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INFINITELY MANY SOLUTIONS FOR A NONLOCAL PROBLEM

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Abstract Consider a class of nonlocal problems

$$\begin{cases} -(a-b\int_\Omega |\nabla u|^2dx)\Delta u = f(x,u), & x\in\Omega,\\ u=0, & x\in\partial\Omega, \end{cases}$$

where $a>0, b>0$, $\Omega\subset\mathbb{R}^N$ is a bounded open domain, $f:\overline{\Omega}\times\mathbb{R}\longrightarrow\mathbb{R}$ is a Carathéodory function. Under suitable conditions, the equivariant link theorem without the $(P.S.)$ condition due to Willem is applied to prove that the above problem has infinitely many solutions, whose energy increasingly tends to $a^2/(4b)$, and they are neither large nor small.

Keywords Nonlocal problems, infinitely many solutions, the $(P.S.)_c$ condition, the equivariant link theorem.

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1. Introduction

Consider the following nonlocal problem

$$\begin{cases} -(a-b\int_\Omega |\nabla u|^2dx)\Delta u = f(x,u), & x\in\Omega,\\ u=0, & x\in\partial\Omega, \end{cases} \tag{1.1}$$

where $a>0, b>0$, $\Omega\subset\mathbb{R}^N$ is a bounded domain, $N\geq 3$, $f:\overline{\Omega}\times\mathbb{R}\longrightarrow\mathbb{R}$ is a Carathéodory function, that is, $f(x,t)$ is measurable in x for every $t\in\mathbb{R}$ and continuous in t for a.e. $x\in\Omega$.

In recent years, some scholars researched problem (1.1) with $f(x,t)=g(x)|t|^{p-2}t$ by using variational method, where $1\leq p<2^*:=\frac{2N}{N-2}$, $g\in L^{\frac{2^*}{2^*-p}}(\Omega)$. In the case that $g(x)=1, 2<p<2^*$, Yin and Liu in [10] obtained nonnegative nontrivial solutions and nonpositive nontrivial solutions for problem (1.1). In [3], for the case that $1<p<2$ and $g(x)=\lambda h_+(x)+h_-(x)$, where $\lambda>0$, $h_\pm(x)=\pm\max\{\pm h(x),0\}\not\equiv 0$ and $h\in L^\infty(\Omega)$, Lei, Liao and Suo obtained two positive solutions when $\lambda>0$ is small enough. Lei, Chu and Suo in [4] considered the case that $g(x)=\lambda(>0)$ and $0<p<1$, they obtained two positive solutions when λ is small enough. In [1], Duan, Sun and Li considered the case that $1\leq p<2^*$, and obtained the existence and the multiplicity of solutions. Motivated by the papers mentioned above,

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we will consider the existence of infinitely many solutions for problem (1.1) under more general situation. Suppose that f satisfies the following conditions.

(f_1) $f:\overline{\Omega}\times\mathbb{R}\longrightarrow\mathbb{R}$ is a Carathéodory function and $f(x,t)t>0$ for every $t\in\mathbb{R}\setminus\{0\}$ and a.e. $x\in\Omega$.

(f_2) There exist $h\in L^{\frac{2^*}{2^*-1}}(\Omega)$, $1\le p<2^*$ and $g\in L^{\frac{2^*}{2^*-p}}(\Omega)$ such that

$$|f(x,t)|\le h(x)+g(x)|t|^{p-1}$$

for every $t\in\mathbb{R}$ and a.e. $x\in\Omega$.

(f_3) $f(x,-t)=-f(x,t)$ for every $t\in\mathbb{R}$ and a.e. $x\in\Omega$.

The main result is the following theorem.

Theorem 1.1. *Assume that $a>0,b>0$ and (f_1), (f_2) and (f_3) hold. Then problem* (1.1) *has infinitely many solutions whose energy increasingly tends to $a^2/(4b)$.*

Corollary 1.1. *Let $a>0,b>0$ and $f(x,t)=g(x)|t|^{p-2}t$, if $1\le p<2^*$ and $g\in L^{\frac{2^*}{2^*-p}}(\Omega)$ is nonzero and nonnegative, then problem* (1.1) *has infinitely many solutions whose energy increasingly tends to $a^2/(4b)$.*

Corollary 1.2. *Suppose that $a>0,b>0$. Then the following problem*

$$-(a-b\int_\Omega|\nabla u|^2dx)\Delta u=\lambda u,\ x\in\Omega;\quad u=0,\ x\in\partial\Omega \tag{1.2}$$

has infinitely many solutions for every $\lambda>0$.

