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ON THE EXISTENCE OF FULL DIMENSIONAL KAM TORUS FOR FRACTIONAL NONLINEAR SCHRÖDINGER EQUATION

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Abstract In this paper, we study fractional nonlinear Schrödinger equation (FNLS) with periodic boundary condition

$$\mathbf{i}u_t = -(-\Delta)^{s_0}u - V*u - \epsilon f(x)|u|^4 u, \quad x \in \mathbb{T}, \quad t \in \mathbb{R}, \quad s_0 \in (\frac{1}{2}, 1), \quad (0.1)$$

where $(-\Delta)^{s_0}$ is the Riesz fractional differentiation defined in [21] and $V*$ is the Fourier multiplier defined by $\widehat{V*u}(n) = V_n\widehat{u}(n)$, $V_n \in [-1, 1]$, and $f(x)$ is Gevrey smooth. We prove that for $0 \le |\epsilon| \ll 1$ and appropriate V, the equation (0.1) admits a full dimensional KAM torus in the Gevrey space satisfying $\frac{1}{2}e^{-rn^\theta} \le |q_n| \le 2e^{-rn^\theta}, \theta \in (0, 1)$, which generalizes the results given by [8–10] to fractional nonlinear Schrödinger equation.

Keywords KAM theory, almost periodic solution, Gevrey space, fractional nonlinear Schrödinger equation.

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1. Introduction and main results

In this paper, we focus on the fractional nonlinear Schrödinger equation (FNLS) with periodic boundary conditions

$$\mathbf{i}u_t = -(-\Delta)^{s_0}u - V*u - \epsilon f(x)|u|^4 u, \ x \in \mathbb{T}, \quad s_0 \in (\frac{1}{2}, 1), \tag{1.1}$$

where $V*$ is a Fourier multiplier defined by

$$V*u = \sum_{n\in\mathbb{Z}} V_n\widehat{u}_n e^{\mathbf{i}nx}, \ V_n \in [-1, 1],$$

and $f(x)$ is Gevrey smooth. Written in Fourier modes $(q_n)_{n\in\mathbb{Z}}$, then (1.1) can be rewritten as

$$\dot{q}_n = \mathbf{i}\frac{\partial H}{\partial \bar{q}_n} \tag{1.2}$$

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with the Hamiltonian

$$H(q,\overline{q})=\sum_{n\in\mathbb{Z}}(C(s_0)n^{2s_0}+V_n)|q_n|^2+\epsilon\sum_{n\in\mathbb{Z}}\sum_{\substack{n_1-n_2+n_3-n_4\\+n_5-n_6=-n}}\widehat{f}(n)q_{n_1}\overline{q}_{n_2}q_{n_3}\overline{q}_{n_4}q_{n_5}\overline{q}_{n_6}, \tag{1.3}$$

where $C(s_0)=-\frac{1}{\cos s_0\pi}$ and $n^{2s_0}=(n^2)^{s_0}$. The details about eigenvalues and eigenfunctions of operator $(-\Delta)^{s_0}$ have been carefully calculated by Li in [26]. Our aim is to show the existence of almost periodic solutions for such a family of FNLS.

For the best understanding of FNLS, we firstly consider the following equation ($s_0=\frac{1}{4}$) with periodic boundary condition

$$\mathbf{i}u_t-|\partial_x|^{\frac{1}{2}}u+V*u+\epsilon f(|u|^2)u=0,\ x\in\mathbb{T}, \tag{1.4}$$

where $V*$ is a Fourier multiplier and f is real analytic near $0\in\mathbb{C}$. The model is motivated by the water wave problem posed in a fluid of infinite depth by Craig and Worfolk [12] and Zakharov [33]. We also mention that in the physics literature the fractional Schrödinger equation was introduced by Laskin [28] to describe frational quantum mechanics. From then on, there have been many works on FNLS, see [14, 19, 28] for more details. More recently, by using KAM technique, Li in [26] showed that there were many quasi-periodic solutions of a class of space fractional nonlinear Schrödinger equations ($1/2<s_0<1$) with the Riesz fractional differentiation and Xu in [32] obtained a family of small-amplitude quasi-periodic solutions with linear stability for FNLS (1.4). In fact, the KAM technique is a powerful tool to obtain quasi-periodic (or almost-periodic) solutions for Hamiltonian partial differential equations (PDEs). The KAM results, such as the existence results of quasi-periodic solutions for Hamiltonian PDEs have attracted a great deal of attention over years and is well understood in [1, 2, 5–7, 11, 13, 15, 20, 22–25, 27, 34] and references therein. In the all above works, the obtained KAM tori are lower (finite) dimension.

In this paper, we are interested in the construction of almost periodic solutions of (1.1), we will investigate the full dimensional tori by proceeding along the 'usual' KAM scheme where the perturbation is eventually removed by consecutive symplectic transformations of phase space. There are some papers focusing on the existence of the full dimensional KAM tori for Hamiltonian PDEs. The first related result is given by Fröhlich-Spencer-Wayne in [16] and Pöschel [30], where the infinite dimensional Hamiltonian system with short range was considered. Such infinite-dimensional Hamiltonian systems are well approximated by finite-dimensional ones and one can show that the classical KAM proof also works in this case, if we choose proper norm. Later, the almost periodic solutions on a full set frequencies for one dimensional NLS and NLW were constructed by Bourgain in [4] (Also see the almost periodic solutions for one dimensional NLS in [17, 18, 31] and higher-dimensional beam equations by Niu-Geng in [29]). These invariant tori were obtained by imposing hyper-exponentially decay on actions I_n ($I_n\sim e^{-|n|^S},\ S>1$). An open problem raised by Kuksin is whether there exist the full dimensional tori with suitable decay, such as $I_n\sim|n|^{-S}$. The first result in this direction for Hamiltonian PDEs was given by Bourgain who in [8] proved that 1-dimensional NLS has a full dimensional KAM torus of prescribed frequencies where the actions of the torus obey the estimates

$$\frac{1}{2}e^{-r|n|^{\theta}}\le I_n\le 2e^{-r|n|^{\theta}},\ n\in\mathbb{Z},\ r>0, \tag{1.5}$$

with $\theta = 1/2$. Recently, Bourgain's results have been extended to any $\theta \in (0,1)$ in [9] and to the case that the nonlinear perturbation depends explicitly on the space variable x in [10]. Biasco-Massetti-Procesi in [3] generalized and rederive Bourgain's results in [8] with a more geometric point of view and construct many elliptic tori independent of their dimension. Indeed, our work was initiated and inspired by Bourgain and it turns out the KAM scheme is still applicable in (1.1) due to special arithmetical features in [8]. Roughly speaking, an important observation by Bourgain is the following: Let (n_i) be a finite set of modes satisfying $|n_1| \geq |n_2| \geq \cdots$ and then unless $n_1 = n_2$, one has

$$|n_1| + |n_2| \leq C\sum_{j\geq 3}|n_j|, \tag{1.6}$$

which follows from the relations

$$n_1 - n_2 + n_3 - \cdots = 0, \tag{1.7}$$

and

$$n_1^2 - n_2^2 + n_3^2 - \cdots = o(1). \tag{1.8}$$

The estimate (1.6) is essential to control the small divisors which arise during the KAM iteration. However, there are many Hamiltonian PDEs do not satisfy (1.6) such as the 1-dimensional nonlinear wave equation. The reason is that (1.8) is no longer true since the linear growth of the frequencies. The quadratic growth of the frequencies (which is also considered as the separation property) is important to overcome the small divisors specially for high-dimensional PDEs. When FNLS with $s_0 \in (1/2, 1)$ is considered, the main part $C(s_0)n^{2s_0}$ of the frequencies for any n will goes n^2 as $s_0 \to 1$ and tends to $|n|$ as $s_0 \to 1/2$. It is helpful to understand how the separation property is useful to overcome the small divisors. In this paper, the relation (1.8) is replaced by

$$[C(s_0)n_1^{2s_0}] - [C(s_0)n_2^{2s_0}] + [C(s_0)n_3^{2s_0}] - \cdots = o(1), \tag{1.9}$$

where $[C(s_0)n^{2s_0}]$ represents the integer part of $C(s_0)n^{2s_0}$. Here, we see clearly that the estimate (1.6) fails when $s_0 = 1/2$. Actually, the main part $C(s_0)n^{2s_0}$ of the n-frequency is no longer an integer which is also a problem to define the Diophantine conditions newly. Since $C(s_0)n^{2s_0} - [C(s_0)n^{2s_0}] \in (0,1)$ and $V_n \in [-1,1]$, the new variable $\widetilde{V}_n(V)$ can be considered as new parameters by denoting $\widetilde{V}_n(V) = V_n + C(s_0)n^{2s_0} - [C(s_0)n^{2s_0}]$ for any n, which finally derive the relation (1.9). Obviously, the condition (1.9) is weaker than (1.8). On the other hand, different from Bourgain [8] in which zero-moment condition is satisfied, we need the sub-exponentially decaying of $\widehat{f}(n)$ together with relatively faster (of course slower than Schrodinger equation case) of normal modes to do that. For this purpose, we impose Gevrey smooth condition on the function $f(x)$, that is, one has

$$|\widehat{f}(n)| \leq Ce^{-\mu|n|^{\theta}},\ \mu > 0,\ \theta \in (0,1). \tag{1.10}$$

Thus we can use the properties (1.9) and (1.10) to guarantee $|n_1| + |n_2|$ can be controlled by $\sum_{j\geq 3}|n_j| + |n|$, which leads to the result of this paper.

To state our result precisely, we will give some definitions firstly.

