# IMPACT OF NOISE IN A PHYTOPLANKTON-ZOOPLANKTON SYSTEM\*

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Abstract In this paper, we investigate the dynamics of a delayed toxic phytoplankton-two zooplankton system incorporating the effects of Lévy noise and white noise. The value of this study lies in two aspects: Mathematically, we first prove the existence of a unique global positive solution of the system, and then we investigate the sufficient conditions that guarantee the stochastic extinction and persistence in the mean of each population. Ecologically, via numerical simulations, we find that the effect of white noise or Lévy noise on the stochastic extinction and persistence of the two noises on the stochastic extinction and persistence of the two noises on the stochastic extinction and persistence of the two noises on the stochastic extinction and persistence of the two noises on the stochastic extinction. In addition, an increase in the toxin liberation rate or the intraspecific competition rate of zooplankton was found to be capable to increase the biomass of the phytoplankton but decrease the biomass of zooplankton. These results may help us to better understand the phytoplankton-zooplankton dynamics in the fluctuating environments.

Keywords Noise, phytoplankton, zooplankton, extinction, persistence.

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## 1. Introduction

Phytoplankton blooms, especially harmful algal blooms, occur frequently in various water bodies all over the world, such as Lake Kasumigaura in Japan [19], Lake Taihu in China [41], the Baltic Sea in Northern Europe [11], and the Caspian Sea in West Asia [40], etc., which seriously threaten the ecological balance, drinking water safety and human healthy, and cause huge economic losses to people [15,18]. However, a clear mechanism, which may provide a convincing explanation for the

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emergence of these phenomena, is still under investigation. Consequently, studying the dynamic mechanisms of interacting plankton in the aquatic environments is of considerable interest.

Actually, a growing effort has been devoted to seeking for a fundamental mechanism that underlies the possible formation or termination of phytoplankton blooms by different approaches [12,14,22,34]. Mathematical models have been revealed as a powerful tool for investigating the dynamics of aquatic plankton ecological systems qualitatively and quantitatively, as the research results may be applied to reveal the growth mechanisms of phytoplankton and to determine some key factors inducing phytoplankton blooms [8, 13, 21, 33, 42, 43, 48, 52].

In recent years, a number of mathematical models have been formulated to study the phytoplankton-zooplankton dynamics, and many physical and biological processes related to dynamic mechanisms of interacting plankton in the aquatic environments have been revealed [8, 12, 13, 21, 22, 33, 42–44, 48, 49, 52]. For example, Li et al. [22] showed that phytoplankton refuge and toxin have a significant impact on the occurring and terminating of algal blooms in the freshwater lakes. Dai et al. [13] found that delay cannot only induce instability, but can also promote the formation of spatial pattern via a Turing-like instability. Nevertheless, most of the remarkable results obtained under a framework of the deterministic environments [8, 12, 13, 21, 22, 33, 42, 43, 48, 52], which have been viewed as being unrealistic to some extent due to the fact that the environments in which plankton live are always uncertain and stochastic in the natural world.

In the existing literature, for all population models in the deterministic environments [7, 8, 12, 13, 21, 22, 33, 42, 43, 45, 47, 48, 50-52, 54], it is usually assumed that the parameters involved in the models are well-defined constants, but in fact they will fluctuate around some average value due to continuous fluctuation in the environments. In the aquatic ecosystems, there exist many objective environmental disturbance factors such as the unpredictability of photosynthetically active radiation, light availability, nutrition loading, water temperature variation, etc., which may have great impacts on the growth rate, the death rate and intraspecific competition rate in the plankton ecosystems [44, 49]. These factors are usually neglected in deterministic mathematical modelling of plankton and are difficult to predict in reality, but they can be described by white noise [16]. A fact has been further pointed out by May [39] that the birth rates, carrying capacity, competition coefficient, and other parameters involved in the system exhibit random fluctuations to a greater or lesser extent because of the effects of white noise. As a result, some mathematical models have been formulated to describe the effects of white noise on the dynamics of interacting plankton in recent years [2, 20, 24, 35, 44, 49, 53]. For example, in the work of Wu et al. [49], they showed that the environmental white noise disturbances are able to destabilize biological populations. Bandyopadhyay et al. [2] demonstrated that environmental noise forces play a crucial role in determining the stability or instability of two interacting phytoplankton species. These works strongly indicate that stochastic differential equation models may be a more appropriate way of modelling plankton ecosystems in many circumstances. In addition, the application of noise effects in other population dynamics or infectious disease dynamics can be found in references [4-6, 29, 46].

However, population ecosystems may also suffer sudden random environmental disturbances [1, 31], such as hurricanes, tsunamis, epidemics, etc., which can not be described by white noise. In other words, the nature of dynamics of population

can not be fully revealed by stochastic population ecosystems with only taking into account the effect of white noise. As indicated by Refs. [3, 28], one may use a Lévy process to model these phenomena and then these limitations in stochastic population systems can be overcome. In the pioneering work of Bao et al. [3], they initially studied the dynamics of Lotka-Volterra system with taking into account the effects of Lévy noises, and their results indicated that the sudden random environmental disturbances can be reasonably described by Lévy noise. In view of this point, a number of population ecosystems considering the effects of Lévy noise have been reported recently [25, 26, 55, 56]. For example, Zhang et al. [55]showed that Lévy noise is capable to stabilize the proposed population system. Zhao et al. [56] summarized that the Lévy noise may seriously affect the asymptotical behaviors of the ecosystem. The role of Lévy noise plays, in the dynamics of population ecosystems, which has been clearly demonstrated by these results.

In aquatic ecosystems, plankton populations, including phytoplankton and zooplankton, may also be affected by Lévy noise. In spite of much attention has been devoted to exploring various factors that may affect the dynamics of plankton ecological systems, the impacts of Lévy noise and white noise on the dynamics of interacting plankton in the aquatic environments are still rarely studied. So, naturally, some questions arise: How do Lévy noise and white noise affect the phytoplankton-zooplankton dynamics in the aquatic environments? Can these noises affect the extinction and persistence of phytoplankton and zooplankton? To answer the above questions, in the present paper, we extend a delayed toxin producing phytoplankton-two zooplankton system originally developed by Lv et al. [33] into a new stochastic version by incorporating the effects of white noise and Lévy noise, and we will investigate the dynamics of stochastic extinction and persistence in the mean of phytoplankton and zooplankton, which determine these plankton will die or survive in the future.

