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BIFURCATIONS AND EXACT TRAVELLING WAVE SOLUTIONS FOR A SHALLOW WATER WAVE MODEL WITH A NON-STATIONARY BOTTOM SURFACE*

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Abstract We consider a shallow water wave model with a non-stationary bottom surface. By applying dynamical system approach to the model problem, we are able to obtain all possible bounded solutions (compactons, solitary wave solutions and periodic wave solutions) under different parameter conditions. More than 19 exact parametric representations are provided explicitly.

Keywords Solitary wave solution, compacton solution, periodic wave solution, shallow water wave model, bifurcation.

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1. Introduction

It is well known that the equations for shallow water waves were developed using approximations, and there have been a number of different formulations that were developed in the literature. The interesting history of various formulations was discussed in [2]. Recently, in [4], the author pointed out that the Serre's nonlinear shallow water wave equations developed in ([9, 10]) for uniform depth and later generalized by Seabra-Santos et al. [11] for non-uniform depth are limited to a stationary bottom surface and a uniform pressure applied to the top surface, while the Green-Naghdi nonlinear shallow water wave equations developed by Green and Naghdi [3] are valid for a non-stationary, non-uniform bottom surface and a non-uniform pressure on the top surface. To be specific, in 2019, Kogelbauer and Rubin [4] developed a class of exact nonlinear traveling wave solutions of the Green-Naghdi equations for a non-stationary and non-uniform bottom surface. The traveling wave equation is given by (see [4], (3.9)):

$$\left[\frac{(1+a)}{F^2\tilde{H}^3}\phi^2+\frac{2}{\phi}+\left(\frac{2}{3}+a\right)\phi\left(\frac{\phi_\xi}{\phi}\right)_\xi\right]_\xi=-a(1+2a)\phi_\xi\left(\frac{\phi_\xi}{\phi}\right)_\xi, \tag{1.1}$$

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where ϕ is the current depth of the fluid depending on $\xi = x - ct$, the constants a, $\tilde{H}$ parameterize the class, with a specified and $\tilde{H}$ determined by a critical value of the depth ϕ, the Froude number F defined by $F^2 = \frac{k_1^2}{\tilde{g}\tilde{H}^3}$.

For convenience, we write $\tilde{\beta} = \frac{(1+a)}{F^2\tilde{H}^3}$. Expanding equation (1.1) and dividing the result by 2ϕ, one has (see [4], (3.11)):

$$\left[\tilde{\beta}\phi^2 + \frac{1}{2\phi^2} + \frac{1}{2}\left(a^2 - \frac{1}{3}\right)\left(\frac{\phi_\xi}{\phi}\right)^2 + \left(\frac{1}{3} + \frac{1}{2}a\right)\frac{\phi_{\xi\xi}}{\phi}\right]_\xi = 0. \tag{1.2}$$

Integrating equation (2) once with respect to ξ, we obtain an equivalent planar dynamical system

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{(1-3a^2)y^2 - \beta\phi^4 + 6g\phi^2 - 3}{2\left(1 + \frac{3}{2}a\right)\phi}, \tag{1.3}$$

where g is an integral constant and $\beta = 6\tilde{\beta}$. Clearly, system (1.3) is a singular traveling wave system of the first class of the same type as [6–8, 12], with the singular straight line $\phi = 0$.

system (1.3) has a first integral:

$$\begin{aligned} H_a(\phi,\ y) =& \phi^{\frac{6a^2-2}{2+3a}}\Big[y^2 + \frac{1}{3(a+1)^2(3a^2-1)(3a^2+3a+1)}(\beta(9a^4+9a^3-3a-1)\phi^4 \\ &- g(54a^4+108a^3+36a^2-36a-18)\phi^2 + 9(3a^4+9a^3+10a^2+5a+1))\Big] = h \end{aligned} \tag{1.4}$$

when $a \neq -1,\ \pm\frac{1}{\sqrt{3}}$,

$$H_{-1}(\phi,\ y) = \frac{y^2}{\phi^4} + \frac{3}{2\phi^4} - \frac{6g}{\phi^2} = h \tag{1.5}$$

when $a = -1$, i.e., $\beta = 0$, and

$$H_{\frac{1}{\sqrt{3}}}(\phi,\ y) = y^2 + \frac{\beta\phi^4}{2(2+\sqrt{3})} - \frac{6g\phi^2}{2+\sqrt{3}} + \frac{6\ln(\phi)}{2+\sqrt{3}} = h, \tag{1.6}$$

$$H_{-\frac{1}{\sqrt{3}}}(\phi,\ y) = y^2 - \frac{\beta\phi^4}{2(-2+\sqrt{3})} + \frac{6g\phi^2}{-2+\sqrt{3}} - \frac{6\ln(\phi)}{-2+\sqrt{3}} = h \tag{1.7}$$

when $a = \pm\frac{1}{\sqrt{3}}$.

