

Since $D(Y) = 4D(Z) = 4E(Z^2)$, the point estimation $\hat{\alpha}_3$ of parameter α is the root of the following equation

$$\frac{8}{\pi}\int_0^{+\infty}\left(\ln\frac{\alpha t + \sqrt{\alpha^2 t^2 + 4}}{2}\right)^2 \frac{1}{1+t^2}\mathrm{d}t = \overline{Y^2} - \bar{Y}^2,$$

that is,

$$\int_0^{+\infty}\left(\ln\frac{\alpha t + \sqrt{\alpha^2 t^2 + 4}}{2}\right)^2 \frac{1}{1+t^2}\mathrm{d}t = \frac{\pi}{8}(\overline{Y^2} - \bar{Y}^2).$$

The above equation is an integral equation, and it is complex to solve it. Then we prove that it has unique positive root.

Lemma 3.1 ([9]). *Suppose that $g(x)$ is a non-negative function in the interval $[a, +\infty)$, it is integrable in $[a, b]$ for any $b > a$. If $\lim\limits_{x\to+\infty}\frac{\ln g(x)}{\ln x} = p$, then $\int_a^{+\infty} g(x)dx$ is convergent for $-\infty \le p < -1$, while $\int_a^{+\infty} g(x)dx$ is divergent for $-1 < p \le +\infty$.*

Lemma 3.2. *The equation $\int_0^{+\infty}\left(\ln\frac{\alpha t+\sqrt{\alpha^2t^2+4}}{2}\right)^2\frac{1}{1+t^2}\mathrm{d}t = \frac{\pi}{8}(\overline{Y^2} - \bar{Y}^2)$ has unique positive root with respect to α.*

Proof. Let the function be $g(\alpha) = \int_0^{+\infty}\left(\ln\frac{\alpha t+\sqrt{\alpha^2t^2+4}}{2}\right)^2\frac{1}{1+t^2}\mathrm{d}t, \alpha > 0$. Firstly we prove that the function $g(\alpha)$ is convergent. Since

$$\lim_{t\to+\infty}\frac{1}{\ln t}\ln\left[\left(\ln\frac{\alpha t+\sqrt{\alpha^2t^2+4}}{2}\right)^2\right] = 2\lim_{t\to+\infty}\frac{1}{\ln t}\ln\left(\ln\frac{\alpha t+\sqrt{\alpha^2t^2+4}}{2}\right)$$
$$=2\lim_{t\to+\infty}\left(\ln\frac{\alpha t+\sqrt{\alpha^2t^2+4}}{2}\right)^{-1}\frac{2}{\alpha t+\sqrt{\alpha^2t^2+4}}\frac{1}{2}\left(\alpha+\frac{\alpha^2 t}{\sqrt{\alpha^2t^2+4}}\right)t$$
$$=2\lim_{t\to+\infty}\left(\ln\frac{\alpha t+\sqrt{\alpha^2t^2+4}}{2}\right)^{-1}\frac{\alpha t\sqrt{\alpha^2t^2+4}+\alpha^2t^2}{\alpha t\sqrt{\alpha^2t^2+4}+\alpha^2t^2+4} = 0,$$
$$\lim_{t\to+\infty}\frac{\ln(1+t^2)}{\ln t} = \lim_{t\to+\infty}\frac{2t^2}{1+t^2} = 2,$$

we have $\lim_{t\to+\infty}\frac{1}{\ln t}\left\{\ln\left[\left(\ln\frac{\alpha t+\sqrt{\alpha^2t^2+4}}{2}\right)^2\right]-\ln(1+t^2)\right\} = -2$, that is,

$$\lim_{t\to+\infty}\frac{1}{\ln t}\left\{\ln\left[\frac{1}{1+t^2}\left(\ln\frac{\alpha t+\sqrt{\alpha^2t^2+4}}{2}\right)^2\right]\right\} = -2.$$

Then according to Lemma 3.1, we know that the function $g(\alpha)$ is convergent.