Based on the remarkable research in [33], the deterministic plankton system mentioned above can be given by

$$\begin{cases} \frac{dP(t)}{dt} = P(t) \left( r - \frac{r}{K} P(t) - \frac{\mu_1 Z_1(t)}{\alpha_1 + P(t)} - \frac{\mu_2 Z_2(t)}{\alpha_2 + P(t)} \right), \\ \frac{dZ_1(t)}{dt} = Z_1(t) \left( \frac{\beta_1 P(t)}{\alpha_1 + P(t)} - \frac{\rho_1 P(t - \tau)}{\alpha_1 + P(t - \tau)} - d_1 - g_1 Z_1(t) \right), \\ \frac{dZ_2(t)}{dt} = Z_2(t) \left( \frac{\beta_2 P(t)}{\alpha_2 + P(t)} - \frac{\rho_2 P(t - \tau)}{\alpha_2 + P(t - \tau)} - d_2 - g_2 Z_2(t) \right), \end{cases}$$
(1.1)

subject to the initial condition  $\Delta := \{(P_0, Z_{10}, Z_{20}) = (\Psi_1(\zeta), \Psi_2(\zeta), \Psi_3(\zeta)) \geq 0, \Psi_i \in C([-\tau, 0]; \mathbb{R}^3_+), \Psi_i(0) > 0, i = 1, 2, 3\}$ , where  $C([-\tau, 0]; \mathbb{R}^3_+)$  stands for the family of all bounded and continuous functions from  $[-\tau, 0]$  to  $\mathbb{R}^3_+ := \{(P, Z_1, Z_2) \in \mathbb{R}^3 | P \geq 0, Z_1 \geq 0, Z_2 \geq 0\}$ ; P(t) represents the density of toxin producing phytoplankton population at any instant of time t and  $\tau$  indicates that the liberation of toxic substances by the phytoplankton is not an instantaneous process but is mediated by some time lag required for maturity of the species;  $Z_j(t)(j = 1, 2)$  denotes the densities of zooplankton population at any instant of time t. For the biological significance of the other parameters, we refer to [33]. Recently, Shi et al. [43] considered the gestation delays of zooplankton into the system (1.1) and investigated the existence of Hopf bifurcation. To the best of our knowledge, up to now, no body has explored the impacts of random environmental fluctuations on the system.

Now, we introduce two types of noise fluctuations mentioned above into the deterministic plankton system (1.1). In the existing literature, there are many ways to introduce noise disturbances into population ecosystem. Following the method used in [30], we assume that white noise is proportional to the size of phytoplankton population; for zooplankton population, taking the method introduced in [32], we assume the white noise disturbances mainly affect the intraspecific competition rate  $g_i$  and death rate  $d_i(i = 1, 2)$ , that is,  $-d_1 \rightarrow -d_1 + \delta_2 dB_2(t), -g_1 \rightarrow -g_1 + \delta_3 dB_3, -d_2 \rightarrow -d_2 + \delta_4 dB_4$  and  $-g_2 \rightarrow -g_2 + \delta_5 dB_5$ . For the Lévy noise, we follow the method used in [27] and let Lévy noise be proportional to the size of plankton population, then system (1.1) becomes

$$\begin{cases} dP(t) = P(t^{-}) \left[ r(1 - \frac{P(t^{-})}{K}) - \frac{\mu_1 Z_1(t^{-})}{\alpha_1 + P(t^{-})} - \frac{\mu_2 Z_2(t^{-})}{\alpha_2 + P(t^{-})} \right] dt \\ + \delta_1 P(t^{-}) dB_1(t) + \int_{\mathbb{Y}} P(t^{-}) \gamma_1(u) \tilde{N}(dt, du), \\ dZ_1(t) = Z_1(t^{-}) \left[ \frac{\beta_1 P(t^{-})}{\alpha_1 + P(t^{-})} - \frac{\rho_1 P(t - \tau)}{\alpha_1 + P(t - \tau)} - d_1 - g_1 Z_1(t^{-}) \right] dt \\ + \delta_2 Z_1(t^{-}) dB_2(t) + \delta_3 Z_1^2(t^{-}) dB_3(t) + \int_{\mathbb{Y}} Z_1(t^{-}) \gamma_2(u) \tilde{N}(dt, du), \\ dZ_2(t) = Z_2(t^{-}) \left[ \frac{\beta_2 P(t^{-})}{\alpha_2 + P(t^{-})} - \frac{\rho_2 P(t - \tau)}{\alpha_2 + P(t - \tau)} - d_2 - g_2 Z_2(t^{-}) \right] dt \\ + \delta_4 Z_2(t^{-}) dB_4(t) + \delta_5 Z_2^2(t^{-}) dB_5(t) + \int_{\mathbb{Y}} Z_2(t^{-}) \gamma_3(u) \tilde{N}(dt, du), \end{cases}$$
(1.2)

where  $P(t^{-})$  denotes the left limit of P(t) and  $Z_i(t^{-})$  means the left limit of  $Z_i(t)(i = 1, 2)$ . N represents a Poisson counting measure with compensator  $\tilde{N}$  and characteristic measure  $\lambda$  on a measurable subset  $\mathbb{Y}$  of  $(0, \infty)$  satisfying  $\lambda(\mathbb{Y}) < \infty$ , and  $\tilde{N}(dt, du) = N(dt, du) - \lambda(du)dt, \gamma_i : \mathbb{Y} \times \Omega \to \mathbb{R}(i = 1, 2, 3)$  is bounded and continuous with respect to  $\lambda$  and is  $\mathcal{B}(\mathbb{Y}) \times \mathcal{F}_t$ -measurable.  $B_j(t)$  are mutually independent standard Brownian motions defined on the probability space  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$  with a filtration  $\{\mathcal{F}_t\}_{t\geq 0}$  satisfying the usual conditions (i.e., it is increasing and right continuous with  $\mathcal{F}_0$  contains all  $\mathbb{P}$ -null sets),  $dB_s(t)(s = 1, 2, 3, 4, 5)$  represents the white noise and  $\delta_j^2$  denotes the intensity of the white noise, j = 1, 2, 3, 4, 5. The initial conditions and the biological significance of all the parameters in system (1.2) are the same as shown in system (1.1). Furthermore, we assume that there exists a constant c > 0 such that  $(A) : \int_{\mathbb{Y}} (\ln(1 + \gamma_i(u)) \vee \ln(1 + \gamma_i(u))^2) < c, 1 + \gamma_i(u) > 0, u \in \mathbb{Y}, i = 1, 2, 3$ . From a biological viewpoint, it denotes that the intensities of Lévy jumps are not very large.

The rest of this article is organized as follows: In Section 2, the existence and uniqueness of global positive solution in system (1.2) is proved. Section 3 is devoted to studying the sufficient conditions that guarantee the stochastic extinction and persistence in the mean of each population. Section 4, numerical simulations are carried out to verify the analytical results. Finally, we give a brief discussion and a summary of our main results.

## 2. The existence of a unique global positive solution

In this section, by using the Lyapunov analysis method [36], we verify that the solution of system (1.2) is global and positive.

**Theorem 2.1.** Let (A) hold, for any initial value  $(P_0, Z_{10}, Z_{20}) \in \Delta$ , then system (1.2) admits a unique global positive solution  $(P(t), Z_1(t), Z_2(t)) \in \mathbb{R}^3_+$  on  $t \geq -\tau$  with probability one.