Especially, at the time that $a = 0, -\frac{1}{2}$ and $a = \tilde{a}_\pm = \frac{1}{2}(1 \pm \sqrt{5})$, for the first integral becomes

$$H_0(\phi,\ y) = \frac{1}{\phi}\left(y^2 + \frac{1}{3}\beta\phi^4 - 6g\phi^2 - 3\right) = h, \tag{1.8}$$

$$H_{-\frac{1}{2}}(\phi,\ y) = \frac{1}{\phi}\left(y^2 + \frac{4}{3}\beta\phi^4 - 24g\phi^2 - 12\right) = h \tag{1.9}$$

and

$$H_{\tilde{a}_+}(\phi,\ y) = \phi^2\left[y^2 + \frac{(7-3\sqrt{5})}{6}(\beta\phi^4 - 9g\phi^2 + 9)\right] = h, \tag{1.10}$$

$$H_{\tilde{a}_-}(\phi,\ y) = \phi^2 \left[y^2 + \frac{(7+3\sqrt{5})}{6}(\beta\phi^4 - 9g\phi^2 + 9) \right] = h, \tag{1.11}$$

respectively.

We would like to point out that in [4], Kogelbauer and Rubin did not investigate the dynamical behavior of system (1.3), while is the main focus in this work. More precisely, by considering the dynamics of the travelling wave solutions determined by system (1.3), we shall generally give all possible exact travelling wave solutions explicitly for equation (1.1) under different parameter conditions (see, e.g., [5–8,12]). More than 19 exact parametric representations are obtained by using the elliptic functions and hyperbolic functions.

The rest of this paper is organized as follows. In section 2, we discuss bifurcations of phase portraits of system (1.3). In section 3 and section 4, corresponding to all bounded orbits given in section 2, we give all possible exact parametric representations of the travelling wave solutions for equation (1.1) explicitly.

2. Bifurcations of the phase portraits of system (1.3)

As we know, the associated regular system of system (1.3) has the form

$$\frac{d\phi}{d\zeta} = (2+3a)y\phi, \quad \frac{dy}{d\zeta} = (1-3a^2)y^2 - \beta\phi^4 + 6g\phi^2 - 3, \tag{2.1}$$

where $d\xi = (2+3a)\phi d\zeta$ for $\phi \neq 0$.

For convenience in the following discussion, we introduce $f(\phi) = \beta\phi^4 - 6g\phi^2 + 3$, and always assume that $\Delta = 9g^2 - 3\beta > 0$. Then:

(1) When $\beta > 0$, i.e., $a > -1$ and $g > 0$, $f(\phi)$ has four real zeros $\phi_j,\ j = 1, \cdots, 4$ satisfying $\phi_1 < \phi_2 < 0 < \phi_3 < \phi_4$ and $\phi_1 = -\phi_4,\ \phi_2 = -\phi_3$, where

$$\phi_3 = \left(\frac{3g - \sqrt{\Delta}}{\beta} \right)^{\frac{1}{2}},\ \phi_4 = \left(\frac{3g + \sqrt{\Delta}}{\beta} \right)^{\frac{1}{2}}. \tag{2.2}$$

(2) When $\beta < 0$, i.e., $a < -1$, for any given g, the function $f(\phi)$ has two real zeros $-\phi_1$ and ϕ_1, where $\phi_1 = \left(\frac{3g-\sqrt{\Delta}}{\beta}\right)^{\frac{1}{2}}$.

(3) When $\beta > 0$, $g < 0$, the function $f(\phi)$ has no real zero.

Clearly, if ϕ_j is a real zero of $f(\phi)$, then, the point $E_j(\phi_j,\ 0)$ in the $\phi-$axis of the $(\phi,\ y)-$phase plane is an equilibrium point of system (1.3). In addition, if $|a| < \frac{1}{\sqrt{3}}$, on the singular straight line $\phi = 0$, there exist two equilibrium points $S_\pm(0,\ \pm y_s)$ of system (2.1), where $y_s^2 = \frac{3}{1-3a^2}$.

Let $M(\phi_j,\ 0)$ be the coefficient matrix of the linearized system of system (2.1) at an equilibrium point $(\phi_j,\ 0)$ and $J(\phi_j,\ 0)$ be its Jacobin determinant. Then, we have $\mathrm{Trace}(M(\phi_j,\ 0)) = 0$ and

$$J(\phi_j,\ 0) = (2+3a)\phi_j f'(\phi_j), \quad J(0,\ y_s) = 6(2+3a),$$

$$\mathrm{Trace}(M(0,\ y_s))^2 - 4J(0,\ y_s) = \frac{27a^2(2a+1)^2}{1-3a^2}.$$

By the theory of planar dynamical systems, we know that for an equilibrium point of a planar integrable system:

(i) if $J < 0$, then the equilibrium point is a saddle point;