Next we prove that the equation $g(\alpha) = 0$ has unique positive root. Since

$$\lim_{\alpha\to 0} g(\alpha) = 0, \lim_{\alpha\to+\infty} g(\alpha) = +\infty,$$
$$g'(\alpha) =2\int_0^{+\infty}\left(\ln\frac{\alpha t+\sqrt{\alpha^2t^2+4}}{2}\right)\frac{2}{\alpha t+\sqrt{\alpha^2t^2+4}}t\frac{\alpha t+\sqrt{\alpha^2t^2+4}}{2\sqrt{\alpha^2t^2+4}}\frac{1}{1+t^2}\mathrm{d}t$$
$$=2\int_0^{+\infty}\left(\ln\frac{\alpha t+\sqrt{\alpha^2t^2+4}}{2}\right)\frac{t}{\sqrt{\alpha^2t^2+4}}\frac{1}{1+t^2}\mathrm{d}t > 0,$$

and $\overline{Y^2}-\bar{Y}^2>0$, we know that the equation $g(\alpha)=0$ has unique positive root. □

Lemma 3.3 ([11]). *Suppose that $X_1, X_2, \cdots, X_n$ is a simple random sample from the population X with sample size n, it is denoted by $E(X) = \mu; D(X) = \sigma^2 < +\infty$, and the forth central moment $\nu_4 = E(X-EX)^4$ of population X is limited. If the forth derivative of the function $h(x)$ is existent and limited, then we have*

$$E[h(\bar{X})] = h(\mu) + \frac{1}{2n}h''(\mu)\sigma^2 + O(n^{-2}),$$
$$D[h(\bar{X})] = \frac{1}{n}[h'(\mu)]^2\sigma^2 + \frac{1}{n^2}\left\{h'(\mu)h''(\mu)\nu_3 + \frac{1}{2}[h''(\mu)]^2\sigma^4 + h'(\mu)h'''(\mu)\sigma^4\right\} + O(n^{-3}).$$

Theorem 3.1. *$\hat{\beta}_3$ is asymptotic unbiased estimation and consistent estimation of β.*

Proof. Since $Y = 2Z+\mu, E(Y) = \mu, Y-E(Y) = 2Z$, we know that the first to fourth central moment of Y are $\nu_1 = E[Y-E(Y)] = 0, \nu_3 = E[Y-E(Y)]^3 =$

$8E(Z^3) = 0,$

$$\nu_2 = E[Y - E(Y)]^2 = 4E(Z^2) = \frac{8}{\pi}\int_0^{+\infty}\left(\ln\frac{\alpha t + \sqrt{\alpha^2 t^2 + 4}}{2}\right)^2 \frac{1}{1+t^2}\mathrm{d}t,$$

$$\nu_4 = E[Y - E(Y)]^4 = 16E(Z^4) = \frac{32}{\pi}\int_0^{+\infty}\left(\ln\frac{\alpha t + \sqrt{\alpha^2 t^2 + 4}}{2}\right)^4 \frac{1}{1+t^2}\mathrm{d}t.$$

Let the function be $h(x) = e^x$, and any order derivation of $h(x)$ is still e^x . Then we have

$$E(\hat{\beta}_3) = E(e^{\bar{Y}}) = e^{\mu} + \frac{1}{2n}e^{\mu}\cdot 4E(Z^2) + O(n^{-2}) = \beta + \frac{2}{n}\beta E(Z^2) + O(n^{-2}),$$

$$\begin{aligned} D(\hat{\beta}_3) =& D(e^{\bar{Y}}) \\ =& \frac{1}{n}e^{2\mu}\cdot 4E(Z^2) + \frac{1}{n^2}\left\{\frac{1}{2}e^{2\mu}\cdot 16\big[E(Z^2)\big]^2 + e^{2\mu}\cdot 16\big[E(Z^2)\big]^2\right\} + O(n^{-3}) \\ =& \frac{4}{n}\beta^2 E(Z^2) + \frac{24}{n^2}\beta^2\big[E(Z^2)\big]^2 + O(n^{-3}). \end{aligned}$$

It is obvious that $\lim\limits_{n\to+\infty} E(\hat{\beta}_3) = \beta, \lim\limits_{n\to+\infty} D(\hat{\beta}_3) = 0.$

Therefore $\hat{\beta}_3$ is asymptotic unbiased estimation and consistent estimation of β . $\square$

