

LIMIT CYCLE BIFURCATIONS IN DISCONTINUOUS PLANAR SYSTEMS WITH MULTIPLE LINES*

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Abstract In this paper, the limit cycle bifurcation problem is investigated for a class of planar discontinuous perturbed systems with n parallel switch lines. Under the assumption that the unperturbed system has a family of periodic orbits crossing all of the lines, an explicit expression of the first order Melnikov function along the periodic orbits is presented, which plays an important role in studying the problem of limit cycle bifurcations. As an application of the established method, the maximal number of limit cycles of a discontinuous system is considered.

Keywords Discontinuous planar system, limit cycle, bifurcation, periodic orbit, Melnikov function.

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1. Introduction

Consider a piecewise smooth system of the form

$$\dot{x} = F(x, y), \quad \dot{y} = G(x, y), \quad (1.1)$$

where

$$F(x, y) = \begin{cases} F^+(x, y), & x \geq 0, \\ F^-(x, y), & x < 0, \end{cases} \quad G(x, y) = \begin{cases} G^+(x, y), & x \geq 0, \\ G^-(x, y), & x < 0 \end{cases}$$

and $F^\pm(x, y)$, $G^\pm(x, y)$ are C^∞ functions. Then, system (1.1) has two subsystems

$$\dot{x} = F^+(x, y), \quad \dot{y} = G^+(x, y) \quad (1.1a)$$

and

$$\dot{x} = F^-(x, y), \quad \dot{y} = G^-(x, y), \quad (1.1b)$$

which are called the right and left subsystems respectively. The notations of the subsystems were first introduced in [13] by Han and Zhang.

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Recently, system (1.1) has been receiving great attention because many problems, from control theory, electronics, physics, medicine as well as biology, can be modeled by it, see [3, 4, 8] and references therein. The non-smoothness of system (1.1) at the y -axis yields many complicated and interesting behaviors which cannot occur in smooth case, such as sliding homoclinic bifurcation and sliding-crossing bifurcation, border-collision bifurcation, stick-slip oscillation and so on, see [4, 6, 8, 9, 26] and references cited therein.

A lot of theory has been developed to investigate Poincaré, Hopf and Homoclinic loop bifurcations on system (1.1). Poincaré bifurcation is to study the maximal number of limit cycles emerging from a period annulus. To the best of our knowledge, there are two main methods to investigate this problem. One is the Melnikov function method proposed in [17], see [10, 18, 29, 30] and references quoted therein. Another one is the averaging method developed in [11, 22], see [23] and references therein. However, the authors [21] proved that the two methods are equivalent to each other. Several cases of Hopf bifurcations were investigated by the papers [5, 13, 25, 29]. On homoclinic loop bifurcation, there are also two ways to study. The first one is the Melnikov function method, see [18–20]. The second one is the method of stability-changing of a homoclinic loop, which, in some cases, can find more limit cycles than the first one. These limit cycles, which are not covered by the Melnikov function method, are called alien limit cycles, see [31]. Furthermore, Poincaré and Homoclinic loop bifurcations are also called global bifurcations, while Hopf bifurcation is called local bifurcation.

However, as mentioned in [1], discontinuities may occur on multiply lines or even nonlinear curves or surfaces. In the literature [1, 16, 21, 24, 34], the authors studied the problem of limit cycle bifurcation for a planar discontinuous system by considering discontinuities on finitely many nonlinear curves emanating from a vertex. In the paper [27], the author considered the number of limit cycles for a piecewise-linear Liénard system with n parallel switch lines and obtained that it can have $2n$ limit cycles, which proved a conjecture in the paper [28]. In this paper we study the limit cycle bifurcation problem of a class of differential systems having discontinuities on finitely many parallel straight lines. The rest of this paper is organized as follows. Some definitions, assumptions and main results are presented in Section 2. The proof of Theorem 2.1 is given in Section 3.

2. Preliminaries and main results

First, introduce some notations as follows

$$I_i = (x_{i+1}, x_i), \quad i = 0, 1, \dots, n, \quad n \in \mathbb{N}^+$$

where

$$x_{n+1} < x_n < \dots < x_1 < x_0, \quad x_{n+1} = -\infty, \quad x_0 = +\infty.$$

Therefore, the set \mathbb{R} is divided into $n+1$ subintervals by the points x_1, x_2, \dots, x_n , and

$$\begin{aligned} \mathbb{R} &= I_0 \cup I_1 \cup \dots \cup I_n \cup \{x_1, x_2, \dots, x_n\} \\ &= \bigcup_{i=0}^n I_i \cup \{x_1, x_2, \dots, x_n\}. \end{aligned}$$

Then, we define vector functions $\Phi_i(x, y) : I_i \mapsto \mathbb{R}^4, i = 0, 1, \dots, n,$

$$\Phi_i(x, y) = (H_{iy}(x, y), -H_{ix}(x, y), p_i(x, y), q_i(x, y)), (x, y) \in I_i \times \mathbb{R}, i = 0, 1, \dots, n,$$

where $H_i(x, y), p_i(x, y), q_i(x, y)$ are C^∞ functions in (x, y) . Further, let us use smooth vector functions $\Phi_i(x, y)$ to construct a piecewise smooth vector function $\Phi(x, y)$ in what follows

$$\Phi(x, y) = \begin{cases} \Phi_0(x, y), & x \in I_0, \\ \Phi_1(x, y), & x \in I_1, \\ \vdots & \vdots \\ \Phi_n(x, y), & x \in I_n \end{cases} \triangleq (H_y(x, y), -H_x(x, y), p(x, y), q(x, y)). \quad (2.1)$$

Now, we utilize the piecewise smooth function $\Phi(x, y)$ to define a piecewise smooth planar system of the form

$$\dot{x} = H_y(x, y) + \varepsilon p(x, y), \quad \dot{y} = -H_x(x, y) + \varepsilon q(x, y), \quad (2.2)$$

where $\varepsilon > 0$ is a small parameter, and $H_y(x, y), H_x(x, y), p(x, y), q(x, y)$ are given in (2.1). For $\varepsilon = 0$, system (2.2) reads

$$\dot{x} = H_y(x, y), \quad \dot{y} = -H_x(x, y). \quad (2.3)$$

It is easy to see that system (2.2) has $n + 1$ subsystems

$$\dot{x} = H_{iy}(x, y) + \varepsilon p_i(x, y), \quad \dot{y} = -H_{ix}(x, y) + \varepsilon q_i(x, y), \quad i = 0, 1, \dots, n, \quad (2.4)$$

which are all C^∞ near-Hamiltonian systems. Similarly, system (2.3) also has $n + 1$ subsystems

$$\dot{x} = H_{iy}(x, y), \quad \dot{y} = -H_{ix}(x, y), \quad i = 0, 1, \dots, n, \quad (2.5)$$

which are all C^∞ Hamiltonian systems. Therefore, one can give a definition as follows.