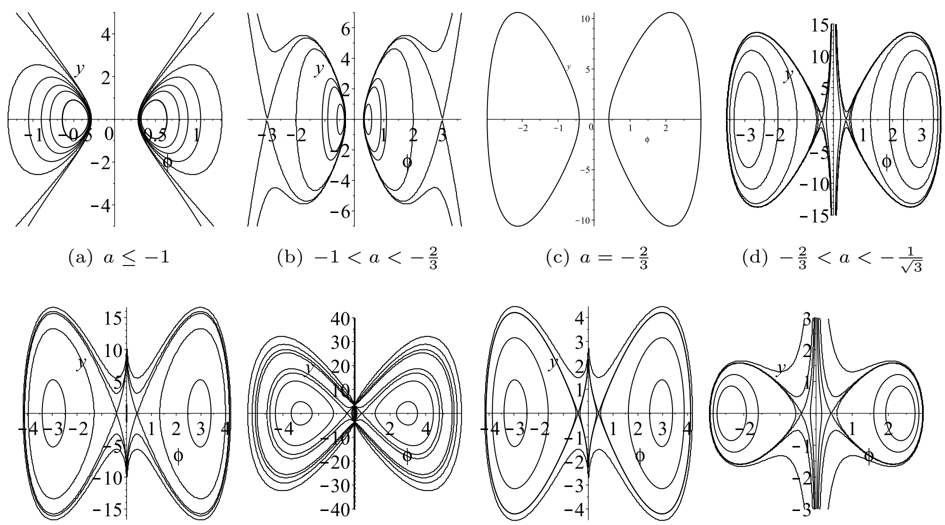
(ii) if $J > 0$ and $\mathrm{Trace}(\mathrm{M}(\phi_\mathrm{j},\ 0))^2 - 4\mathrm{J}(\phi_\mathrm{j},\ 0) < 0$, then it is a center point;

(iii) if $J > 0$ and $\mathrm{Trace}(\mathrm{M}(\phi_\mathrm{j},\ 0))^2 - 4\mathrm{J}(\phi_\mathrm{j},\ 0) > 0$, then it is a node;

(iv) if $J = 0$ and the Poincare index of the equilibrium point is 0, then it is a cusp.

Thus, if there exist two equilibrium points $S_{\mp}(0,\ \mp y_s)$, then, they are both nodes because of $(1 - 3a^2) > 0$.

We write that $h_j = H(\phi_j,\ 0)$ for $j = 1, \cdots, 4$. Taking the above fact into account, we know that for all $g > 0$ satisfying $\Delta > 0$, the parameter a can be taken as a bifurcation parameter such that $a = -1,\ -\frac{2}{3},\ \mp\frac{1}{\sqrt{3}}$ are bifurcation values. As a increases, we have the bifurcations of phase portraits of system (1.3) shown in Fig.1 (a)-(h). Notice that when $a = -\frac{2}{3}$, equation (1.2) becomes that $y^2 + 3\beta\phi^4 - 18g\phi^2 + 9 = 0$, where $\beta = \frac{1}{3F^2H^2} > 0$. For $g > 0$, it gives rise to the two families of closed orbits (see Fig.1 (c)).



(a) $a \le -1$ (b) $-1 < a < -\frac{2}{3}$ (c) $a = -\frac{2}{3}$ (d) $-\frac{2}{3} < a < -\frac{1}{\sqrt{3}}$

(e) $a = -\frac{1}{\sqrt{3}}$ (f) $-\frac{1}{\sqrt{3}} < a < \frac{1}{\sqrt{3}}$ (g) $a = \frac{1}{\sqrt{3}}$ (h) $a > \frac{1}{\sqrt{3}}$

Figure 1. The bifurcations of phase portraits of system (1.3) when a varies.

3. Exact parametric representations of traveling wave solutions of equation (1.1) for $a = -1, -\frac{2}{3}$ and $a = \frac{1}{2}(1 - \sqrt{5})$

In this section, we consider the exact parametric representations of traveling wave solutions of equation (1.1) given by the bounded orbits in Fig.1.

3.1. The case $a = -1$ (see Fig.1 (a))

In this case, system (1.3) has two equilibrium points $E_1(-\phi_1,\ 0)$ and $E_2(\phi_1,\ 0)$, where $\phi_1 = \frac{1}{2\sqrt{g}}$. The level curves defined by $H_{-1}(\phi,\ y) = h,\ h \in (h_1,\ 0)$

in equation (1.5) give the two families of periodic orbits of system (1.3), where $h_1 = -6g^2$. We see from equation (1.5) and the first equation of system (1.3) that $\sqrt{|h|}\xi = \int_b^{\phi} \frac{d\phi}{\sqrt{(a^2-\phi^2)(\phi^2-b^2)}}$, where $a^2 = \frac{1}{|h|}\left(3g + \sqrt{9g^2 + \frac{3}{2}|h|}\right)$ and $b^2 = \frac{1}{|h|}\left(3g - \sqrt{9g^2 + \frac{3}{2}|h|}\right)$. Thus, we obtain the following exact parametric representations of the two families of periodic wave solutions of equation (1.1):

$$\phi(\xi) = \pm \frac{b}{\mathrm{dn}\left(a\sqrt{|h|}\xi,\ k\right)}, \tag{3.1}$$

where $k^2 = \frac{a^2-b^2}{a^2}$.

