Journal of Applied Analysis and Computation
Volume 10, Number 4, August 2020, 1545–1560
Website:http://jaac-online.com
DOI:10.11948/20190249

DISCONTINUOUS STURM-LIOUVILLE PROBLEMS INVOLVING AN ABSTRACT LINEAR OPERATOR

Oktay Sh. Mukhtarov[1,2,†] and Kadriye Aydemir[3]

Abstract In this paper we introduce to consideration a new type boundary value problems consisting of an "Sturm-Liouville" equation on two disjoint intervals as

$$-p(x)y'' + q(x)y + \mathfrak{B}y|_x = \mu y, x \in [a,c) \cup (c,b]$$

together with two end-point conditions whose coefficients depend linearly on the eigenvalue parameter, and two supplementary so-called transmission conditions, involving linearly left-hand and right-hand values of the solution and its derivatives at point of interaction $x = c$, where $\mathfrak{B} : L_2(a,c) \oplus L_2(c,b) \to L_2(a,c) \oplus L_2(c,b)$ is an abstract linear operator, non-selfadjoint in general. For self-adjoint realization of the pure differential part of the main problem we define "alternative" inner products in Sobolev spaces, "incorporating" with the boundary-transmission conditions. Then by suggesting an own approaches we establish such properties as topological isomorphism and coercive solvability of the corresponding nonhomogeneous problem and prove compactness of the resolvent operator in these Sobolev spaces. Finally, we prove that the spectrum of the considered eigenvalue problem is discrete and derive asymptotic formulas for the eigenvalues. Note that the obtained results are new even in the case when the equation is not involved an abstract linear operator $\mathfrak{B}$.

Keywords Sturm-Liouville problems, transmission conditions, coerciveness, spectrum, resolvent operator.

MSC(2010) 34B24, 34L05, 34L10,34L20.

1. Introduction

Boundary value problem for the Sturm-Liouville equation with discontinuous leading coefficients arises in geophysics, electromagnetics, elasticity, and other fields of engineering and physics; for example, modeling toroidal vibrations and free vibrations of the earth, reconstructing the discontinuous material properties of a nonabsorbingmedia, as a rule leads to direct and inverse problems for the Sturm-Liouville

[†]the corresponding author. Email address:omukhtarov@yahoo.com(O. Sh. Mukhtarov)
[1]Department of Mathematics, Faculty of Art and Science, Gaziosmanpaşa University, 60250 Tokat, Turkey
[2]Institute of Mathematics and Mechanics, Azerbaijan National, Academy of Sciences, Baku, Azerbaijan
[3]Department of Mathematics, Faculty of Art and Science, Amasya University, Amasya, Turkey

equation with discontinuous coefficients. The applications of Sturm-Liouville problems in physics and engineering are numerous. Their use in problems of vibration, heat transfer, quantum mechanics, and a host of other areas have proven successful for many years, and indeed go back to the early 18th century. For example, consider the initial-boundary value problem for the heat equation

$$
\begin{aligned}
&u_t = u_{xx} - q(x)u, \quad 0 < x < 1, \ t > 1, \\
&u(0,t) = u(1,t) = 0, \quad t \geq 1 \quad u(x,0) = f(x), \ 0 \leq x \leq 1,
\end{aligned}
$$

where q is a given coefficient function. This problem describes the temperature of a heat conducting bar with a nonuniform heat loss term given by $-q(x)u$. Applying the method of separation of variables we obtain the Sturm-Liouville problem