Definition 2.1. If a planar system has n subsystems and each subsystem is a near-Hamiltonian (resp. Hamiltonian) system on the plane, then we call this system an n -piecewise near-Hamiltonian (resp. Hamiltonian) system.

From Definition 2.1, one can know that system (2.2) (resp. (2.3)) is an $(n + 1)$ -piecewise near-Hamiltonian (resp. Hamiltonian) system. And system $(2.2)|_{\varepsilon=0}$ or (2.3) has a piecewise Hamiltonian function defined by $H(x, y)$, i.e.

$$H(x, y) = \begin{cases} H_0(x, y), & x \in I_0, \\ H_1(x, y), & x \in I_1, \\ \vdots & \vdots \\ H_n(x, y), & x \in I_n. \end{cases} \quad (2.6)$$

Regarding system (2.3), we make the following three assumptions:

(A1) System (2.3) has a period annulus \mathcal{A} consisting of a one-parameter family of clockwise periodic orbits

$$\Gamma_h, \quad h \in \mathcal{J} = (h_1, h_2), \quad h_1 < h_2.$$

For $h \in \mathcal{J}$, each Γ_h crosses the straight line $l_i : x = x_i$ two times clockwise, having the intersection points denoted by $A_i(h) = (x_i, a_i(h))$, $B_i(h) = (x_i, b_i(h))$, $a_i(h) > b_i(h)$, $i = 1, 2, \dots, n$, see Figure 1.

(A2) There exist $2n - 1$ C^∞ functions $\alpha_i(h)$, $\beta_i(h)$, $i = 1, 2, \dots, n - 1$ and $\alpha_n(h)$ such that for $h \in \mathcal{J}$

$$\begin{aligned} H_0(A_1(h)) = H_0(B_1(h)) = h, \quad H_n(A_n(h)) = H_n(B_n(h)) = \alpha_n(h), \\ H_i(A_i(h)) = H_i(A_{i+1}(h)) = \alpha_i(h), \quad H_i(B_i(h)) = H_i(B_{i+1}(h)) = \beta_i(h), \end{aligned}$$

where $i = 1, 2, \dots, n - 1$.

(A3) For $h \in \mathcal{J}$, we have

$$\begin{aligned} H_{0y}(A_1)H_{0y}(B_1)H_{ny}(A_n)H_{ny}(B_n) \neq 0, \\ H_{jy}(A_j)H_{jy}(A_{j+1})H_{jy}(B_j)H_{jy}(B_{j+1}) \neq 0, \quad j = 1, 2, \dots, n - 1. \end{aligned}$$

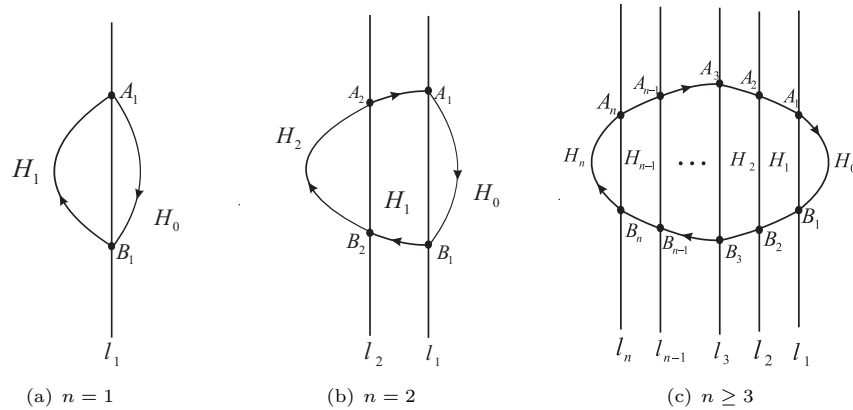


Figure 1. The periodic orbit Γ_h of system (2.3).

From the above assumptions, one can see that the (x, y) -plane has been split into $n + 1$ subregions by the straight lines l_1, l_2, \dots, l_n . That is to say,

$$\begin{aligned} \mathbb{R} \times \mathbb{R} &= I_0 \times \mathbb{R} \cup I_1 \times \mathbb{R} \cup \dots \cup I_n \times \mathbb{R} \cup l_1 \times \mathbb{R} \cup l_2 \times \mathbb{R} \dots \cup l_n \times \mathbb{R} \\ &= \bigcup_{i=0}^n I_i \times \mathbb{R} \cup \bigcup_{i=1}^n l_i \times \mathbb{R}. \end{aligned}$$

On each subregion $I_i \times \mathbb{R}$, it defines a C^∞ near-Hamiltonian system. And there exists a family of periodic Γ_h , $h \in \mathcal{J}$ passing through each subregion with a clockwise orientation.

Now, we investigate the unperturbed system (2.2) under the conditions (A1)-(A3). By continuous dependency of discontinuous planar systems on initial data established in [2], the positive orbit of system (2.2) starting from A_1 must intersect the

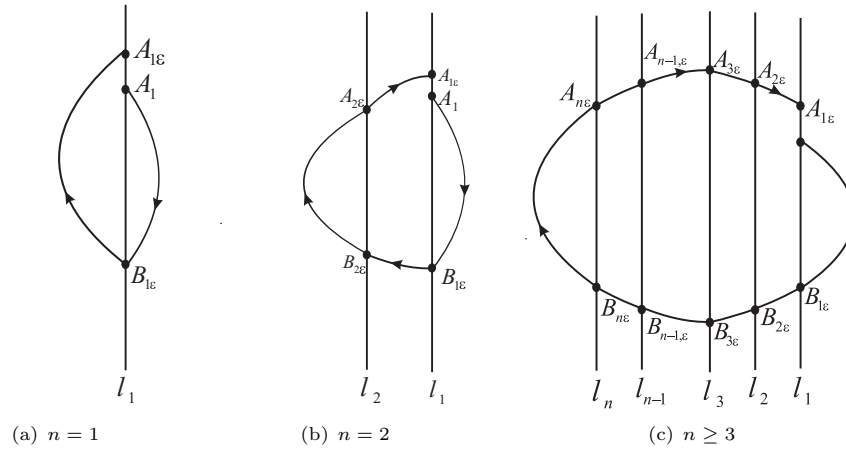


Figure 2. Poincaré map of system (2.2).