3.2. The case $a = -\frac{2}{3}$ (see Fig.1 (c))

In this case, it deduces from equation (1.2) that

$$\tilde{\beta}\phi^2 + \frac{1}{2\phi^2} + \frac{1}{18}\left(\frac{\phi_\xi}{\phi}\right)^2 = g. \tag{3.2}$$

Thus, from $\beta = 6\tilde{\beta}$ and first equation of system (1.3), equation (3.2) gives rise to the curve equation

$$y^2 + 3\beta\phi^4 - 18g\phi^2 + 9 = 0. \tag{3.3}$$

Clearly, for $\beta > 0,\ g > 0$, equation (3.2) defines two closed curves enclosing the points $E_\pm\left(\pm\sqrt{\frac{3g\pm\Delta}{\beta}},\ 0\right)$, respectively. Equation (3.2) can be rewritten as $y^2 = 3\beta(\phi_4^2 - \phi^2)(\phi^2 - \phi_3^2)$, where $\phi_3,\ \phi_4$ given by (2.2). Hence, we have the exact parametric representations of the two families of periodic wave solutions of equation (1.1) as follows:

$$\phi(\xi) = \pm \frac{\phi_3}{\mathrm{dn}\left(\phi_4\sqrt{3\beta}\xi,\ k\right)}, \tag{3.4}$$

where $k^2 = \frac{\phi_4^2-\phi_3^2}{\phi_4^2}$.

3.3. The case $a = \frac{1}{2}(1-\sqrt{5}) = -0.618\cdots$ (see Fig.1 (d))

In this case, for $H_{\tilde{a}_-}(\phi,\ y) = h$ given by equation (1.11), we have $h_3 = \frac{1}{2\beta}(7 + 3\sqrt{5})(g\sqrt{\Delta} - 3g^2 + 2\beta)\phi_3^2$ and $h_4 = -\frac{1}{2\beta}(7+3\sqrt{5})(g\sqrt{\Delta} + 3g^2 - 2\beta)\phi_4^2$. When h varies, the changes of the level curves defined by $H_{\tilde{a}_-}(\phi,\ y) = h$ are shown in Fig.2 (a)-(d).

We see from equation (1.11) that

$$\begin{aligned} y^2 &= \frac{h}{\phi^2} - \frac{(7+3\sqrt{5})}{6}[9 - 9g\phi^2 + \beta\phi^4] \\ &= \frac{(7+3\sqrt{5})\beta}{6\phi^2}\left[\frac{6h}{(7+3\sqrt{5})\beta} - \left(\frac{9}{\beta} - \frac{9g}{\beta}\phi^2 + \phi^4\right)\phi^2\right]. \end{aligned}$$

By using the first equation of system (1.3), we have

$$\omega_0\xi = \int_{\phi_0}^{\phi} \frac{\phi d\phi}{\sqrt{\frac{h}{\omega_0^2} - (\frac{9}{\beta} - \frac{9g}{\beta}\phi^2 + \phi^4)\phi^2}}, \tag{3.5}$$

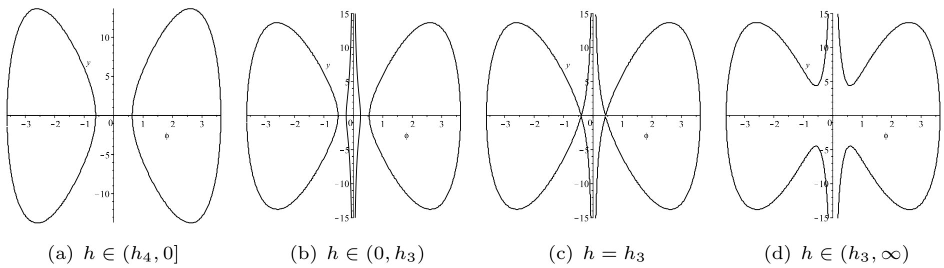


Figure 2. The changes of the level curves defined by $H_{\tilde{a}_-}(\phi, y) = h$ when $\tilde{a}_- = \frac{1}{2}(1-\sqrt{5})$.

where $\omega_0 = \sqrt{\frac{(7+3\sqrt{5})\beta}{6}}$. We can use equation (3.5) to calculate all parametric representations of the bounded orbits of system (1.3) in Fig.2.

(i) Corresponding to the two periodic orbit families defined by $H_{\tilde{a}_-}(\phi,\ y) = h,\ h \in (h_4,\ 0]$, letting $\psi = \phi^2$, equation (3.5) becomes $2\omega_0\xi = \int_{r_2}^{\psi} \frac{d\psi}{\sqrt{(r_1-\psi)(\psi-r_2)(\psi-r_3)}}$, where $r_3 < 0 < r_2 < \phi_4^2 < r_1$. Hence, we obtain the exact parametric representations of the two families of periodic wave solutions of equation (1.1) (see Fig.3 (a)):

$$\phi(\xi) = \pm\left(\frac{r_2 - k^2 r_3 \mathrm{sn}^2(\Omega_1\xi,\ k)}{\mathrm{dn}^2(\Omega_1\xi,\ k)}\right)^{\frac{1}{2}}, \tag{3.6}$$

where $\Omega_1 = \omega_0\sqrt{r_1 - r_3}$ and $k^2 = \frac{(r_1-r_2)}{(r_1-r_3)}$.

