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INVERSES AND EIGENPAIRS OF TRIDIAGONAL TOEPLITZ MATRIX WITH OPPOSITE-BORDERED ROWS*

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Abstract In this paper, tridiagonal Toeplitz matrix (type I, type II) with opposite-bordered rows are introduced. Main attention is paid to calculate the determinants, the inverses and the eigenpairs of these matrices. Specifically, the determinants of an $n \times n$ tridiagonal Toeplitz matrix with opposite-bordered rows can be explicitly expressed by using the $(n-1)$th Fibonacci number, the inversion of the tridiagonal Toeplitz matrix with opposite-bordered rows can also be explicitly expressed by using the Fibonacci numbers and unknown entries from the new matrix. Besides, we give the expression of eigenvalues and eigenvectors of the tridiagonal Toeplitz matrix with opposite-bordered rows. In addition, some algorithms are presented based on these theoretical results. Numerical results show that the new algorithms have much better computing efficiency than some existing algorithms studied recently.

Keywords Tridiagonal Toeplitz matrix, opposite-bordered, Fibonacci number, determinant, inverse, eigenpairs.

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1. Introduction

We start by introducing the main research object of this paper. Two new special matrices are introduced at beginning. Let $\mathcal{A} \in \mathbb{C}^{n \times n}$ be a square tridiagonal

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Toeplitz matrix with opposite-bordered rows type I,

$$\mathcal{A}=\begin{pmatrix} p & b_{n-2} & b_{n-3} & b_{n-4} & \cdots & b_3 & b_2 & b_1 & q \\ 0 & -d & -d & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & d & -d & -d & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & & \ddots & d & -d & -d & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & d & -d & 0 \\ s & a_{n-2} & a_{n-3} & a_{n-4} & \cdots & a_3 & a_2 & a_1 & t \end{pmatrix}, \tag{1.1}$$

where d ($d \neq 0$), p, q, s, t and a_i, b_i ($i = 1, 2, \ldots, n-2$) are arbitrary complex numbers.

An $n \times n$ tridiagonal Toeplitz matrix with opposite-bordered rows $\mathcal{B}$ (type II) over a field $\mathbb{C}$ is

$$\mathcal{B} = \hat{\mathbf{I}}_n \mathcal{A} \hat{\mathbf{I}}_n, \tag{1.2}$$

where $\hat{\mathbf{I}}_n$ is a "reverse unit matrix" of order n, having ones along the secondary diagonal and zeros elsewhere.

The special banded matrices such as Toeplitz matrices [21–24, 35, 36], especially tridiagonal matrices, etc. have been widely used in various application areas ranging from engineering to economics [3, 12] as well as in the computation of special functions, number theory [26] and partial differential equations. The one-dimensional linear hyperbolic equation

$$\frac{\partial u(x,t)}{\partial t} + v\frac{\partial u(x,t)}{\partial x} = g$$

considered by Holmgren and Otto [13] as an example to study certain matrices occurred in discretized partial differential equations, where $0 < x \leq 1,\ t > 0,\ u(0,t) = f(-at),\ u(x,0) = f(x), g = (v-a)f'$. Here v and a are positive constants and f is a scalar function with derivative f'. Let k and h denote the time step and spatial step, respectively. The linear hyperbolic equations is discretized based on trapezoidal rule in time and center difference in space, respectively, whose coefficient matrix is a tridiagonal matrix with perturbed last row [1]

$$\mathfrak{T}=\begin{pmatrix} 4 & \alpha & 0 & \cdots & \cdots & 0 \\ -\alpha & \ddots & \ddots & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & -\alpha & 4 & \alpha \\ 0 & \cdots & \cdots & 0 & -2\alpha & 4+2\alpha \end{pmatrix}_{n\times n},$$

where $\alpha = vk/h$. In addition, various features of tridiagonal matrices are employed to solve the systems of linear equations that arise from these applications [11, 25].

In [5,7–10,30,31,33], the authors had studied LU decompositions, determinant, inverse and eigen properties of various tridiagonal or periodic tridiagonal matrices. The inverse formula for periodic tridiagonal Toeplitz matrices was proposed by Shehawey in [6] who generalized the method proposed by Huang and McColl in [14]. Based on the block diagonalization technique, Jia et al. put forward some algorithms [18–20] for the k-tridiagonal matrix. Recently, Jia proposed a new breakdown-free recursive algorithm for computing the determinants of periodic tridiagonal matrices via a three-term recurrence in [16]. In addition, an explicit formula for the determinant of the periodic tridiagonal matrix with Toeplitz structure is also discussed. More in [17], Jia and Li introduced the solution of opposite-bordered tridiagonal (OBT) systems of linear equations. They present two efficient algorithms which used reliable tridiagonal linear solver and column operation, respectively. And the computational accuracy and efficiency of the proposed algorithms are illustrated in the paper. Also, by using the Doolittle LU factorization, El-Mikkawy and Atlan proposed a symbolic algorithm for computing the inverse of the k-tridiagonal matrix in [4]. In [29], Tim and Emrah used backward continued fractions to obtain the LU factorization of periodic tridiagonal matrix and then derived the explicit formula for its inverse. Furthermore, on the basis of symbolic calculus for difference equations, many authors had done a lot of research on the eigenvalues and eigenvectors of tridiagonal matrices or periodic tridiagonal matrices, see [2,7,32]. In addition, some scholars were attracted by the fact that one could view periodic tridiagonal Toeplitz matrices as a special case of periodic tridiagonal matrices.