Table 2. Simulation comparisons of point estimations

Sample size	Parameters	Quantile estimation and MLE $\hat{\alpha}_1, \hat{\beta}_1$		Regression estimation $\hat{\alpha}_2, \hat{\beta}_2$		logarithmic moment estimation $\hat{\beta}_3$	
		Mean value	Mean square error	Mean value	Mean square error	Mean value	Mean square error
10	$\hat{\beta}$	1.2018	0.7515	4.6318	597.548	1.3321	1.3688
	$\hat{\alpha}$	0.9626	0.2692	1.9327	5.3221	–	–
15	$\hat{\beta}$	1.1145	0.4255	4.6632	758.174	1.2107	0.8028
	$\hat{\alpha}$	0.9752	0.1668	2.1033	16.9944	–	–
20	$\hat{\beta}$	1.0556	0.1778	3.2599	146.272	1.1261	0.4219
	$\hat{\alpha}$	0.9889	0.1110	1.94312	8.5407	–	–
25	$\hat{\beta}$	1.0526	0.1301	3.3713	165.071	1.0990	0.2721
	$\hat{\alpha}$	0.9930	0.0877	1.9720	7.2284	–	–
30	$\hat{\beta}$	1.0431	0.1060	3.4698	476.644	1.0740	0.2405
	$\hat{\alpha}$	0.9872	0.0745	1.9224	6.7699	–	–

In order to compare the precisions of various point estimations of parameters α, β , we choose the sample size $n = 10(5)30$ and the truth values of parameters $\alpha = 1, \beta = 1$. Then we obtain the samples from $GBS - Cauchy(\alpha, \beta)$ by 1000 Monte-Carlo simulations, and calculate mean values and mean square errors of various point estimations for parameters α, β. The results are shown in Table 2. It can be concluded that quantile estimation and maximum likelihood estimation $\hat{\alpha}_1, \hat{\beta}_1$ are best.

Remark 3.1. Since logarithmic moment estimation $\hat{\alpha}_3$ of parameter α refers to solving complex integral equation, it is not compared in the simulations here.

4. Approximate Interval Estimations of Parameters for Two-parameter Distribution $GBS\!-\!Cauchy(\alpha,\beta)$

Let the parameter be $\mu=\ln\beta$, and it is denoted by $Y=\ln X, Y_i=\ln X_i, i=1,2,\cdots,n$. Then $Y_1,Y_2,\cdots,Y_n$ is a simple random sample from the distribution function $F_Y(y)=\Phi\{\frac{1}{\alpha}\left[\exp\left(\frac{y-\mu}{2}\right)-\exp\left(-\frac{y-\mu}{2}\right)\right]\}$ with sample size n, and its order statistics are denoted by $Y_{(1)},Y_{(2)},\cdots,Y_{(n)}$.

The first order Taylor expansion of $\frac{1}{\alpha}\left[\exp\left(\frac{y-\mu}{2}\right)-\exp\left(-\frac{y-\mu}{2}\right)\right]$ at the point $y=\mu$ is

$$\frac{1}{\alpha}\left[\exp\left(\frac{y-\mu}{2}\right)-\exp\left(-\frac{y-\mu}{2}\right)\right]\approx\frac{y-\mu}{\alpha}.$$

Then $F_Y(y)=\Phi\{\frac{1}{\alpha}\left[\exp(\frac{y-\mu}{2})-\exp(-\frac{y-\mu}{2})\right]\}$ is approximately

$$F_Y(y)\approx\Phi\left(\frac{y-\mu}{\alpha}\right)=\frac{1}{2}+\frac{1}{\pi}\arctan\left(\frac{y-\mu}{\alpha}\right).$$

That is, $Y=\ln X$ can be approximately regarded as two-parameter Cauchy distribution with location-scale parameters.

Let $Z=\frac{Y-\mu}{\alpha}, Z_i=\frac{Y_i-\mu}{\alpha}, i=1,2,\cdots,n$, and then Z approximately follows standard Cauchy distribution $C(0,1)$. $Z_1,Z_2,\cdots,Z_n$ follow the same distribution as a simple random sample from standard Cauchy distribution $C(0,1)$ with sample size n , and it is sorted from small to large, which is denoted by $Z_{(1)},Z_{(2)},\cdots,Z_{(n)}$.

