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ANALYTICAL RESULTS FOR QUADRATIC INTEGRAL EQUATIONS WITH PHASE–LAG TERM

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Abstract In the present paper, we are concerning with a quadratic integral equation with phase–lag term. In the following pages, sufficient conditions are given for the existence of positive continuous solution to quadratic integral equations. The method used here depends on both Tychonoff fixed point principle and Arzelà–Ascoli theorem. A concrete example illustrating the mentioned applicability is also included.

Keywords Quadratic integral equation, phase–lag, Tychonoff fixed point theorem, Banach space, fixed point theorem.

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1. Introduction

Phase lag has a very important role in our applied science and there are currently one, dual and three phases and each phase has a different applications. For example the three–phase–lag model incorporates the microstructural interaction effect in the fast–transient process of heat transport. It describes the finite time required for the various microstructural interactions to take place, including the phonon–electron interaction in metals, the phonon scattering in dielectric crystals, insulators, and semiconductors, and the activation of molecules at extremely low temperature, by the resulting phase lag (time delay) in the process of heat transport see [9, 11, 13, 23]. Integral equations with phase lag term are the mathematical model of many evolutionary in problems chemistry, engineering, quantum mechanics, biology, optimal control systems, mathematical physics and so on. For example, integral equations for the dual lag model of heat transfer.

Integral equations create a very important and significant part of mathematical analysis and their applications to real–world problems. On the other hand, normality and continuity are very useful tools in the wide area of functional analysis such as the metric fixed–point theory and the theory of operator equations in Banach spaces. They are also used in the studies of functional equations, ordinary and partial differential equations, fractional partial differential equations, integral

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and integro–differential equations, optimal control theory, etc., see [1–8, 14, 17–22]. In our investigations, we apply the Tychonoff fixed point principle [12] to prove the existence and uniqueness of solution of the quadratic integral equation with phase–lag term.

In this paper, the existence of at least one continuous solution for the quadratic integral equation of phase–lag term (in short QIEPLT),

$$\psi(t+\delta t) = a(t) + \psi(t+\delta t)\int_0^1 k(t+\delta t,\tau)g(\tau,\psi(\tau))\mathrm{d}\tau; \quad (0<\delta t<<1), \tag{1.1}$$

will be proved, where δt is the phase–lag constant, the function $\psi(t)$ is unknown in the Banach space and continuous with their derivative with respect to time. The kernel $k(t,\tau)$ is positive and continuous, the functions $a(t)$, $g(\tau,\psi)$ are continuous its derivatives with respect to time. Let $I=[0,1]$, denote by $E=C(I)$ the space of continuous functions defined on I with norm $\|\psi\| = \max_{t\in I}|\psi(t)|$.
Using Taylor expansion after neglecting the second derivative in Eq. (1.1), we get

$$\psi(t)+\delta t\frac{\partial\psi}{\partial t}(t) = a(t)+\left(\psi(t)+\delta t\frac{\partial\psi}{\partial t}(t)\right)\int_0^1\left(k(t,\tau)+\delta t\frac{\partial k}{\partial t}(t,\tau)\right)g(\tau,\psi(\tau))\mathrm{d}\tau, \tag{1.2}$$

with initial condition,

$$\psi(0)=\psi_0. \tag{1.3}$$

Equation (1.2) with initial condition (1.3) is called quadratic integro–differential equation. The quadratic integro–differential equation (QIDE) is a kind of functional equation that has associate integral and derivatives of an unknown function. These equations were named after the leading mathematicians who have first studied them, such as quadratic Fredholm, quadratic Volterra. quadratic Fredholm and Volterra equations are the most encountered types.
By comparing the expressions with the same powers of parameter δt, we receive the relations

$$\psi(t) = a(t)+\psi(t)\int_0^1 k(t,\tau)g(\tau,\psi(\tau))\mathrm{d}\tau, \tag{1.4}$$

and

$$\frac{\partial\psi}{\partial t}(t) = \psi(t)\int_0^1\frac{\partial k}{\partial t}(t,\tau)g(\tau,\psi(\tau))\mathrm{d}\tau + \frac{\partial\psi}{\partial t}(t)\int_0^1 k(t,\tau)g(\tau,\psi(\tau))\mathrm{d}\tau. \tag{1.5}$$