We notice that the existence of uncountably infinitely many bounded breaking wave solutions is an important property of a singular travelling wave system. For example, we consider the two families of open curves defined by $H_{\tilde{a}_-}(\phi,\ y) = h$ with $h \in (0,\ h_3)$, $(h_3,\ \infty)$ (see Fig.2 (b), (d)), which lie on the two sides of the singular straight line $\phi = 0$, respectively. Obviously, along each open curve as $|\xi|$ increases, $\phi(\xi)$ approaches to $\phi = 0$ and $\phi(\xi) \to 0,\ |y(\xi)| \to \infty$. By the theory of singular traveling wave systems developed in [6], we know that these open curves give rise to uncountably infinitely many bounded two-sided breaking wave solutions of $\phi(\xi)$. In other words, these traveling wave solutions have compact supports, that is, they vanish identically outside finite core regions. Such compact support solutions are called compactons.

(ii) Corresponding to the two periodic orbit families and the two open curve families which approach the straight line $\phi = 0$ when $|y| \to \infty$ (see Fig.2 (b)), defined by $H_{\tilde{a}_-}(\phi,\ y) = h,\ h \in (0,\ h_3)$, equation (3.5) can be rewritten as $2\omega_0\xi = \int_{r_2}^{\psi} \frac{d\psi}{\sqrt{(r_1-\psi)(\psi-r_2)(\psi-r_3)}}$ and $2\omega_0\xi = \int_{\psi}^{r_3} \frac{d\psi}{\sqrt{(r_1-\psi)(r_2-\psi)(r_3-\psi)}}$, where $0 < r_3 < \phi_3^2 < r_2 < \phi_4^2 < r_1$.

For the two periodic wave solution families of system (1.3), we have the same exact parametric representations of the two families of periodic wave solutions of equation (1.1) as (3.6).

For the two open curve families, we obtain the two families of compacton solution of equation (1.1) as follows (see Fig.3 (c)):

$$\phi(\xi) = \pm\left(\frac{r_3 - r_2 \mathrm{sn}^2(\Omega_1\xi,\ k)}{\mathrm{cn}^2(\Omega_1\xi,\ k)}\right)^{\frac{1}{2}}, \tag{3.7}$$

where $k^2 = \frac{r_1 - r_2}{r_1 - r_3}$ and $\xi \in \left(-\frac{1}{\Omega_1}\mathrm{sn}^{-1}\left(\sqrt{\frac{r_3}{r_2}},\ k\right),\ \frac{1}{\Omega_1}\mathrm{sn}^{-1}\left(\sqrt{\frac{r_3}{r_2}},\ k\right)\right)$.

(iii) Corresponding to the level curves defined by $H_{\tilde{a}_-}(\phi,\ y) = h_3$, there are two homoclinic orbits of system (1.3) to the equilibrium points $E_3(\phi_3,\ 0)$ and $E_2(\phi_2,\ 0)$, enclosing the equilibrium points $E_4(\phi_4,\ 0)$ and $E_1(\phi_1,\ 0)$, respectively. Now, equation (3.5) becomes $2\omega_0\xi = \int_\psi^{\psi_M} \frac{d\psi}{(\psi-\psi_3)\sqrt{(\psi_M-\psi)}}$, where $\psi_3 = \phi_3^2$, $\psi_M = \phi_M^2$. Thus, we obtain the following parametric representations of the two solitary wave solutions of equation (1.1) (see Fig.3 (c)):

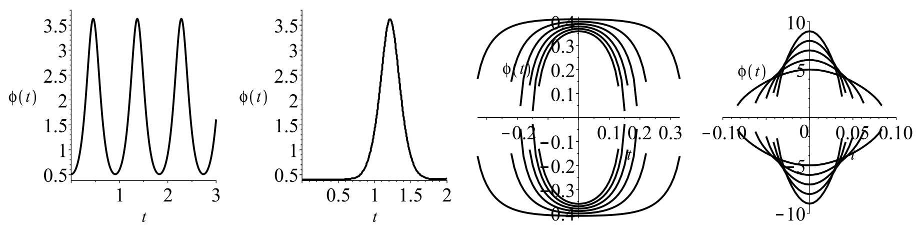
$$\phi(\xi) = \pm\left(\phi_3^2 + (\phi_M^2 - \phi_3^2)\mathrm{sech}^2(\omega_1\xi)\right)^{\frac{1}{2}}, \tag{3.8}$$

where $\omega_1 = \sqrt{2(\phi_M^2 - \phi_3^2)}$.