**Proof.** Since the coefficients of system (1.2) satisfy the local Lipschitz condition, then for any initial values  $(P_0, Z_{10}, Z_{20}) \in \Delta$ , there exists a unique local solution  $(P(t), Z_1(t), Z_2(t))$  on  $t \in [-\tau, \tau_e)$ , where  $\tau_e$  denotes the explode time [36]. To show that this solution is global in  $\mathbb{R}^3_+$ , we need to show that  $\tau_e = \infty$  a.s.. To this end, we choose a sufficiently large non-negative number  $\epsilon_0$  such that  $P_0, Z_{10}$  and  $Z_{20}$  lie with in the interval  $[\frac{1}{\epsilon_0}, \epsilon_0]$ . For each integer  $\epsilon$ , we can define the stopping time

$$\tau_{\epsilon} = \inf\left\{t \in [-\tau, \tau_{e}) : P(t) \notin (\frac{1}{\epsilon}, \epsilon) \text{ or } Z_{1}(t) \notin (\frac{1}{\epsilon}, \epsilon) \text{ or } Z_{2}(t) \notin (\frac{1}{\epsilon}, \epsilon)\right\},$$

where  $\inf \emptyset = \infty$  (as usual  $\emptyset$  denotes the empty set).  $\tau_{\epsilon}$  is increasing as  $\epsilon \to \infty$ . Set  $\tau_{\infty} = \lim_{\epsilon \to \infty} \tau_{\epsilon}$ , then  $\tau_{\infty} \leq \tau_{\epsilon} a.s.$ 

In the following, we need to show that  $\tau_e = \infty \ a.s.$  If this statement is violated, there exists two constants T > 0 and  $\sigma \in (0, 1)$  such that

$$P\{\tau_{\infty} \le T\} > \sigma.$$

Hence we can find an integer  $\epsilon_1 \geq \epsilon_0$  such that

$$P\{\tau_{\epsilon} \le T\} \ge \sigma,$$

for all  $\epsilon > \epsilon_1$ . Define a  $C^2$ -function  $V : \mathbb{R}^3_+ \to \mathbb{R}_+$  by

$$V(P, Z_1, Z_2) = (\sqrt{P} - 1 - 0.5 \ln P) + (\sqrt{Z_1} - 1 - 0.5 \ln Z_1) + (\sqrt{Z_2} - 1 - 0.5 \ln Z_2) + 0.5 \int_{t-\tau}^t \frac{\rho_1 P(s)}{\alpha_1 + P(s)} ds + 0.5 \int_{t-\tau}^t \frac{\rho_2 P(s)}{\alpha_2 + P(s)} ds.$$

Considering that  $(\sqrt{A_1} - 1 - 0.5 \ln A_1) \ge 0$  for all  $A_1 > 0$ , the function  $V(\cdot)$  is positive definite for all  $(P, Z_1, Z_2) \in \mathbb{R}^3_+$ . Calculating the differential of V along the solution trajectories of system (1.2) by using Itô's formula, we get

$$\begin{split} dV(P,Z_1,Z_2) =& LV(P,Z_1,Z_2)dt + 0.5\delta_1(\sqrt{P}-1)dB_1(t) \\ &+ \int_{\mathbb{Y}} \left[ (\ln(1+\gamma_1(u))^{\frac{1}{2}}-1)P^{\frac{1}{2}} - 0.5\ln(1+\gamma_1(u)) \right] \tilde{N}(dt,du) \\ &+ 0.5\delta_2(\sqrt{Z_1}-1)dB_2(t) + 0.5\delta_3Z_1(\sqrt{Z_1}-1)dB_3(t) \\ &+ \int_{\mathbb{Y}} \left[ (\ln(1+\gamma_2(u))^{\frac{1}{2}}-1)Z_1^{\frac{1}{2}} - 0.5\ln(1+\gamma_2(u)) \right] \tilde{N}(dt,du) \\ &+ 0.5\delta_4(\sqrt{Z_2}-1)dB_4(t) + 0.5\delta_5Z_2(\sqrt{Z_2}-1)dB_5(t) \\ &+ \int_{\mathbb{Y}} \left[ (\ln(1+\gamma_3(u))^{\frac{1}{2}}-1)Z_2^{\frac{1}{2}} - 0.5\ln(1+\gamma_3(u)) \right] \tilde{N}(dt,du), \end{split}$$

where

$$\begin{split} LV(P,Z_1,Z_2) = 0.5(\sqrt{P}-1)\Big(r-\frac{r}{K}P-\frac{\mu_1Z_1}{\alpha_1+P}-\frac{\mu_2Z_2}{\alpha_2+P}\Big) + 0.125(2-\sqrt{P})\delta_1^2 \\ &+ \int_Y [((1+\gamma_1(u))^{\frac{1}{2}}-1-0.5\gamma_1(u))P^{\frac{1}{2}}+0.5(\gamma_1(u) \\ &-\ln(1+\gamma_1(u)))]\lambda(du) \\ &+ 0.5(\sqrt{Z_1}-1)\bigg(\frac{\beta_1P}{\alpha_1+P}-\frac{\rho_1P(t-\tau)}{\alpha_1+P(t-\tau)}-d_1-g_1Z_1\bigg) \\ &+ 0.125(2-\sqrt{Z_1})\delta_2^2 \\ &+ 0.125Z_1^2\delta_3^2(2-\sqrt{Z_1})+\int_Y [((1+\gamma_2(u))^{\frac{1}{2}}-1-0.5\gamma_2(u))Z_1^{\frac{1}{2}} \\ &+ 0.5(\gamma_2(u)-\ln(1+\gamma_2(u)))]\lambda(du) + 0.5(\sqrt{Z_2}-1)\bigg(\frac{\beta_2P}{\alpha_2+P} \\ &-\frac{\rho_2P(t-\tau)}{\alpha_2+P(t-\tau)}-d_2-g_2Z_2\bigg) + 0.125(2-\sqrt{Z_2})\delta_4^2 \\ &+ 0.125Z_2^2\delta_5^2(2-\sqrt{Z_2}) \\ &+ \int_Y [((1+\gamma_3(u)))^{\frac{1}{2}}-1-0.5\gamma_1(u))Z_2^{\frac{1}{2}}+0.5(\gamma_3(u) \\ &-\ln(1+\gamma_3(u)))]\lambda(du) + 0.5\frac{\rho_1P(t)}{\alpha_2+P(t)} \\ &- 0.5\frac{\rho_1P(t-\tau)}{\alpha_1+P(t-\tau)}+0.5\frac{\rho_2P(t)}{\alpha_2+P}P^{\frac{1}{2}}-r+\frac{r}{K}P+\frac{\mu_1Z_1}{\alpha_1+P} \\ &+\frac{\mu_2Z_2}{\alpha_2+P}\bigg) + 0.25\delta_1^2-0.125\delta_1^{\frac{1}{2}}+0.5\bigg(\frac{\beta_1P}{\alpha_1+P}Z_1^{\frac{1}{2}} \\ &- \frac{\rho_1P(t-\tau)}{\alpha_1+P(t-\tau)}Z_1^{\frac{1}{2}}-d_1Z_1^{\frac{1}{2}} \\ &- g_1Z_1^{\frac{3}{2}}-\frac{\beta_1P}{\alpha_1+P}+\frac{\rho_1P(t-\tau)}{\alpha_1+P(t-\tau)} \\ &+ d_1+g_1Z_1\bigg) + 0.25\delta_2^2-0.125\delta_2^2Z_1^{\frac{1}{2}} \\ &- \frac{\rho_2P(t-\tau)}{\alpha_2+P(t-\tau)}Z_2^{\frac{1}{2}}-d_2Z_2^{\frac{1}{2}}-g_2Z_2^{\frac{1}{2}} \\ &- \frac{\beta_2P}{\alpha_2+P}+\frac{\rho_2P(t-\tau)}{\alpha_2+P(t-\tau)}+d_2+g_2Z_2) + 0.25\delta_4^2 \\ &- 0.125\delta_4^2Z_2^{\frac{1}{2}} + 0.25\delta_5^2Z_2^{\frac{1}{2}} \\ &- 0.25\delta_5^2Z_2^{\frac{1}{2}} + 0.25\delta_5^2Z_2^{\frac{1}{2}} \\ &- 0.5\frac{\rho_1P(t-\tau)}{\alpha_1+P(t-\tau)} - 0.5\frac{\rho_2P(t-\tau)}{\alpha_2+P(t-\tau)} \\ \end{array}$$