Remark 1.1. Problem (1.1) is reduced to Kirchhoff type problems if $b<0$ and to semilinear elliptic equations if $b=0$, infinitely many solutions of which were obtained by using variational methods, but they are either small negative energy solutions or large energy solutions (see [2,6] for Kirchhoff type problems and see [5,8,9,12] for semilinear elliptic equations, and their references therein). Unusually, it is interesting that the infinitely many solutions in Theorem 1.1 are neither large nor small energy solutions. Comparing Corollary 1.1 with the results in [1,3,10], under the same conditions we obtain infinitely many nonzero solutions, but they obtained at least two solutions. Corollary 1.2 shows that the appearance of the nonlocal term may cause the change of the properties of the equation. In fact, as Corollary 1.2 states, when $b>0$, problem (1.2) has infinitely many solutions for every $\lambda>0$. But when $b=0$, problem (1.2) has infinitely many solutions only for $\lambda=a\lambda_k,k=1,2,\cdots$, where $\{\lambda_k\}$ are the eigenvalues of $-\Delta$ in $H_0^1(\Omega)$.

2. Proof of the main result

In this section, we will prove the main result by using the equivariant link theorem without the $(P.S.)$ condition(see [7]). Let $H_0^1(\Omega)$ be the usual Hilbert space with norm and inner product

$$\|u\|=\left(\int_\Omega|\nabla u|^2dx\right)^{\frac{1}{2}}\quad\text{and}\quad(u,v)=\int_\Omega\nabla u\cdot\nabla v dx.$$

We choose an orthonormal basis $\{e_j\}$ of $H_0^1(\Omega)$, Set $X_j=\mathbb{R}e_j(j=1,2,\cdots)$, $V=\mathbb{R}$, $G=\mathbb{Z}/2$, $Y_k=X_1\oplus\cdots\oplus X_k$, $Z_k=X_k\oplus Y_k^\perp$.

Define the functional $\varphi : H_0^1(\Omega) \to R$ as follows:

$$\varphi(u) = \frac{a}{2}\|u\|^2 - \frac{b}{4}\|u\|^4 - \int_\Omega F(x,u)dx \quad \text{for every } u \in H_0^1(\Omega),$$

where $F(x,t) \equiv \int_0^t f(x,s)ds$ for all $t \in \mathbb{R}$ and a.e. $x \in \Omega$. From (f_2) and (f_3), by a standard argument, we have $\varphi \in C^1(H_0^1(\Omega), R)$ and $\varphi(-u) = \varphi(u)$ for any $u \in H_0^1(\Omega)$. It is well-known that the weak solutions of problem (1.1) correspond to the critical points of the functional φ, and we have

$$\langle \varphi'(u), v\rangle = \left(a - b\|u\|^2\right)(u,v) - \int_\Omega f(x,u)v dx \quad \text{for any } u, v \in H_0^1(\Omega).$$

Lemma 2.1. *Suppose that conditions $(f_1) - (f_3)$ hold. Let $c \neq \frac{a^2}{4b}$, then the functional φ satisfies the $(P.S.)_c$ condition.*

Proof. Suppose that $\{u_n\}$ is a $(P.S.)_c$ sequence of the functional φ, that is,

$$\varphi(u_n) \to c, \quad \varphi'(u_n) \to 0 \ \text{ as } \ n \to \infty.$$

It follows from (f_1) that $F(x,t) \geq 0$ for every $t \in \mathbb{R}$ and a.e. $x \in \Omega$. Hence by $\varphi(u_n) \to c$, one has

$$\begin{aligned} \frac{a}{2}\|u_n\|^2 &= \frac{b}{4}\|u_n\|^4 + \int_\Omega F(x,u_n)dx + \varphi(u_n) \\ &\geq \frac{b}{4}\|u_n\|^4 + c - 1 \end{aligned}$$

for large n, which implies that $\{u_n\}$ is bounded. Hence, up to subsequences, there exists $u_0 \in H_0^1(\Omega)$ such that

$$u_n \rightharpoonup u_0 \text{ in } H_0^1(\Omega), \quad u_n \to u_0 \text{ in } L^p(\Omega), \quad u_n(x) \to u_0(x) \text{ for a.e. } x \in \Omega.$$