Unlike tridiagonal or periodic tridiagonal matrices which have received much attention and obtained very mature theoretical results, a few researchers know about the tridiagonal Toeplitz matrix with opposite-bordered rows. This motivates us to attempt to study some problems, including the determinant, the inverse matrix and the eigenpairs of the tridiagonal Toeplitz matrix with opposite-bordered rows. In this paper, combining the Schur determinant formula, Fibonacci numbers, the determinant of tridiagonal Toeplitz matrix with opposite-bordered rows can be obtained easily. Using an additional circulant matrix, inverse of block matrix and Fibonacci numbers, the explicit expression of inverse matrix can be obtained. Based on an additional circulant matrix and some properties of the similar matrices, eigenpairs of tridiagonal Toeplitz matrix with opposite-bordered rows can be obtained. In addition, some efficient computational algorithms for these theoretical results are given.

On the other hand, one kind of famous number that play an important role in the new theoretical results is introduced. The Fibonacci sequence $\{F_n\}$ satisfies the following recurrence relation [28]

$$F_0 = 0,\ F_1 = 1,\ F_n = F_{n-1} + F_{n-2}\ (n \geq 2),\ F_{-n} = (-1)^{n+1} F_n. \tag{1.3}$$

Notations. For convenience, let D_n be the n-order determinant of A, $\mathbb{C}$ represent the complex set and $\mathbb{C}^{n\times n}$ denote $n \times n$ complex matrices set. Let $\det A$ denote the determinant of any matrix A, $\operatorname{circ}(x_1, x_2, \ldots, x_n)$ be a circulant matrix generated by vector $(x_1, x_2, \ldots, x_n)$ as the first row entries, diag represent a diagonal matrix composed by diagonal entries. $(y_1, y_2, \ldots, y_n)^T$ is the transpose of vector $(y_1, y_2, \ldots, y_n)$.

The paper is organized as follows. Section 2 is devoted to computing the determinants of the tridiagonal Toeplitz matrix with opposite-bordered rows. In addition, the corresponding algorithm of the main theoretical result is given. In Section 3, the inverse matrices of the tridiagonal Toeplitz matrix with opposite-bordered rows are presented. And also the algorithm is presented for main theorem. In Section 4, the eigenvalues and eigenvectors of the tridiagonal Toeplitz matrix with opposite-bordered rows are introduced. Two numerical experiments are given to show the performance of the new algorithms in the following section. Finally, we end this paper with some conclusions at Section 6.

2. The determinants

In this section, we introduce the determinants of the tridiagonal Toeplitz matrices with opposite-bordered rows. The corresponding algorithm for main theoretical result is given.

Theorem 2.1. *Let an* $n \times n$ $(n \geq 3)$ *matrix* $\mathcal{A}$ *be defined in* (1.1). *Then the determinant of* $\mathcal{A}$ *is*

$$\det \mathcal{A} = (-d)^{n-2}(pt - qs)F_{n-1}, \tag{2.1}$$

where F_{n-1} *is the* $(n-1)$*th Fibonacci number.*

Proof. For $n \geq 3$, consider an additional circulant matrix

$$C = \mathrm{circ}(0, 1, 0, \cdots, 0). \tag{2.2}$$

Clearly,

$$\det C = (-1)^{n-1}, \tag{2.3}$$

which means that C is nonsingular. Furthermore,

$$\begin{cases} C^{-1} = C^T, \\ \det C^{-1} = \det C^T = \det C = (-1)^{n-1}. \end{cases} \tag{2.4}$$

Multiplying $\mathcal{A}$ by C^{-1} and C from the left and right, respectively and dividing $C^{-1}\mathcal{A}C$ into the following two by two blocks

$$C^{-1}\mathcal{A}C = \left(\begin{array}{cc|cccccccc} t & s & a_{n-2} & a_{n-3} & a_{n-4} & \cdots & a_3 & a_2 & a_1 \\ q & p & b_{n-2} & b_{n-3} & b_{n-4} & \cdots & b_3 & b_2 & b_1 \\ \hline 0 & 0 & -d & -d & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & d & -d & -d & \ddots & & & \vdots \\ 0 & 0 & 0 & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & & & \ddots & d & -d & -d \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & d & -d \end{array}\right)_{n \times n} \tag{2.5}$$

$$= \left(\begin{array}{c|c} \mathcal{A}_1 & \mathcal{A}_2 \\ \hline \mathcal{A}_3 & \mathcal{A}_4 \end{array}\right).$$