$$
\mathcal{A}u := -u'' + qu = \lambda u, \quad u(0) = u(1) = 0.
$$

It is important to find a complete set of eigenvectors of $\mathcal{A}$, or, equivalently, to diagonalize $\mathcal{A}$ in suitable infinite dimensional Hilbert space. The problem of diagonalizing a linear map on an infinite-dimensional space arises in many other ways, and is part of what is called spectral theory. Spectral theory provides a powerful way to understand linear operators by decomposing the space on which they act into invariant subspaces on which their action is simple. In the finite-dimensional case, the spectrum of a linear operator consists of its eigenvalues. The action of the operator on the subspace of eigenvectors with a given eigenvalue is just multiplication by the eigenvalue. Spectral theory of bounded linear operators on infinite-dimensional spaces is more involved. For example, an operator may have a continuous spectrum, in addition to, or instead of, a point spectrum of eigenvalues. A particularly simple and important case is that of compact, self-adjoint operators since such operators may be approximated by finite-dimensional operators, and their spectral theory is close to that of finite-dimensional symmetric operators. Sturm-liouville problems of spectral analysis consist in recovering operators from their spectral characteristics. Many thousands of papers, by Mathematicians and by others, have been published on this topic since then. Although the history of the subject is long, it remains an active area of research as new applications and concepts as well as computational difficulties continue to arise. The general results on the eigenvalue distribution of the eigenvalues of ordinary differential operators were obtained by Birkhoff [6], and for partial differential operators by Weyl [39]. In 1910 Weyl proved that the essential spectrum, which in this case is just the set of accumulation points of the spectrum, is stable when the boundary condition is modified. Tamarkin [31] introduced a concept of regular boundary conditions and proved that the system of root functions, i.e. eigenfunctions and associated functions of the regular boundary value problem is complete. In 1957 several remarkable papers were published. Rosenblum [27] and Kato [12] proved stability of absolutely continuous spectra for self-adjoint operators under trace class perturbations and Aronszajn [4] showed that the absolutely continuous Darts of spectral measures of Sturm-Liouville problems corresponding to different boundary conditions are equivalent, whereas their singular parts are mutually singular measures. Keldysh [14] elaborated expansions over root functions for weak perturbations of compact self-adjoint operators. Different challenges emerged with the development of Sturm theory and a corresponding awareness of the importance of distinguishing the absolutely continuous component from other parts of the essential spectrum, in connection with existence and completeness of the wave operators [3, 13, 40]. The most complete and sharp results for compact perturbations and for the so-called β-subordinate perturbations of self-adjoint operators are due to Markus and Matsaev [16] (see more details in [17]). Some recent developments of higher order differential operators whose boundary conditions depend on the eigenvalue parameter, including spectral asymptotics and basis properties, have been investigated in [18, 30]. General characterizations of self-adjoint boundary conditions have been presented in [38] for singular and regular problems. Gesztesy and Simon [8] found new uniqueness results with partial information on the spectrum for Sturm-Liouville operators with scalar and matrix coefficients, respectively. They showed that more information on the potential can compensate for less information about the spectrum. Martinyuk and Pivovarchik [19] proposed a new method for reconstructing the potential on half the interval. Sakhnovich [28] studied the existence of solutions of half inverse problems. Singular potentials were studied by Hryniv and Mykytyuk [9]. Buterin studied half inverse problem for differential pencils with the spectral parameter in boundary conditions [7]. Trooshin and Yamamoto [35] obtained Hochstadt-Lieberman type theorems for nonsymmetric first order systems. For quadratic pencils of Sturm-Liouville operators without the spectral parameter and transmission conditions Yang and Zettl [41] proved that if $p(x)$ and $q(x)$ are known on half of the domain interval, then one spectrum suffices to determine them uniquely on the other half. These references are certainly not intended to be comprehensive but are given to indicate the wide interest in and variety of half inverse type problems. For the background and applications of the boundary value problems to different areas, we refer the reader to the monographs and some recent contributions as [13, 16, 20, 25, 26, 31, 34, 37, 40].

In this study we consider a "Sturm-Liouville" equation involving an abstract linear operator $\mathfrak{B}$, namely the differential-operator equation

$$\Psi y := -p(x)y'' + q(x)y + \mathfrak{B}y|_x = \mu y \tag{1.1}$$

on $[a, c) \cup (c, b]$, together with eigendependent boundary conditions at the end-points $x = a$ and $x = b$

$$\Psi_1(\mu)y := \alpha_{10}y(a) - \alpha_{11}y'(a) - \mu(\alpha'_{10}y(a) - \alpha'_{11}y'(a)) = 0, \tag{1.2}$$

$$\Psi_2(\mu)y := \alpha_{20}y(b) - \alpha_{21}y'(b) + \mu(\alpha'_{20}y(b) - \alpha'_{21}y'(b)) = 0 \tag{1.3}$$

and transmission conditions at one interior point $x = c$

$$\Psi_3 y := \beta^-_{11}y'(c-) + \beta^-_{10}y(c-) + \beta^+_{11}y'(c+) + \beta^+_{10}y(c+) = 0, \tag{1.4}$$

$$\Psi_4 y := \beta^-_{21}y'(c-) + \beta^-_{20}y(c-) + \beta^+_{21}y'(c+) + \beta^+_{20}y(c+) = 0, \tag{1.5}$$

where $p(x) = p_1 > 0$ for $x \in [a, c)$, $p(x) = p_2 > 0$ for $x \in (c, b]$, the potential $q(x)$ is real-valued function which continuous in each of the intervals $[a, c)$ and $(c, b]$, and has a finite limits $q(c \mp 0)$, μ is a complex spectral parameter, α_{ij}, $\beta^{\pm}_{ij}$, α'_{ij} ($i = 1, 2$ and $j = 0, 1$) are real numbers. $\mathfrak{B}$ is an abstract linear operator in Hilbert space $L_2(a, c) \oplus L_2(c, b)$ (non-selfadjoint in general) .