By the boundedness of $\{u_n\}$, without loss of generality, we may assume that

$$m = \lim_{n\to\infty} \|u_n\|^2.$$

It follows from $\varphi'(u_n) \to 0$ and the boundedness of $\{u_n\}$ that $\langle \varphi'(u_n), u_n\rangle \to 0$. Hence, from (f_2), we have

$$(a - bm)\, m - \int_\Omega f(x,u_0)u_0 dx = 0. \tag{2.1}$$

By the fact that $\langle \varphi'(u_n), u_0\rangle \to 0$ and (f_2), we have

$$(a - bm)\, \|u_0\|^2 - \int_\Omega f(x,u_0)u_0 dx = 0. \tag{2.2}$$

From (2.1) and (2.2), we obtain

$$(a - bm)\,(m - \|u_0\|^2) = 0.$$

Assume that $m \neq \|u_0\|^2$, then $m = \frac{a}{b}$, by (2.1), one has

$$\int_\Omega f(x,u_0)u_0 dx = 0,$$

which implies that $u_0 = 0$ by (f_1). Moreover by $\varphi(u_n) \to c$, we have

$$c = \frac{a}{2}m - \frac{b}{4}m^2 - \int_\Omega F(x, u_0)dx = \frac{a^2}{4b},$$

which contradicts to the condition $c \neq \frac{a^2}{4b}$. Hence $m = \|u_0\|^2$, it follows that

$$\lim_{n\to\infty} \|u_n\|^2 = \|u_0\|^2.$$

Combining with $u_n \rightharpoonup u_0$ in $H_0^1(\Omega)$, one has $u_n \to u_0$ in $H_0^1(\Omega)$. The proof is completed. $\square$

Lemma 2.2. *Suppose that $u_k \in Z_k (k = 1, 2, \cdots)$, if $\{u_k\}$ is bounded, then $\{u_k\}$ weakly converges to zero in $H_0^1(\Omega)$ as $k \to \infty$.*

Proof. For $k > n$, by the orthogonality of Y_n and Z_k, one has

$$\langle u_k, v\rangle = 0$$

for every $v \in Y_n$, which implies that

$$\lim_{k\to\infty} \langle u_k, v\rangle = 0$$

for every $v \in Y_n$. It follows from the boundedness of $\{u_k\}$, the density of $\cup_{n=1}^\infty Y_n$ in $H_0^1(\Omega)$ and Theorem 5.1.3 in [11] that $\{u_k\}$ weakly converges to zero. $\square$

In the following, we give a slight generalization of Lemma 2.13 in [7].

Lemma 2.3. *Suppose that $N \geq 3$, $1 \leq p < 2^*$ and $g \in L^{\frac{2^*}{2^*-p}}(\Omega)$. Then the functional $\xi(u) = \int_\Omega g(x)|u|^p dx$ is weakly continuous on $H_0^1(\Omega)$.*

Proof. Assume that sequence $\{u_n\}$ weakly converges to u in $H_0^1(\Omega)$. By the compactness of the embedding $H_0^1(\Omega) \to L^p(\Omega)$, we have $u_n \to u$ in $L^p(\Omega)$, which implies that $|u_n|^p \to |u|^p$ in $L^1(\Omega)$. Hence $\int_\Omega v(x)|u_n|^p dx \to \int_\Omega v(x)|u|^p dx$ as $n \to \infty$ for every $v \in L^\infty(\Omega)$. It follows from the boundedness of $\{u_n\}$ in $L^{2^*}(\Omega)$, the density of $L^\infty(\Omega)$ in $L^{\frac{2^*}{2^*-p}}(\Omega)$ and Theorem 5.1.3 in [11] that $\int_\Omega g(x)|u_n|^p dx \to \int_\Omega g(x)|u|^p dx$ as $n \to \infty$. $\square$

Now we can prove our main result.