straight lines l_1, \dots, l_n successively with points $B_{1\epsilon}(h) = (x_1, b_{1\epsilon}(h)), \dots, B_{n\epsilon}(h) = (x_n, b_{n\epsilon}(h))$, then it turns and crosses the lines l_n, \dots, l_1 again with points $A_{n\epsilon}(h) = (x_n, a_{n\epsilon}(h)), \dots, A_{1\epsilon}(h) = (x_1, a_{1\epsilon}(h))$ respectively, see Figure 2. Obviously, in view of (A3), one knows that

$$B_{i\epsilon}(h) = B_i(h) + O(\epsilon) \in C^\infty, \quad A_{i\epsilon}(h) = A_i(h) + O(\epsilon) \in C^\infty, \quad i = 1, 2, \dots, n. \quad (2.7)$$

It is not hard to see that from the point $A_1(h)$ to the point $A_{1\epsilon}(h)$ on the straight line l_1 , it results in a return map or Poincaré map, denoted by $\mathcal{P}(h, \epsilon)$,

$$\mathcal{P}(h, \epsilon) : A_1(h) \mapsto A_{1\epsilon}(h), \quad h \in \mathcal{J}.$$

Based on these, we can define a function below

$$\mathcal{B}(h, \epsilon) = H_0(A_{1\epsilon}(h)) - H_0(A_1(h)) = \epsilon F(h, \epsilon) = \sum_{i \geq 1} \epsilon^i M_i(h). \quad (2.8)$$

On account of the definition of $H_0(x, y)$, together with (2.7), one can find that the functions $M_i(h), i \geq 1$ are C^∞ functions in $h \in \mathcal{J}$. On the other hand, by the differential mean value theorem,

$$\begin{aligned} \mathcal{B}(h, \epsilon) &= DH_0(A_1 + O(A_{1\epsilon} - A_1))(A_{1\epsilon} - A_1) \\ &= (DH_0(A_1) + O(\epsilon))(A_{1\epsilon} - A_1) \\ &= (H_{0y}(A_1) + O(\epsilon))(a_{1\epsilon} - a_1). \end{aligned} \quad (2.9)$$

Using (A3) again, we obtain that $\mathcal{B}(h, \epsilon) = 0$ if and only if $a_{1\epsilon} = a_1$ (or $A_{1\epsilon} = A_1$). Recall that, similar to the case of continuous systems, a limit cycle is an isolated periodic orbit of the discontinuous systems. Therefore, for $\epsilon > 0$ small, system (2.2) has a limit cycle if and only if F in (2.8) has an isolated zero in h . Thus, one can call the function $\mathcal{B}(h, \epsilon)$ in (2.8) a bifurcation function of system (2.2). The function M_i in (2.8) can be called the i th order Melnikov function.

It is easy to prove that, if the first non-zero $M_i(h)$ has a zero of odd multiplicity in h , then for $\epsilon > 0$ small enough, $F(h, \epsilon)$ also has a zero having an odd multiplicity

near h . This means that one can investigate the number of isolated zeros of the first non-zero $M_i(h)$ to obtain the number of limit cycles emerging from the period annulus \mathcal{A} . Recently, the authors of [17] have derived the explicit expression of $M_1(h)$ for a 2-piecewise near-Hamiltonian system and gave the corresponding applications.

This paper focuses on presenting the general expression of $M_1(h)$ for an $(n+1)$ -piecewise near-Hamiltonian system, $n \geq 1$, $n \in \mathbb{N}$. As an application, we study the limit cycle bifurcation for a 3-piecewise near-Hamiltonian system.

For convenience, let $M(h) = M_1(h)$. Also, we define for $j \geq 1$

$$\prod_{i=j}^k a_i = \begin{cases} 1, & \text{for } k = j - 1, \\ a_j a_{j+1} \cdots a_k, & \text{for } k \geq j. \end{cases}$$

Then, using the same idea of [17] or by Theorem 2.2 in [16], it is easy to obtain the following lemma.

Lemma 2.1. *Suppose that (A1)-(A3) hold. Then, we have for $n \geq 1$, $n \in \mathbb{N}$,*

$$\begin{aligned} M(h) &= \int_{\widehat{A_1 B_1}} q_0 dx - p_0 dy + \sum_{j=1}^{n-1} \prod_{k=1}^j \frac{H_{ky}(B_{k+1})}{H_{ky}(B_k)} \frac{H_{0y}(B_1)}{H_{jy}(B_{j+1})} \int_{\widehat{B_j B_{j+1}}} q_j dx - p_j dy \\ &+ \sum_{j=1}^{n-1} \prod_{k=1}^{j-1} \frac{H_{ky}(A_{k+1})}{H_{ky}(A_k)} \frac{H_{0y}(A_1)}{H_{jy}(A_j)} \int_{\widehat{A_{j+1} A_j}} q_j dx - p_j dy \\ &+ \prod_{k=1}^{n-1} \frac{H_{ky}(A_{k+1})}{H_{ky}(A_k)} \frac{H_{0y}(A_1)}{H_{ny}(A_n)} \int_{\widehat{B_n A_n}} q_n dx - p_n dy. \end{aligned}$$

Particularly, we have for $n = 1$,

$$M(h) = \int_{\widehat{A_1 B_1}} q_0 dx - p_0 dy + \frac{H_{0y}(A_1)}{H_{1y}(A_1)} \int_{\widehat{B_1 A_1}} q_1 dx - p_1 dy$$

and for $n = 2$, we have

$$\begin{aligned} M(h) &= \int_{\widehat{A_1 B_1}} q_0 dx - p_0 dy + \frac{H_{0y}(B_1)}{H_{1y}(B_1)} \int_{\widehat{B_1 B_2}} q_1 dx - p_1 dy \\ &+ \frac{H_{0y}(A_1)}{H_{1y}(A_1)} \int_{\widehat{A_2 A_1}} q_1 dx - p_1 dy \\ &+ \frac{H_{1y}(A_2)H_{0y}(A_1)}{H_{1y}(A_1)H_{2y}(A_2)} \int_{\widehat{B_2 A_2}} q_2 dx - p_2 dy. \end{aligned} \quad (2.10)$$

Furthermore, if $M(h_0) = 0$ and $M'(h_0) \neq 0$ for some $h_0 \in \mathcal{J}$, then for $\varepsilon > 0$ small system (2.2) has a unique limit cycle near Γ_{h_0} . If h_0 is a zero of $M(h)$ having an odd multiplicity, then for $\varepsilon > 0$ small, system (2.2) has at least one limit cycle near Γ_{h_0} .