Let $q = (q_n)_{n\in\mathbb{Z}}$ and its complex conjugate $\bar{q} = (\bar{q}_n)_{n\in\mathbb{Z}}$. Introduce $I_n = |q_n|^2$ and $J_n = I_n - I_n(0)$ as notations but not as new variables, where $I_n(0)$ will be considered as the initial data. Then the Hamiltonian R has the form

$$R(q, \bar{q}) = \sum_{a,k,k' \in \mathbb{N}^{\mathbb{Z}}} B_{akk'} \mathcal{M}_{akk'}$$

with

$$\mathcal{M}_{akk'} = \prod_{n\in\mathbb{Z}} I_n(0)^{a_n} q_n^{k_n} \bar{q}_n^{k'_n},$$

and $B_{akk'}$ are the coefficients.

Define by

$$\text{supp } \mathcal{M}_{akk'} = \{n : 2a_n + k_n + k'_n \neq 0\}, \tag{1.11}$$

and define the momentum of $\mathcal{M}_{akk'}$ by

$$\text{momentum } \mathcal{M}_{akk'} := m(k, k') = \sum_{n\in\mathbb{Z}} (k_n - k'_n) n. \tag{1.12}$$

Moreover, denote by

$$n_1^* = \max\{|n| : a_n + k_n + k'_n \neq 0\},$$

and

$$m^*(k, k') = |m(k, k')|.$$

Now we define the norm of the Hamiltonian as follows

Definition 1.1. For any given $\rho > 0, \mu > 0$ and $0 < \theta < 1$, define the norm of the Hamiltonian R by

$$\|R\|_{\rho,\mu} = \sup_{a,k,k'\in\mathbb{N}^{\mathbb{Z}}} \frac{|B_{akk'}|}{e^{\rho\sum_n (2a_n+k_n+k'_n)|n|^\theta - 2\rho(n_1^*)^\theta - \mu m^*(k,k')^\theta}}. \tag{1.13}$$

Definition 1.2. Given $0 < \theta < 1$ and $r > 0$, we define the Banach space $\mathfrak{H}_{r,\infty}$ consisting of all complex sequences $q = (q_n)_{n\in\mathbb{Z}}$ with

$$\|q\|_{r,\infty} = \sup_{n\in\mathbb{Z}} |q_n| e^{r|n|^\theta} < \infty. \tag{1.14}$$

Definition 1.3. Denote $\|x\| = \text{dist}(x, \mathbb{Z})$. A vector $\omega = (\omega_n)_{n\in\mathbb{Z}}$ is called to be Diophantine if there exists a real number $\gamma > 0$ such that the following resonance issues

$$\left\| \sum_{n\in\mathbb{Z}} l_n \omega_n \right\| \geq \gamma \prod_{n\in\mathbb{Z}} \frac{1}{1 + l_n^2 |n|^4} \tag{1.15}$$

hold, where $0 \neq l = (l_n)_{n\in\mathbb{Z}}$ is a finitely supported sequence of integers and

$$|n| = \max\{1, n, -n\}.$$

Lemma 1.1. *Let (ω_n) be as above. Then, (1.15) holds except on a set of small measure in $[-1, 1]^{\mathbb{Z}}$.*

Proof. See Lemma 4.1 in [8] for more details. □

Theorem 1.1. *Given $r > 0$, $0 < \theta < 1$ and a Diophantine vector $\omega = (\omega_n)_{n\in\mathbb{Z}}$ satisfying the non-resonant conditions (1.15), then for any $\mu > r$, sufficiently small $\epsilon > 0$, there exist $V = (V_n)_{n\in\mathbb{Z}}$ with $V_n \in [-1,1]$, such that (1.1) has a full dimensional invariant torus $\mathcal{E}$ with amplitude in $\mathfrak{H}_{r,\infty}$ satisfying:*

(1) the amplitude of $\mathcal{E}$ is restricted as

$$\frac{1}{2}e^{-r|n|^{\theta}} \leq |q_n| \leq 2e^{-r|n|^{\theta}};$$

(2) the frequency on $\mathcal{E}$ was prescribed to be $\left([C(s_0)n^{2s_0}] + \omega_n\right)_{n\in\mathbb{Z}}$;

(3) the invariant torus $\mathcal{E}$ is linearly stable.

Remark 1.1. The statement holds for most $(\omega_n)_{n\in\mathbb{Z}} \in [-1,1]^{\mathbb{Z}}$ here. Bourgain in [8] has claimed that the statement held for most $(V_n)_{n\in\mathbb{Z}} \in [-1,1]^{\mathbb{Z}}$, but not proven yet.

2. KAM theorem

2.1. Some notations and the norm of the Hamiltonian

For any $k \in \mathbb{N}^{\mathbb{Z}}$, define

$$\text{supp } k = \{n : k_n \neq 0\}. \tag{2.1}$$

Rewrite R as

$$R = R_0 + R_1 + R_2 \tag{2.2}$$

where

$$R_0 = \sum_{\substack{a,k,k'\in\mathbb{N}^{\mathbb{Z}} \\ \text{supp } k \bigcap \text{supp } k' = \emptyset}} B_{akk'}\mathcal{M}_{akk'}, \tag{2.3}$$

$$R_1 = \sum_{m\in\mathbb{Z}} J_m \left(\sum_{\substack{a,k,k'\in\mathbb{N}^{\mathbb{Z}} \\ \text{supp } k \bigcap \text{supp } k' = \emptyset}} B^{(m)}_{akk'}\mathcal{M}_{akk'} \right), \tag{2.4}$$

$$R_2 = \sum_{m_1,m_2\in\mathbb{Z}} J_{m_1}J_{m_2} \left(\sum_{\substack{a,k,k'\in\mathbb{N}^{\mathbb{Z}} \\ \text{no assumption}}} B^{(m_1,m_2)}_{akk'}\mathcal{M}_{akk'} \right). \tag{2.5}$$

Given $r > 0$, let

$$D = \left\{ q = (q_n)_{n\in\mathbb{Z}} : \frac{1}{2}e^{-r|n|^{\theta}} \leq |z_n| \leq 2e^{-r|n|^{\theta}} \right\},$$

and

$$\Pi = \left\{ V = (V_n)_{n\in\mathbb{Z}} : V_n \in [-1,1] \right\}.$$

Then we have the following result:

Theorem 2.1. *For $0<\theta<1$ and $\mu>r>\frac{100}{2-2^{\theta}}\rho>0$, suppose the Hamiltonian*

$$H(q,\bar{q})=N(q,\bar{q})+R(q,\bar{q})$$

is real analytic on the domain $D\times\Pi$, where

$$N(q,\bar{q})=\sum_{n\in\mathbb{Z}}(C(s_0)n^{2s_0}+V_n)|q_n|^2,$$

and $R(q,\bar{q})$ satisfies

$$||R||_{\rho,\mu}\leq\epsilon.$$

Then given any $\omega=(\omega_n)_{n\in\mathbb{Z}}$ satisfying the non-resonant condition (1.15) and for sufficiently small ϵ depending on r,ρ,μ,θ and γ, there exist $V_*\in\Pi$ and a real analytic symplectic coordinate transformation $\Phi:D_*\times\{V_*\}\to D$, where

$$D_*=\left\{q=(q_n)_{n\in\mathbb{Z}}:\frac{2}{3}e^{-r|n|^{\theta}}\leq|q_n|\leq\frac{5}{6}e^{-r|n|^{\theta}}\right\}$$

satisfying

$$\sup_{q\in D_*}||(\Phi-id)(q)||_{r,\infty}\leq\epsilon^{0.4}$$

such that for $H_*=H\circ\Phi=N_*+R_{2,*}$ with

$$N_*=\sum_{n\in\mathbb{Z}}([C(s_0)n^{2s_0}]+\omega_n)|q_n|^2$$

and $R_{2,*}$ has the form of (2.5) and satisfies

$$||R_{2,*}||^{+}_{10\rho,\mu-18\rho}\leq\frac{7}{6}\epsilon.$$

2.2. Derivation of homological equations

The proof of Theorem 2.1 employs the rapidly converging iteration scheme of Newton type to deal with small divisor problems introduced by Kolmogorov, involving the infinite sequence of coordinate transformations. At the s-th step of the scheme, a Hamiltonian $H_s=N_s+R_s$ is considered as a small perturbation of some normal form N_s. A transformation Φ_s is set up so that

$$H_s\circ\Phi_s=N_{s+1}+R_{s+1}$$

with another normal form N_{s+1} and a much smaller perturbation R_{s+1}. We drop the index s of H_s,N_s,R_s,Φ_s and shorten the index $s+1$ as $+$.

We desire to eliminate the terms R_0,R_1 in (2.2) by the coordinate transformation Φ, which is obtained as the time-1 map $X_F^t|_{t=1}$ of a Hamiltonian vector field X_F with $F=F_0+F_1$. Let F_0 (resp. F_1) has the form of R_0 (resp. R_1), that is

$$F_0=\sum_{\substack{a,k,k'\in\mathbb{N}^{\mathbb{Z}}\\ \text{supp }k\cap\text{supp }k'=\emptyset}}F_{akk'}\mathcal{M}_{akk'},\tag{2.6}$$

$$F_1=\sum_{m\in\mathbb{Z}}J_m\left(\sum_{\substack{a,k,k'\in\mathbb{N}^{\mathbb{Z}}\\ \text{supp }k\cap\text{supp }k'=\emptyset}}F^{(m)}_{akk'}\mathcal{M}_{akk'}\right),\tag{2.7}$$

and the homological equations become

$$\{N,F\}+R_0+R_1=[R_0]+[R_1], \tag{2.8}$$

where

$$[R_0]=\sum_{a\in\mathbb{N}^{\mathbb{Z}}}B_{a00}\mathcal{M}_{a00}, \tag{2.9}$$

and

$$[R_1]=\sum_{m\in\mathbb{Z}}J_m\sum_{a\in\mathbb{N}^{\mathbb{Z}}}B^{(m)}_{a00}\mathcal{M}_{a00}.$$