Proof of Theorem 1.1. Let $r = \sqrt{a/b}$, $B_{2r} = \{u \in H_0^1(\Omega) \mid \|u\| < 2r\}$ and $S_{2r} = \{u \in H_0^1(\Omega) \mid \|u\| = 2r\}$, for any $k \geq 2$, define

$$c_k = \inf_{\gamma\in\Gamma_k} \max_{u\in Y_k\cap B_{2r}} \varphi(\gamma(u)),$$

where $\Gamma_k = \{\gamma \in C(Y_k \cap B_{2r}, H_0^1(\Omega)) \mid \gamma_k \text{ is odd and } \gamma|_{Y_k\cap S_{2r}} = I\}$. We have

$$c_k \leq \max_{u\in Y_k\cap B_{2r}} \varphi(u) := \alpha_k$$

for $k \geq 2$. Then there exists $w_k \in Y_k$ with $\|w_k\| \leq 2r$ such that $\alpha_k = \varphi(w_k)$. In the case that $w_k = 0$, we have

$$c_k \leq \alpha_k = \varphi(w_k) = \varphi(0) = 0 < \frac{a^2}{4b}.$$

In the case that $w_k \neq 0$, by (f_1), one has

$$c_k \leq \alpha_k = \varphi(w_k) \leq \frac{a^2}{4b} - \int_\Omega F(x, w_k)dx < \frac{a^2}{4b}.$$

Hence, we have

$$c_k < \frac{a^2}{4b} \quad \text{for any } k \geq 2. \tag{2.3}$$

Let

$$a_k = \inf_{u \in Y_k \cap S_{2r}} \varphi(u) \quad \text{and} \quad b_k = \max_{u \in Z_k \cap S_r} \varphi(u),$$

then we have

$$a_k = \frac{a}{2}(2r)^2 - \frac{b}{4}(2r)^4 - \max_{u \in Y_k \cap S_{2r}} \int_\Omega F(x, u)dx \leq -\frac{2a^2}{b},$$

and

$$b_k = \frac{a}{2}r^2 - \frac{b}{4}r^4 - \inf_{u \in Z_k \cap S_r} \int_\Omega F(x, u)dx = \frac{a^2}{4b} - \inf_{u \in Z_k \cap S_r} \int_\Omega F(x, u)dx.$$

Set

$$\beta_k = \inf_{u \in Z_k \cap S_r} \int_\Omega F(x, u)dx,$$

then there exists $u_k \in Z_k$ with $\|u_k\| = r$ such that

$$\beta_k \leq \int_\Omega F(x, u_k)dx \leq \beta_k + \frac{1}{k}.$$

It follows from Lemma 2.2 that $\{u_k\}$ converges to zero weakly in $H_0^1(\Omega)$ as $k \to \infty$. By (f_2) and Lemma 2.3, one has

$$\int_\Omega F(x, u_k)dx \to 0,$$

which implies that $\lim_{k\to\infty} \beta_k = 0$. Hence

$$\lim_{k\to\infty} b_k = \frac{a^2}{4b}, \tag{2.4}$$

which implies that

$$b_k > a_k$$

for k large enough. It follows from Definition 3.2, Lemma 3.4 and Theorem 3.5 in [7] that $c_k \geq b_k$ and there exists a sequence $\{u_n\} \subset H_0^1(\Omega)$ such that

$$\varphi(u_n) \to c_k, \quad \varphi'(u_n) \to 0$$

as $n \to \infty$ for k large enough. It is easy to see that c_k is a critical value of φ by Lemma 2.1. By (2.3), (2.4) and the fact that $b_k \leq c_k$, φ has infinitely many distinct critical values c_k such that $c_k \to \frac{a^2}{4b}$ as $n \to \infty$, which implies that problem (1.1) has infinitely many distinct solutions. The proof of Theorem 1.1 is completed. $\square$

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