This Sturm-Liouville problem is a non-classical eigenvalue problem since it contains an abstract linear operator in the equation, eigenvalue parameter appears also in the boundary conditions and two new conditions added to boundary conditions (so-called transmission conditions). Naturally the spectral theory of this problem is more involved. Boundary value problems with the spectral parameter in boundary conditions and/or with supplementary transmission conditions arise in various problems of mathematics and physical as well as in applications. For example, some boundary value problems with transmission conditions arise in heat and mass transfer problems [15], in vibrating string problems when the string loaded additionally with point masses [32], in diffraction problems [37] and etc. Such properties, as isomorphism, coerciveness with respect to the spectral parameter, completeness and Abel bases of a system of root functions of the similar boundary value problems with transmission conditions and its applications to the corresponding initial boundary value problems for parabolic equations have been investigated in [22] and [23]. Also some problems with transmission conditions which arise in mechanics (thermal conduction problem for a thin laminated plate) were studied in the article [33]. Similar problems for differential equations with discontinuous coefficients were investigated by Rasulov in monograph [26]. Detailed studies on spectral problems for ordinary differential operators depending on the parameter and/or with transmission conditions can be found in various publications, see e.g. [1,2,5,10,11,21,22,24,29,36,42], where further references and links to applications can be found.

Remark 1.1. Note that such generalization of Sturm-Liouville problems involving abstract linear operator in the equation has been investigated by us for the first time in literature. Naturally, the considered problem (1.1)–(1.5) covered a wide class of classical and nonclassical BVP's. For instance, our results is applicable to the following type nonstandard equations together with the same boundary-transmission conditions (1.2)–(1.5).

$$i) - p(x)y''(x) + q(x)y(x) + r_0(x)y(\xi_0) + r_1(x)y'(\xi_1) = \mu y(x),\ \ x \in [a, c) \cup (c, b]$$

where the functions $r_i(x)(i = 0, 1)$ are satisfy the same conditions as $q(x)$, $\xi_i(i = 0, 1) \in (a, c) \cup (c, b)$ are any interior points.

$$ii)\ \ -p(x)y''(x) + q(x)y(x) + \sum_{j=0}^{1} \left(\int_a^{c-0} K_{1j}(x, \xi)y^{(j)}(\xi)d\xi \right.$$
$$\left. + \int_{c+0}^{b} K_{2j}(x, \xi)y^{(j)}(\xi)d\xi \right) = \mu y(x)$$

where the Kernels $K_{1j}(x, \xi)$ and $K_{2j}(x, \xi)(j = 0, 1)$ are defined and continuous in $[a, b] \times [a, c]$ and $[a, b] \times [c, b]$, respectively.

2. Sobolev spaces with "alternative" inner-products "incorporating" with the considered problem

At first we shall introduce an "alternative" inner product in classical Sobolev spaces such a way that the pure differential part of the considered problem can be interpreted as self-adjoint problem in these spaces.

Denote the determinant of the boundary-matrix $B_i = \begin{bmatrix} \alpha_{i1} & \alpha_{i0} \\ \alpha'_{i1} & \alpha'_{i0} \end{bmatrix}$ by $\theta_i(i = 1, 2)$ and the determinant of the k-th and j-th columns of the transmission-matrix

$T = \begin{bmatrix} \beta_{10}^{+} & \beta_{11}^{+} & \beta_{10}^{-} & \beta_{11}^{-} \\ \beta_{20}^{+} & \beta_{21}^{+} & \beta_{20}^{-} & \beta_{21}^{-} \end{bmatrix}$ by $\Delta_{kj}(1 \le k < j \le 4)$. Throughout in this study we shall assume that the conditions

$$\theta_1 > 0,\ \theta_2 > 0,\ \Delta_{12} > 0 \ \text{ and } \Delta_{34} > 0 \tag{2.1}$$