Integrating Eq. (1.5) twice and using initial condition (1.3), we get

$$\psi(t)=\psi_0+\int_0^t\psi(\tau)\int_0^1\frac{\partial k}{\partial\tau}(\tau,s)g(s,\psi(s))\mathrm{d}s\mathrm{d}\tau+\int_0^t\frac{\partial\psi}{\partial\tau}(\tau)\int_0^1 k(\tau,s)g(s,\psi(s))\mathrm{d}s\mathrm{d}\tau, \tag{1.6}$$

integration by parts and applying the Leibniz's rule, we obtain

$$\begin{aligned}\psi(t) =&\psi_0+\int_0^t\psi(\tau)\mathrm{d}\tau\left(\int_0^1\frac{\partial k}{\partial t}(t,s)g(s,\psi(s))\mathrm{d}s-\int_0^t\int_0^1\frac{\partial^2 k}{\partial\tau^2}(\tau,s)g(s,\psi(s))\mathrm{d}s\mathrm{d}\tau\right)\\ &+(\psi(t)-\psi_0)\left(\int_0^1 k(t,s)g(s,\psi(s))\mathrm{d}s-\int_0^t\int_0^1\frac{\partial k}{\partial\tau}(\tau,s)g(s,\psi(s))\mathrm{d}s\mathrm{d}\tau\right).\end{aligned} \tag{1.7}$$

Equation (1.4) is called the Fredholm quadratic integral equation and (1.7) the Volterra–Fredholm quadratic integral equation, in order to guarantee discuss the existence and uniqueness of solution of the equation (1.1), we must establish the existence and the uniqueness of solutions of the Eqs. (1.4) and (1.7).
The contribution of this work can be summarized in the following four points:

- Introducing the preliminaries and auxiliary results about the fixed point theorem needed in the following points of the paper.
- The existence and uniqueness of the solution of a quadratic integral equation of Volterra type (1.4), under certain conditions, will be discussed and proved using Banach's fixed point method in the space $E = C(I)$.
- We study existence of solution of quadratic integral equation (1.7) by using Tychonoff fixed point theorem.
- An example is given to show the applications of our results.

2. Preliminaries

In this section, the existence results will be based on the following fixed–point theorems and definitions that are used in the paper.

Definition 2.1 (Convex set [16]). A set $S \subset E$ is said to be a convex set if $\forall\lambda \in [0,1]$ and $\forall\phi, \psi \in S$, $\lambda\varphi + (1-\lambda)\psi \in S$.

Theorem 2.1 (Banach's Fixed Point Theorem [10]). *If E be a Banach space and $T : E \to E$ be a contraction mapping, then T has a unique fixed point in E.*

Theorem 2.2 (Tychonoff's Fixed Point Theorem [12]). *Suppose E is a complete, locally convex linear space and S_r is a closed convex subset of E. Let the mapping $T : S_r \to S_r$ be continuous and $T(E) \subset E$. If the closure of $T(E)$ is compact, then T has a fixed–point in E.*

Notice that a normed vector space is a locally convex topological vector space so this theorem extends the Schauder fixed point theorem.

Theorem 2.3 (Arzelà–Ascoli Theorem [15]). *Let E be a compact metric space and $C(E)$ the Banach space of real or complex valued continuous functions normed by*

$$\|\psi\| = \max_{t\in E} |\psi(t)|.$$

If $F = \{f_n\}$ is a sequence in $C(E)$ such that is uniformly bounded and equi–continuous, then the closure of F is compact.

3. Existence of positive continuous solution

In this section, we study the existence of at least one solution of the integral equation (1.1), to achieve this, the existence and uniqueness of the Eqs. (1.4) and (1.7) were discussed in the following subsections:

3.1. The existence and uniqueness of solution of Eq. (1.4)

Here, we prove the existence of positive continuous solution for Eq. (1.4). To facilitate our discussion, let us first state the following assumptions:

(i) The kernel $k(t,\tau) \in C([0,1])$, $t,\tau \in [0,1]$ satisfies $|k(t,\tau)| < k^*$, k^* is a constant.

(ii) $M = \sup\{|g(t,0)| : t \in [0,1]\}$.

(iii) $N_i = \max\{|\psi_i(t)| : t \in [0,1]\}$, $i = 1,2$.