Then by taking the determinants on both sides of (2.5) and using the theorem in [34, p.10], we obtain

$$\begin{aligned}\det C^{-1}\det\mathcal{A}\det C &= \det(C^{-1}\mathcal{A}C)\\ &= \det\mathcal{A}_1\det\mathcal{A}_4.\end{aligned}\tag{2.6}$$

Now we turn to study the determinants of $\mathcal{A}_1$, $\mathcal{A}_4$. In (2.5), $\mathcal{A}_1 = \begin{pmatrix} t & s \\ q & p \end{pmatrix}$, obviously,

$$\det\mathcal{A}_1 = pt - qs.\tag{2.7}$$

Following, we evaluate the determinant of $\mathcal{A}_4$. From (2.5), we know that $\mathcal{A}_4$ is an $(n-2)\times(n-2)$ tridiagonal Toeplitz matrix and $\mathcal{A}_4$ is invertible. If $n = 3$, obviously, $\det\mathcal{A}_4 = -d$, then by using (2.4), (2.6) and (2.7), we have $\det\mathcal{A} = d(qs-pt)$ which means that (2.1) is satisfied.

For $n \geq 4$, denoting $\det\mathcal{A}_4$ as D_{n-2}, then based on the Laplace expansion, expanding $\mathcal{A}_4$ along the first column, we have

$$D_{n-2} = -dD_{n-3} + d^2D_{n-4}\ (n \geq 4).\tag{2.8}$$

Combining the recurrence relations of Fibonacci numbers and (2.8), we can calculate and simplify the determinant of $\mathcal{A}_4$ (i.e. D_{n-2}) as follows

$$D_{n-2} = (-1)^i d^i F_{i+1} D_{n-i-2} + (-1)^{i-1} d^{i+1} F_i D_{n-i-3}\ (1 \leq i \leq n-4),\tag{2.9}$$

where F_i and F_{i+1} are the ith and the $(i+1)$th Fibonacci numbers, respectively.

Let $i = n-4$. Then (2.9) can be written as

$$D_{n-2} = (-1)^{n-4} d^{n-4} F_{n-3} D_2 + (-1)^{n-5} d^{n-3} F_{n-4} D_1,\tag{2.10}$$

and it is easy to calculate that

$$D_2 = 2d^2,\quad D_1 = -d.\tag{2.11}$$

Substituting (2.11) into (2.10) and according to the recurrence relations of Fibonacci numbers, we get

$$\begin{aligned}\det\mathcal{A}_4 = D_{n-2} &= (-1)^{n-4} d^{n-2}(2F_{n-3} + F_{n-4})\\ &= (-d)^{n-2} F_{n-1},\end{aligned}\tag{2.12}$$

where F_{n-1} is the $(n-1)$th Fibonacci number. Finally, computing (2.6) by (2.4), (2.7) and (2.12), we obtain the formula for the determinant of matrix $\mathcal{A}$ as (2.1), i.e. the result holds. □

Remark 2.1. The result of Theorem 2.1 is very interesting, the determinant of $\mathcal{A}$ is just determined by p, q, s, t, d and the $(n-1)$th Fibonacci number, but has nothing to do with the unknown entries a_i and b_i $(1 \leq i \leq n-2)$.

Remark 2.2. From (2.1), we know that $F_{n-1} = \frac{\det\mathcal{A}}{(-d)^{n-2}(pt-qs)}$ when $pt - qs \neq 0$, which implies that the Fibonacci number has infinite expression by the determinant of $\mathcal{A}$.

The process of how to compute the determinant of the tridiagonal Toeplitz matrix with opposite-bordered rows is introduced in the following algorithm.

Algorithm 1. Compute the determinant of the matrix $\mathcal{A}$.
Step 1. Input the components a_i, b_i $(1 \leq i \leq n-2)$, p, q, s, t, and d, order n, and generate the $(n-1)$th Fibonacci number by (1.3).
Step 2. Compute the determinant of $\mathcal{A}$ by (2.1).
Step 3. Output the determinant of $\mathcal{A}$: $\det \mathcal{A}$.

From the above algorithm, we know that Algorithm 1 requires $3n+2$ operations for computing the determinant of $\mathcal{A}$. Based on the analysis in [27, p.226-227], the complexity of our method can be reduced to $O(\log n)$. The comparison of the computational cost between LU decomposition and Algorithm 1 is given in Table 1. Comparing the results in Table 1, we can see that the computational cost of our algorithm is less than that of the LU decomposition algorithm.

Table 1. The complexity comparison of computing determinant for different algorithms.

Algorithm	Complexity
LU decomposition	$n^2+10n-12$
Algorithm 1	$3n+2$

Theorem 2.2. *Let an $n\times n$ $(n \geq 3)$ matrix $\mathcal{B}$ defined in* (1.2). *Then the determinant of $\mathcal{B}$ is*

$$\det \mathcal{B} = (-d)^{n-2}(pt-qs)F_{n-1}, \tag{2.13}$$

where F_{n-1} is the $(n-1)$th Fibonacci number.