4.1. Approximate interval estimations of parameter β

The approximate interval estimation of parameter μ is obtained firstly, and then it is easy to obtain the approximate interval estimation of parameter β.

(1) When n is an even number, it is denoted by $j=\frac{n}{2}, n-j=\frac{n}{2}, A=\frac{1}{j}\sum\limits_{i=j+1}^{n}Y_{(i)}$.

Then we have $\sum\limits_{i=j+1}^{n}Y_{(i)}-\sum\limits_{i=1}^{j}Y_{(i)}>0$.

Let the function be $\mathcal{F}(\mu)=\frac{\sum_{i=j+1}^{n}Y_{(i)}-\sum_{i=1}^{j}Y_{(i)}}{\sum_{i=j+1}^{n}(Y_{(i)}-\mu)}, -\infty<\mu<+\infty$, and we know

$$\begin{aligned}\mathcal{F}(\mu)=&\frac{\sum\limits_{i=j+1}^{n}(Y_{(i)}-\mu)-\sum\limits_{i=1}^{j}(Y_{(i)}-\mu)}{\sum\limits_{i=j+1}^{n}(Y_{(i)}-\mu)}=\frac{\sum\limits_{i=j+1}^{n}\frac{Y_{(i)}-\mu}{\alpha}-\sum\limits_{i=1}^{j}\frac{Y_{(i)}-\mu}{\alpha}}{\sum\limits_{i=j+1}^{n}\frac{Y_{(i)}-\mu}{\alpha}}\\=&\frac{\sum\limits_{i=j+1}^{n}Z_{(i)}-\sum\limits_{i=1}^{j}Z_{(i)}}{\sum\limits_{i=j+1}^{n}Z_{(i)}}.\end{aligned}$$

Then $\mathcal{F}(\mu)$ is a pivot that only contains parameter μ. Besides, $\mathcal{F}(\mu)$ is a strictly monotonic increasing function of μ, and we know

$$\lim_{\mu\to-\infty}\mathcal{F}(\mu)=0^+,\ \lim_{\mu\to A^-}\mathcal{F}(\mu)=+\infty,\ \lim_{\mu\to A^+}\mathcal{F}(\mu)=-\infty,\ \lim_{\mu\to+\infty}\mathcal{F}(\mu)=0^-.$$

Hence, for a given significance level α' £¬the upper $1-\alpha'/2, \alpha'/2$ quantiles of the pivot $\mathcal{F}(\mu)$ are denoted by $\mathcal{F}_{1-\alpha'/2}$ and $\mathcal{F}_{\alpha'/2}$. Then it is obvious that the approximate interval estimation of parameter μ at the confidence level $1-\alpha'$ is

$$\left[\frac{(\mathcal{F}_{1-\alpha'/2}-1)\sum\limits_{i=j+1}^{n} Y_{(i)}+\sum\limits_{i=1}^{j} Y_{(i)}}{j\mathcal{F}_{1-\alpha'/2}}, \frac{\left(\mathcal{F}_{\alpha'/2}-1\right)\sum\limits_{i=j+1}^{n} Y_{(i)}+\sum\limits_{i=1}^{j} Y_{(i)}}{j\mathcal{F}_{\alpha'/2}}\right].$$

Furthermore, the approximate interval estimation of parameter β at the confidence level $1-\alpha'$ is $[\beta_L, \beta_U]$, here

$$\begin{aligned}
\beta_L =& \exp\left\{\frac{1}{j\mathcal{F}_{1-\alpha'/2}}\left[(\mathcal{F}_{1-\alpha'/2}-1)\sum_{i=j+1}^{n} Y_{(i)}+\sum_{i=1}^{j} Y_{(i)}\right]\right\},\\
\beta_U =& \exp\left\{\frac{1}{j\mathcal{F}_{\alpha'/2}}\left[\left(\mathcal{F}_{\alpha'/2}-1\right)\sum_{i=j+1}^{n} Y_{(i)}+\sum_{i=1}^{j} Y_{(i)}\right]\right\}.
\end{aligned}$$

(2) When n is an odd number, it is denoted by $j=\frac{n+1}{2}, n-j+1=\frac{n+1}{2}, A=\frac{1}{j}\sum\limits_{i=j}^{n} Y_{(i)}$. Then we have $\sum\limits_{i=j}^{n} Y_{(i)}-\sum\limits_{i=1}^{j} Y_{(i)}>0$.