Usually, the boundary of the period annulus \mathcal{A} is a center or a polycycle. For example, the following 2-piecewise Hamiltonian system

$$(\dot{x}, \dot{y}) = \begin{cases} (y, -x), & x \geq 0, \\ (y, x + 1), & x < 0 \end{cases}$$

has a period annulus $\tilde{\mathcal{A}} = \{L_h = L_h^+ \cup L_h^- \mid h \in (0, \frac{1}{2})\}$ surrounding the origin, where $L_h^+ : \frac{1}{2}y^2 + \frac{1}{2}x^2 = h$ and $L_h^- : \frac{1}{2}y^2 - \frac{1}{2}x^2 - x = h$. The boundary of $\tilde{\mathcal{A}}$ is a generalized homoclinic loop (as $h \rightarrow \frac{1}{2}$) or the origin (as $h \rightarrow 0$). Here, the origin is an elementary center, see Definition 1.4 of [13]. Similar to smooth systems [14, 15, 32, 33], one can investigate the asymptotic expansion of $M(h)$ near the boundary to study the homoclinic loop bifurcation or the Hopf bifurcation, see [18, 19, 29].

Now, we apply Lemma 2.1 to study the limit cycle bifurcation of a 3-piecewise near-Hamiltonian system. For definiteness, take $\Phi(x, y)$ in (2.1) as follows

$$\Phi(x, y) = \begin{cases} (y, x - 1, 0, \sum_{i=0}^{2m} a_i x^i y), & (x, y) \in (1, +\infty) \times \mathbb{R}, \\ (y, -x, 0, \sum_{i=0}^m b_i x^{2i} y), & (x, y) \in (-1, 1) \times \mathbb{R}, \\ (y, x + 1, 0, \sum_{i=0}^{2m} (-1)^i a_i x^i y), & (x, y) \in (-\infty, -1) \times \mathbb{R} \end{cases} \\ \triangleq (y, -g(x), 0, f(x)y). \tag{2.11}$$

Then, system (2.2) becomes

$$\dot{x} = y, \quad \dot{y} = -g(x) + \varepsilon f(x)y, \tag{2.12}$$

where $f(x)$ and $g(x)$ are given in (2.11), i.e.

$$g(x) = \begin{cases} x - 1, & (x, y) \in (1, +\infty) \times \mathbb{R}, \\ -x, & (x, y) \in (-1, 1) \times \mathbb{R}, \\ x + 1, & (x, y) \in (-\infty, -1) \times \mathbb{R}, \end{cases} \\ f(x) = \begin{cases} \sum_{i=0}^{2m} a_i x^i y, & (x, y) \in (1, +\infty) \times \mathbb{R}, \\ \sum_{i=0}^m b_i x^{2i} y, & (x, y) \in (-1, 1) \times \mathbb{R}, \\ \sum_{i=0}^{2m} (-1)^i a_i x^i y, & (x, y) \in (-\infty, -1) \times \mathbb{R}. \end{cases}$$

In view of Lemma 2.1, one can obtain that

Theorem 2.1. *For $\varepsilon > 0$ small, system (2.12) can have $7m + 3$ limit cycles.*

From Theorem 2.1, we conclude that a 3-piecewise linear near-Hamiltonian system can have 3 limit cycles. While, for a 2-piecewise linear near-Hamiltonian, we only find 2 limit cycles, see [7, 29]. This implies that one can find more limit cycles by the Melnikov function method by adding a switch line. The proof of Theorem 2.1 will be given in the next section.

3. Proof of Theorem 2.1

Clearly, system (2.12) has the following three subsystems

$$\dot{x} = y, \quad \dot{y} = -(x - 1) + \varepsilon \sum_{i=0}^{2m} a_i x^i y, \tag{3.1}$$

$$\dot{x} = y, \quad \dot{y} = x + \varepsilon \sum_{i=0}^m b_i x^{2i} y, \quad (3.2)$$

and

$$\dot{x} = y, \quad \dot{y} = -(x+1) + \varepsilon \sum_{i=0}^{2m} (-1)^i a_i x^i y. \quad (3.3)$$

Systems (3.1), (3.2) and (3.3) are polynomial Liénard systems. For $\varepsilon = 0$, systems (3.1), (3.2) and (3.3) are Hamiltonian systems with the Hamiltonian functions, respectively

$$H_1(x, y) = \frac{1}{2}y^2 + \frac{1}{2}(x-1)^2, \quad H(x, y) = \frac{1}{2}y^2 - \frac{1}{2}x^2, \quad H_2(x, y) = \frac{1}{2}y^2 + \frac{1}{2}(x+1)^2. \quad (3.4)$$

It is easy to see that system (2.12)| $_{\varepsilon=0}$ has three families of periodic orbits given by

$$\begin{aligned} \Gamma_{1h} &= \{(x, y) \mid H_1(x, y) = h + \frac{1}{2}, 1 \leq x \leq 2\} \\ &\quad \cup \{(x, y) \mid H(x, y) = h, 0 < x < 1\}, \\ \Gamma_{2h} &= \{(x, y) \mid H_2(x, y) = h + \frac{1}{2}, 1 \leq -x \leq 2\} \\ &\quad \cup \{(x, y) \mid H(x, y) = h, 0 < -x < 1\}, \\ \Gamma_h &= \{(x, y) \mid H_1(x, y) = h + \frac{1}{2}, x \geq 1\} \cup \{(x, y) \mid H_2(x, y) = h + \frac{1}{2}, x \leq -1\} \\ &\quad \cup \{(x, y) \mid H(x, y) = h, -1 < x < 1\}, \quad h \in (-\frac{1}{2}, 0), \end{aligned} \quad (3.5)$$

As $h \rightarrow -\frac{1}{2}$, Γ_{1h} (resp. Γ_{2h}) approaches to a generalized elementary center (1, 0) (resp. (-1, 0)), and as $h \rightarrow 0$, Γ_{1h} (resp. Γ_{2h}) approaches to a generalized homoclinic loop denoted by Γ_1 (resp. Γ_2). Further, as $h \rightarrow 0$, Γ_h tends to a generalized double homoclinic loop denoted by Γ . Obviously, we have

$$\Gamma = \Gamma_1 \cup \Gamma_2.$$

The closed curve Γ_{1h} (resp. Γ_{2h}) crosses the straight line $x = 1$ (resp. $x = -1$) clockwise with two points denoted by $C_1(h) = (1, \sqrt{2h+1})$ and $D_1 = (1, -\sqrt{2h+1})$ (resp. $C_2(h) = (-1, \sqrt{2h+1})$ and $D_2 = (-1, -\sqrt{2h+1})$) respectively, where $h \in (-\frac{1}{2}, 0)$. The periodic orbit Γ_h intersects the straight lines $x = \pm 1$ clockwise with four points in turn, denoted by $A_1(h) = (1, \sqrt{2h+1})$, $B_1(h) = (1, -\sqrt{2h+1})$, $B_2 = (-1, -\sqrt{2h+1})$ and $A_2(h) = (-1, \sqrt{2h+1})$ respectively, where $h \in (0, +\infty)$. Figure 3 shows the phase portrait of system (2.12)| $_{\varepsilon=0}$.