Proof. For $n \geq 3$, from (1.2), it follows that

$$\det \mathcal{B} = \det(\hat{\mathbf{I}}_n \mathcal{A} \hat{\mathbf{I}}_n) = \det \hat{\mathbf{I}}_n \det \mathcal{A} \det \hat{\mathbf{I}}_n.$$

Then we can obtain (2.13) by using Theorem 2.1 and $\det \hat{\mathbf{I}}_n = (-1)^{\frac{n(n-1)}{2}}$. □

Remark 2.3. The Theorem 2.2 is interesting that only p, q, s, t, d and the $(n-1)$th Fibonacci number are enough to explicitly express the determinant of $\mathcal{B}$.

Remark 2.4. From (2.13), we know that $F_{n-1} = \frac{\det \mathcal{B}}{(-d)^{n-2}(pt-qs)}$ when $pt-qs \neq 0$, which means that the Fibonacci number has infinite expression by the determinant of $\mathcal{B}$.

The corresponding algorithm for computing the determinant of $\mathcal{B}$ can be easily obtained according to the Theorem 2.2 and Algorithm 1. And the operation for computing the determinant of $\mathcal{B}$ is the same with the Algorithm 1, i.e. $3n+2$.

3. The inverse matrices

We now turn our attention to compute the inverse matrices of the tridiagonal Toeplitz matrix with opposite-bordered rows $\mathcal{A}$ and $\mathcal{B}$.

Theorem 3.1. *Let an $n \times n$ $(n \geq 3)$ matrix $\mathcal{A}$ be defined in (1.1). Assuming that $pt - qs \neq 0$, then we have*

$$\mathcal{A}^{-1} = \left(\begin{array}{c|cccccc|c} \frac{t}{pt-qs} & h_1 & h_2 & \cdots & \cdots & \cdots & h_{n-2} & \frac{-q}{pt-qs} \\ \hline 0 & a_{1,1} & a_{1,2} & \cdots & \cdots & \cdots & a_{1,n-2} & 0 \\ 0 & a_{2,1} & a_{2,2} & \cdots & \cdots & \cdots & a_{2,n-2} & 0 \\ \vdots & \vdots & \vdots & & & & \vdots & \vdots \\ \vdots & \vdots & \vdots & & & & \vdots & \vdots \\ 0 & a_{n-3,1} & a_{n-3,2} & \cdots & \cdots & \cdots & a_{n-3,n-2} & 0 \\ 0 & a_{n-2,1} & a_{n-2,2} & \cdots & \cdots & \cdots & a_{n-2,n-2} & 0 \\ \hline \frac{-s}{pt-qs} & k_1 & k_2 & \cdots & \cdots & \cdots & k_{n-2} & \frac{p}{pt-qs} \end{array}\right)_{n\times n}, \tag{3.1}$$

where

$$a_{i,j} = \begin{cases} -\frac{F_j F_{n-i-1}}{dF_{n-1}}, & 1 \leq j \leq i \leq n-2, \\ -\frac{(-1)^{j-i} F_i F_{n-j-1}}{dF_{n-1}}, & 1 \leq i < j \leq n-2, \end{cases} \tag{3.2}$$

F_k $(k = 1, 2, \ldots, n-1)$ *are Fibonacci numbers. And*

$$\begin{cases} h_j = \frac{\sum_{i=1}^{n-2}[(sb_{n-i-1} - pa_{n-i-1})a_{i,j}]}{pt-qs}, & 1 \leq j \leq n-2, \\ k_j = \frac{\sum_{i=1}^{n-2}[(qa_{n-i-1} - tb_{n-i-1})a_{i,j}]}{pt-qs}, & 1 \leq j \leq n-2. \end{cases} \tag{3.3}$$

Proof. Firstly, the following equation holds

$$\mathcal{A}^{-1} = CC^{-1}\mathcal{A}^{-1}CC^{-1} = C(C^{-1}\mathcal{A}C)^{-1}C^{-1}, \tag{3.4}$$

where C and C^{-1} are the same as that given in (2.2) and (2.4). From (3.4), we know that calculating the inverse of $\mathcal{A}$ means calculating the inverse of $C^{-1}\mathcal{A}C$.

For $n \geq 3$, from (2.5) and the theorem in [34, p.13], we have

$$(C^{-1}\mathcal{A}C)^{-1} = \begin{pmatrix} \mathcal{A}_1^{-1} & -\mathcal{A}_1^{-1}\mathcal{A}_2\mathcal{A}_4^{-1} \\ 0 & \mathcal{A}_4^{-1} \end{pmatrix}. \tag{3.5}$$

Now, we turn our attention to calculate $\mathcal{A}_1^{-1}$, $\mathcal{A}_4^{-1}$, $-\mathcal{A}_1^{-1}\mathcal{A}_2\mathcal{A}_4^{-1}$. It is obvious that

$$\mathcal{A}_1^{-1} = \begin{pmatrix} \frac{p}{pt-qs} & \frac{-s}{pt-qs} \\ \frac{-q}{pt-qs} & \frac{t}{pt-qs} \end{pmatrix}. \tag{3.6}$$