(iv) Corresponding to the level curves defined by $H_{\tilde{a}_-}(\phi,\ y) = h,\ h \in (h_3,\ \infty)$, there exist the two families of open orbits of system (1.3) which tend to the straight line $\phi = 0$, when $|y| \to \infty$ (see Fig.2 (d)). Now, equation (3.5) can be rewritten as $2\omega_0\xi = \int_\psi^{r_1} \frac{d\psi}{\sqrt{(r_1-\psi)[(\psi-b_1)^2+a_1^2]}}$. It gives rise to the following parametric representations of the two compacton solution families of equation (1.1) (see Fig.3 (d)):

$$\phi(\xi) = \pm\left(\frac{r_1 - A_1 + (r_1 + A_1)\mathrm{cn}(\Omega_2\xi,\ k)}{1 + \mathrm{cn}(\Omega_2\xi,\ k)}\right)^{\frac{1}{2}}, \quad \xi \in (-\xi_0,\ \xi_0), \tag{3.9}$$

provided $A_1^2 = (b_1 - r_1)^2 + a_1^2$, $k^2 = \frac{A_1 - b_1 + r_1}{2A_1}$, $\Omega_2 = 2\omega_0\sqrt{A_1}$ and $\xi_0 = \mathrm{cn}^{-1}\left(\frac{A_1 - r_1}{A_1 + r_1}, k\right)$.



(a) Periodic wave (b) Solitary wave (c) Two compactons (d) Two compactons

Figure 3. The profiles of solitary wave, periodic wave and compactons of equation (1.1).

Similarly, for $a = \frac{1}{2}(1+\sqrt{5})$, we have $H_{\tilde{a}_+}(\phi,\ y) = h$ given by equation (1.10). Instead of ω_0 by $\sqrt{\frac{(7-3\sqrt{5})\beta}{6}}$, we arrive at the same results for the case $a = \frac{1}{2}(1-\sqrt{5})$.

To sum up, we have the following conclusions.

Theorem 3.1. *(i) When* $a = -1$ *and* $g > 0$*, equation* (1.1) *has the two exact periodic wave solution families given by* (3.1).

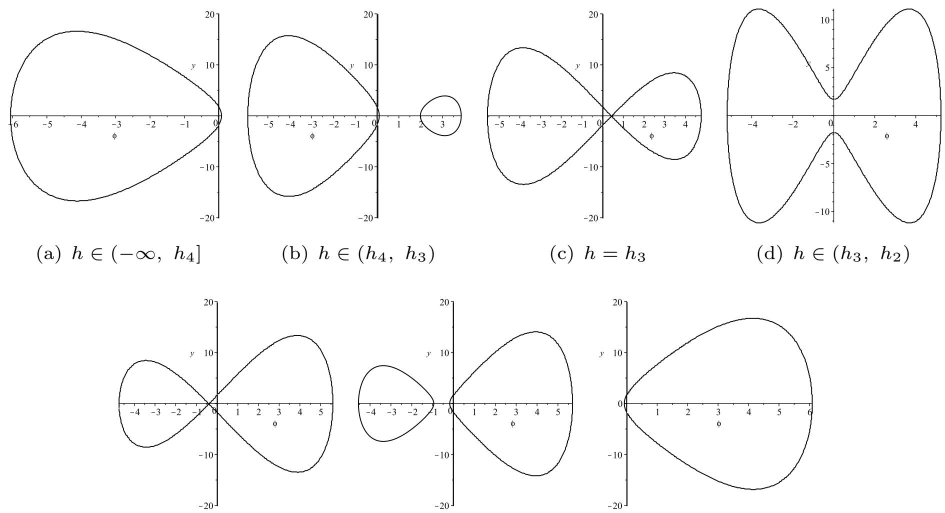
(ii) When $a = -\frac{2}{3}$ *and* $g > 0$*, equation* (1.1) *has the two exact periodic wave solutions families given by* (3.4).

(iii) When $a = \frac{1}{2}(1-\sqrt{5}),\ \frac{1}{2}(1+\sqrt{5})$ *and* $\Delta > 0$*, equation* (1.1) *has the two exact periodic wave solution families given by* (3.6)*, the two exact solitary wave solutions given by* (3.8)*, and equation* (1.1) *has the four exact compacton solution families given by* (3.7) *and* (3.9).

4. Exact parametric representations of traveling wave solutions of equation (1.1) for $a = 0$ and $a = -\frac{1}{2}$

In this section, we consider the exact parametric representations of traveling wave solutions of equation (1.1) given by the bounded orbits in Fig.1 (f). In this case, we have $h_3 = -\frac{1}{\phi_3}\left(3 - \frac{1}{3}\beta\phi_3^4 + 6g\phi_3^2\right)$, $h_4 = -\frac{1}{\phi_4}\left(3 - \frac{1}{3}\beta\phi_4^4 + 6g\phi_4^2\right)$, $h_1 = -h_4$ and $h_2 = -h_3$ from equation (1.8).

When h varies, the changes of the level curves defined by $H_0(\phi,\ y) = h$ are shown in Fig.4 (a)-(g).