Then, associated to the three families of periodic orbits, one has three the first order Melnikov functions as follows

$$\begin{aligned} M_1(h) &= \int_{\widehat{C_1 D_1}} \sum_{i=0}^{2m} a_i x^i y dx + \int_{\widehat{D_1 C_1}} \sum_{i=0}^m b_i x^{2i} y dx, \\ M_2(h) &= \int_{\widehat{C_2 D_2}} \sum_{i=0}^m b_i x^{2i} y dx + \int_{\widehat{D_2 C_2}} \sum_{i=0}^{2m} (-1)^i a_i x^i y dx, \end{aligned} \quad h \in (-\frac{1}{2}, 0) \quad (3.6)$$

and

$$\begin{aligned} M(h) &= \int_{\widehat{A_1 B_1}} \sum_{i=0}^{2m} a_i x^i y dx + \int_{\widehat{B_1 B_2} \cup \widehat{A_2 A_1}} \sum_{i=0}^m b_i x^{2i} y dx + \int_{\widehat{B_2 A_2}} \sum_{i=0}^{2m} (-1)^i a_i x^i y dx, \\ &h \in (0, +\infty). \end{aligned} \quad (3.7)$$

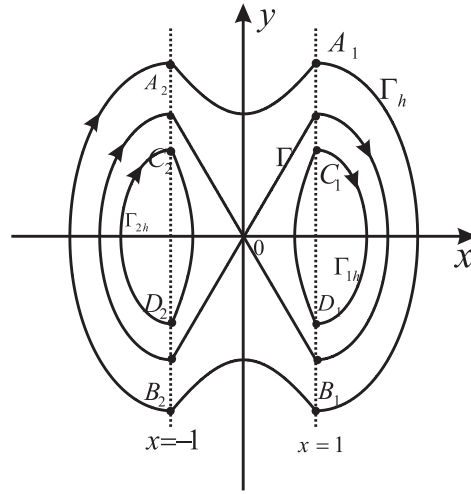


Figure 3. Phase portrait of system (2.12)|_{\epsilon=0}.

Note that system (2.12) is symmetric with respect to the origin. Then, from (3.6) and (3.7), we obtain that

$$M_1(h) = M_2(h), \quad h \in \left(-\frac{1}{2}, 0\right),$$

$$M(h) = 2 \int_{\widehat{A_1 B_1}} \sum_{i=0}^{2m} a_i x^i y dx + 2 \int_{\widehat{A_2 A_1}} \sum_{i=0}^m b_i x^{2i} y dx, \quad h \in (0, +\infty). \tag{3.8}$$

Thus, it suffices to compute the functions $M_1(h)$ and $M(h)$. First, we have

Lemma 3.1. *Let (3.4) and (3.5) hold. Then, the function $M_1(h)$ in (3.6) can be written as*

$$M_1(h) = \sum_{i=0}^{2m} A_i (2h+1)^{\frac{i}{2}+1} + \sum_{i=0}^{m-1} B_i h^i (2h+1)^{\frac{3}{2}} + \sum_{i=0}^m \tilde{B}_i h^i A(h), \quad h \in \left(-\frac{1}{2}, 0\right), \tag{3.9}$$

where

$$A(h) = \sqrt{2h+1} + 2h \ln(1 + \sqrt{2h+1}) - h \ln 2 - h \ln |h|,$$

$$A_i = \sum_{k=i}^{2m} a_k \alpha_{ki}, \quad i = 0, 1, \dots, 2m,$$

$$B_i = \sum_{j=i}^{m-1} b_{j+1} \beta_{i,j+1}, \quad i = 0, 1, \dots, m-1,$$

$$\tilde{B}_0 = b_0, \quad \tilde{B}_i = b_i \rho_i, \quad i = 1, 2, \dots, m \tag{3.10}$$

and each $\alpha_{ki} > 0$, $\beta_{ij} \neq 0$ and $\rho_i \neq 0$.

Proof. Along the curve $\widehat{C_1 D_1}$, one has $\frac{1}{2}y^2 + \frac{1}{2}(x-1)^2 = h + \frac{1}{2}$, which intersects

the x -axis with a point $(1 + \sqrt{2h+1}, 0)$. Then, from (3.6), it gives that

$$\int_{\widehat{C_1 D_1}} \sum_{i=0}^{2m} a_i x^i y dx = 2 \sum_{i=0}^{2m} a_i \int_1^{1+\sqrt{2h+1}} x^i \sqrt{(2h+1) - (x-1)^2} dx. \quad (3.11)$$

Make a transformation $x = \sqrt{2h+1} \sin \theta + 1$ to the above integral. Then, (3.11) can be carried into

$$\begin{aligned} \int_{\widehat{C_1 D_1}} \sum_{i=0}^{2m} a_i x^i y dx &= 2 \sum_{i=0}^{2m} a_i (2h+1) \int_0^{\frac{\pi}{2}} (\sqrt{2h+1} \sin \theta + 1)^i \cos^2 \theta d\theta \\ &= 2 \sum_{i=0}^{2m} a_i (2h+1) \int_0^{\frac{\pi}{2}} \sum_{r=0}^i C_i^r (2h+1)^{\frac{r}{2}} \sin^r \theta \cos^2 \theta d\theta \\ &= \sum_{i=0}^{2m} a_i \sum_{r=0}^i \alpha_{ir} (2h+1)^{\frac{r}{2}+1} \\ &= \sum_{i=0}^{2m} \sum_{k=i}^{2m} a_k \alpha_{ki} (2h+1)^{\frac{i}{2}+1}, \quad h \in (-\frac{1}{2}, 0), \end{aligned} \quad (3.12)$$

where

$$\alpha_{ir} = 2C_i^r \int_0^{\frac{\pi}{2}} \sin^r \theta \cos^2 \theta d\theta = C_i^r \mathbf{B}\left(\frac{r+1}{2}, \frac{3}{2}\right) > 0$$

and $\mathbf{B}(\bullet)$ is a Beta function.