Based on the equation (2.1)-(2.6) in [5, p.713] and Fibonacci numbers, we can calculate and simplify $\mathcal{A}_4^{-1}$ as follows

$$\mathcal{A}_4^{-1} = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & \cdots & a_{1,n-2} \\ a_{2,1} & a_{2,2} & \cdots & \cdots & a_{2,n-2} \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ a_{n-2,1} & a_{n-2,2} & \cdots & \cdots & a_{n-2,n-2} \end{pmatrix}_{(n-2)\times(n-2)}, \tag{3.7}$$

where $a_{i,j}$ $(1 \le i,j \le n-2)$ are the same as that given in (3.2). Then using (2.5), (3.6) and (3.7), we obtain

$$-\mathcal{A}_1^{-1}\mathcal{A}_2\mathcal{A}_4^{-1} = \begin{pmatrix} h_1 & h_2 & h_3 & \cdots & h_{n-3} & h_{n-2} \\ k_1 & k_3 & k_3 & \cdots & k_{n-3} & k_{n-2} \end{pmatrix}_{2\times n-2}, \tag{3.8}$$

where the expression of h_j and k_j $(1 \le j \le n-2)$ are the same with (3.3).

Substituting (3.6), (3.7) and (3.8) into (3.5), we obtain

$$(C^{-1}\mathcal{A}C)^{-1} =$$

$$\left(\begin{array}{cc:cccccc} \frac{p}{pt-qs} & \frac{-s}{pt-qs} & h_1 & h_2 & \cdots & \cdots & \cdots & h_{n-2} \\ \frac{-q}{pt-qs} & \frac{t}{pt-qs} & k_1 & k_2 & \cdots & \cdots & \cdots & k_{n-2} \\ \hdashline 0 & 0 & a_{1,1} & a_{1,2} & \cdots & \cdots & \cdots & a_{1,n-2} \\ 0 & 0 & a_{2,1} & a_{2,2} & \cdots & \cdots & \cdots & a_{2,n-2} \\ \vdots & \vdots & \vdots & \vdots & & & & \vdots \\ \vdots & \vdots & \vdots & \vdots & & & & \vdots \\ \vdots & \vdots & \vdots & \vdots & & & & \vdots \\ 0 & 0 & a_{n-2,1} & a_{n-2,2} & \cdots & \cdots & \cdots & a_{n-2,n-2} \end{array}\right)_{n\times n}. \tag{3.9}$$

Finally, computing (3.4) by (2.2), (2.4) and (3.9), we get $\mathcal{A}^{-1}$ and the proof is completed. $\square$

Remark 3.1. The result of Theorem 3.1 is amazing, when a_i and b_i $(1 \le i \le n-2)$ are changed, the inverse of $\mathcal{A}$ is only changed $2n-4$ entries, i.e. h_j and k_j $(1 \le j \le n-2)$.

The process of how to calculate the inverse of the tridiagonal Toeplitz matrix with opposite-bordered rows is presented in the following algorithm.

Algorithm 2. Compute the inverse of the matrix $\mathcal{A}$.

Step 1. Input order n, the components a_i, b_i $(1 \le i \le n-2)$, p, q, s, t, and d.

Step 2. Compute the entries of $\mathcal{A}^{-1}$

(1) Generate Fibonacci numbers F_k $(k = 1, 2, \ldots, n-1)$ by (1.3);

(2) Compute $a_{i,j}$ $(1 \le i,j \le n-2)$ via (3.2);

(3) Compute h_j and k_j $(1 \le j \le n-2)$ via (3.3);
(4) Compute the remaining entries of $\mathcal{A}^{-1}$ by (3.1).

Step 3. Output the inverse matrix $\mathcal{A}^{-1}$.

The computational cost (i.e., the number of basic arithmetic operations) for Algorithm 2 is $13n^2-49n+60$. The comparison of the algorithmic complexity between Algorithm 2 and LU decomposition algorithm is showed in Table 2. Comparing the results in Table 2, we can see that the computational cost of our algorithm is less than that of the LU decomposition algorithm.

Table 2. The complexity comparison of computing inverse for different algorithms.