The solutions of the homological equations (2.8) are given by

$$F_{akk'}=\frac{B_{akk'}}{\sum_{n\in\mathbb{Z}}(k_n-k'_n)([C(s_0)n^{2s_0}]+\widetilde{V}_n)}, \tag{2.10}$$

and

$$F^{(m)}_{akk'}=\frac{B^{(m)}_{akk'}}{\sum_{n\in\mathbb{Z}}(k_n-k'_n)([C(s_0)n^{2s_0}]+\widetilde{V}_n)}, \tag{2.11}$$

where $\widetilde{V}_n$ denote the modulated frequencies by readjusting the multiplier (V_n) in (1.1) to ensure at each stage $\widetilde{V}_n=\omega_n$ with $\omega=(\omega_n)$ a fixed frequency.
The new Hamiltonian H_+ has the form

$$\begin{aligned}H_+&=H\circ\Phi\\&=N+\{N,F\}+R_0+R_1\\&\quad+\int_0^1\{(1-t)\{N,F\}+R_0+R_1,F\}\circ X_F^t\,\mathrm{d}t+R_2\circ X_F^1\\&=N_++R_+,\end{aligned} \tag{2.12}$$

where

$$N_+=N+[R_0]+[R_1], \tag{2.13}$$

and

$$R_+=\int_0^1\{(1-t)\{N,F\}+R_0+R_1,F\}\circ X_F^t\,\mathrm{d}t+R_2\circ X_F^1. \tag{2.14}$$

2.3. The solvability of the homological equations (2.8)

In this subsection, we will estimate the solutions of the homological equations (2.8). To this end, we define the new norm for the Hamiltonian R of the form as follows:

$$||R||^+_{\rho,\mu}=\max\{||R_0||^+_{\rho,\mu},||R_1||^+_{\rho,\mu}|,||R_2||^+_{\rho,\mu}\}, \tag{2.15}$$

where

$$||R_0||^+_{\rho,\mu}=\sup_{a,k,k'\in\mathbb{N}^{\mathbb{Z}}}\frac{|B_{akk'}|}{e^{\rho(\sum_n(2a_n+k_n+k'_n)|n|^\theta-2(n_1^*)^\theta)-\mu m^*(k,k')^\theta}}, \tag{2.16}$$

$$||R_1||^+_{\rho,\mu}=\sup_{\substack{a,k,k'\in\mathbb{N}^{\mathbb{Z}}\\m\in\mathbb{Z}}}\frac{\left|B^{(m)}_{akk'}\right|}{e^{\rho(\sum_n(2a_n+k_n+k'_n)|n|^\theta+2|m|^\theta-2(n_1^*)^\theta)-\mu m^*(k,k')^\theta}}, \tag{2.17}$$

$$||R_2||^+_{\rho,\mu}=\sup_{\substack{a,k,k'\in\mathbb{N}^{\mathbb{Z}}\\m_1,m_2\in\mathbb{Z}}}\frac{\left|B^{(m_1,m_2)}_{akk'}\right|}{e^{\rho(\sum_n(2a_n+k_n+k'_n)|n|^\theta+2|m_1|^\theta+2|m_2|^\theta-2(n_1^*)^\theta)-\mu m^*(k,k')^\theta}}.$$

Moreover, one has the following estimates:

Lemma 2.1. *Given any $\mu > \delta > 0, \rho > 0$, one has*

$$||R||^{+}_{\rho+\delta,\mu-\delta} \leq \left(\frac{1}{\delta}\right)^{C(\theta)\delta^{-\frac{1}{\theta}}} ||R||_{\rho,\mu} \tag{2.18}$$

and

$$||R||_{\rho+\delta,\mu-\delta} \leq \frac{C(\theta)}{\delta^2}||R||^{+}_{\rho,\mu}, \tag{2.19}$$

where $C(\theta)$ is a positive constant depending on θ only.

Proof. The details of the proof had been given in [10] of Lemma 2.2. □

Lemma 2.2. *Let $(\widetilde{V}_n)_{n\in\mathbb{Z}}$ be Diophantine with $\gamma > 0$ (see (1.15)). Then for any $\rho, \mu > 0, 0 < \delta \ll 1$ (depending only on θ), the solutions of the homological equations (2.8), which are given by (2.10) and (2.11), satisfy*

$$||F_i||^{+}_{\rho+\delta,\mu-2\delta} \leq \frac{1}{\gamma} \cdot e^{C(\theta)\delta^{-\frac{6s_0-1}{(2s_0-1)\theta}}} ||R_i||^{+}_{\rho,\mu}, \tag{2.20}$$

where $i = 0, 1$ and $C(\theta)$ is a positive constant depending on θ only.

Proof. We distinguish two cases:

Case. 1.

$$\left|\sum_{n\in\mathbb{Z}}(k_n - k'_n)[C(s_0)n^{2s_0}]\right| > 10\sum_{n\in\mathbb{Z}}|k_n - k'_n|.$$

Since $|\widetilde{V}_n| \leq 2$, we have

$$\left|\sum_{n\in\mathbb{Z}}(k_n - k'_n)([C(s_0)n^{2s_0}] + \widetilde{V}_n)\right| > 10\sum_{n\in\mathbb{Z}}|k_n - k'_n| - 2\sum_{n\in\mathbb{Z}}|k_n - k'_n| \geq 1,$$

where the last inequality is based on supp $k \bigcap$ supp $k' = \emptyset$. There is no small divisor and (2.20) holds trivially.

Case. 2.

$$\left|\sum_{n\in\mathbb{Z}}(k_n - k'_n)[C(s_0)n^{2s_0}]\right| \leq 10\sum_{n\in\mathbb{Z}}|k_n - k'_n|.$$

In this case, we always assume

$$\left|\sum_{n\in\mathbb{Z}}(k_n - k'_n)([C(s_0)n^{2s_0}] + \widetilde{V}_n)\right| \leq 1,$$

otherwise there is no small divisor.

Firstly, one has

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}}|k_n - k'_n||n|^{\frac{(2s_0-1)\theta}{2s_0}} \\
&\leq 7\cdot 12^{\frac{\theta}{2s_0}}\left(\sum_{i\geq 3}(n_i^*)^\theta + m^*(k,k')^\theta\right) \qquad \text{(in view of Lemma 3.3)} \\
&\leq \frac{7\cdot 12^{\frac{\theta}{2s_0}}}{2-2^\theta}\left(\sum_{n\in\mathbb{Z}}(2a_n + k_n + k'_n)|n|^\theta - 2(n_1^*)^\theta + 2m^*(k,k')^\theta\right),
\end{aligned} \tag{2.21}$$

where the last inequality is based on Lemma 3.2.

Since $C(s_0)n^{2s_0}-[C(s_0)n^{2s_0}]\in(0,1)$ and $V_n\in[-1,1]$, one has

$$\widetilde{V}_n(V)=V_n+C(s_0)n^{2s_0}-[C(s_0)n^{2s_0}]=\omega_n$$

by choosing $V_n=\omega_n-C(s_0)n^{2s_0}+[C(s_0)n^{2s_0}]$. It follows from $\sum_n(k_n-k_n')[C(s_0)n^{2s_0}]\in\mathbb{Z}$ and (1.15) that

$$\left\|\sum_{n\in\mathbb{Z}}(k_n-k_n')([C(s_0)n^{2s_0}]+\widetilde{V}_n)\right\|\geq\frac{\gamma}{2}\prod_{n\in\mathbb{Z}}\frac{1}{1+|k_n-k_n'|^2|n|^4}. \tag{2.22}$$

Hence,

$$\begin{aligned}
&|F_{akk'}|e^{-(\rho+\delta)(\sum_n(2a_n+k_n+k_n')|n|^\theta-2(n_1^*)^\theta)+(\mu-2\delta)m^*(k,k')^\theta}\\
\leq{}&2\gamma^{-1}||R_0||^+_{\rho,\mu}\prod_n\left(1+|k_n-k_n'|^2|n|^4\right)e^{-\delta\left(\sum_n(2a_n+k_n+k_n')|n|^\theta-2(n_1^*)^\theta+2m^*(k,k')^\theta\right)}\\
&\text{(in view of (2.10), (2.16) and (2.22))}\\
\leq{}&2\gamma^{-1}||R_0||^+_{\rho,\mu}e^{\sum_n\ln(1+|k_n-k_n'|^2|n|^4)}e^{-\frac{2-2^\theta}{7\cdot12^{\theta/2s_0}}\delta\sum_n\left(|k_n-k_n'||n|^{\frac{(2s_0-1)\theta}{2s_0}}\right)}\\
&\text{(in view of (2.21) and noting }\tilde{\delta}=\tfrac{2-2^\theta}{7\cdot12^{\theta/2s_0}}\delta)\\
={}&2\gamma^{-1}||R_0||^+_{\rho,\mu}e^{\sum\limits_{n:k_n\neq k_n'}\ln(1+|k_n-k_n'|^2|n|^4)-\tilde{\delta}\sum\limits_{n:k_n\neq k_n'}\left(|k_n-k_n'||n|^{\frac{(2s_0-1)\theta}{2s_0}}\right)}\\
\leq{}&2\gamma^{-1}||R_0||^+_{\rho,\mu}e^{8\left(\sum\limits_{n:k_n\neq k_n'}\ln(|k_n-k_n'||n|)\right)+3-\tilde{\delta}\sum\limits_{n:k_n\neq k_n'}\left(|k_n-k_n'|^{\frac{(2s_0-1)\theta}{2s_0}}|n|^{\frac{(2s_0-1)\theta}{2s_0}}\right)}\\
={}&\frac{2e^3}{\gamma}||R_0||^+_{\rho,\mu}e^{\sum\limits_{|n|\leq N:k_n\neq k_n'}\left(8\ln(|k_n-k_n'||n|)-\tilde{\delta}|k_n-k_n'|^{\frac{(2s_0-1)\theta}{2s_0}}|n|^{\frac{(2s_0-1)\theta}{2s_0}}\right)}\\
&+\frac{2e^3}{\gamma}||R_0||^+_{\rho,\mu}e^{\sum\limits_{n>N:k_n\neq k_n'}\left(8\ln(|k_n-k_n'||n|)-\tilde{\delta}|k_n-k_n'|^{\frac{(2s_0-1)\theta}{2s_0}}|n|^{\frac{(2s_0-1)\theta}{2s_0}}\right)}\\
&\text{(where }N=\left(\tfrac{16s_0}{(2s_0-1)\theta\tilde{\delta}}\right)^{\frac{4s_0}{(2s_0-1)\theta}})\\
={}&:A.
\end{aligned}$$