Let the function be $\mathcal{F}(\mu)=\frac{\sum_{i=j}^{n} Y_{(i)}-\sum_{i=1}^{j} Y_{(i)}}{\sum_{i=j}^{n}(Y_{(i)}-\mu)}, -\infty<\mu<+\infty$, and we know

$$\begin{aligned}
\mathcal{F}(\mu) =& \frac{\sum\limits_{i=j}^{n}(Y_{(i)}-\mu)-\sum\limits_{i=1}^{j}(Y_{(i)}-\mu)}{\sum\limits_{i=j}^{n}(Y_{(i)}-\mu)} = \frac{\sum\limits_{i=j}^{n}\frac{Y_{(i)}-\mu}{\alpha}-\sum\limits_{i=1}^{j}\frac{Y_{(i)}-\mu}{\alpha}}{\sum\limits_{i=j}^{n}\frac{Y_{(i)}-\mu}{\alpha}}\\
=& \frac{\sum\limits_{i=j}^{n} Z_{(i)}-\sum\limits_{i=1}^{j} Z_{(i)}}{\sum\limits_{i=j}^{n} Z_{(i)}}.
\end{aligned}$$

Then $\mathcal{F}(\mu)$ is a pivot that only contains parameter μ. Besides, $\mathcal{F}(\mu)$ is a strictly monotonic increasing function of μ, and we know

$$\lim_{\mu\to-\infty}\mathcal{F}(\mu)=0^+,\ \lim_{\mu\to A^-}\mathcal{F}(\mu)=+\infty,\ \lim_{\mu\to A^+}\mathcal{F}(\mu)=-\infty,\ \lim_{\mu\to+\infty}\mathcal{F}(\mu)=0^-.$$

Hence, for a given significance level α', the upper $1-\alpha'/2, \alpha'/2$ quantiles of the pivot $\mathcal{F}(\mu)$ are denoted by $\mathcal{F}_{1-\alpha'/2}$ and $\mathcal{F}_{\alpha'/2}$. Then it is obvious that the approximate interval estimation of parameter μ at the confidence level $1-\alpha'$ is

$$\left[\frac{(\mathcal{F}_{1-\alpha'/2}-1)\sum\limits_{i=j}^{n} Y_{(i)}+\sum\limits_{i=1}^{j} Y_{(i)}}{j\mathcal{F}_{1-\alpha'/2}}, \frac{(\mathcal{F}_{\alpha'/2}-1)\sum\limits_{i=j}^{n} Y_{(i)}+\sum\limits_{i=1}^{j} Y_{(i)}}{j\mathcal{F}_{\alpha'/2}}\right].$$