(iv) Function $g(\tau,\psi(\tau))$ satisfy the Lipschitz condition with Lipschitz constant l

$$|g(\tau,\psi_1(\tau)) - g(\tau,\psi_2(\tau))| \le l|\psi_1(\tau) - \psi_2(\tau)|.$$

To prove the existence and the uniqueness solution of Eq. (1.4), we use the continuity of the integral operator, with the help of Banach fixed point. For this the integral equation (1.4) can be written in the integral operator form:

$$(H\psi)(t) = a(t) + \psi(t)\int_0^1 k(t,\tau)g(\tau,\psi(\tau))\mathrm{d}\tau. \tag{3.1}$$

Theorem 3.1. *If the conditions (i) – (iv) are satisfied and the integral operator (3.1) is a continuous, then equation (1.4) has an unique solution $\psi(t)$ in the Banach space $C([0,1])$, under the condition,*

$$k^*(M + lN_1 + lN_2) < 1.$$

Proof. For the continuity, we assume the two functions $\psi_1(t)$ and $\psi_2(t)$ in the space $C([0,1])$ satisfy the integral operator then,

$$\begin{aligned}
&(H\psi_1)(t)-(H\psi_2)(t)\\
=&\psi_1(t)\int_0^1 k(t,\tau)g(\tau,\psi_1(\tau))\mathrm{d}\tau-\psi_2(t)\int_0^1 k(t,\tau)g(\tau,\psi_2(\tau))\mathrm{d}\tau,\\
=&\psi_1(t)\int_0^1 k(t,\tau)g(\tau,\psi_1(\tau))\mathrm{d}\tau-\psi_2(t)\int_0^1 k(t,\tau)g(\tau,\psi_1(\tau))\mathrm{d}\tau\\
&+\psi_2(t)\int_0^1 k(t,\tau)g(\tau,\psi_1(\tau))\mathrm{d}\tau-\psi_2(t)\int_0^1 k(t,\tau)g(\tau,\psi_2(\tau))\mathrm{d}\tau,\\
=&[\psi_1(t)-\psi_2(t)]\int_0^1 k(t,\tau)g(\tau,\psi_1(\tau))\mathrm{d}\tau\\
&+\psi_2(t)\int_0^1 k(t,\tau)[g(\tau,\psi_1(\tau))-g(\tau,\psi_2(\tau))]\mathrm{d}\tau.
\end{aligned}$$

Using the properties of the norm and the conditions (i)–(iv), we get

$$\begin{aligned}
\|(H\psi_1)(t) - (H\psi_2)(t)\| \le& \max_{t\in I}|[\psi_1(t) - \psi_2(t)]\int_0^1 k(t,\tau)g(\tau,\psi_1(\tau))\mathrm{d}\tau|\\
&+\max_{t\in I}|\psi_2(t)\int_0^1 k(t,\tau)[g(\tau,\psi_1(\tau)) - g(\tau,\psi_2(\tau))]\mathrm{d}\tau|,\\
\le& \max_{t\in I}|\psi_1(t) - \psi_2(t)|\int_0^1 |k(t,\tau)||g(\tau,\psi_1(\tau))|\mathrm{d}\tau
\end{aligned}$$

$$+\max_{t\in I}|\psi_2(t)|\int_0^1 |k(t,\tau)||g(\tau,\psi_1(\tau))-g(\tau,\psi_2(\tau))|\mathrm{d}\tau,$$

$$\begin{aligned}\|(H\psi_1)(t)-(H\psi_2)(t)\| &\le k^*\max_{t\in I}|\psi_1(t)-\psi_2(t)|\int_0^1(|g(\tau,\psi_1(\tau))-g(\tau,0)|\\ &\quad+|g(\tau,0)|)\mathrm{d}\tau + k^*N_2 l\max_{t\in I}|\psi_1(t)-\psi_2(t)|\int_0^1\mathrm{d}\tau,\\ &\le k^*\max_{t\in I}|\psi_1(t)-\psi_2(t)|\left(\int_0^1(l|\psi_1(\tau)|+M)\mathrm{d}\tau+N_2 l\int_0^1\mathrm{d}\tau\right),\\ &\le k^*(M+lN_1+lN_2)\|\psi_1(t)-\psi_2(t)\|,\end{aligned}$$

hence, we have

$$\|(H\psi_1)(t)-(H\psi_2)(t)\|\le\alpha\|\psi_1(t)-\psi_2(t)\|;\qquad \alpha<1. \tag{3.2}$$