It is easy to verify the following two facts that

$$\begin{aligned}
\max_{x\geq1}f(x)&=f\left(\left(\frac{16s_0}{(2s_0-1)\theta\delta}\right)^{\frac{2s_0}{(2s_0-1)\theta}}\right)\\
&=-\frac{16s_0}{(2s_0-1)\theta}+8\ln\left(\left(\frac{16s_0}{(2s_0-1)\theta\delta}\right)^{\frac{2s_0}{(2s_0-1)\theta}}\right)\\
&\leq\frac{16s_0}{(2s_0-1)\theta}\ln\left(\frac{16s_0}{(2s_0-1)\theta\delta}\right)
\end{aligned} \tag{2.23}$$

with $f(x)=(-\delta x^{\frac{(2s_0-1)\theta}{2s_0}}+8\ln x)$, and when $|n|>N=\left(\frac{16s_0}{(2s_0-1)\theta\delta}\right)^{\frac{4s_0}{(2s_0-1)\theta}}$, $k_n\neq$

k_n', one has

$$-\delta\left(|k_n-k_n'|^{\frac{(2s_0-1)\theta}{2s_0}}|n|^{\frac{(2s_0-1)\theta}{2s_0}}\right)+8\ln(|k_n-k_n'||n|)<0\quad(\text{for } 0<\delta\ll 1).\tag{2.24}$$

In view of (2.23) and (2.24), we have

$$\begin{aligned}A&\le\frac{2e^3}{\gamma}||R_0||^+_{\rho,\mu}e^{\left(\frac{16s_0}{(2s_0-1)\theta\delta}\right)^{\frac{4s_0}{(2s_0-1)\theta}}\cdot\frac{32s_0}{(2s_0-1)\theta}\ln\left(\frac{16s_0}{(2s_0-1)\theta\delta}\right)}+\frac{2e^3}{\gamma}||R_0||^+_{\rho,\mu}\\&\le\frac{1}{\gamma}\cdot e^{C(\theta)\delta^{-\frac{6s_0-1}{(2s_0-1)\theta}}}||R_0||^+_{\rho,\mu}\quad(\text{for } 0<\delta\ll 1).\end{aligned}\tag{2.25}$$

Therefore, in view of (2.16) and (2.25), we finish the proof of (2.20) for $i=0$.

Similarly, one can prove (2.20) for $i=1$. □

2.4. The new perturbation R_+ and the new normal form N_+

Firstly, we will prove two lemmas.

Lemma 2.3 (Poisson Bracket). *Let $\theta\in(0,1),\rho,\mu>0$ and $0<\delta_1,\delta_2\ll 1$ (depending on θ,ρ,μ). Then one has*

$$||\{H_1,H_2\}||_{\rho,\mu}\le\frac{1}{\delta_2}\left(\frac{1}{\delta_1}\right)^{C(\theta)\delta_1^{-\frac{1}{\theta}}}||H_1||_{\rho-\delta_1,\mu+2\delta_1}||H_2||_{\rho-\delta_2,\mu+2\delta_2},\tag{2.26}$$

where $C(\theta)$ is a positive constant depending on θ only.

Proof. The details of proof had been given in [10] of Lemma 2.4. □

Lemma 2.4. *Let $\theta\in(0,1),\rho,\mu>0$ and $0<\delta_1,\delta_2\ll 1$ (depending on θ,ρ,μ). Assume further*

$$\frac{1}{\delta_2}\left(\frac{1}{\delta_1}\right)^{C(\theta)\delta_1^{-\frac{1}{\theta}}}||F||_{\rho-\delta_1,\mu+2\delta_1}\ll 1,\tag{2.27}$$

where $C(\theta)$ is the constant given in (2.26) in Lemma 2.3. Then for any Hamiltonian function H, we get

$$||H\circ\Phi_F||_{\rho,\mu}\le\left(1+\frac{1}{\delta_2}\left(\frac{1}{\delta_1}\right)^{C_1(\theta)\delta_1^{-\frac{1}{\theta}}}||F||_{\rho-\delta_1,\mu+2\delta_1}\right)||H||_{\rho-\delta_2,\mu+2\delta_2},\tag{2.28}$$

where $C_1(\theta)$ is a positive constant depending only on θ.

Proof. The details of proof had been given in [10] of Lemma 2.5. □

Recall the new term R_+ is given by (2.14) and write

$$R_+=R_{0+}+R_{1+}+R_{2+}.\tag{2.29}$$

Following the proof of [9], one has

$$||R_{0+}||^{+}_{\rho+3\delta,\mu-\frac{11}{2}\delta} \leq \frac{1}{\gamma}\cdot e^{\delta^{-\frac{12s_0-2}{(2s_0-1)\theta}}}(||R_0||^{+}_{\rho,\mu}+||R_1||^{+}_{\rho,\mu})(||R_0||^{+}_{\rho,\mu}+||R_1||^{+\,2}_{\rho,\mu}), \quad (2.30)$$

$$||R_{1+}||^{+}_{\rho+3\delta,\mu-\frac{11}{2}\delta} \leq \frac{1}{\gamma}\cdot e^{\delta^{-\frac{12s_0-2}{(2s_0-1)\theta}}}(||R_0||^{+}_{\rho,\mu}+||R_1||^{+\,2}_{\rho,\mu}), \quad (2.31)$$

$$||R_{2+}||^{+}_{\rho+3\delta,\mu-\frac{11}{2}\delta} \leq ||R_2||^{+}_{\rho,\mu}+\frac{1}{\gamma}\cdot e^{\delta^{-\frac{12s_0-2}{(2s_0-1)\theta}}}(||R_0||^{+}_{\rho,\mu}+||R_1||^{+}_{\rho,\mu}). \quad (2.32)$$

The new normal form N_+ is given in (2.13). Note that $[R_0]$ (in view of (2.9)) is a constant which does not affect the Hamiltonian vector field. Moreover, in view of (2.9), we denote by

$$\omega_{n+} = [C(s_0)n^{2s_0}] + \widetilde{V}_n + \sum_{a\in\mathbb{N}^{\mathbb{Z}}} B^{(n)}_{a00}\mathcal{M}_{a00}, \quad (2.33)$$

where $\widetilde{V}_n = V_n + C(s_0)n^{2s_0} - [C(s_0)n^{2s_0}]$ and the terms $\sum_{a\in\mathbb{N}^{\mathbb{Z}}} B^{(n)}_{a00}\mathcal{M}_{a00}$ is the so-called frequency shift. The estimate of $\left|\sum_{a\in\mathbb{N}^{\mathbb{Z}}} B^{(n)}_{a00}\mathcal{M}_{a00}\right|$ will be given in the next section (see (3.26) for the details).

Finally, we give the estimate of the Hamiltonian vector field.

Lemma 2.5. *Given a Hamiltonian*

$$H = \sum_{a,k,k'\in\mathbb{N}^{\mathbb{Z}}} B_{akk'}\mathcal{M}_{akk'}, \quad (2.34)$$

then for any $\mu > r > (\frac{1}{2-2^{\theta}}+3)\rho$, $||q||_{r,\infty} < 1$ *and* $||I(0)||_{r,\infty} < 1$, *one has*

$$\sup_{j\in\mathbb{Z}} \left| e^{r|j|^{\theta}} \frac{\partial H}{\partial q_j} \right| \leq C(r,\rho,\mu,\theta)||H||_{\rho,\mu}, \quad (2.35)$$

where $C(r,\rho,\mu,\theta)$ *is a positive constant depending on* r,ρ,μ *and* θ *only, and*

$$||I(0)||_{r,\infty} := \sup_{n\in\mathbb{Z}} |I_n(0)|e^{2r|n|^{\theta}}. \quad (2.36)$$

Proof. The details of the proof had been given in [10] of Lemma 2.6. □

3. Iteration and Convergence

Now we give the precise set-up of iteration parameters. Let $s \geq 1$ be the s-th KAM step.

$\rho_0 = \rho$, $r \geq \frac{100\rho}{2-2^{\theta}}$, $\mu_0 > r$,

$\delta_s = \frac{\rho}{s^2}$,

$\rho_{s+1} = \rho_s + 3\delta_s$,

$\mu_{s+1} = \mu_s - 6\delta_s$,

$\epsilon_s = \epsilon_0^{(\frac{3}{2})^s}$, which dominates the size of the perturbation,

$$\lambda_s = e^{-C(\theta)(\ln\frac{1}{\epsilon_{s+1}})^{\frac{\frac{2s_0}{2s_0-1}+2}{\theta+\frac{2s_0}{2s_0-1}+2}}},\ s_0>\frac{1}{2},$$
$$\eta_{s+1} = \frac{1}{20}\lambda_s\eta_s,$$
$$d_0=0,\ d_{s+1} = d_s + \frac{1}{\pi^2(s+1)^2},$$
$$D_s = \{(q_n)_{n\in\mathbb{Z}} : \tfrac{1}{2}+d_s \le |q_n|e^{r|n|^\theta} \le 1-d_s\}.$$

Denote $\bar V_n = V_n + C(s_0)n^{2s_0} - [C(s_0)n^{2s_0}]$ and the complex cube of size $\lambda>0$:

$$\mathcal{C}_\lambda(V^*) = \left\{(\bar V_n)_{n\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}} : |\bar V_n - V_n^*| \le \lambda\right\}. \tag{3.1}$$