Table 3. The upper quantiles of $\mathcal{F}(\mu)$

n	0.99	0.975	0.95	0.90	0.85	0.15	0.1	0.05	0.025	0.01
3	-97.9306	-32.9726	-13.2367	-4.7529	-2.3301	3.7697	6.1429	13.1427	31.4721	98.2586
4	-97.2331	-33.8379	-13.2479	-2.9288	0.7586	6.2786	10.4199	24.4451	55.1807	157.3630
5	-94.5367	-33.0919	-12.4417	-3.5011	0.7277	5.6883	9.0893	21.6882	52.0811	159.9950
6	-93.7252	-23.1696	-5.9044	0.9615	1.0212	6.7547	10.6474	23.4994	52.2856	141.9530
7	-75.9926	-22.3511	-6.5464	0.9305	1.0104	6.6040	10.8195	24.3312	49.7260	120.8820
8	-55.7718	-10.6179	0.9577	1.0306	1.0826	6.7387	10.2194	21.3462	47.6734	126.6860
9	-62.6324	-12.8775	0.9319	1.0255	1.0831	6.6364	9.6440	20.3638	45.8003	134.1600
10	-26.2255	0.9311	1.0096	1.0682	1.1294	6.0894	9.2479	18.4087	37.0378	99.3003
11	-23.0760	0.9405	1.0106	1.0672	1.1310	6.1586	9.3044	20.1718	39.8467	106.9740
12	-4.1032	1.0050	1.0359	1.0970	1.1603	6.0515	8.7048	16.8060	34.1734	86.4785
13	-8.9870	1.0030	1.0323	1.0998	1.1725	5.9278	8.6204	16.0885	32.0254	79.4370
14	0.9881	1.0177	1.0518	1.1245	1.1953	5.5279	7.9297	14.4097	27.6007	72.2351
15	0.9863	1.0183	1.0545	1.1238	1.1973	5.5983	7.9744	15.0111	26.1032	56.6878
16	1.0034	1.0285	1.0652	1.1373	1.2071	5.2596	7.4297	13.4594	26.4642	67.2128
17	1.0065	1.0287	1.0657	1.1442	1.2254	5.5620	7.8142	13.9454	27.3275	69.2133
18	1.0116	1.0358	1.0772	1.1605	1.2381	5.2894	7.3125	13.1865	24.7251	59.4397
19	1.0110	1.0372	1.0763	1.1540	1.2322	5.0528	7.0531	13.1083	25.2890	59.5523
20	1.0142	1.0453	1.0916	1.1696	1.2453	5.1495	7.0568	13.0393	25.0189	55.4469
21	1.0146	1.0476	1.0896	1.1644	1.2455	5.0581	6.9920	11.4888	20.1672	47.2190
22	1.0209	1.0460	1.0885	1.1685	1.2511	4.8725	6.4805	11.3797	20.7126	42.0599
23	1.0236	1.0520	1.0945	1.1772	1.2560	4.8599	6.7186	11.5188	19.7674	48.8084
24	1.0225	1.0564	1.1020	1.1822	1.2635	4.5993	6.0753	10.2386	18.4129	45.5858
25	1.0171	1.0484	1.0996	1.1860	1.2678	4.6848	6.2981	10.5393	19.8586	42.9610
26	1.0228	1.0620	1.1061	1.1957	1.2829	4.7381	6.3043	10.3140	17.8626	39.5137
27	1.0216	1.0573	1.1041	1.1849	1.2650	4.7270	6.4190	10.8639	18.6931	39.9059
28	1.0216	1.0609	1.1139	1.2058	1.2806	4.5000	5.9244	9.5772	16.8008	37.7600
29	1.0302	1.0628	1.1177	1.2053	1.2878	4.5401	6.0084	10.3410	18.8671	39.5221
30	1.0224	1.0622	1.1128	1.1988	1.2764	4.4943	5.9519	9.7092	15.7022	36.5503

Furthermore, the approximate interval estimation of parameter β at the confidence level $1-\alpha'$ is $[\beta_L, \beta_U]$, here

$$\begin{aligned}\beta_L =& \exp\left\{\frac{1}{j\mathcal{F}_{1-\alpha'/2}}\left[(\mathcal{F}_{1-\alpha'/2}-1)\sum_{i=j}^{n} Y_{(i)} + \sum_{i=1}^{j} Y_{(i)}\right]\right\},\\ \beta_U =& \exp\left\{\frac{1}{j\mathcal{F}_{\alpha'/2}}\left[\left(\mathcal{F}_{\alpha'/2}-1\right)\sum_{i=j}^{n} Y_{(i)} + \sum_{i=1}^{j} Y_{(i)}\right]\right\}.\end{aligned}$$

Let the sample size be $n = 3(1)30$, and through 10000 Monte-Carlo simulations, the upper $0.99, 0.975, 0.95, 0.90, 0.85, 0.15, 0.10, 0.05, 0.025, 0.01$ quantiles of $\mathcal{F}(\mu)$ are shown in Table 3.