Further, the curve $\widehat{D_1 C_1} : \frac{1}{2}y^2 - \frac{1}{2}x^2 = h$, $0 < x < 1$ intersects the x -axis at a point $(\sqrt{-2h}, 0)$. Then,

$$\begin{aligned} \int_{\widehat{D_1 C_1}} \sum_{i=0}^m b_i x^{2i} y dx &= 2 \int_{\sqrt{-2h}}^1 \sum_{i=0}^m b_i x^{2i} \sqrt{2h+x^2} dx \\ &= 2 \sum_{i=0}^m b_i I_i(h), \end{aligned} \quad (3.13)$$

where

$$I_i(h) = \int_{\sqrt{-2h}}^1 x^{2i} (2h+x^2)^{\frac{1}{2}} dx.$$

Recall that

$$\int x^{2i} (2h+x^2)^{\frac{1}{2}} dx = \frac{x^{2i-1} (2h+x^2)^{\frac{3}{2}}}{2i+2} + \frac{(1-2i)h}{i+1} \int x^{2i-2} (2h+x^2)^{\frac{1}{2}} dx, \quad i \geq 1. \quad (3.14)$$

Then, one can obtain the following recurrent relation

$$I_i(h) = \frac{(2h+1)^{\frac{3}{2}}}{2(i+1)} + \frac{(1-2i)h}{i+1} I_{i-1}(h), \quad i \geq 1,$$

which follows that

$$\begin{aligned} I_i(h) &= \frac{(2h+1)^{\frac{3}{2}}}{2(i+1)} \left(1 + \sum_{k=1}^{i-1} \prod_{j=0}^{k-1} \frac{2j+1-2i}{i-j} h^k\right) + \prod_{j=0}^{i-1} \frac{2j+1-2i}{i+1-j} h^i I_0(h) \\ &= \frac{1}{2} (2h+1)^{\frac{3}{2}} \sum_{k=0}^{i-1} \beta_{ki} h^k + \rho_i h^i I_0(h), \end{aligned}$$

where

$$\beta_{ki} = \frac{1}{i+1} \prod_{j=0}^{k-1} \frac{2j+1-2i}{i-j}, \quad k \geq 0, \quad \rho_i = \prod_{j=0}^{i-1} \frac{2j+1-2i}{i+1-j}, \quad i \geq 1. \quad (3.15)$$

Therefore, from (3.13), we get that

$$\begin{aligned} \int_{\widehat{D_1 C_1}} \sum_{i=0}^m b_i x^{2i} y dx &= 2b_0 I_0 + 2 \sum_{i=1}^m b_i I_i(h) \\ &= 2b_0 I_0(h) + 2 \sum_{i=1}^m b_i \left[\frac{1}{2} (2h+1)^{\frac{3}{2}} \sum_{k=0}^{i-1} \beta_{ki} h^k + \rho_i h^i I_0(h) \right] \\ &= \sum_{i=0}^{m-1} b_{i+1} \sum_{k=0}^i \beta_{k,i+1} h^k (2h+1)^{\frac{3}{2}} + 2b_0 I_0 + 2 \sum_{i=1}^m b_i \rho_i h^i I_0(h) \\ &= \sum_{i=0}^{m-1} \sum_{j=i}^{m-1} b_{j+1} \beta_{i,j+1} h^i (2h+1)^{\frac{3}{2}} + 2b_0 I_0 + 2 \sum_{i=1}^m b_i \rho_i h^i I_0(h). \end{aligned} \quad (3.16)$$

We note that for $a > 0$

$$\int \sqrt{x^2 - a^2} dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \ln |x + \sqrt{x^2 - a^2}| + C,$$

where C is a constant. Then, for $h \in (-\frac{1}{2}, 0)$

$$\begin{aligned} I_0(h) &= \int_{\sqrt{-2h}}^1 \sqrt{2h+x^2} dx = \frac{x}{2} \sqrt{2h+x^2} + h \ln |x + \sqrt{2h+x^2}| \Big|_{\sqrt{-2h}}^1 \\ &= \frac{1}{2} \sqrt{1+2h} + h \ln(1 + \sqrt{1+2h}) - \frac{\ln 2}{2} h - \frac{1}{2} h \ln |h|. \end{aligned}$$

Combining (3.6), (3.12), (3.16) and the above formula, we easily derive the expression of $M_1(h)$ in (3.9). This completes the proof. \square

Lemma 3.2. *Let (3.4) and (3.5) hold. Then, the function $M(h)$ in (3.7) can be expressed as*

$$M(h) = 2 \sum_{i=0}^{2m} A_i (2h+1)^{\frac{i}{2}+1} + 2 \sum_{i=0}^{m-1} B_i h^i (2h+1)^{\frac{3}{2}} + 2 \sum_{i=0}^m \tilde{B}_i h^i A(h), \quad h \in (0, +\infty), \quad (3.17)$$

where A_i , B_i , \tilde{B}_i and $A(h)$ are given in (3.10).

Proof. Similar to handling the integral in (3.11), by (3.12), one obtains that

$$\int_{\widehat{A_1 B_1}} \sum_{i=0}^{2m} a_i x^i y dx = \sum_{i=0}^{2m} \sum_{k=i}^{2m} a_k \alpha_{ki} (2h+1)^{\frac{i}{2}+1}, \quad h \in (0, +\infty). \quad (3.18)$$

Further, along the curve $\widehat{A_2 A_1} : y^2 - x^2 = 2h, \quad -1 < x < 1$. Then,

$$\int_{\widehat{A_2 A_1}} \sum_{i=0}^m b_i x^{2i} y dx = \int_{-1}^1 \sum_{i=0}^m b_i x^{2i} \sqrt{2h+x^2} dx = \sum_{i=0}^m b_i \tilde{I}_i(h), \quad (3.19)$$

where

$$\tilde{I}_i(h) = \int_{-1}^1 x^{2i} \sqrt{2h+x^2} dx.$$

Using (3.14), we can derive that

$$\tilde{I}_i(h) = \frac{(2h+1)^{\frac{3}{2}}}{i+1} + \frac{(1-2i)h}{i+1} \tilde{I}_{i-1}(h), \quad i \geq 1.$$

It implies that

$$\tilde{I}_i(h) = (2h+1)^{\frac{3}{2}} \sum_{k=0}^{i-1} \beta_{ki} h^k + \rho_i h^i \tilde{I}_0(h), \quad i \geq 1,$$

where β_{ki} and ρ_i are defined in (3.15). Inserting the above into (3.19), we achieve that

$$\begin{aligned} \int_{A_2 A_1} \sum_{i=0}^m b_i x^{2i} y dx &= b_0 \tilde{I}_0(h) + \sum_{i=1}^m b_i \tilde{I}_i(h) \\ &= b_0 \tilde{I}_0(h) + \sum_{i=1}^m b_i \left[(2h+1)^{\frac{3}{2}} \sum_{k=0}^{i-1} \beta_{ki} h^k + \rho_i h^i \tilde{I}_0(h) \right] \\ &= \sum_{i=0}^{m-1} \sum_{j=i}^m b_{j+1} \beta_{i,j+1} h^i (2h+1)^{\frac{3}{2}} + b_0 \tilde{I}_0(h) + \sum_{i=1}^m b_i \rho_i h^i \tilde{I}_0(h). \end{aligned} \tag{3.20}$$