Algorithm	Complexity
LU decomposition	$\frac{5n^3}{6}+\frac{n^2}{2}+\frac{53n}{3}+18$
Algorithm 2	$13n^2-49n+60$

Theorem 3.2. *Let an $n \times n$ $(n \ge 3)$ $\mathcal{B}$ be tridiagonal Toeplitz matrix with opposite-bordered rows given in* (1.2). *If $pt-qs \neq 0$, we have*

$$\mathcal{B}^{-1}=\left(\begin{array}{c|cccccc|c}
\frac{p}{pt-qs} & h_{n-2} & h_{n-3} & \cdots & \cdots & h_2 & h_1 & \frac{-s}{pt-qs} \\
\hline
0 & a_{n-2,n-2} & a_{n-2,n-3} & \cdots & \cdots & a_{n-2,2} & a_{n-2,1} & 0 \\
\vdots & \vdots & \vdots & & & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & & & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & & & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & & & \vdots & \vdots & \vdots \\
0 & a_{1,n-2} & a_{1,n-3} & \cdots & \cdots & a_{1,2} & a_{1,1} & 0 \\
\hline
\frac{-q}{pt-qs} & k_{n-2} & k_{n-3} & \cdots & \cdots & k_2 & k_1 & \frac{t}{pt-qs}
\end{array}\right)_{n\times n}, \tag{3.10}$$

where $a_{i,j}$, h_j and k_j $(1 \le i,j \le n-2)$ are same as that given in (3.2) *and* (3.3), *respectively.*

Proof. For $n \ge 3$, we prove (3.10) by using

$$\mathcal{B}^{-1}=\hat{I}_n^{-1}\mathcal{A}^{-1}\hat{I}_n^{-1}=\hat{I}_n\mathcal{A}^{-1}\hat{I}_n$$

and Theorem 2.2. □

Remark 3.2. The inverse of $\mathcal{B}$ is amazing, only $2n-4$ entries, i.e. h_j and k_j $(1 \le j \le n-2)$ are changed when we change a_i and b_i $(1 \le i \le n-2)$.

One could obtain the corresponding algorithm of the inverse matrix of $\mathcal{B}$ by using Theorem 3.2 and Algorithm 2. We can see that the computational cost for computing the inverse matrix of $\mathcal{B}$ is also $13n^2-49n+60$, since just the position is changed of the inverse matrix of $\mathcal{B}$.

4. The eigenvalues and eigenvectors

In this section, our attention is devoted to computing eigenvalues and eigenvectors of tridiagonal Toeplitz matrix with opposite-bordered rows $\mathcal{A}$ and $\mathcal{B}$. The corresponding algorithm for the main theorem is given.

Theorem 4.1. *Supposed that an* $n \times n$ $(n \geq 6)$ *matrix* $\mathcal{A}$ *is the same as that given in* (1.1)*, then the eigenvalues of* $\mathcal{A}$ *are*

$$\begin{cases} \lambda_1 = \frac{p+t+\sqrt{(p-t)^2+4sq}}{2}, \\ \lambda_2 = \frac{p+t-\sqrt{(p-t)^2+4sq}}{2}, \\ \lambda_j = -d - 2d\mathbf{i}\cos\frac{(j-2)\pi}{n-1}, \quad j = 3, 4, \ldots, n, \end{cases} \tag{4.1}$$

where $\mathbf{i}$ *is the imaginary unity* $(\mathbf{i}^2 = -1)$. *And the corresponding eigenvectors are* u_j*, where*

$$\begin{cases} u_1 = \left[1, 0, \ldots, 0, \frac{t-p+\sqrt{(p-t)^2+4qs}}{2q}\right]^T, \\ u_2 = \left[1, 0, \ldots, 0, \frac{t-p-\sqrt{(p-t)^2+4qs}}{2q}\right]^T, \\ u_j = (v_{2,j}, v_{3,j}, \ldots, v_{n,j}, v_{1,j})^T, \quad j = 3, 4, \ldots, n, \end{cases} \tag{4.2}$$

$v_{n,j}$ *is free, and*

$$\begin{cases} v_{1,j} = \frac{d(\lambda_j - p) - ds[-2\cos\frac{(j-2)\pi}{n-1}\mathbf{i}x_{n-4,j} + x_{n-5,j}]}{(\lambda_j - p)(\lambda_j - t) - qs} v_{n,j}, \\ v_{2,j} = \frac{qd - d(\lambda_j - t)[-2\cos\frac{(j-2)\pi}{n-1}\mathbf{i}x_{n-4,j} + x_{n-5,j}]}{(\lambda_j - p)(\lambda_j - t) - qs} v_{n,j}, \\ v_{l,j} = x_{l,j} v_{n,j}, \quad l = 3, 4, \ldots, n-1, \\ x_{l,j} = \begin{cases} -2\cos\frac{(j-2)\pi}{n-1}\mathbf{i}, & l = n-1, \\ -4[\cos\frac{(j-2)\pi}{n-1}]^2 + 1, & l = n-2, \\ -2\cos\frac{(j-2)\pi}{n-1}\mathbf{i}x_{l+1,j} + x_{l+2,j}, & l = n-3, n-2, \ldots, 3. \end{cases} \end{cases} \tag{4.3}$$

Proof. Let λ and u be the corresponding eigenvalue and eigenvector of matrix $\mathcal{A}$, $\mathbb{B}$ be given in (2.5), i.e.

$$\mathbb{B} = C^{-1}\mathcal{A}C.$$

Let μ and $v = (v_1, \ldots, v_n)^T$ be the corresponding eigenvalue and eigenvector of $\mathbb{B}$, respectively, we know that $\mathbb{B}v = \mu v$. And $C\mathbb{B}v = \mu Cv$ implies that $\mathcal{A}Cv = \mu Cv$, hence μ and Cv is the corresponding eigenvalue and eigenvector of $\mathcal{A}$, respectively, i.e.

$$\lambda = \mu, \ u = Cv.$$

From the above discussion, we just need to compute μ and v. It is easy to see that the eigenvalues of $\mathbb{B}$ are the union of the eigenvalues of $\mathcal{A}_1$ and $\mathcal{A}_4$.