$$\begin{split} &+ \int_{\mathbb{Y}} [((1+\gamma_1(u))^{\frac{1}{2}} - 1 - 0.5\gamma_1(u))P^{\frac{1}{2}} + 0.5(\gamma_1(u) \\ &- \ln(1+\gamma_1(u)))]\lambda(du) \\ &+ \int_{\mathbb{Y}} [((1+\gamma_2(u))^{\frac{1}{2}} - 1 - 0.5\gamma_2(u))Z_1^{\frac{1}{2}} + 0.5(\gamma_2(u) \\ &- \ln(1+\gamma_2(u)))]\lambda(du) \\ &+ \int_{\mathbb{Y}} [((1+\gamma_3(u))^{\frac{1}{2}} - 1 - 0.5\gamma_3(u))Z_3^{\frac{1}{2}} + 0.5(\gamma_3(u) \\ &- \ln(1+\gamma_3(u)))]\lambda(du) \\ &\leq - 0.125\delta_3^2 Z_1^{\frac{5}{2}} - 0.125\delta_5^2 Z_2^{\frac{5}{2}} + 0.25\delta_3^2 Z_1^2 + 0.25\delta_5^2 Z_2^2 \\ &+ 0.5(\frac{r}{K} + \frac{\rho_1}{\alpha_1} + \frac{\rho_2}{\alpha_2})P \\ &+ 0.5(\frac{\mu_1}{\alpha_1} + g_1)Z_1 + 0.5(\frac{\mu_2}{\alpha_2} + g_2)Z_2 - 0.5\frac{r}{K}P^{\frac{3}{2}} - 0.5g_1Z_1^{\frac{3}{2}} \\ &- 0.5g_2 Z_2^{\frac{3}{2}} + (0.5r - 0.125\delta_1^2)P^{\frac{1}{2}} + (0.5\frac{\beta_1}{\alpha_1} - 0.5d_1 - 0.125\delta_2^2)Z_1^{\frac{1}{2}} \\ &+ (0.5\frac{\beta_2}{\alpha_2} - 0.5d_2 - 0.125\delta_4^2)Z_2^{\frac{1}{2}} \\ &+ 0.25(\delta_1^2 + \delta_2^2 + \delta_4^2) + 0.5(d_1 + d_2 - r) \\ &+ 0.5\int_{\mathbb{Y}} (|\gamma_1(u)| + |\ln(1+\gamma_1(u))|)\lambda(du) \\ &+ 0.5\int_{\mathbb{Y}} (|\gamma_2(u)| + |\ln(1+\gamma_3(u))|\lambda(du) \\ &+ 0.5\int_{\mathbb{Y}} |\gamma_3(u)| + |\ln(1+\gamma_3(u))|\lambda(du) \\ &\leq \Upsilon, \end{split}$$

where  $\Upsilon$  is a positive constant. The reminder of proof is similar to those works in the study [37] and hence is omitted here. This completes the proof.

## 3. Stochastic extinction and persistence in the mean

In this section, we are committed to investigating the stochastic extinction and persistence in the mean of each population in system (1.2). For the sake of convenience, we give the following notations and definition before presenting the main results.

$$\begin{split} \Phi_1 &= \lim_{t \to \infty} \sup 0.5 \delta_3^2 t^{-1} \int_0^t Z_1^2(s) ds, \\ \Phi_2 &= \lim_{t \to \infty} \sup 0.5 \delta_5^2 t^{-1} \int_0^t Z_2^2(s) ds, \\ Q &= r - 0.5 \delta_1^2 + \int_{\mathbb{Y}} (\ln(1 + \gamma_1(u)) - \gamma_1(u)) \lambda(du), \\ Q_1 &= \beta_1 - d_1 - 0.5 \delta_2^2 + \int_{\mathbb{Y}} (\ln(1 + \gamma_2(u)) - \gamma_2(u)) \lambda(du), \end{split}$$

$$\begin{aligned} Q_{2} &= \beta_{2} - d_{2} - 0.5\delta_{4}^{2} + \int_{\mathbb{Y}} (\ln(1+\gamma_{3}(u)) - \gamma_{3}(u))\lambda(du), \\ Q_{3} &= \rho_{1} + \frac{\alpha_{1}^{2} - \beta_{1}^{2}}{2\alpha_{1}} + \int_{\mathbb{Y}} (\ln(1+\gamma_{2}(u)) - \gamma_{2}(u))\lambda(du), \\ Q_{4} &= \rho_{2} + \frac{\alpha_{2}^{2} - \beta_{2}^{2}}{2\alpha_{2}} + \int_{\mathbb{Y}} (\ln(1+\gamma_{3}(u)) - \gamma_{3}(u))\lambda(du), \\ M_{1}(t) &= \int_{0}^{t} \delta_{1} dB_{i}(s), \quad M_{2}(t) = \int_{0}^{t} \delta_{2} dB_{2}, \\ M_{3}(t) &= \int_{0}^{t} \delta_{3} Z_{1}(s) dB_{3}(s), \quad M_{4}(t) = \int_{0}^{t} \delta_{4} dB_{4}(s), \\ M_{5}(t) &= \int_{0}^{t} \delta_{5} Z_{2}(s) dB_{5}(s), \quad K_{i}(t) = \int_{0}^{t} \int_{\mathbb{Y}} \ln(1+\gamma_{i}) \tilde{N}(dt, du), i = 1, 2, 3, \\ \langle f(t) \rangle &= t^{-1} \int_{0}^{t} f(s) ds, \quad \langle f(t) \rangle_{*} = \inf \frac{1}{t} \int_{0}^{t} f(s) ds, \quad \langle f(t) \rangle^{*} = \sup \frac{1}{t} \int_{0}^{t} f(s)(s) ds. \end{aligned}$$

**Definition 3.1** ([57]). (1) The population x(t) is said to be extinct if  $\lim_{t\to\infty} x(t) = 0$  a.s.

(2) If  $\lim_{t\to\infty} \langle x(t) \rangle_* > 0$  a.s., the population x(t) is said to be strongly persistent in the mean.