4.2. Approximate interval estimation of parameter α

The following pivot that only contains parameter α is constructed

$$\mathcal{T}(\alpha) = \frac{1}{\alpha}\sum_{i=2}^{n}(n-i+1)(Y_{(i)} - Y_{(i-1)}), \alpha > 0,$$

and we know

$$\mathcal{T}(\alpha) = \sum_{i=2}^{n}(n-i+1)\left(\frac{Y_{(i)}-\mu}{\alpha} - \frac{Y_{(i-1)}-\mu}{\alpha}\right) = \sum_{i=2}^{n}(n-i+1)(Z_{(i)} - Z_{(i-1)}).$$

Then $\mathcal{T}(\alpha)$ is a pivot that only contains parameter α, and $\mathcal{T}(\alpha)$ is a strictly monotonic decreasing function of α.

For a given significance level α' the upper $1-\alpha'/2, \alpha'/2$ quantiles of the pivot $\mathcal{T}(\alpha)$ are denoted by $\mathcal{T}_{1-\alpha'/2}$ and $\mathcal{T}_{\alpha'/2}$. Then the interval estimation of parameter α at the confidence level $1-\alpha'$ is

$$\left[\frac{\sum\limits_{i=2}^{n}(n-i+1)(Y_{(i)}-Y_{(i-1)})}{\mathcal{T}_{\alpha'/2}}, \frac{\sum\limits_{i=2}^{n}(n-i+1)(Y_{(i)}-Y_{(i-1)})}{\mathcal{T}_{1-\alpha'/2}}\right].$$

Let the sample size be $n = 3(1)30$, and through 10000 Monte-Carlo simulations, the upper $0.99, 0.975, 0.95, 0.90, 0.85, 0.15, 0.10, 0.05, 0.025, 0.01$ quantiles of $\mathcal{T}(\alpha)$ are shown in Table 4.

Table 4. The upper quantiles of $\mathcal{T}(\alpha)$

n	0.99	0.975	0.95	0.90	0.85	0.15	0.10	0.05	0.025	0.01
3	0.4293	0.6852	0.9714	1.4481	1.8400	19.3900	28.9623	54.3122	109.881	273.504
4	1.1548	1.6625	2.1726	3.0410	3.7541	35.3298	51.7930	99.5265	192.218	452.630
5	2.3213	3.0899	3.9448	5.2302	6.2337	57.2133	84.2922	162.638	316.717	816.952
6	3.4179	4.6534	5.7501	7.4268	8.9644	76.2889	116.359	220.559	464.608	1074.67
7	5.1810	6.6567	8.1231	10.2454	12.3108	105.793	157.466	301.045	639.400	1575.50
8	6.8888	8.6553	10.5361	13.4144	15.8507	140.936	209.971	439.396	857.461	2246.24
9	8.6707	10.8650	13.1704	16.4520	19.5813	180.670	272.593	548.893	1083.18	2676.91
10	10.6758	13.3248	16.1845	20.3753	23.9314	209.054	319.680	618.745	1268.79	3238.48
11	13.1564	16.3692	19.4900	24.2333	28.6042	260.870	408.413	793.911	1566.32	3986.88
12	15.4392	19.1869	23.2190	28.9234	33.4408	305.902	457.615	920.696	1914.25	4890.63
13	18.4075	22.2706	26.5914	33.7140	40.0629	354.591	530.912	1013.49	2165.41	5369.13
14	21.4358	25.6088	30.7273	38.5922	45.2654	419.647	619.056	1211.04	2494.86	6041.82
15	23.8531	29.2273	34.6951	43.5928	50.9930	467.259	697.422	1470.47	3039.50	7805.31
16	27.1974	32.5867	39.3899	49.3388	57.8647	524.718	800.923	1556.80	2989.33	7680.14
17	30.3740	37.1464	43.9699	55.9326	65.8398	617.915	915.742	1935.32	3933.79	9805.47
18	32.0711	39.2773	47.2288	58.5766	69.8789	676.375	1030.02	2155.36	4354.29	10747.8
19	37.7366	45.1091	54.1401	67.6419	80.1402	753.754	1147.27	2288.89	4667.68	12118.7
20	41.5028	49.3480	58.4619	72.6495	86.0159	817.671	1212.21	2463.56	4929.69	12823.3
21	45.8929	54.7748	65.0308	81.3781	97.2452	864.327	1348.02	2671.61	4923.80	11687.2
22	48.5223	58.2815	69.5566	86.8719	102.070	985.779	1494.46	2900.61	6006.69	14348.4
23	52.7743	63.3927	76.1213	95.6073	112.002	1037.67	1561.80	3221.60	6626.94	19180.4
24	57.6928	69.3932	82.6387	105.861	123.905	1147.19	1682.43	3433.35	7196.83	18946.3
25	64.4626	75.4329	89.7008	111.809	132.822	1285.99	1946.52	4149.46	8050.44	18802.6
26	68.5224	81.3941	97.1826	120.940	142.551	1377.46	2110.65	4413.53	8404.97	21430.1
27	72.4984	85.3810	102.862	126.489	149.790	1454.42	2231.65	4634.69	9383.39	20839.8
28	79.5776	94.6180	111.637	140.225	165.500	1553.42	2463.74	4933.20	9767.80	24246.5
29	80.8976	98.0319	116.716	147.492	173.589	1752.99	2699.43	5214.68	10497.9	24064.4
30	85.9036	104.518	125.719	153.776	183.468	1779.02	2820.16	5704.12	11149.6	30787.5