Lemma 3.1. *Suppose $H_s = N_s + R_s$ is real analytic on $D_s\times\mathcal{C}_{\eta_s}(V_s^*)$, where*

$$N_s = \sum_{n\in\mathbb{Z}}([n^{2s_0}] + \widetilde V_{n,s})|q_n|^2$$

is a normal form with coefficients satisfying

$$\widetilde V_s(V_s^*) = \omega, \tag{3.2}$$
$$\left|\left|\frac{\partial \widetilde V_s}{\partial \bar V} - I\right|\right|_{l^\infty\to l^\infty} < d_s\epsilon_0^{\frac{1}{10}}, \tag{3.3}$$

and $R_s = R_{0,s}+R_{1,s}+R_{2,s}$ satisfying

$$||R_{0,s}||^+_{\rho_s,\mu_s} \le \epsilon_s, \tag{3.4}$$
$$||R_{1,s}||^+_{\rho_s,\mu_s} \le \epsilon_s^{0.6}, \tag{3.5}$$
$$||R_{2,s}||^+_{\rho_s,\mu_s} \le (1+d_s)\epsilon_0. \tag{3.6}$$

Then for all $\bar V\in\mathcal{C}_{\eta_s}(V_s^*)$ satisfying $\widetilde V_s(\bar V)\in\mathcal{C}_{\lambda_s}(\omega)$, there exist real analytic symplectic coordinate transformations $\Phi_{s+1}: D_{s+1}\to D_s$ satisfying

$$||\Phi_{s+1} - id||_{r,\infty} \le \epsilon_s^{0.5}, \tag{3.7}$$
$$||D\Phi_{s+1} - I||_{(r,\infty)\to(r,\infty)} \le \epsilon_s^{0.5}, \tag{3.8}$$

such that for $H_{s+1} = H_s\circ\Phi_{s+1} = N_{s+1}+R_{s+1}$, the same assumptions as above are satisfied with '$s+1$' in place of 's', where $\mathcal{C}_{\eta_{s+1}}(V_{s+1}^*)\subset \widetilde V_s^{-1}(\mathcal{C}_{\lambda_s}(\omega))$ and

$$||\widetilde V_{s+1} - \widetilde V_s||_\infty \le \epsilon_s^{0.5}, \tag{3.9}$$

$$||V_{s+1}^* - V_s^*||_\infty \le 2\epsilon_s^{0.5}. \tag{3.10}$$

Proof. In the step $s\to s+1$, there is saving of a factor

$$e^{-\delta_s\left(\sum_n(2a_n+k_n+k_n')|n|^\theta - 2|n_1^*|^\theta + 2m^*(k,k')^\theta\right)}. \tag{3.11}$$

By (3.39), one has

$$(3.11) \le e^{-(2-2^\theta)\delta_s\left(\sum_{i\ge3}|n_i|^\theta\right) - \delta_s m^*(k,k')^\theta} \le e^{-(2-2^\theta)\delta_s\left(\sum_{i\ge3}|n_i|^\theta + m^*(k,k')^\theta\right)}.$$

Recalling after this step, we need

$$||R_{0,s+1}||^{+}_{\rho_{s+1},\mu_{s+1}} \le \epsilon_{s+1},$$
$$||R_{1,s+1}||^{+}_{\rho_{s+1},\mu_{s+1}} \le \epsilon_{s+1}^{0.6}.$$

Consequently, in $R_{i,s}$ $(i=0,1)$, it suffices to eliminate the non-resonant monomials $\mathcal{M}_{akk'}$ for which

$$e^{-(2-2^{\theta})\delta_s(\sum_{i\ge 3}|n_i|^{\theta}+m^*(k,k')^{\theta})} \ge \epsilon_{s+1},$$

that is

$$\sum_{i\ge 3}|n_i|^{\theta}+m^*(k,k')^{\theta} \le \frac{s^2}{(2-2^{\theta})\rho}\ln\frac{1}{\epsilon_{s+1}}. \tag{3.12}$$

On the other hand, in the small divisors analysis (see Lemma 3.3), one has

$$\begin{aligned}\sum_{n\in\mathbb{Z}}|k_n-k_n'||n|^{\frac{(2s_0-1)\theta}{2s_0}} &\le 7\cdot 12^{\theta/2s_0}\left(\sum_{i\ge 3}|n_i|^{\theta}+m^*(k,k')^{\theta}\right)\\ &\le \frac{7\cdot 12^{\theta/2s_0}\cdot s^2}{(2-2^{\theta})\rho}\ln\frac{1}{\epsilon_{s+1}} \quad \text{(in view of (3.12))}\\ &:= B_s.\end{aligned}$$

Hence we need only impose condition on $(\widetilde{V}_n)_{|n|\le\mathcal{N}_s}$, where

$$\mathcal{N}_s \sim B_s^{\frac{2s_0}{(2s_0-1)\theta}}. \tag{3.13}$$

Correspondingly, the Diophantine condition becomes

$$\left\|\sum_{|n|\le\mathcal{N}_s}(k_n-k_n')\widetilde{V}_{n,s}\right\| \ge \gamma\prod_{|n|\le\mathcal{N}_s}\frac{1}{1+(k_n-k_n')^2|n|^4}. \tag{3.14}$$

We finished the truncation step.

Next we will show (3.14) preserves under small perturbation of $(\widetilde{V}_n)_{|n|\le\mathcal{N}_s}$ and this is equivalent to get lower bound on the right hand side of (3.14). Let

$$M_s \sim \left(\frac{B_s}{\ln B_s}\right)^{\frac{\frac{2s_0}{2s_0-1}+0.5}{\theta+\frac{2s_0}{2s_0-1}+0.5}},$$

then we have

$$\begin{aligned}&\prod_{|n|\le\mathcal{N}_s}\frac{1}{1+(k_n-k_n')^2|n|^4}\\ &= e^{\sum_{|n|\le M_s}\ln\left(\frac{1}{1+(k_n-k_n')^2|n|^4}\right)+\sum_{|n|>M_s}\ln\left(\frac{1}{1+(k_n-k_n')^2|n|^4}\right)}\\ &\ge e^{-C(\theta)\sum_{|n|\le M_s,k_n\ne k_n'}\ln\left(|k_n-k_n'||n|^{\frac{(2s_0-1)\theta}{2s_0}}\right)-\sum_{|n|>M_s,k_n\ne k_n'}4\left(|k_n-k_n'|\cdot\ln|n|\right)}\\ &\ge e^{-C(\theta)M_s\ln B_s-4\sum_{|n|>M_s,k_n\ne k_n'}\left(|k_n-k_n'||n|^{\frac{(2s_0-1)\theta}{2s_0}}(|n|^{-\frac{(2s_0-1)\theta}{2s_0}}\ln|n|)\right)}\end{aligned}$$

$$
\begin{aligned}
&\geq e^{-C(\theta)M_s\ln B_s - C(\theta)(M_s^{-\frac{(2s_0-1)\theta}{2s_0}}\ln M_s)B_s}\\
&\geq e^{-C(\theta)M_s\ln B_s - C(\theta)M_s^{-\frac{\theta}{\frac{2s_0}{2s_0-1}+0.5}}B_s}\\
&\geq e^{-C(\theta)B_s^{\frac{\frac{2s_0}{2s_0-1}+1}{\theta+\frac{2s_0}{2s_0-1}+1}}}\\
&\geq e^{-C(\theta)s^{\frac{\frac{4s_0}{2s_0-1}+2}{\theta+\frac{2s_0}{2s_0-1}+1}}(\ln\frac{1}{\epsilon_{s+1}})^{\frac{\frac{2s_0}{2s_0-1}+1}{\theta+\frac{2s_0}{2s_0-1}+1}}}\\
&> e^{-C(\theta)(\ln\frac{1}{\epsilon_{s+1}})^{\frac{\frac{2s_0}{2s_0-1}+2}{\theta+\frac{2s_0}{2s_0-1}+2}}} = \lambda_s, \qquad (3.15)
\end{aligned}
$$

where the last inequality is based on ϵ_0 is small enough.

Assuming $V_n + C(s_0)n^{2s_0} - [C(s_0)n^{2s_0}] \in \mathcal{C}_{\lambda_s}(\omega_n)$ for $n \in \mathbb{Z}$, i.e. $\bar{V} \in \mathcal{C}_{\lambda_s}(\omega)$, from the lower bound (3.15), the relation (3.14) remains true if we substitute $\bar{V}$ for ω. Moreover, there is analyticity on $\mathcal{C}_{\lambda_s}(\omega)$. The transformations Φ_{s+1} is obtained as the time-1 map $X^t_{F_s}|_{t=1}$ of the Hamiltonian vector field X_{F_s} with $F_s = F_{0,s}+F_{1,s}$. Taking $\rho = \rho_s$, $\delta = \delta_s$ in Lemma 2.2, we get

$$
||F_{i,s}||^+_{\rho_s+\delta_s,\mu_s-2\delta_s} \leq \frac{1}{\gamma}\cdot e^{C(\theta)\delta_s^{-\frac{6s_0-1}{(2s_0-1)\theta}}}||R_{i,s}||^+_{\rho_s,\mu_s}, \qquad (3.16)
$$

where $i = 0, 1$. By Lemma 2.1, we get

$$
||F_{i,s}||_{\rho_s+2\delta_s,\mu_s-3\delta_s} \leq \frac{C(\theta)}{\delta_s^2}||F_{i,s}||^+_{\rho_s+\delta_s,\mu_s-2\delta_s}. \qquad (3.17)
$$

Combining (3.4), (3.5), (3.16) and (3.17), we get

$$
||F_s||_{\rho_s+2\delta_s,\mu_s-3\delta_s} \leq \frac{C(\theta)}{\gamma\delta_s^2}e^{C(\theta)\delta_s^{-\frac{6s_0-1}{(2s_0-1)\theta}}}(\epsilon_s+\epsilon_s^{0.6}). \qquad (3.18)
$$

By Lemma 2.5, we get

$$
\begin{aligned}
\sup_{||q||_{r,\infty}<1}||X_{F_s}||_{r,\infty} &\leq C(r,\rho,\mu,\theta)||F_s||_{\rho_s+2\delta_s,\mu_s-3\delta_s}\\
&\leq \frac{C(r,\rho,\mu,\theta)}{\gamma\delta_s^2}e^{C(\theta)\delta_s^{-\frac{6s_0-1}{(2s_0-1)\theta}}}(\epsilon_s+\epsilon_s^{0.6})\\
&\leq \epsilon_s^{0.55}, \qquad (3.19)
\end{aligned}
$$

where noting that $0 < \epsilon_0 \ll 1$ small enough and depending on r, ρ, μ, θ only.