Since, for $a > 0$,

$$\int \sqrt{x^2+a^2} dx = \frac{x}{2} \sqrt{x^2+a^2} + \frac{a^2}{2} \ln(x + \sqrt{2h+x^2}) + \tilde{C},$$

where \tilde{C} is a constant, we have for $h \in (0, +\infty)$

$$\begin{aligned} \tilde{I}_0(h) &= \int_{-1}^1 \sqrt{2h+x^2} dx = \frac{x}{2} \sqrt{2h+x^2} + h \ln |x + \sqrt{2h+x^2}| \Big|_{-1}^1 \\ &= \sqrt{1+2h} + 2h \ln(1 + \sqrt{1+2h}) - h \ln 2 - h \ln |h|. \end{aligned}$$

Therefore, from (3.18), (3.20) and the above, we can obtain the formula in (3.17). This finishes the proof. \square

Now, we are in the position of proving Theorem 2.1.

Proof of Theorem 2.1. First, we consider the expansions of $M_1(h)$ and $M(h)$ around $h = 0$. To do this, denote by

$$M_1^*(h) = \frac{M_1(h)}{2h+1}, \quad M^*(h) = \frac{M(h)}{2h+1}. \tag{3.21}$$

Then, from (3.9)

$$\begin{aligned}
 M_1^*(h) &= \sum_{i=0}^{2m} A_i(2h+1)^{\frac{i}{2}} + \sum_{i=0}^{m-1} B_i h^i (2h+1)^{\frac{1}{2}} + \sum_{i=0}^m \tilde{B}_i \frac{h^i A(h)}{2h+1} \\
 &= \sum_{i=0}^m A_{2i}(1+2h)^i + \sum_{i=0}^{m-1} A_{2i+1}(2h+1)^{i+\frac{1}{2}} + \sum_{i=0}^{m-1} B_i h^i (2h+1)^{\frac{1}{2}} + \sum_{i=0}^m \tilde{B}_i \frac{h^i A(h)}{2h+1} \\
 &= \sum_{i=0}^m A_{2i} \sum_{r=0}^i C_i^r 2^r h^r + \sum_{i=0}^{m-1} A_{2i+1} \sum_{r=0}^i C_i^r 2^r h^r (2h+1)^{\frac{1}{2}} + \sum_{i=0}^{m-1} B_i h^i (2h+1)^{\frac{1}{2}} \\
 &\quad + \sum_{i=0}^m \tilde{B}_i \frac{h^i A(h)}{2h+1} \\
 &= \sum_{i=0}^m A_i^* h^i + \sum_{i=0}^{m-1} \tilde{A}_i h^i (1+2h)^{\frac{1}{2}} + \sum_{i=0}^m \tilde{B}_i \frac{h^i A(h)}{2h+1}, \quad h \in (-\frac{1}{2}, 0), \tag{3.22}
 \end{aligned}$$

where

$$\begin{aligned}
 A_i^* &= \sum_{j=i}^m 2^i A_{2j} C_j^i, \quad i = 0, 1, \dots, m, \\
 \tilde{A}_i &= \sum_{j=i}^{m-1} 2^i A_{2j+1} C_j^i + B_i, \quad i = 0, 1, \dots, m-1.
 \end{aligned} \tag{3.23}$$

Similarly, in view of (3.17), one derives that

$$M^*(h) = 2 \sum_{i=0}^m A_i^* h^i + 2 \sum_{i=0}^{m-1} \tilde{A}_i h^i (1+2h)^{\frac{1}{2}} + 2 \sum_{i=0}^m \tilde{B}_i \frac{h^i A(h)}{2h+1}, \quad h \in (0, +\infty), \tag{3.24}$$

where A_i^* , \tilde{A}_i and \tilde{B}_i are given in (3.23).

By (3.10), it is easy to see that A_i , $i = 0, 1, \dots, 2m$ are independent of each other. Thus, on account of (3.23), $A_0^*, A_1^*, \dots, A_m^*, \tilde{A}_0, \tilde{A}_1, \dots, \tilde{A}_{m-1}$ are independent of each other. This implies that $A_0^*, A_1^*, \dots, A_m^*, \tilde{A}_0, \tilde{A}_1, \dots, \tilde{A}_{m-1}, \tilde{B}_0, \tilde{B}_1, \dots, \tilde{B}_m$ can be taken as free parameters.

For $|h| > 0$ small,

$$(1+2h)^{\frac{1}{2}} = 1 + \sum_{i \geq 1} \eta_i h^i, \quad \eta_1 = 1, \quad \eta_i = (-1)^{i-1} \frac{\prod_{r=0}^{i-2} (2r+1)}{\prod_{r=0}^{i-2} (r+2)}, \quad r \geq 2, \tag{3.25}$$

$$\frac{A(h)}{1+2h} = 1 - h \ln|h| + O(|h|).$$

Then, for $0 < -h \ll 1$, $M_1^*(h)$ in (3.22) can be expanded as

$$\begin{aligned}
 M_1^*(h) &= \sum_{i=0}^m A_i^* h^i + \sum_{i=0}^{m-1} \tilde{A}_i h^i (1 + \sum_{i \geq 1} \eta_i h^i) + \sum_{i=0}^m \tilde{B}_i h^i (1 - h \ln|h| + O(|h|)) \\
 &= \sum_{i=0}^m (C_i + D_i h \ln|h|) h^i + \sum_{i \geq m+1} C_i h^i, \tag{3.26}
 \end{aligned}$$

where

$$\begin{aligned}
 C_i &= A_i^* + \sum_{k=0}^i \tilde{A}_k \eta_{i-k} + O(|\tilde{B}_0, \tilde{B}_1, \dots, \tilde{B}_m|), \quad i = 0, 1, \dots, m-1, \\
 C_m &= A_m^* + \sum_{k=0}^{m-1} \tilde{A}_k \eta_{m-k} + O(|\tilde{B}_0, \tilde{B}_1, \dots, \tilde{B}_m|), \\
 C_i &= \sum_{k=0}^{m-1} \tilde{A}_k \eta_{i-k} + O(|\tilde{B}_0, \tilde{B}_1, \dots, \tilde{B}_m|), \quad i \geq m+1, \\
 D_i &= -\tilde{B}_i, \quad i = 0, 1, \dots, m.
 \end{aligned}
 \tag{3.27}$$

Similarly, $M^*(h)$ can be expressed as for $0 < h \ll 1$

$$M^*(h) = 2 \sum_{i=0}^m (C_i + D_i h \ln |h|) h^i + 2 \sum_{i \geq m+1} C_i h^i,
 \tag{3.28}$$

where C_i and D_i are the same as in (3.27).