(a) $h \in (-\infty,\ h_4]$ (b) $h \in (h_4,\ h_3)$ (c) $h = h_3$ (d) $h \in (h_3,\ h_2)$

(e) $h = h_2$ (f) $h \in (h_2,\ h_1)$ (g) $h \in (h_1,\ \infty)$

Figure 4. The changes of the level curves defined by $H_0(\phi,\ y) = h$ when provided $a = 0$.

It follows from equation (1.8) that $y^2 = \frac{1}{3}\beta\left(\frac{9}{\beta} + \frac{3h}{\beta}\phi + \frac{18g}{\beta}\phi^2 - \phi^4\right)$. Thus, it deduces from the first equation of system (1.3) that

$$\sqrt{\frac{\beta}{3}}\xi = \int_{\phi_0}^{\phi} \frac{d\phi}{\sqrt{\frac{9}{\beta} + \frac{3h}{\beta}\phi + \frac{18g}{\beta}\phi^2 - \phi^4}}. \tag{4.1}$$

By using equation (4.1), we can obtain all parametric representations of the bounded orbits of system (1.3).

We see from Fig.4 that there exist two node points of the associated regular system (2.1) on the singular straight line $\phi = 0$, then the singular system has no peakon, periodic peakon and compacton solutions. In this case, the traveling wave system has no curve triangle surrounding a periodic annulus of a center. Instead, there exist smooth periodic waves, solitary waves of the singular system, because the singular system and its associated regular system define different vector fields respectively. As shown by Fig.1 (f), on the left-hand side of the singular straight line $\phi = 0$, the direction of the orbits of the vector field defined by the singular system (1.3) is the inverse direction of the orbits of the vector field defined by its associated regular system (2.1).

(i) Corresponding to the level curves defined by $H_0(\phi,\ y)=h,\ h\in(-\infty,\ h_4]$, there exists a family of periodic orbits of system (1.3) (see Fig.4 (a)), with its boundary passing through two equilibrium points $S_{\pm}(0,\ \pm y_s)$. Now, equation (4.1) can be written as $\sqrt{\frac{\beta}{3}}\xi=\int_{r_2}^{\phi}\frac{d\phi}{\sqrt{(r_1-\phi)(\phi-r_2)[(\phi-b_1)^2+a_1^2]}}$. Therefore, we obtain the following exact parametric representation of a periodic wave solution family of equation (1.1):

$$\phi(\xi)=\frac{r_1B_1+r_2A_1-(r_1B_1-r_2A_1)\mathrm{cn}(\Omega_3\xi,\ k)}{(A_1+B_1)+(A_1-B_1)\mathrm{cn}(\Omega_3\xi,\ k)},\tag{4.2}$$

where $A_1^2=(r_1-b_1)^2+a_1^2,\ B_1^2=(r_2-b_1)^2+a_1^2,\ k^2=\frac{(r_1-r_2)^2-(A_1-B_1)^2}{4A_1B_1}$ and $\Omega_3=\sqrt{\frac{\beta A_1B_1}{3}}$.

(ii) Corresponding to the level curves defined by $H_0(\phi,\ y)=h,\ h\in(h_4,\ h_3)$, there exist the two families of periodic orbits of system (1.3), enclosing the equilibrium points $E_1(\phi_1,\ 0)$ and $E_4(\phi_4,\ 0)$, respectively (see Fig.4 (b)). For the right family of periodic orbits, equation (4.1) can be written as $\sqrt{\frac{\beta}{3}}\xi=\int_{r_2}^{\phi}\frac{d\phi}{\sqrt{(r_1-\phi)(\phi-r_2)(\phi-r_3)(\phi-r_4)}}$. Thus, it gives rise to the following a periodic wave solution family of equation (1.1):

$$\phi(\xi)=\frac{r_2-r_3\tilde{\alpha}_1^2\mathrm{sn}^2(\Omega_4\xi,\ k)}{1-\tilde{\alpha}_1^2\mathrm{sn}^2(\Omega_4\xi,\ k)},\tag{4.3}$$

where $\tilde{\alpha}_1^2=\frac{r_1-r_2}{r_1-r_3},\ k^2=\frac{\tilde{\alpha}_1^2(r_3-r_4)}{(r_2-r_4)}$ and $\Omega_4=\sqrt{\frac{\beta(r_1-r_3)(r_2-r_4)}{12}}$.

For the left family of periodic orbits, now equation (4.1) can be written as

$$\sqrt{\frac{\beta}{3}}\xi=\int_{r_4}^{\phi}\frac{d\phi}{\sqrt{(r_1-\phi)(r_2-\phi)(r_3-\phi)(\phi-r_4)}}.$$

Hence, one has

$$\phi(\xi)=\frac{r_4-r_1\tilde{\alpha}_2^2\mathrm{sn}^2(\Omega_4\xi,\ k)}{1-\tilde{\alpha}_2^2\mathrm{sn}^2(\Omega_4\xi,\ k)},\tag{4.4}$$

where $\tilde{\alpha}_2^2=\frac{r_4-r_3}{r_1-r_3},\ k^2=\frac{-\tilde{\alpha}_2^2(r_1-r_2)}{r_2-r_4}$ and $\Omega_4=\sqrt{\frac{\beta(r_1-r_3)(r_2-r_4)}{12}}$.