Upon simple calculation, we find the eigenvalues of $\mathcal{A}_1$ are

$$\mu_1 = \frac{p + t + \sqrt{(p-t)^2 + 4sq}}{2},$$

$$\mu_1 = \frac{p + t - \sqrt{(p-t)^2 + 4sq}}{2}.$$

From the equation (1) in [15], the eigenvalues of $\mathcal{A}_4$ are

$$\mu_j = -d - 2d\mathbf{i}\cos\frac{(j-2)\pi}{n-1},\ j = 3, 4, \ldots, n.$$

Next, we compute the corresponding eigenvector v_j of $\mathbb{B}$. We just need to compute such a equation that $(\mu_j I - \mathbb{B})v_j = 0$ for all eigenvalues μ_j. For μ_1 and μ_2, we have

$$(\mu_1 I - \mathbb{B})v_1 = 0,$$
$$(\mu_2 I - \mathbb{B})v_2 = 0.$$

Performing a series of elementary transformations on $\mu_1 I - \mathbb{B}$ and $\mu_2 I - \mathbb{B}$, respectively, we get

$$v_1 = \left[\frac{t-p+\sqrt{(p-t)^2+4qs}}{2q}, 1, 0, \ldots, 0\right]^T,$$
$$v_2 = \left[\frac{t-p-\sqrt{(p-t)^2+4qs}}{2q}, 1, 0, \ldots, 0\right]^T,$$

then

$$u_1 = Cv_1 = \left[1, 0, \ldots, 0, \frac{t-p+\sqrt{(p-t)^2+4qs}}{2q}\right]^T,$$
$$u_2 = Cv_2 = \left[1, 0, \ldots, 0, \frac{t-p-\sqrt{(p-t)^2+4qs}}{2q}\right]^T.$$

For $\mu_j = -d - 2d\mathbf{i}\cos\frac{(j-2)\pi}{n-1}$, $(\mu_j I - \mathbb{B})v_j = 0$, $j = 3, 4, \ldots, n$. Let $v_j = (v_{1,j}, v_{2,j}, \ldots, v_{n,j})^T$ be the corresponding eigenvector of μ_j, then performing a series of elementary transformations on $\mu_j I - \mathbb{B}$, we have

$$v_{1,j} = \frac{d(\lambda_j - p) - ds[-2\cos\frac{(j-2)\pi}{n-1}\mathbf{i}x_{n-4,j} + x_{n-5,j}]}{(\lambda_j - p)(\lambda_j - t) - qs} v_{n,j},$$
$$v_{2,j} = \frac{qd - d(\lambda_j - t)[-2\cos\frac{(j-2)\pi}{n-1}\mathbf{i}x_{n-4,j} + x_{n-5,j}]}{(\lambda_j - p)(\lambda_j - t) - qs} v_{n,j},$$
$$v_{l,j} = x_{l,j}v_{n,j}, \qquad l = 3, 4, \ldots, n-1,$$
$$x_{l,j} = \begin{cases} -2\cos\frac{(j-2)\pi}{n-1}\mathbf{i}, & l = n-1, \\ -4[\cos\frac{(j-2)\pi}{n-1}]^2 + 1, & l = n-2, \\ -2\cos\frac{(j-2)\pi}{n-1}\mathbf{i}x_{l+1,j} + x_{l+2,j}, & l = n-3, n-2, \ldots, 3, \end{cases}$$

$v_{n,j}$ is free, then

$$u_j = Cv_j = (v_{2,j}, v_{3,j}, \ldots, v_{n,j}, v_{1,j})^T, \quad j = 3, 4, \ldots, n.$$

Thus we complete the proof. □

The process of how to calculate the eigenvalue and eigenvector of the tridiagonal Toeplitz matrix with opposite-bordered rows is presented in the following algorithm.

Algorithm 3. Compute the eigenvalue and eigenvector of the matrix $\mathcal{A}$.

Step 1. Input the components a_i, b_i $(1 \le i \le n-2)$, p, q, s, t, and d, order n.

Step 2. Calculate λ_k and u_j

(1) Calculate eigenvalues $\lambda_1, \lambda_2 \dots, \lambda_n$ according to (4.1);

(2) Calculate $v_{1,j}, v_{2,j}, \dots, v_{n-1,j}$ by using the formulae (4.3);

(3) Calculate eigenvectors $u_1, u_2, \dots, u_n$ based on the formulae (4.2).

Step 3. Output eigenvalue λ and eigenvector u.

From the above algorithm, we can see that Algorithm 3 involves a total of $5n^2 - 3n - 6$ arithmetic operations.