Hence, H is a contraction operator in the space $C([0,1])$; therefore, by Banach's fixed point theorem, H has a unique fixed point. If the reader uses the continuity of the integral operator, with the help of Banach's fixed point, we arrive to the existence and uniqueness of the Eq. (1.4). □

3.2. The existence and uniqueness of solution of Eq. (1.7)

Equation (1.7) can be written in the following integral operator from:

$$(V\psi)(t)=\psi_0+(T\psi)(t)((K_tG)(t)-(K_{\tau\tau}G)(t))+(\psi(t)-\psi_0)((KG)(t)-(K_\tau G)(t)), \tag{3.3}$$

where

$$\begin{aligned}(T\psi)(t)&=\int_0^t\psi(\tau)\mathrm{d}\tau,\\ (KG)(t)&=\int_0^1 k(t,s)g(s,\psi(s))\mathrm{d}s,\\ (K_tG)(t)&=-\int_0^1\frac{\partial k}{\partial t}(t,s)g(s,\psi(s))\mathrm{d}s,\\ (K_\tau G)(t)&=-\int_0^t\int_0^1\frac{\partial k}{\partial\tau}(\tau,s)g(s,\psi(s))\mathrm{d}s\mathrm{d}\tau,\\ (K_{\tau\tau}G)(t)&=-\int_0^t\int_0^1\frac{\partial^2 k}{\partial\tau^2}(\tau,s)g(s,\psi(s))\mathrm{d}s\mathrm{d}\tau.\end{aligned}$$

Assume that g is a real function defined on the set $I\times R_+$, we consider the superposition operator $(G\psi)(t)=g(t,\psi(t))$ under the some following assumptions.

(a) g is continuous on the set $I\times R_+$.

(b) The function $t\to g(t,\psi)$ is nondecreasing on I for any fixed $\psi\in R_+$.

(c) For any fixed $t\in I$ the function $\psi\to g(t,\psi)$ is nondecreasing on R_+.

(d) The function $g=g(t,\psi)$ satisfies the Lipschitz condition with respect to the variable ψ, i.e. there exists a constant $l>0$ such that for any $t\in I$ and for ψ_1, $\psi_2\in R_+$ the following inequality holds

$$|g(t,\psi_1)-g(t,\psi_2)|\le l|\psi_1-\psi_2|. \tag{3.4}$$

Then the following result is implied.

Theorem 3.2. *Assume that the hypotheses (a)–(d) are satisfied and* $\psi \in \Psi_I \subseteq C(I)$*. Then*

$$d(G\psi) \leq ld(\psi).$$

The above theorem follows that when the function g satisfies the Lipschitz condition with a constant $l < 1$ (cf. the assumption (d)) the superposition operator G generated by the function g improves the degree of monotonicity of any subset Ψ of Ψ_I with the coefficient l.

Corollary 3.1. *Suppose the function* $g(t,\psi) = g : I \times R_+ \to R_+$ *satisfies the assumptions* (a), (b)*. Moreover, we assume that* g *has partial derivative* g_ψ *which is nonnegative and bounded on the set* $I \times R_+$*. Then* g *satisfies the assumptions* (c) *and* (d) *with the Lipschitz constant* l *defined as follows*

$$l = \sup\{g_\psi(t,\psi) : (t,\psi) \in I \times R_+\}.$$

In order that discuss the existence and uniqueness solution of Eq. (1.7), we assume the following assumptions:

- **(i)** $k : I \times I \to R_+$ is continuous and the functions $s \to k(t,s)$ and $t \to k(t,s)$ are nondecreasing on R_+ for fixed $t \in I$ and $s \in I$, respectively such that $|-k_\tau(\tau,s)| < k_1$, $|-k_{\tau\tau}(\tau,s)| < k_2$, $\forall t \in I$, where k_1, k_2 are positive constants.
- **(ii)** The operator $T : C(I) \to C(I)$ is continuous and satisfies the $|(T\psi)(t)| \leq |\psi|$.
- **(iii)** The function $g : I \times R_+ \to R_+$ satisfies the conditions $(a) - (d)$, and there exists a nondecreasing function $m : R_+ \to R_+$ such that $|g(s,\psi(s))| \leq m(|\psi|)$.
- **(iv)** The unknown function $\psi(t)$ satisfies $|\psi(t) - \psi_0| \leq |\psi|$ in the space $C(I)$.
- **(v)** The inequality
$$\psi_0 + rm(r)[k_2 + k^*] \leq r,$$
where $k^* = \max\{k(t,s) : t,s \in I\}$.