(3) If  $\lim_{t\to\infty} \langle x(t) \rangle^* > 0$  a.s., the population x(t) is said to be weakly persistent in the mean.

Using the above notations and definition, we can get the following results regarding the persistence or extinction of plankton for system (1.2).

**Theorem 3.1.** Let (A) hold, the solution of system (1.2) satisfies the following properties.

(1) For any solution  $(P(t), Z_1(t), Z_2(t) \in \mathbb{R}^3_+$  of system (1.2), we get that

$$\lim_{t \to \infty} \sup \frac{\ln P(t)}{t} \le Q \quad a.s., \quad \lim_{t \to \infty} \sup \frac{\ln Z_1(t)}{t} \le Q_1 \quad a.s.,$$
$$\lim_{t \to \infty} \sup \frac{\ln Z_2(t)}{t} \le Q_2 \quad a.s.$$

(2) If  $Q < 0, Q_1 < 0$  and  $Q_2 < 0$ , then

$$\lim_{t \to \infty} P(t) = 0 \quad a.s., \quad \lim_{t \to \infty} Z_1(t) = 0 \quad a.s., \quad \lim_{t \to \infty} Z_2(t) = 0 \quad a.s.$$

*i.e.*, the phytoplankton and zooplankton populations will go to extinction with probability one.

(3) If Q > 0,  $Q_1 > 0$  and  $Q_2 > 0$ , then

$$\lim_{t \to \infty} \langle P(t) \rangle^* \le \frac{QK}{r} \quad a.s., \quad \lim_{t \to \infty} \langle Z_1(t) \rangle^* \le \frac{Q_1}{g_1} \quad a.s., \quad \lim_{t \to \infty} \langle Z_2(t) \rangle^* \le \frac{Q_2}{g_2} \quad a.s.,$$

i.e., both phytoplankton and zooplankton populations for system (1.2) are weakly persistent in the mean with probability one.

persistent in the mean with probability one. (4) If  $Q > \frac{\mu_1 Q_1}{\alpha_1 g_1} + \frac{\mu_2 Q_2}{\alpha_2 g_2}$ ,  $Q_1 > 0$  and  $Q_2 > 0$ , we have

$$\lim_{t \to \infty} \langle P(t) \rangle_* \ge \frac{QK\alpha_1\alpha_2 g_1 g_2 - KQ_1 \mu_1 \alpha_2 g_2 - KQ_2 \mu_2 \alpha_1 g_1}{rg_1 g_2 \alpha_1 \alpha_2} \quad a.s.,$$

*i.e.*, the phytoplankton population is strongly persistent in the mean with probability one. (5) If  $Q_3 > d_1 + 0.5\delta_2^2 + \Phi_1 + \frac{\rho_1 K Q}{\alpha \cdot r}$  and Q > 0, we get

$$\lim_{t \to \infty} \langle Z_1(t) \rangle_* \ge \frac{Q_3 \alpha_1 r - d_1 \alpha_1 r - 0.5 \delta_2^2 \alpha_1 r - \Phi_1 \alpha_1 r - \rho_1 Q K}{\alpha_1 g_1 r} \quad a.s.,$$

*i.e.*, the zooplankton population  $Z_1$  is strongly persistent in the mean with probability one.

(6) If  $Q_4 > d_2 + 0.5\delta_4^2 + \Phi_2 + \frac{\rho_2 QK}{\alpha_2 r}$  and Q > 0, we derive

$$\lim_{t \to \infty} \langle Z_2(t) \rangle_* \ge \frac{Q_4 \alpha_2 r - d_2 \alpha_2 r - 0.5 \delta_4^2 \alpha_2 r - \Phi_2 \alpha_2 r - \rho_2 K Q}{g_2 \alpha_2 r} \quad a.s.,$$

*i.e.*, the zooplankton population  $Z_2$  is strongly persistent in the mean with probability one.

**Proof.** Making use of the generalized It $\hat{o}$ 's formula to the system (1.2) yields

$$\ln P(t) - \ln P(0) = \int_0^t \left[ r - 0.5\delta_1^2 + \int_{\mathbb{Y}} (\ln(1 + \gamma_1(u)) - \gamma_1(u))\lambda(du), - \frac{r}{K}P(s) - \frac{\mu_1 Z_1(s)}{\alpha_1 + P(s)} - \frac{\mu_2 Z_2(s)}{\alpha_2 + P(s)} \right] ds, + \int_0^t \delta_1 dB_1(s) + \int_0^t \int_{\mathbb{Y}} \ln(1 + \gamma_1(u))\tilde{N}(ds, du).$$
(3.1)

From Eq.(3.1), it is not difficult to derive

$$t^{-1}\ln\frac{P(t)}{P(0)} \le Q - \frac{r}{K}\langle P(s)\rangle + \frac{M_1(t)}{t} + \frac{K_1(t)}{t}.$$
(3.2)

Let  $f_i(P) = \frac{\beta_i P}{\alpha_i + P}$  (i = 1, 2), and then by applying the positivity of P, we have that  $f_i(P) = \beta_i - \frac{\beta_i \alpha_i}{\alpha_i + P} \leq \beta_i$ . Similarly, we can get

$$t^{-1} \ln \frac{Z_1(t)}{Z_1(0)} \leq Q_1 - g_1 \langle Z_1(s) \rangle - 0.5 \delta_3^2 t^{-1} \int_0^t Z_1^2(s) ds, + \frac{M_3(t)}{t} + \frac{M_2(t)}{t} + \frac{K_2(t)}{t},$$
(3.3)

$$t^{-1} \ln \frac{Z_2(t)}{Z_2(0)} \leq Q_2 - g_2 \langle Z_2(s) \rangle - 0.5 \delta_5^2 t^{-1} \int_0^t Z_2^2(s) ds, + \frac{M_5(t)}{t} + \frac{M_4(t)}{t} + \frac{K_3(t)}{t}.$$
(3.4)