In order to investigate the precisions of approximate α, β interval estimations of parameters , we choose the sample size $n = 10(1)15$ and the truth values of parameters $\alpha = 1, \beta = 1$. Then we obtain the samples from $GBS-Cauchy(\alpha, \beta)$ by 1000 Monte-Carlo simulations, and calculate mean lower limit, mean upper limit, mean interval length and the number of intervals that contain the truth values of approximate interval estimations for parameters α, β at the confidence level 0.95. The results are shown in Table 5.

Table 5. Simulation results of approximate interval estimations

Sample capacity	approximate interval estimation of β				approximate interval estimation of α			
	mean lower limit	mean upper limit	mean interval length	number contains truth value	mean lower limit	mean upper limit	mean interval length	number contains truth value
10	0.2716	6.8005	6.5289	955	0.0289	2.7562	2.7273	953
11	0.3014	5.5689	5.2675	964	0.0271	2.5952	2.5680	954
12	0.3042	6.2001	5.8958	973	0.0251	2.5004	2.4753	954
13	0.3150	5.1164	4.8014	971	0.0248	2.4199	2.3951	945
14	0.3059	5.7695	5.4636	974	0.0239	2.3348	2.3109	946
15	0.3190	4.8452	4.5262	978	0.0217	2.2528	2.2323	941

5. Simulation Examples

Example 5.1. A simple random sample is generated from the distribution $GBS-Cauchy(0.5, 2)$ with sample size 20 by Monte-Carlo simulations: $0.242714, 1.86936, 1.52568, 2.68807, 2.67345, 1.73983, 5.99763, 1.76747, 0.234591, 8.21912, 2.30862, 2.1332, 3.95174, 3.53211, 4.08302, 1.37021, 1.80273, 6.28822, 1.12606, 0.803841$. By using the proposed methods in this paper, the point estimations of parameters α, β are respectively

$$\hat{\alpha}_1 = 0.4262, \hat{\beta}_1 = 2.0013, \hat{\alpha}_2 = 0.4367, \hat{\beta}_2 = 1.7878, \hat{\alpha}_3 = 0.1338, \hat{\beta}_3 = 1.9597.$$

At the confidence level $1 - \alpha' = 0.95$, approximate interval estimation of parameter α is $[0.0086, 0.8603]$, and approximate interval estimation of parameter β is $[1.0708, 3.6027]$.

Example 5.2. A simple random sample is generated from the distribution $GBS-Cauchy(1, 1)$ with sample size 10 by Monte-Carlo simulations: $7.04891, 0.0138007, 1.14273, 0.455985, 0.281722, 0.25294, 0.804234, 0.924005, 3.2357, 11.0822$. By using the proposed methods in this paper, the point estimations of parameters α, β are respectively

$$\hat{\alpha}_1 = 1.0096, \hat{\beta}_1 = 0.8641, \hat{\alpha}_2 = 1.0353, \hat{\beta}_2 = 0.5421, \hat{\alpha}_3 = 0.5696, \hat{\beta}_3 = 0.7913.$$

At the confidence level $1 - \alpha' = 0.95$, approximate interval estimation of parameter α is $[0.0319, 3.0387]$, and approximate interval estimation of parameter β is $[0.1677, 2.8417]$.

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