Now we can formulate the main existence theorem:

Theorem 3.3. *Let the assumptions (i)–(v) be satisfied. Then the quadratic functional integral equation (1.7) has at least one solution* $\psi = \psi(t)$ *in the space* $C(I)$*.*

Proof. Let S_r be the subset of the space $C([0,1])$ defined as follows:

$$S_r = \{\psi \in C(I) : |\psi(t)| \leq r \; for \; t \in I\}.$$

The space $C([0,1])$ is a complete locally convex linear space that has been proved in [12], it is clear that the set S_r is nonempty, bounded and closed, but we will prove that the set S_r convex.

Let ψ_1, $\psi_2 \in S_r$ and $\lambda \in [0,1]$ then we have

$$\begin{aligned}\|\lambda\psi_1 + (1-\lambda)\psi_2\| &\leq \lambda\|\psi_1\| + (1-\lambda)\|\psi_2\|,\\ &\leq \lambda r + (1-\lambda)r,\\ &\leq \lambda r + r - \lambda r = r.\end{aligned}$$

Then $\lambda\psi_1 + (1-\lambda)\psi_2 \in S_r$ which means that is convex set. Let us consider the operator V is defined on the space $C(I)$ by the formula:

$$\begin{aligned}(V\psi)(t) =&\psi_0+(T\psi)(t)\left(\int_0^1\frac{\partial k}{\partial t}(t,s)g(s,\psi(s))\mathrm{d}s-\int_0^t\int_0^1\frac{\partial^2 k}{\partial \tau^2}(\tau,s)g(s,\psi(s))\mathrm{d}s\mathrm{d}\tau\right)\\&+(\psi(t)-\psi_0)\left(\int_0^1 k(t,s)g(s,\psi(s))\mathrm{d}s-\int_0^t\int_0^1\frac{\partial k}{\partial \tau}(\tau,s)g(s,\psi(s))\mathrm{d}s\mathrm{d}\tau\right).\end{aligned}$$

To show the operator V transforms the space S_r into itself. For that let $\psi \in S_r$, then

$$\begin{aligned}|(V\psi)(t)|\leq&\left|\psi_0+(T\psi)(t)\left(\int_0^1\frac{\partial k}{\partial t}(t,s)g(s,\psi(s))\mathrm{d}s-\int_0^t\int_0^1\frac{\partial^2 k}{\partial \tau^2}(\tau,s)g(s,\psi(s))\mathrm{d}s\mathrm{d}\tau\right)\right|\\&+\left|(\psi(t)-\psi_0)\left(\int_0^1 k(t,s)g(s,\psi(s))\mathrm{d}s-\int_0^t\int_0^1\frac{\partial k}{\partial \tau}(\tau,s)g(s,\psi(s))\mathrm{d}s\mathrm{d}\tau\right)\right|.\end{aligned}$$

Using the properties of the norm and the conditions (i)–(v), we get

$$\begin{aligned}|(V\psi)(t)| \leq&\psi_0+|(T\psi)(t)|\left(\int_0^1-\left|-\frac{\partial k}{\partial t}(t,s)\right||g(s,\psi(s))|\mathrm{d}s\right.\\&\left.+\int_0^t\int_0^1\left|-\frac{\partial^2 k}{\partial \tau^2}(\tau,s)\right||g(s,\psi(s))|\mathrm{d}s\mathrm{d}\tau\right)\\&+|(\psi(t)-\psi_0)|\left(\int_0^1|k(t,s)||g(s,\psi(s))|\mathrm{d}s\right.\\&\left.+\int_0^t\int_0^1\left|-\frac{\partial k}{\partial \tau}(\tau,s)\right||g(s,\psi(s))|\mathrm{d}s\mathrm{d}\tau\right),\\ \leq&\psi_0+|\psi(t)|m(|\psi|)[k_2+k^*],\end{aligned}$$

hence, we obtain

$$|(V\psi)(t)| \leq \psi_0 + rm(r)[k_2+k^*] \leq r.$$

From the above estimate and assumption (v), then $(V\psi)(t) \in S_r$ implies $VS_r \subset S_r$.