Based on the strong law of large number for local martingale [23] and Assumption (A) in the Introduction part, we can know that  $\lim_{t\to\infty} \frac{M_i(t)}{t} = 0$  a.s., i = 1, 2, 4 and  $\lim_{t\to\infty} \frac{K_j(t)}{t} = 0$  a.s., j = 1, 2, 3. From (3.2), we have that

$$\frac{\ln P(t)}{t} \le Q + \frac{\ln P(0)}{t} + \frac{M_1}{t} + \frac{K_1(t)}{t}.$$

Taking the limit on both side of the above inequality, we can derive

$$\lim_{t \to \infty} \sup \frac{\ln P(t)}{t} \le Q \quad a.s.$$

Therefore, it is clear that  $\lim_{t\to\infty} P(t) = 0$  a.s. if Q < 0 holds. By applying Lemma 2 in [27], if Q > 0, we have

$$\lim_{t \to \infty} \langle P(t) \rangle^* \le \frac{QK}{r} \quad a.s.$$

On the other hand, the quadratic variation of  $M_3(t)$  is

$$\langle M_3, M_3 \rangle(t) = \delta_3^2 \int_0^t Z_1^2(s) ds.$$

In view of the exponential martingale inequality [36], we can observe that

$$\mathbb{P}\{\sup_{0\leq t\leq \epsilon} [M_3(t) - 0.5\langle M_3, M_3\rangle(t)] \geq 2\ln\epsilon\} \leq \frac{1}{\epsilon^2}.$$

By the Borel-Cantelli Lemma [38], we can obtain that for almost all  $\omega_1 \in \Omega$ , there is a random integer  $\epsilon_0 = \epsilon(\omega_1)$  such that for  $\epsilon > \epsilon_0$ ,

$$\sup_{0 \le t \le \epsilon} [M_3(t) - 0.5 \langle M_3, M_3 \rangle(t)] \ge 2 \ln \epsilon.$$

Namely, we have shown

$$M_3(t) \le 2\ln\epsilon + 0.5\langle M_3, M_3(t) \rangle = 2\ln\epsilon + 0.5\delta_3^2 \int_0^t Z_1^2(s)ds,$$
(3.5)

for all  $0 \le t \le \epsilon$  and  $\epsilon > \epsilon_0$  a.s. By substituting (3.5) into (3.3) leads to

$$t^{-1}\ln\frac{Z_1(t)}{Z_1(0)} \le Q_1 - g_1 \langle Z_1(t) \rangle + \frac{M_2(t)}{t} + \frac{K_2(t)}{t} + \frac{2\ln\epsilon}{\epsilon - 1},$$
(3.6)

for  $0 < \epsilon - 1 \le t \le \epsilon$  and  $\epsilon \ge \epsilon_0$  a.s.

From (3.4), similarly, we have

$$t^{-1}\ln\frac{Z_2(t)}{Z_2(0)} \le Q_2 - g_2 \langle Z_2(t) \rangle + \frac{M_4(t)}{t} + \frac{K_3(t)}{t} + \frac{2\ln\epsilon}{\epsilon - 1},$$
(3.7)

for all  $0 < \epsilon - 1 \le t \le \epsilon$  and  $\epsilon \ge \epsilon_0$  a.s.

Using the same analysis method, in view of (3.6) and (3.7), and then we obtain

$$\lim_{t\to\infty}\sup\frac{\ln Z_1(t)}{t}\leq Q_1\quad a.s.,\quad \lim_{t\to\infty}\sup\frac{\ln Z_2(t)}{t}\leq Q_2\quad a.s.$$

Moreover, if  $Q_i < 0$ , we can get that  $\lim_{t\to\infty} Z_i(t) = 0$  a.s., i = 1, 2. By applying Lemma 2 in [27], if  $Q_1 > 0$  and  $Q_2 > 0$  hold, we have

$$\lim_{t \to \infty} \langle Z_1(t) \rangle^* \le \frac{Q_1}{g_1} \quad a.s., \quad \lim_{t \to \infty} \langle Z_2(t) \rangle^* \le \frac{Q_2}{g_2} \quad a.s.$$

By virtue of  $\beta_i - \rho_i - d_i > 0$  and the positivity of P, we have that  $f_i(P) > \rho_i + \frac{\alpha_i^2 - \beta_i^2}{2\alpha_i}$ , i = 1, 2. Supposing that  $Q > 0, Q_1 > 0$  and  $Q_2 > 0$ , and making use of the Itô's formula to system (1.2) again and then we can get

$$t^{-1}\ln\frac{P(t)}{P(0)} \ge Q - \frac{r}{K}\langle P(t)\rangle - \frac{\mu_1}{\alpha_1}\langle Z_1(t)\rangle^* - \frac{\mu_2}{\alpha_2}\langle Z_2(t)\rangle^* + \frac{M_1(t)}{t} + \frac{K_1(t)}{t}, \quad (3.8)$$

$$t^{-1} \ln \frac{Z_1(t)}{Z_1(0)} \ge Q_3 - d_1 - 0.5\delta_2^2 - \Phi_1 - \frac{\rho_1}{\alpha_1} \langle P(t) \rangle^* - g_1 \langle Z_1(t) \rangle, + \frac{\rho_1}{\alpha_1} t^{-1} (\int_{t-\tau}^t P(s)dt - \int_{-\tau}^t P(s)dt) + \frac{M_2(t)}{t} + \frac{M_3(t)}{t} + \frac{K_2(t)}{t},$$
(3.9)

$$t^{-1} \ln \frac{Z_2(t)}{Z_2(0)} \ge Q_4 - d_2 - 0.5\delta_4^2 - \Phi_2 - \frac{\rho_2}{\alpha_2} \langle P(t) \rangle^* - g_2 \langle Z_2(t) \rangle, + \frac{\rho_2}{\alpha_2} t^{-1} (\int_{t-\tau}^t P(s)dt - \int_{-\tau}^t P(s)dt) + \frac{M_4(t)}{t} + \frac{M_5(t)}{t} + \frac{K_3(t)}{t}.$$
(3.10)

From (3.8), if  $Q - \frac{\mu_1 Q_1}{\alpha_1 g_1} - \frac{\mu_2 Q_2}{\alpha_2 g_2} > 0$  and  $Q_i > 0 (i = 1, 2)$ , based on Lemma 2 in [27], we have

$$\lim_{t \to \infty} \langle P(t) \rangle_* \ge \frac{QK\alpha_1 \alpha_2 g_1 g_2 - K\mu_1 Q_1 \alpha_2 g_2 - K\mu_2 Q_2 \alpha_1 g_1}{r g_1 g_2 \alpha_1 \alpha_2} \quad a.s.$$