(iii) Corresponding to the level curves defined by $H_0(\phi,\ y)=h_3$, there exist the two homoclinic orbits of system (1.3) enclosing the equilibrium point $E_4(\phi_4,\ 0)$ and $E_1(\phi_1,\ 0)$, respectively (see Fig.4 (c)). Now, equation (4.1) becomes $\sqrt{\frac{\beta}{3}}\xi=\int_{\phi}^{\phi_M}\frac{d\phi}{(\phi-\phi_3)\sqrt{(\phi_M-\phi)(\phi-\phi_m)}}$ and $\sqrt{\frac{\beta}{3}}\xi=\int_{\phi_m}^{\phi}\frac{d\phi}{(\phi_3-\phi)\sqrt{(\phi_M-\phi)(\phi-\phi_m)}}$, respectively. From this, we have the following solitary wave solutions of equation (1.1):

$$\phi(\xi)=\phi_3+\frac{2(\phi_M-\phi_3)(\phi_3-\phi_m)}{(\phi_M-\phi_m)\cosh(\omega_2\xi)-(\phi_m+\phi_M-2\phi_3)}\tag{4.5}$$

and

$$\phi(\xi)=\phi_3-\frac{2(\phi_M-\phi_3)(\phi_3-\phi_m)}{(\phi_M-\phi_m)\cosh(\omega_2\xi)+(\phi_m+\phi_M-2\phi_3)},\tag{4.6}$$

where $\omega_2=\frac{1}{\sqrt{3}}\sqrt{\beta(\phi_M-\phi_3)(\phi_3-\phi_m)}$.

(iv) Corresponding to the level curves defined by $H(\phi,\ y)=h,\ h\in(h_3,\ h_2)$, there is a family of periodic orbits of system (1.3) (see Fig.4 (d)), with its boundary passing through two equilibrium points $S_\pm(0,\ \pm y_s)$, enclosing the four equilibrium points $E_j(\phi_j,\ 0),\ j=1,\cdots,4$. It gives rise to a periodic wave solution family of equation (1.1), which has the same parametric representation as equation (4.2).

(v) Corresponding to the level curves defined by $H_0(\phi,\ y)=h_2$, there are two homoclinic orbits of system (1.3) enclosing two equilibrium points $E_4(\phi_4,\ 0)$ and $E_1(\phi_1,\ 0)$, respectively (see Fig.4 (e)). Now, equation (4.1) can be written as $\sqrt{\frac{\beta}{3}}\xi=\int_{\phi}^{\phi_M}\frac{d\phi}{(\phi-\phi_2)\sqrt{(\phi_M-\phi)(\phi-\phi_m)}}$ and $\sqrt{\frac{\beta}{3}}\xi=\int_{\phi_m}^{\phi}\frac{d\phi}{(\phi_2-\phi)\sqrt{(\phi_M-\phi)(\phi-\phi_m)}}$, respectively. Thus, we have the following solitary wave solutions of equation (1.1):

$$\phi(\xi)=\phi_2+\frac{2(\phi_M-\phi_2)(\phi_2-\phi_m)}{(\phi_M-\phi_m)\cosh(\omega_3\xi)-(\phi_m+\phi_M-2\phi_2)} \tag{4.7}$$

and

$$\phi(\xi)=\phi_2-\frac{2(\phi_M-\phi_2)(\phi_2-\phi_m)}{(\phi_M-\phi_m)\cosh(\omega_3\xi)+(\phi_m+\phi_M-2\phi_2)}, \tag{4.8}$$

where $\omega_3=\frac{1}{\sqrt{3}}\sqrt{\beta(\phi_M-\phi_2)(\phi_2-\phi_m)}$.

(vi) Corresponding to the level curves defined by $H(\phi,\ y)=h,\ h\in(h_2,\ h_1)$, there exist the two families of periodic orbits of system (1.3) enclosing two equilibrium points $E_4(\phi_4,\ 0)$ and $E_1(\phi_1,\ 0)$, respectively. The two families of periodic wave solutions of equation (1.1) have the same parametric representations as equation (4.3) and (4.4).

(vii) Corresponding to the level curves defined by $H(\phi,\ y)=h,\ h\in(h_1,\ \infty)$, there is a family of periodic orbits of system (1.3) enclosing three equilibrium points, which gives rise to a family of periodic wave solutions of equation (1.1) with the exact parametric representations as equation (4.2).

As for $a=-\frac{1}{2}$, we have similar results as the above conclusions.

With the above estimates at hand, we give the following main results.

Theorem 4.1. *Assume* $a=0,-\frac{1}{2}$, $\Delta>0$ *and* $g>0$. *Then*

(i) Equation (1.1) *has the exact periodic wave solution families given by* (4.2)-(4.4).

(ii) Equation (1.1) *has the exact solitary wave solutions given by* (4.5)-(4.8).

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