Now, let the fix arbitrarily $\delta > 0$ and choose $t_1, t_2 \in I$ such that $|t_2 - t_1| \leq \delta,\ t_2 \geq t_1$. Then, keeping in mind our assumptions, we obtain

$$\begin{aligned}|(V\psi)(t_2)-(V\psi)(t_1)| \leq&|(T\psi)(t_2)|\left|\int_{t_1}^{t_2}\int_0^1\left|-\frac{\partial^2 k}{\partial \tau^2}(\tau,s)\right|g(s,\psi(s))\mathrm{d}s\mathrm{d}\tau\right|\\&+|(\psi)(t_2)-\psi_0|\left|\int_{t_1}^{t_2}\int_0^1\left|-\frac{\partial k}{\partial \tau}(\tau,s)\right|g(s,\psi(s))\mathrm{d}s\mathrm{d}\tau\right|\\&+|(T\psi)(t_2)-(T\psi)(t_1)|\left|\int_0^1-\left|-\frac{\partial k}{\partial t}(t,s)\right|g(s,\psi(s))\mathrm{d}s\right.\\&\left.+\int_0^{t_1}\int_0^1\left|-\frac{\partial^2 k}{\partial \tau^2}(\tau,s)\right|g(s,\psi(s))\mathrm{d}s\mathrm{d}\tau\right|\\&+|(\psi)(t_2)-(\psi)(t_1)|\left|\int_0^1 k(t,s)g(s,\psi(s))\mathrm{d}s\right.\\&\left.+\int_0^{t_1}\int_0^1\left|-\frac{\partial k}{\partial \tau}(\tau,s)\right|g(s,\psi(s))|\mathrm{d}s\mathrm{d}\tau\right|,\end{aligned}$$

using the properties of the norm and the conditions (i)–(v), we obtain

$$\begin{aligned}|(V\psi)(t_2)-(V\psi)(t_1)| \le & m(|\psi|)[|(T\psi)(t_2)|k_2+|(\psi)(t_2)-\psi_0|k_1](t_2-t_1)\\ &+|(T\psi)(t_2)-(T\psi)(t_1)|m(|\psi|)[-k_1+k_2t_1]\\ &+|(\psi)(t_2)-(\psi)(t_1)|m(|\psi|)[k^*+k_1t_1].\end{aligned}$$

Hence, keeping in mind our assumptions and the above-established facts, we arrive at the following relation:

$$|(V\psi)(t_2)-(V\psi)(t_1)|\to 0 \quad as \quad |t_2-t_1|\to 0.$$

This means that the function VS_r is equi–continuous on I. By using Arzelà–Ascoli theorem [13], we can say that is VS_r compact.

Now, Tychonoff fixed point theorem is satisfied all its conditions, then Eq. (1.7) has at least one solution $\psi \in C(I)$. This completes the proof. □

4. Example

In this section, we will discuss the following example and applying theories 3.1 and 3.3, then check the results.

Example 4.1. Consider the following quadratic integral equation with phase lag term:

$$\psi(t+\delta t)=a(t)+\psi(t+\delta t)\int_0^1 \frac{(t+\delta t)^2}{2e^{((t+\delta t)+\tau)}}\ln(1+\tau|\psi(\tau)|)\mathrm{d}\tau; \quad (\psi(0)=0). \tag{4.1}$$