Using Lemma 1 in [53] or Lemma 2.2 in [31], we have that  $\lim_{t\to\infty} \frac{M_3(t)}{t} = 0$  a.s. and  $\lim_{t\to\infty} \frac{M_5(t)}{t} = 0$  a.s.. A similar proof to the Lemma 2 in [17], we get that  $\lim_{t\to\infty} t^{-1} \int_{t-\tau}^t P(s)dt = \lim_{t\to\infty} t^{-1} (\int_0^t P(s)ds - \int_0^{t-\tau} P(s)ds) = 0$  a.s.. Therefore, according to (3.9) and in view of Lemma 2 in [27], if  $Q_3 > d_1 + 0.5\delta_2^2 + \Phi_1 + \frac{\rho_1 K Q}{\alpha_1 r}$  holds, we derive

$$\lim_{t \to \infty} \langle Z_1(t) \rangle_* \ge \frac{Q_3 \alpha_1 r - d_1 \alpha_1 r - 0.5 \delta_2^2 \alpha_1 r - \Phi_1 \alpha_1 r - \rho_1 Q K}{\alpha_1 g_1 r} \quad a.s$$

From (3.10), similarly, we have

$$\lim_{t \to \infty} \langle Z_2(t) \rangle_* \ge \frac{Q_4 \alpha_2 r - d_2 \alpha_2 r - 0.5 \delta_4^2 \alpha_2 r - \Phi_2 \alpha_2 r - \rho_2 K Q}{g_2 \alpha_2 r} \quad a.s.$$

provided  $Q_4 > d_2 + 0.5\delta_4^2 + \Phi_2 + \frac{\rho_2 QK}{\alpha_2 r}$ . This competes the proofs.

**Remark 3.1.** From Theorem 3.1, it is clearly indicated that  $\{Q, Q_1, Q_2\}$  is a threshold determining whether plankton is weakly persistent or extinct (Q for phytoplankton P(t) and  $Q_i$  for zooplankton  $Z_i(t)$ , i = 1, 2). That is, if  $\{Q, Q_1, Q_2\} > 0$ , the plankton is weakly persistent and extinct if  $\{Q, Q_1, Q_2\} < 0$ .

### 4. Numerical simulations

In this section, we carry out some numerical simulations to show the effects of noises on the resulting dynamics of stochastic system (1.2). As an example, we obtain such

set of parameters [33]: r = 1, K = 20,  $\mu_1 = 5$ ,  $\mu_2 = 6$ ,  $\alpha_1 = 6$ ,  $\alpha_2 = 10$ ,  $\beta_1 = 4$ ,  $\beta_2 = 5, \tau = 1.2$ ,  $\rho_1 = 1$ ,  $\rho_2 = 1.4$ ,  $d_1 = 1$ ,  $d_2 = 1.11$ ,  $g_1 = 2$  and  $g_2 = 5$ , which are used for the following numerical simulations. For this choices, Lv et al. [33] showed that there exists a unique positive equilibrium  $E_* = (16.30, 0.595, 0.224)$  of system (1.1) and it is globally asymptotically stable (see Fig.4 in [33]). In what follows, we shall present numerically that the stochastic system (1.2) admits much richer and more interesting dynamics because of the influences of white noise and Lévy noise.



Figure 1. The solutions of stochastic system (1.2) with time step  $\Delta t = 0.001$  and different white noise intensities.a for  $(\delta_1, \delta_2, \delta_3, \delta_4, \delta_5) = (0.40, 0.41, 0.42, 0.43, 0.44)$ . b for  $(\delta_1, \delta_2, \delta_3, \delta_4, \delta_5) = (0.80, 0.81, 0.82, 0.83, 0.84)$ . c for  $(\delta_1, \delta_2, \delta_3, \delta_4, \delta_5) = (1.80, 1.81, 1.82, 1.83, 1.84)$ .



Figure 2. The solutions of stochastic system (1.2) and its deterministic system (1.1) with time step  $\Delta t = 0.001$ . a for phytoplankton P(t). b for zooplankton  $Z_1(t)$ . c for zooplankton  $Z_2(t)$ .



Figure 3. The solutions of the stochastic system (1.2) and its deterministic system (1.1) with time step  $\Delta t = 0.001$ . a for phytoplankton P(t). b for zooplankton  $Z_1(t)$ . c for zooplankton  $Z_2(t)$ .

Letting the Lévy noise intensities  $\gamma_i(u) = 0$  (i = 1, 2, 3) and only varying the intensities of white noise, we first investigate the effects of white noise on the phytoplankton-zooplankton dynamics and the numerical method can be found in [17]. The numerical results are shown in Fig.1. By comparing Fig.1a and Fig.1b, it can be found that the irregular random variation and the fluctuation range of phytoplankton and zooplankton in the system (1.2) increase as the white noise intensities increase. However, they eventually go extinct as the white noise intensities continue to increase, as is shown in Fig.1c. We make white noise intensities

 $\delta_j = 0(j = 1, 2, 3, 4, 5)$  and only change the intensities of Lévy noise, by following the method used in [9, 10], a result similar to Fig.1 can be achieved, and so, we don't show the pictures of this result.



Figure 4. The sensitivity analysis of the threshold for phytoplankton P(t) in system (1.2) with respect to its growth rate and noise intensities. a for r and  $\delta_1$ . b for  $\delta_1$  and  $\gamma_1(u)$ . The space region I denotes the persistence of phytoplankton, the space region II indicates the extinction of phytoplankton, and the planes (pink and green) represent the critical plane between the persistence and extinction of phytoplankton.

Next, by following the same method used in [9,10] again, we show the synergistic effects of white noise and Lévy noise on the phytoplankton-zooplankton dynamics. Let  $\mathbb{Y} = (0, +\infty)$ ,  $\lambda(\mathbb{Y}) = 1$ , we start our numerical simulation with the following noise intensities:

 $(\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \gamma_1(u), \gamma_2(u), \gamma_3(u)) = (0.11, 0.12, 0.13, 0.14, 0.15, 0.16, 0.17, 0.18),$  $(\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \gamma_1(u), \gamma_2(u), \gamma_3(u)) = (0.11, 0.12, 0.13, 0.14, 0.15, 3.16, 3.17, 3.18),$ 

which satisfy the conditions of Assumption (A). For the first cases, by direct calculations, it is not difficult to derive  $\{Q, Q_1, Q_2\} > 0$ , in view of Theorem 3.1, it can be got that phytoplankton  $P_1(t)$  and zooplankton  $Z_i(t)(i = 1, 2)$  in the stochastic system (1.2) never go to extinction and they are weakly persistent in the mean, which is demonstrated in Fig.2. By comparing Figs.1a-1b and Fig.2, it is obvious that the irregular random variation of phytoplankton and zooplankton under the synergistic effects of the two noises is stronger than that of phytoplankton and zooplankton under the effects of single noise. For the second case, a simple calculation shows  $\{Q, Q_1, Q_2\} < 0$ . From Theorem 3.1, it is found that with the increase of noise intensities, these plankton populations eventually go extinct with probability one, which does not occur in the deterministic system (1.1). It is worth noting that the only difference between Fig.2 and Fig.3 is that the Lévy noise intensities are different. By further comparison of Fig.2 and Fig.3, one can find that an increase in the Lévy noise intensity can result in the extinction of persistent plankton.