Since $\epsilon_s^{0.55} \ll \frac{1}{\pi^2(s+1)^2} = d_{s+1} - d_s$, we have $\Phi_{s+1} : D_{s+1} \to D_s$ with

$$
\|\Phi_{s+1} - id\|_{r,\infty} \leq \sup_{q\in D_s}\|X_{F_s}\|_{r,\infty} \leq \epsilon_s^{0.55} < \epsilon_s^{0.5}, \qquad (3.20)
$$

which is the estimate (3.7). Moreover, from (3.20) we get

$$
\sup_{q\in D_s}||DX_{F_s} - I||_{r,\infty} \leq \frac{1}{d_s}\epsilon_s^{0.55} \ll \epsilon_s^{0.5}, \qquad (3.21)
$$

and thus the estimate (3.8) follows.

Moreover, under the assumptions (3.4)–(3.6) at stage s, we get from (2.30), (2.31) and (2.32) that

$$\begin{aligned}
||R_{0,s+1}||^{+}_{\rho_{s+1},\mu_{s+1}} &\leq e^{\frac{s^{\frac{24s_0-4}{(2s_0-1)\theta}}}{\rho^{\frac{12s_0-2}{(2s_0-1)\theta}}}} \left(\epsilon_0^{(\frac{3}{2})^s} + \epsilon_0^{0.9(\frac{3}{2})^{s-1}}\right)\left(\epsilon_0^{(\frac{3}{2})^s} + \epsilon_0^{1.8(\frac{3}{2})^{s-1}}\right)\\
&= e^{\frac{s^{\frac{24s_0-4}{(2s_0-1)\theta}}}{\rho^{\frac{12s_0-2}{(2s_0-1)\theta}}}} \left(\epsilon_0^{2.2(\frac{3}{2})^s} + \epsilon_0^{1.8(\frac{3}{2})^s} + \epsilon_0^{1.6(\frac{3}{2})^s} + \epsilon_0^{2(\frac{3}{2})^s}\right)\\
&\leq 4e^{\frac{s^{\frac{24s_0-4}{(2s_0-1)\theta}}}{\rho^{\frac{12s_0-2}{(2s_0-1)\theta}}}} \epsilon_0^{1.6(\frac{3}{2})^s}\\
&< \epsilon_0^{1.5(\frac{3}{2})^s} \quad \text{for } 0<\epsilon_0\ll 1 \text{ (depending on } \rho,\mu,\theta \text{ only)}\\
&= \epsilon_{s+1},\\
||R_{1,s+1}||^{+}_{\rho_{s+1},\mu_{s+1}} &\leq e^{\frac{s^{\frac{24s_0-4}{(2s_0-1)\theta}}}{\rho^{\frac{12s_0-2}{(2s_0-1)\theta}}}} \left(\epsilon_0^{(\frac{3}{2})^s} + \epsilon_0^{1.8(\frac{3}{2})^{s-1}}\right)\\
&= e^{\frac{s^{\frac{24s_0-4}{(2s_0-1)\theta}}}{\rho^{\frac{12s_0-2}{(2s_0-1)\theta}}}} \left(\epsilon_0^{(\frac{3}{2})^s} + \epsilon_0^{1.2(\frac{3}{2})^s}\right)\\
&\leq 2e^{\frac{s^{\frac{24s_0-4}{(2s_0-1)\theta}}}{\rho^{\frac{12s_0-2}{(2s_0-1)\theta}}}} \epsilon_0^{(\frac{3}{2})^s}\\
&< \epsilon_{s+1}^{0.6} \quad \text{for } 0<\epsilon_0\ll 1 \text{ (depending on } \rho,\mu,\theta \text{ only)},
\end{aligned}$$

and

$$\begin{aligned}
||R_{2,s+1}||^{+}_{\rho_{s+1},\mu_{s+1}} &\leq ||R_{2,s}||^{+}_{\rho_s,\mu_s} + e^{\frac{s^{\frac{24s_0-4}{(2s_0-1)\theta}}}{\rho^{\frac{12s_0-2}{(2s_0-1)\theta}}}} \left(\epsilon_0^{(\frac{3}{2})^s} + \epsilon_0^{0.6(\frac{3}{2})^s}\right)\\
&\leq (1+d_s)\epsilon_0 + 2e^{\frac{s^{\frac{24s_0-4}{(2s_0-1)\theta}}}{\rho^{\frac{12s_0-2}{(2s_0-1)\theta}}}} \epsilon_0^{0.6(\frac{3}{2})^s}\\
&\leq (1+d_{s+1})\epsilon_0 \quad \text{for } 0<\epsilon_0\ll 1 \text{ (depending on } \rho,\mu,\theta \text{ only)},
\end{aligned}$$

which are just the assumptions (3.4)–(3.6) at stage $s+1$.

If $\bar{V} \in \mathcal{C}_{\frac{\eta_s}{2}}(V_s^*) \subset \mathcal{C}_{\eta_s}(V_s^*)$ and using Cauchy's estimate, for any m one has

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \left|\frac{\partial \widetilde{V}_{m,s}}{\partial \bar{V}_n}(\bar{V})\right| &\leq \frac{2}{\eta_s}||\widetilde{V}_s||_\infty\\
&< 10\eta_s^{-1} \quad (\text{since } ||\widetilde{V}_s||_\infty \leq 1). \qquad (3.22)
\end{aligned}$$

Let $\bar{V} \in \mathcal{C}_{\frac{1}{10}\lambda_s\eta_s}(V_s^*)$, then

$$\begin{aligned}
||\widetilde{V}_s(\bar{V}) - \omega||_\infty &= ||\widetilde{V}_s(\bar{V}) - \widetilde{V}_s(V_s^*)||_\infty\\
&\leq \sup_{\mathcal{C}_{\frac{1}{10}\lambda_s\eta_s}(V_s)} \left|\left|\frac{\partial \widetilde{V}_s}{\partial \bar{V}}\right|\right|_{l^\infty\to l^\infty} ||\bar{V} - V_s^*||_\infty
\end{aligned}$$

$$< 10\eta_s^{-1} \cdot \frac{1}{10}\lambda_s\eta_s \quad \text{(in view of (3.22))}$$
$$= \lambda_s,$$

that is

$$\widetilde{V}_s\left(\mathcal{C}_{\frac{1}{10}\lambda_s\eta_s}(\bar{V}_s)\right) \subseteq \mathcal{C}_{\lambda_s}(\omega).$$

Note that

$$\begin{aligned}\left|B_{a00}^{(m)}\right| &\leq ||R_{1,s+1}||^{+}_{\rho_{s+1},\mu_{s+1}} e^{2\rho_{s+1}\left(\sum_n a_n|n|^\theta+|m|^\theta-(n_1^*)^\theta\right)} \\ &< \epsilon_0^{0.6(\frac{3}{2})^s} e^{2\rho_{s+1}\left(\sum_n a_n|n|^\theta+|m|^\theta-(n_1^*)^\theta\right)}.\end{aligned} \tag{3.23}$$

Assuming further

$$I_n(0) \leq e^{-2r|n|^\theta} \tag{3.24}$$

and for any s,

$$\rho_s < \frac{1}{2}r, \tag{3.25}$$

we obtain

$$\begin{aligned}\left|\sum_{a\in\mathbb{N}^{\mathbb{Z}}} B_{a00}^{(m)}\mathcal{M}_{a00}\right| &\leq \epsilon_0^{0.6(\frac{3}{2})^s} \sum_{a\in\mathbb{N}^{\mathbb{Z}}} e^{2\rho_{s+1}\left(\sum_n a_n|n|^\theta+|m|^\theta-(n_1^*)^\theta\right)} \prod_{n\in\mathbb{Z}} I_n(0)^{a_n} \\ &\leq \epsilon_0^{0.6(\frac{3}{2})^s} \sum_{a\in\mathbb{N}^{\mathbb{Z}}} e^{2\rho_{s+1}\left(\sum_n a_n|n|^\theta\right)} \prod_{n\in\mathbb{Z}} I_n(0)^{a_n} \\ &\leq \epsilon_0^{0.6(\frac{3}{2})^s} \sum_{a\in\mathbb{N}^{\mathbb{Z}}} e^{\sum_n 2\rho_{s+1}a_n|n|^\theta-\sum_n 2ra_n|n|^\theta} \quad \text{(in view of (3.24))} \\ &\leq \epsilon_0^{0.6(\frac{3}{2})^s} \sum_{a\in\mathbb{N}^{\mathbb{Z}}} e^{-r\left(\sum_n a_n|n|^\theta\right)} \quad \text{(in view of (3.25))} \\ &\leq \epsilon_0^{0.6(\frac{3}{2})^s} \prod_{n\in\mathbb{Z}} \left(1-e^{-r|n|^\theta}\right)^{-1} \\ &\leq \left(\frac{1}{r}\right)^{C(\theta)r^{-\frac{1}{\theta}}} \epsilon_0^{0.6(\frac{3}{2})^s}.\end{aligned} \tag{3.26}$$