Note that

$$\begin{aligned}
 &\det \frac{\partial(D_0, D_1, \dots, D_m, C_0, C_1, \dots, C_m, C_{m+1}, \dots, C_{2m})}{\partial(\tilde{B}_0, \tilde{B}_1, \dots, \tilde{B}_m, A_0^*, A_1^*, \dots, A_m^*, \tilde{A}_0, \dots, \tilde{A}_{m-1})} \\
 &= \begin{vmatrix} -I & 0 & 0 \\ \mathcal{D}_{11} & \mathcal{D}_{12} & \mathcal{D}_{13} \\ \mathcal{D}_{21} & 0 & \mathcal{D}_{23} \end{vmatrix} \triangleq |\mathcal{D}|,
 \end{aligned}
 \tag{3.29}$$

where I is a $(m+1) \times (m+1)$ identity matrix, \mathcal{D}_{11} , $\tilde{\mathcal{D}}_{13}$ and $\tilde{\mathcal{D}}_{21}$ are $(m+1) \times (m+1)$, $(m+1) \times m$ and $m \times (m+1)$ matrices, respectively, \mathcal{D}_{12} is a $(m+1) \times (m+1)$ upper triangular matrix whose elements on the diagonal are all 1, and

$$\mathcal{D}_{23} = \begin{pmatrix} \eta_{m+1} & \eta_m & \eta_{m-1} & \cdots & \eta_2 \\ \eta_{m+2} & \eta_{m+1} & \eta_m & \cdots & \eta_3 \\ \eta_{m+3} & \eta_{m+2} & \eta_m & \cdots & \eta_4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \eta_{2m} & \eta_{2m-1} & \eta_{2m-2} & \cdots & \eta_{m+1} \end{pmatrix}
 \tag{3.30}$$

and η_i are defined in (3.25). Then, we have from (3.29)

$$|\mathcal{D}| = (-1)^{m+1} |\mathcal{D}_{23}|,$$

where \mathcal{D}_{23} is given in (3.30). Since

$$\eta_i = (-1)^{i-1} \frac{\prod_{r=0}^{i-2} (2r+1)}{\prod_{r=0}^{i-2} (2r+4)} \times 2^{i-1} = 2^{i-1} \eta_{i-1}^*, \quad i \geq 2,$$

where

$$\eta_k^* = (-1)^k \frac{\prod_{r=0}^{k-1} (2r+1)}{\prod_{r=0}^{k-1} (2r+4)}, \quad k \geq 1.$$

Therefore, one finds that from (3.30)

$$\begin{aligned} |\mathcal{D}_{23}| &= \begin{vmatrix} 2^m \eta_m^* & 2^{m-1} \eta_{m-1}^* & 2^{m-2} \eta_{m-2}^* & \cdots & 2 \eta_1^* \\ 2^{m+1} \eta_{m+1}^* & 2^m \eta_m^* & 2^{m-1} \eta_{m-1}^* & \cdots & 2^2 \eta_2^* \\ 2^{m+2} \eta_{m+2}^* & 2^{m+1} \eta_{m+1}^* & 2^m \eta_m^* & \cdots & 2^3 \eta_3^* \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2^{2m-1} \eta_{2m-1}^* & 2^{2m-2} \eta_{2m-2}^* & 2^{2m-3} \eta_{2m-3}^* & \cdots & 2^m \eta_m^* \end{vmatrix} \\ &= 2^{m+m-1+m-2+\cdots+1} \begin{vmatrix} \eta_m^* & \eta_{m-1}^* & \eta_{m-2}^* & \cdots & \eta_1^* \\ 2 \eta_{m+1}^* & 2 \eta_m^* & 2 \eta_{m-1}^* & \cdots & 2 \eta_2^* \\ 2^2 \eta_{m+2}^* & 2^2 \eta_{m+1}^* & 2^2 \eta_m^* & \cdots & 2^2 \eta_3^* \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2^{m-1} \eta_{2m-1}^* & 2^{m-1} \eta_{2m-2}^* & 2^{m-1} \eta_{2m-3}^* & \cdots & 2^{m-1} \eta_m^* \end{vmatrix} \\ &= 2^{m+m-1+m-2+\cdots+1} \times 2^{1+2+\cdots+m-1} \begin{vmatrix} \eta_m^* & \eta_{m-1}^* & \eta_{m-2}^* & \cdots & \eta_1^* \\ \eta_{m+1}^* & \eta_m^* & \eta_{m-1}^* & \cdots & \eta_2^* \\ \eta_{m+2}^* & \eta_{m+1}^* & \eta_m^* & \cdots & \eta_3^* \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \eta_{2m-1}^* & \eta_{2m-2}^* & \eta_{2m-3}^* & \cdots & \eta_m^* \end{vmatrix} \\ &= 2^{m^2} |\mathcal{A}_{m,1}(2, 4, 2)|, \end{aligned}$$

where $\mathcal{A}_{m,1}(2, 4, 2)$ is given in (3.2) of [30]. From Lemma 3.2 of [30], one can see that $|\mathcal{D}_{23}| \neq 0$. This means that $|\mathcal{D}| \neq 0$. In other words, from (3.29), $D_0, D_1, \dots, D_m, C_0, C_1, \dots, C_{2m}$ in (3.26) can be taken as free parameters. Then, one can choose them satisfying

$$\begin{aligned} 0 &\ll -C_0 \ll D_0 \ll (-1)^2 C_1 \ll (-1) D_1 \ll \cdots \ll (-1)^m C_{m-1} \ll (-1)^{m-1} D_{m-1} \\ &\ll (-1)^{m+1} C_m \ll (-1)^m D_m \ll (-1)^{m+2} C_{m+1} \ll (-1)^{m+2} C_{m+2} \\ &\ll \cdots \ll (-1)^{m+2} C_{2m} \ll 1 \end{aligned}$$

such that the signs of $M_1^*(h)$ and $M^*(h)$ have been changed $3m+1$ and $m+1$ times, respectively. In view of (3.21), one can know that $M_1(h)$ can have $3m+1$ simple zeros for $0 < -h \ll 1$, at the same time, $M(h)$ can have $m+1$ simple zeros for $0 < h \ll 1$. On the other hand, $M_2(h) = M_1(h)$. Thus, system (2.2) can have $2(3m+1) + m+1 = 7m+3$ limit cycles near the generalized double loop L . This ends the proof. \square

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