In order to further study the effects of some key factors such as the growth rate and noise intensities on the survival of phytoplankton and to show the intuitive response of these factors to the survival of phytoplankton, we carry out some sensitivity analysis of the survival probability of phytoplankton by taking the same



Figure 5. The effects of toxin liberation rate  $\rho_i$  on the dynamics of plankton in the stochastic system (1.2).

parameters as in Fig.2, and the results are shown in Fig.4. In Fig.4a, by varying the parameters r and  $\delta_1$ , and fixing the other parameters, one can find that with the increase of noise intensity  $\delta_1$ , the threshold value decreases, while it increases as the growth rate r increases. Clearly, increasing the noise intensity  $\delta_1$  is disadvantage for the survival of phytoplankton, and the increase of r is advantage for the survival of phytoplankton. By changing the parameters  $\delta_1$  and  $\gamma_1(u)$ , and keep the other parameters fixed in Fig.4b, it can be found that the increase of  $\delta_1$  or  $\gamma_1(u)$  is able to reduce the threshold value of phytoplankton. Consequently, the large random environmental fluctuations such as hurricanes, infectious diseases and tsunamis are always harmful to the survival of phytoplankton.



Figure 6. The effects of intraspecific coefficient  $g_i$  (i = 1, 2) on the dynamics of plankton in the stochastic system (1.2).

In addition, toxin liberate rate and intraspecific competition rate are also important factors influencing the interactions between phytoplankton and zooplankton. Therefore, we investigate numerically the effects of toxin liberate rate  $\rho_i (i = 1, 2)$ and intraspecific competition rate  $g_i$  (i = 1, 2) on the dynamics of plankton in the stochastic system (1.2). Keeping the parameters in Fig.2 unchanged, we only vary the toxin liberation rate  $\rho_1$  and  $\rho_2$ , it is worth noting that with the increase of the  $\rho_j (j = 1, 2)$ , the biomass of phytoplankton increases but that of zooplankton  $Z_j(t)(j = 1, 2)$  decreases, as is demonstrated in Fig.5a, Fig.5b and Fig.5c, respectively. Furthermore, by changing the values of  $g_s(s = 1, 2)$  and fixing the other parameters, we achieve a similar result as the  $g_s(s = 1, 2)$  increases (see Fig.6). Hence, it can be concluded that toxin liberate rate of phytoplankton and intraspecific competition rate of zooplankton have a great influence on the variation of plankton biomass.

## 5. Concluding remarks

In this paper, an extension of the delayed plankton system consisting of toxin producing phytoplankton and two zooplankton originally developed in [33] was derived to explore the effects of environmental white noise and Lévy noise fluctuations on the phytoplankton-zooplankton dynamics. In spite of some stochastic predator-prey models with Lévy noise have been reported in the existing literature, a delayed toxin producing phytoplankton-two zooplankton system taking into account the impacts of white noise and Lévy noise has been rarely studied. Hence, the study of the influences of white noise and Lévy noise on the phytoplankton-zooplankton dynamics, especially the stochastic extinction and persistence of these plankton, which have been considered as two important topics in plankton ecology, is of great significance and importance. Mathematically, we first proved the existence of a unique global positive solution of the system, and then the survival analysis was carried out to investigate the sufficient conditions for stochastic extinction and persistence in the mean of each population, which in turn provided a theoretical basis for numerical simulations. Numerical analysis indicated that white noise and Lévy noise have a great influence on the phytoplankton-zooplankton dynamics in the aquatic environments.

Actually, when the stochastic system (1.2) with only white noise or only Lévy noise or both of them, it can be found that the persistence of phytoplankton and zooplankton remains unchanged with the weak noise intensities (see Figs.1a-1b and Fig.2). For the strong noise intensities, it was to be noted that both phytoplankton and zooplankton can not avoid the fate of extinction (see Fig.1c). This results indicated that the weak white noise or L e v v noise could not affect the persistence of phytoplankton and zooplankton, whereas the strong white noise or Lévy noise were able to result in the extinction of these plankton, which did not occur in the deterministic system. In spite of the weak noise intensity could not drive phytoplankton and zooplankton to go extinct, it was worth noting that they can cause the irregular random variation of phytoplankton and zooplankton in the system (1.2). In fact, the same weak intensity of white noise and  $L e^{i} v v$  noise may generate the same intensity of irregular random variation for phytoplankton and zooplankton, but the random variation of phytoplankton and zooplankton under the synergistic effects of white noise and Lévy noise was found to be stronger than that of the phytoplankton and zooplankton under the effects of single noise (see Fig.1 and Fig.2). which indicated that the small random environmental perturbations can not result in the extinction of phytoplankton and zooplankton, but they are bound to affect the growth process of these plankton. Furthermore, the rapidly increase or decrease in the irregular random variation could cause the appearance or disappearance of the phytoplankton (see Fig.2a and Fig.3a), and hence, the control of the rapidity of the random fluctuations may be a key factor for the termination of phytoplankton blooms. Consequently, it could be concluded that the effects of white noise or Lévy noise on the stochastic extinction and persistence of phytoplankton and zooplankton are similar, but the synergistic effect of the two noises on the stochastic extinction and persistence of these plankton is stronger than that of single noise. In view of the above results, the questions mentioned in the Introduction have been partially answered.

Additionally, it was observed that an increase in the toxin liberation rate can increase the biomass of the phytoplankton but decrease the biomass of zooplankton, which implied that toxin released by phytoplankton is conducive to the growth of phytoplankton but not to the growth of zooplankton (see Fig.5). More interestingly, with the increase in the intraspecific competition rate of zooplankton, a similar result has been found (see Fig.6). However, in the work of Lv et al. [33], where the deterministic plankton system (1.1) was studied, one of their results indicated that the intraspecific competition of zooplankton reduces the biomass of both phytoplankton and zooplankton populations. As can be seen from condition of (4) in Theorem 3.1, the result of this paper was very obvious and was also apparent by the ecological significance. In fact, the increase in intraspecific competition among zooplankton naturally reduces the grazing pressure of phytoplankton by zooplankton, which may lead to the rapid growth of phytoplankton and thus increase their biomass. Consequently, our conclusion may be more realistic than those in [33].

Although some interesting results have achieved in this paper, it should be noted that the extinction of phytoplankton can be caused by the random environmental noise perturbations, but not all noises involved in the plankton equations can have great influence on the phytoplankton-zooplankton dynamics. Actually, some noises involved the plankton equations can be ignored while some noises of the plankton equations must be taken into accounts (see  $Q,Q_1$  and  $Q_2$ ). Further, one may consider more realistic but complex system to study more interesting topics, such as incorporating the effect of impulsive perturbation or colored noise into the system (1.2). These will leave our future works.

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