Equation (4.1), has the exact solution $\psi(t)=t$. Using numerical treatment of the equation (4.1) and comparing the expressions with the same powers of parameter δt, we obtained

$$\psi(t)=a(t)+\psi(t)\int_0^1 \frac{t^2}{2e^{t+\tau}}\ln(1+\tau|\psi(\tau)|)\mathrm{d}\tau; \quad (t\in I=[0,1]), \tag{4.2}$$

and

$$\begin{aligned}\psi(t)=&(T\psi)(t)\left(\int_0^1 -\frac{t}{2}(-2+t)e^{(-s-t)}\ln(1+s|\psi(s)|)\mathrm{d}s\right.\\ &\left.+\int_0^t\int_0^1 \frac{\tau}{4}(10-8\tau+\tau^2)e^{(-s-\tau)}\ln(1+s|\psi(s)|)\mathrm{d}s\mathrm{d}\tau\right)\\ &+\psi(t)\left(\int_0^t\int_0^1 \frac{\tau}{2}(-2+\tau)e^{(-s-\tau)}\ln(1+s|\psi(s)|)\mathrm{d}s\mathrm{d}\tau\right.\\ &\left.+\int_0^1 \frac{t^2}{2e^{t+s}}\ln(1+s|\psi(s)|)\mathrm{d}s\right).\end{aligned} \tag{4.3}$$

In this example, comparing with equation (1.4) and assumptions (i)–(iv) in Subsect. (3.1), we have

(i) The kernel $k(t,\tau)=t^2/2e^{t+\tau}$, $t,\tau\in[0,1]$ satisfies $|k(t,\tau)|<1/2e^2$, $(k^*=1/2e^2)$.

(ii) $M = \sup\{|g(t,0)| : t \in [0,1]\} = \ln(1) = 0.$

(iii) $N_1 = \max\{|t| : t \in [0,1]\} = 1$ and $N_2 = \max\{|t^2| : t \in [0,1]\} = 1.$

(iv) The function $g(\tau, \psi(\tau)) = \ln(1 + \tau|\psi(\tau)|)$ satisfy the Lipschitz condition with Lipschitz constants

$$l = \sup\{\frac{\partial}{\partial \psi}(\ln(1 + \tau|\psi(\tau)|)) : (\tau, \psi) \in I \times R_+\} = 1.$$

Under the following inequality:

$$k^*(M + lN_1 + lN_2) = (\frac{1}{2e^2})(2) = (\frac{1}{e^2}) < 1,$$

the theorem 3.1 is true, and comparing with equation (1.7) and assumptions (i)–(v) in Subsect. (3.2), we have the operator T is defined as $(T\psi)(t) = \int_0^t \psi(\tau)\mathrm{d}\tau$ which is also continuous with norm $\|T\| = 1$, the function $k_\tau(\tau, s) = -[\tau(-2 + \tau)e^{(-s-\tau)}]/2$, where $|k_\tau(\tau, s)| \leq 1/2e^2$, $(k_1 = 1/2e^2)$, Also, we have $k_{\tau\tau}(\tau, s) = -[(10 - 8\tau + \tau^2)e^{(-s-\tau)}]/4$, where $|k_{\tau\tau}(\tau, s)| \leq 7/e^2$, $(k_2 = 7/e^2)$, the function $g(\tau, \psi(\tau)) = \ln(1 + \tau|\psi(\tau)|)$, which satisfies the assumption (iii) with $|g(\tau, \psi(\tau))| = |\ln(1 + \tau|\psi(\tau)|)| \leq |\psi(\tau)|$. Thus, we get $m(r) = r$.
Further, let us consider the inequality

$$\psi_0 + rm(r)[k_2 + k^*] \leq r,$$

or equivalently,

$$r^2[\frac{7}{e^2} + \frac{1}{2e^2}] \leq r. \tag{4.4}$$

Using the standard methods we can verify that the function $\rho(r) = r(7/e^2 + 1/2e^2)$ attains its maximum at the point $r = 0.985$ and $\rho(0.985) = 0.985(7/e^2 + 1/2e^2) \leq 1$. So, the number $r = 0.985$ is a positive solution of the inequality (4.4), hence the theorem 3.3 is true.

Finally, taking into account all the above–established facts and theories 3.1 and 3.3 we conclude that the equation (1.1) has at least one solution $\psi = \psi(t)$ defined and continuous on the interval I. Moreover, $|\psi| \leq r = 0.985$.

5. Conclusion

In this work, from the above results and discussion, the following may be conclude, the quadratic integral equation with phase–lag term (1.1) possesses at least one solution $\psi(t)$ in the space $C([0,1])$, under all the assumptions of theories 3.1 and 3.3. Fixed point theories are one of the best methods to prove the existence and uniqueness of any equation.

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