By (3.26), we have

$$\begin{aligned}\left|\widetilde{V}_{m,s+1}-\widetilde{V}_{m,s}\right| &< \left(\frac{1}{r}\right)^{C(\theta)r^{-\frac{1}{\theta}}} \epsilon_0^{0.6(\frac{3}{2})^s} \\ &< \epsilon_s^{0.5} \quad \text{(for } \epsilon_0 \text{ small enough)},\end{aligned} \tag{3.27}$$

which verifies (3.9). Further applying Cauchy's estimate on $\mathcal{C}_{\lambda_s\eta_s}(V_s^*)$, one gets

$$\begin{aligned}\sum_{n\in\mathbb{Z}}\left|\frac{\partial\widetilde{V}_{m,s+1}}{\partial\bar{V}_n}-\frac{\partial\widetilde{V}_{m,s}}{\partial\bar{V}_n}\right| &\leq C(\theta)\frac{||\widetilde{V}_{s+1}-\widetilde{V}_s||_\infty}{\lambda_s\eta_s} \\ &\leq C(\theta)\frac{\epsilon_s^{0.5}}{\lambda_s\eta_s}\end{aligned}$$

$$\begin{aligned}
&\leq e^{C(\theta)(\ln\frac{1}{\epsilon_{s+1}})^{\frac{\frac{2s_0}{2s_0-1}+2}{\theta+\frac{2s_0}{2s_0-1}+2}}-\frac{1}{3}\ln\frac{1}{\epsilon_{s+1}}}\left(\frac{1}{\eta_s}\right)\\
&\leq e^{-\frac{1}{4}\ln\frac{1}{\epsilon_{s+1}}}\left(\frac{1}{\eta_s}\right)\quad\text{(for ϵ_0 small enough)}\\
&=\frac{1}{\eta_s}\epsilon_0^{\frac{1}{4}(\frac{3}{2})^{s+1}}. \qquad (3.28)
\end{aligned}$$

Since

$$\eta_{s+1}=\frac{1}{20}\lambda_s\eta_s,$$

it follows that

$$\begin{aligned}
\eta_{s+1}&\geq \eta_s e^{-C(\theta)(\ln\frac{1}{\epsilon_0})^{\frac{\frac{2s_0}{2s_0-1}+2}{\theta+\frac{2s_0}{2s_0-1}+2}}(\frac{3}{2})^{\frac{\frac{2s_0}{2s_0-1}+2}{\theta+\frac{2s_0}{2s_0-1}+2}(s+1)}}\\
&\geq \eta_s e^{-C(\theta)\ln\frac{1}{\epsilon_0}\cdot(\frac{3}{2})^{\frac{\frac{2s_0}{2s_0-1}+3}{\theta+\frac{2s_0}{2s_0-1}+3}s}}\quad\text{(for ϵ_0 small enough)}\\
&=\eta_s\epsilon_0^{C(\theta)(\frac{3}{2})^{\frac{\kappa s}{\theta+\kappa}}},\quad\left(\text{letting }\kappa=\tfrac{2s_0}{2s_0-1}+3\right) \qquad (3.29)
\end{aligned}$$

and hence by iterating (3.29) implies

$$\begin{aligned}
\eta_s&\geq \eta_0\epsilon_0^{C(\theta)\sum_{i=0}^{s-1}(\frac{3}{2})^{\frac{\kappa i}{\theta+\kappa}}}\\
&=\eta_0\epsilon_0^{C(\theta)\frac{(\frac{3}{2})^{\frac{\kappa s}{\theta+\kappa}}-1}{(\frac{3}{2})^{\frac{\kappa}{\theta+\kappa}}-1}}\\
&>\epsilon_0^{C(\theta)(\frac{3}{2})^{\frac{\kappa s}{\theta+\kappa}}}\\
&\geq \epsilon_0^{\frac{1}{100}(\frac{3}{2})^s}\quad\text{(for ϵ_0 small enough).} \qquad (3.30)
\end{aligned}$$

On $\mathcal{C}_{\frac{1}{10}\lambda_s\eta_s}(V_s^*)$ and for any m, we deduce from (3.28), (3.30) and the assumption (3.3) that

$$\begin{aligned}
\sum_{n\in\mathbb{Z}}\left|\frac{\partial\widetilde{V}_{m,s+1}}{\partial\bar{V}_n}-\delta_{mn}\right|&\leq\sum_{n\in\mathbb{Z}}\left|\frac{\partial\widetilde{V}_{m,s+1}}{\partial\bar{V}_n}-\frac{\partial\widetilde{V}_{m,s}}{\partial\bar{V}_n}\right|+\sum_{n\in\mathbb{Z}}\left|\frac{\partial\widetilde{V}_{m,s}}{\partial\bar{V}_n}-\delta_{mn}\right|\\
&\leq\epsilon_0^{(\frac{3}{8}-\frac{1}{100})(\frac{3}{2})^s}+d_s\epsilon_0^{\frac{1}{10}}\\
&<d_{s+1}\epsilon_0^{\frac{1}{10}},
\end{aligned}$$

and consequently

$$\left\|\frac{\partial\widetilde{V}_{s+1}}{\partial\bar{V}}-I\right\|_{l^\infty\to l^\infty}<d_{s+1}\epsilon_0^{\frac{1}{10}}, \qquad (3.31)$$

which verifies (3.3) for $s+1$.

Finally, we will freeze ω by invoking an inverse function theorem. From (3.31) and the standard inverse function theorem, we can see that the functional equation

$$\widetilde{V}_{s+1}(V_{s+1}^*)=\omega, V_{s+1}^*\in\mathcal{C}_{\frac{1}{10}\lambda_s\eta_s}(V_s^*), \qquad (3.32)$$

has a solution V^*_{s+1}, which verifies (3.2) for $s+1$. Rewriting (3.32) as

$$V^*_{s+1} - V^*_s = (I - \widetilde{V}_{s+1})(V^*_{s+1}) - (I - \widetilde{V}_{s+1})(V^*_s) + (\widetilde{V}_s - \widetilde{V}_{s+1})(V^*_s), \tag{3.33}$$

and by using (3.27), (3.31) implies

$$||V^*_{s+1} - V^*_s||_\infty \le (1 + d_{s+1})\epsilon_0^{\frac{1}{10}} ||V^*_{s+1} - V^*_s||_\infty + \epsilon_s^{0.5} < 2\epsilon_s^{0.5} \ll \lambda_s \eta_s, \tag{3.34}$$

which verifies (3.10) and completes the proof of the iterative lemma. □

We are now in a position to prove the convergence. To apply iterative lemma with $s = 0$, set

$$V_{n,0} = \omega_n - C(s_0)n^{2s_0} + [C(s_0)n^{2s_0}],\ \widetilde{V}_0 = id,$$
$$\eta_0 = 1 - \sup_{n\in\mathbb{Z}} |\omega_n|,\ \mu_0 = r + \rho_0,\ \epsilon_0 = C\epsilon,$$

and consequently (3.2)–(3.6) with $s = 0$ are satisfied. Hence, the iterative lemma applies, and we obtain a decreasing sequence of domains $D_s \times \mathcal{C}_{\eta_s}(V^*_s)$ and a sequence of transformations

$$\Phi^s = \Phi_1 \circ \cdots \circ \Phi_s : \quad D_s \times \mathcal{C}_{\eta_s}(V^*_s) \to D_0 \times \mathcal{C}_{\eta_0}(V^*_0),$$

such that $H \circ \Phi^s = N_s + P_s$ for $s \ge 1$. Moreover, the estimates (3.7)–(3.10) hold. Thus we can show V^*_s converge to a limit V_* with the estimate

$$||V_* - \omega||_\infty \le \sum_{s=0}^{\infty} 2\epsilon_s^{0.5} < \epsilon_0^{0.4},$$

and Φ^s converge uniformly on $D_* \times \{V_*\}$, where $D_* = \{(q_n)_{n\in\mathbb{Z}} : \frac{2}{3} \le |q_n| e^{r|n|^\theta} \le \frac{5}{6}\}$, to $\Phi : D_* \times \{V_*\} \to D_0$ with the estimates

$$||\Phi - id||_{r,\infty} \le \epsilon_s^{0.4},$$
$$||D\Phi - I||_{(r,\infty)\to(r,\infty)} \le \epsilon_s^{0.4}.$$

Hence

$$H_* = H \circ \Phi = N_* + R_{2,*}, \tag{3.35}$$

where

$$N_* = \sum_{n\in\mathbb{Z}} ([C(s_0)n^{2s_0}] + \omega_n)|q_n|^2 \tag{3.36}$$

and

$$||R_{2,*}||^+_{10\rho,\mu-18\rho} \le \frac{7}{6}\epsilon_0. \tag{3.37}$$

By (2.35), the Hamiltonian vector field $X_{R_{2,*}}$ is a bounded map from $\mathfrak{H}_{r,\infty}$ into $\mathfrak{H}_{r,\infty}$. Taking

$$I_n(0) = \frac{3}{4} e^{-2r|n|^\theta}, \tag{3.38}$$

we get an invariant torus $\mathcal{T}$ with frequency $([C(s_0)n^{2s_0}] + \omega_n)_{n\in\mathbb{Z}}$ for X_{H_*}. Finally, by $X_H \circ \Phi = D\Phi \cdot X_{H_*}$, $\Phi(\mathcal{T})$ is the desired invariant torus for the FNLS (1.1). Moreover, we deduce the torus $\Phi(\mathcal{T})$ is linearly stable from the fact that (3.35) is a normal form of order 2 around the invariant torus.