Theorem 4.2. *Let an $n \times n$ $(n \geq 6)$ matrix $\mathcal{B}$ be the same as the matrix that given in* (1.2)*, ν_j and $\hat{u}_j$ $(j = 1, 2, \dots, n)$ are the corresponding eigenvalues and eigenvectors of $\mathcal{B}$, then $\nu_j = \lambda_j$, $\hat{u}_j = \hat{I}_n u_j$, λ_j and u_j given in* (4.1) *and* (4.2)*, respectively, where $\hat{I}_n$ is the "reverse unit matrix" of order n, having ones along the secondary diagonal and zeros elsewhere.*

Proof. In fact, we have $\mathcal{A}u = \lambda u$. Note that $\mathcal{B} = \hat{I}_n \mathcal{A} \hat{I}_n$, i.e. $\mathcal{A} = \hat{I}_n \mathcal{B} \hat{I}_n$, and $\hat{I}_n \mathcal{A} u = \lambda \hat{I}_n u$ implies that $\mathcal{B} \hat{I}_n u = \lambda \hat{I}_n u$, hence λ is the corresponding eigenvalue of $\mathcal{B}$ and $\hat{I}_n u$ is the corresponding eigenvector of $\mathcal{B}$. According to the Theorem 4.1, we complete the proof. $\square$

We can also propose the corresponding algorithm for computing the eigenvalues and eigenvectors of $\mathcal{B}$ by using the Theorem 4.2 and Algorithm 3.

5. Numerical example

In this section, two numerical examples are given to prove the superiority of the new methods, and all the experiments are performed on a double-precision PC with a MATLAB (R2018b) and a central processor of 3.40 GHz [Intel(R) Core(TM)i7-3770 CPU], 8GB Microsoft Windows 10 operating system.

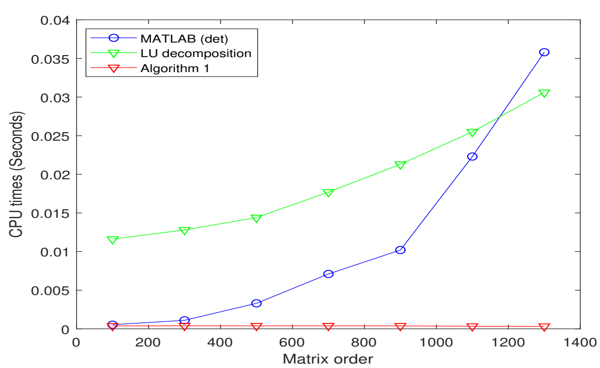

Figure 1. The CPU times comparison of calculating the determinant.

Example 5.1. We consider such a tridiagonal Toeplitz matrix with opposite-bordered rows that the variable entries are selected randomly in (1.1).

In this example, we compare the CPU times of the determinant of matrix $\mathcal{A}$ with different orders among the method in MATLAB, LU decomposition and Algorithm 1 in Figure 1. And as shown in Figure 1, the CPU times of MATLAB(det) and LU decomposition is much higher than Algorithm 1. The result is reasonable because the complexity of Algorithm 1 is much lower than other methods.

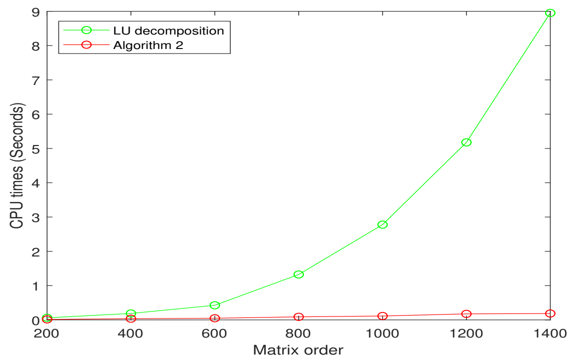

Figure 2. The CPU times comparison of calculating the inverse.

Example 5.2. We consider such a tridiagonal Toeplitz matrix with opposite-bordered rows that the variable entries are selected randomly in (1.1) and ensure that $pt - qs \neq 0$.

For example 5.2, the comparison of CPU times for the inverse of the matrix $\mathcal{A}$ between LU decomposition and Algorithm 2 is showed in Figure 2. From Figure 2, we can see that the CPU times of Algorithm 2 is much less than that of the LU decomposition, especially for the large problem. However, when the order of the matrix $\mathcal{A}$ is much more than 1476, we can not get the Fibonacci number because the memory of MATLAB, thus the inverse of matrix $\mathcal{A}$ can not obtain accurately.

6. Conclusions

In this paper, we firstly give a new class of tridiagonal Toeplitz matrix with opposite-bordered rows. Then we study the determinant, the inverse matrix and the eigenpairs of the tridiagonal Toeplitz matrix with opposite-bordered rows and the corresponding algorithms based on theoretical results are presented. Finally, two numerical examples are given to demonstrate the effectiveness of our algorithms and its competitiveness with other existing algorithms.

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