Appendix

3.1. Technical Lemmas

Lemma 3.2. *Denote $(n_i^*)_{i\ge 1}$ the decreasing rearrangement of*

$$\{|n|:\ \text{where } n \text{ is repeated } 2a_n+k_n+k_n' \text{ times}\}.$$

Then for any $\theta\in(0,1)$, one has

$$\sum_{n\in\mathbb{Z}}(2a_n+k_n+k_n')|n|^\theta-2(n_1^*)^\theta+m^*(k,k')^\theta\ge(2-2^\theta)\left(\sum_{i\ge 3}(n_i^*)^\theta\right). \tag{3.39}$$

Proof. See Lemma 6.1 in [10] for more details. □

Lemma 3.3. *Let $\theta\in(0,1)$ and $k_n,k_n'\in\mathbb{N},|\widetilde{V}_n|\le 2$ for $\forall\ n\in\mathbb{Z}$. Assume further*

$$\left|\sum_{n\in\mathbb{Z}}(k_n-k_n')([C(s_0)n^{2s_0}]+\widetilde{V}_n)\right|\le 1. \tag{3.40}$$

Then one has

$$\sum_{n\in\mathbb{Z}}|k_n-k_n'||n|^{\frac{(2s_0-1)\theta}{2s_0}}\le 7\cdot 12^{\frac{\theta}{2s_0}}\left(\sum_{i\ge 3}|n_i|^\theta+m^*(k,k')^\theta\right), \tag{3.41}$$

where $(n_i)_{i\ge1},|n_1|\ge|n_2|\ge|n_3|\ge\cdots$, denote the system {n: n is repeated k_n+k_n' times}.

Proof. From the definition of $(n_i)_{i\ge1}$, there exists $(\mu_i)_{i\ge1}$ with $\mu_i\in\{-1,1\}$ such that

$$m(k,k')=\sum_{i\ge1}\mu_in_i, \tag{3.42}$$

and

$$\sum_{n\in\mathbb{Z}}(k_n-k_n')[C(s_0)n^{2s_0}]=\sum_{i\ge1}\mu_i[C(s_0)n_i^{2s_0}]. \tag{3.43}$$

In view of (3.40), (3.43) and $|\widetilde{V}_n|\le 2$, one has

$$\left|\sum_{i\ge1}\mu_i[C(s_0)n_i^{2s_0}]\right|\le\left|\sum_{n\in\mathbb{Z}}(k_n-k_n')\widetilde{V}_n\right|+1\le 2\sum_{n\in\mathbb{Z}}(k_n+k_n')+1,$$

which implies

$$\begin{aligned}\left|[C(s_0)n_1^{2s_0}]+\left(\frac{\mu_2}{\mu_1}\right)[C(s_0)n_2^{2s_0}]\right|&\le 2\sum_{i\ge1}1+\sum_{i\ge3}[C(s_0)n_i^{2s_0}]+1\\&\le\sum_{i\ge3}(2+[C(s_0)n_i^{2s_0}])+5.\end{aligned} \tag{3.44}$$

On the other hand, by (3.42), we obtain

$$\left|n_1+\left(\frac{\mu_2}{\mu_1}\right)n_2\right|\le\sum_{i\ge3}|n_i|+m^*(k,k'). \tag{3.45}$$

To prove the inequality (3.41), we will distinguish two cases:

Case. 1. $\frac{\mu_2}{\mu_1} = -1$.

Subcase. 1.1. $n_1 = n_2$.

Then it is easy to show that

$$\sum_{n\in\mathbb{Z}} |k_n - k_n'||n|^{\frac{(2s_0-1)\theta}{2s_0}} \le \sum_{i\ge 3} |n_i|^{\frac{(2s_0-1)\theta}{2s_0}} \le 7\cdot 12^{\frac{\theta}{2s_0}} \left(\sum_{i\ge 3} |n_i|^\theta + m^*(k,k')^\theta\right).$$

Subcase. 1.2. $n_1 = -n_2$.

See the proof of **Case. 2.** below.

Subcase. 1.3. $n_1 \neq -n_2$.

Consider the function

$$f(x) = x^\delta,\ x \in [|n_2|, |n_1|] \text{ and } \delta \in (1,\infty),$$

and by the Lagrange's mean value theorem, one has

$$f(|n_1|) - f(|n_2|) = f'(\xi)(|n_1| - |n_2|),\ \xi \in (|n_2|, |n_1|).$$

That is,

$$|n_1|^\delta - |n_2|^\delta = \delta\xi^{\delta-1}(|n_1| - |n_2|).$$

Since

$$\delta\xi^{\delta-1} = \frac{|n_1^\delta| - |n_2|^\delta}{|n_1| - |n_2|} \ge \delta|n_2|^{\delta-1},$$

we then get

$$|n_2| \le \frac{|n_1|^\delta - |n_2|^\delta}{\delta(|n_1| - |n_2|)} \le \left(\frac{|n_1^\delta| - |n_2|^\delta}{\delta}\right)^{\frac{1}{\delta-1}}.$$

Hence, letting $\delta = 2s_0$, we have

$$\begin{aligned}
|n_1 - n_2|^{2s_0-1} &\le \left(\sum_{i\ge 3} |n_i| + m^*(k,k')\right)^{2s_0-1} \qquad \text{(in view of (3.45))},\\
|n_2|^{2s_0-1} &\le \frac{n_1^{2s_0} - n_2^{2s_0}}{2s_0}\\
&\le \frac{[C(s_0)n_1^{2s_0}] - [C(s_0)n_2^{2s_0}] + 2}{2s_0C(s_0)}\\
&\le \frac{\sum_{i\ge 3}(2 + [C(s_0)n_i^{2s_0}]) + 7}{2s_0C(s_0)} \qquad \text{(in view of (3.44))}\\
&\le \frac{5}{s_0}\cdot\left(\sum_{i\ge 3} |n_i|^{2s_0} + m^*(k,k')^{2s_0}\right).
\end{aligned}$$

Moreover, one has

$$\begin{aligned}|n_1|^{2s_0-1} &= |n_1-n_2+n_2|^{2s_0-1}\\ &\le |n_1-n_2|^{2s_0-1}+|n_2|^{2s_0-1}\\ &\le \frac{6}{s_0}\cdot\left(\sum_{i\ge3}|n_i|^{2s_0}+m^*(k,k')^{2s_0}\right)\\ &\le 12\left(\sum_{i\ge3}|n_i|^{2s_0}+m^*(k,k')^{2s_0}\right).\end{aligned}$$

Therefore

$$\max\{|n_1|,|n_2|\}\le 3\cdot 12^{\frac{1}{2s_0-1}}\left(\sum_{i\ge3}|n_i|^{2s_0}+m^*(k,k')^{2_0}\right)^{\frac{1}{2s_0-1}}.$$

For $j=1,2$, one has

$$\begin{aligned}|n_j|^{\frac{(2s_0-1)\theta}{2s_0}} &\le 3\cdot 12^{\frac{\theta}{2s_0}}\left(\sum_{i\ge3}|n_i|^{2s_0}+m^*(k,k')^{2s_0}\right)^{\frac{\theta}{2s_0}}\\ &\le 3\cdot 12^{\frac{\theta}{2s_0}}\left(\sum_{i\ge3}|n_i|^{\theta}+m^*(k,k')^{\theta}\right),\end{aligned}$$

where the last inequality is based on the fact that the function $|x|^{\frac{\theta}{2s_0}}$ is a concave function for $0<\theta<1$ and $s_0>\frac{1}{2}$. Therefore,

$$|n_1|^{\frac{(2s_0-1)\theta}{2s_0}}+|n_2|^{\frac{(2s_0-1)\theta}{2s_0}}\le 6\cdot 12^{\frac{\theta}{2s_0}}\left(\sum_{i\ge3}|n_i|^{\theta}+m^*(k,k')^{\theta}\right). \tag{3.46}$$

Now one has

$$\begin{aligned}\sum_{n\in\mathbb{Z}}|k_n-k_n'||n|^{\frac{(2s_0-1)\theta}{2s_0}} &\le \sum_{n\in\mathbb{Z}}(k_n+k_n')|n|^{\frac{(2s_0-1)\theta}{2s_0}} = \sum_{i\ge1}|n_i|^{\frac{(2s_0-1)\theta}{2s_0}}\\ &\le \left(|n_1|^{\frac{(2s_0-1)\theta}{2s_0}}+|n_2|^{\frac{(2s_0-1)\theta}{2s_0}}\right)+\sum_{i\ge3}|n_i|^{\theta}\\ &\le (6\cdot 12^{\frac{\theta}{2s_0}}+1)\left(\sum_{i\ge3}|n_i|^{\theta}+m^*(k,k')^{\theta}\right)\quad\text{(in view of (3.46))}\\ &\le 7\cdot 12^{\frac{\theta}{2s_0}}\left(\sum_{i\ge3}|n_i|^{\theta}+m^*(k,k')^{\theta}\right).\end{aligned} \tag{3.47}$$

Case. 2. $\frac{\mu_2}{\mu_1}=1$.

In view of (3.44), one has

$$[C(s_0)n_1^{2s_0}] + [C(s_0)n_2^{2s_0}] \le 7 \sum_{i\ge 3}[C(s_0)n_i^{2s_0}],$$

which implies

$$|n_j|^{\frac{(2s_0-1)\theta}{2s_0}} \le 8^{\frac{(2s_0-1)\theta}{(2s_0)^2}} \left(\sum_{i\ge 3}|n_i|^{2s_0}\right)^{\frac{(2s_0-1)\theta}{(2s_0)^2}} \le 8^{\frac{\theta}{2s_0}} \sum_{i\ge 3}|n_i|^{\theta} \quad (j=1,2).$$

Therefore,

$$|n_1|^{\frac{(2s_0-1)\theta}{2s_0}} + |n_2|^{\frac{(2s_0-1)\theta}{2s_0}} \le 2\cdot 8^{\frac{\theta}{2s_0}} \sum_{i\ge 3}|n_i|^{\theta}. \tag{3.48}$$

Following the proof of (3.47), we have

$$\sum_{n\in\mathbb{Z}} |k_n - k'_n||n|^{\frac{(2s_0-1)\theta}{2s_0}} \le 7\cdot 12^{\frac{\theta}{2s_0}} \left(\sum_{i\ge 3}|n_i|^{\theta} + m^*(k,k')^{\theta}\right).$$

□

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