is hold. For self-adjoint realization we shall introduce some "new" Hilbert spaces with alternative inner products. Recall that the direct sum space $L_2(a,c) \oplus L_2(c,b)$ is the Hilbert space consisting of all functions f on $[a,c) \cup (c,b]$ for which $f_1 := f|_{[a,c)} \in L_2(a,c)$ and $f_2 := f|_{(c,b]} \in L_2(c,b)$ and the direct sum of classical Sobolev spaces $W_2^n(a,c) \oplus W_2^n(c,b)$ is the Hilbert space consisting of all functions $f \in L_2(a,c) \oplus L_2(c,b)$ such that f_1 and f_2 has generalized n-th derivatives (in the sense of distributions) in $L_2(a,c)$ and $L_2(c,b)$, respectively, with the inner product

$$< f,g >_{W_2^n} = \sum_{k=0}^{n} (< f_1^{(k)}, g_1^{(k)} >_{W_2^n(a,c)} + < f_2^{(k)}, g_2^{(k)} >_{W_2^n(c,b)},$$

and the finite norm $\|f\|_{W_2^n} = (< f,f >_{W_2^n})^{\frac{1}{2}}$. Naturally, $W_2^0(a,c) \oplus W_2^0(c,b) \equiv L_2(a,c) \oplus L_2(c,b) \equiv L_2(a,b)$. The standard inner products in direct sum spaces $L_2(a,c) \oplus L_2(c,b)$ and $(L_2(a,c) \oplus L_2(c,b)) \oplus \mathbb{C}^2$ which is given by

$$< f(.), g(.) >_{L_2} := < f_1, g_1 >_{L_2(a,c)} + < f_2, g_2 >_{L_2(c,b)} \tag{2.2}$$

and

$$< F, G >_{L_2 \oplus \mathbb{C}^2} := < f(.), g(.) >_{L_2} + f_1\overline{g_1} + f_2\overline{g_2} \tag{2.3}$$

we shall replace by the "alternative" inner products as

$$\langle f,g \rangle_{\mathcal{H}_1} := \frac{\Delta_{34}}{p_1} \int_a^{c-} f(x)\overline{g(x)}dx + \frac{\Delta_{12}}{p_2} \int_{c+}^{b} f(x)\overline{g(x)}dx$$

and

$$\langle F,G \rangle_{\mathcal{H}} := \langle f,g \rangle_{\mathcal{H}_1} + \frac{\Delta_{34}}{\theta_1} f_1\overline{g_1} + \frac{\Delta_{12}}{\theta_2} f_2\overline{g_2} \tag{2.4}$$

respectively and apply operator theory in the new Hilbert space

$$\mathcal{H} := (L_2(a,c) \oplus L_2(c,b)) \oplus \mathbb{C}^2, < .,. >_{\mathcal{H}}), \tag{2.5}$$

where $F = (f(.), f_1, f_2),\ G = (g(.), g_1, g_2) \in (L_2(a,c) \oplus L_2(c,b)) \oplus \mathbb{C} \oplus \mathbb{C}$.

Remark 2.1. The new inner product (2.4) is equivalent to the inner product (2.3), so $\mathcal{H}$ is also Hilbert space and can be seen as different realization of the Hilbert space $(L_2(a,c) \oplus L_2(c,b)) \oplus \mathbb{C}^2$. But such realization of this direct sum space allow as to interpret the conditions (1.2)–(1.5) as "self-adjoint boundary-transmission conditions."

Let us we define the boundary functionals:

$$B_a(f) := \alpha_{10} f(a) - \alpha_{11} f'(a), \quad B'_a(f) := \alpha'_{10} f(a) - \alpha'_{11} f'(a)$$

$$B_b(f) := \alpha_{20} f(b) - \alpha_{21} f'(b), \quad B'_b(f) := \alpha'_{20} f(b) - \alpha'_{21} f'(b)$$

and differential operator

$$\Phi u := -pu'' + q(x)u. \tag{2.6}$$

A suitable inner-product space in which to search for solution of the equation $\mu y - \Psi y = f(x)$, is the linear space

$$\begin{aligned}\widetilde{\mathcal{H}} = \Big\{F = \Big(f(.), f_1, f_2\Big) : f(.) \in W_2^2(a,c) \oplus W_2^2(c,b),\\ \Psi_3(f) = \Psi_4(f) = 0,\ f_1 = B'_a(f), f_2 = -B'_b(f)\Big\}\end{aligned} \tag{2.7}$$

equipped with the inner product

$$< (f(.), f_1, f_2), (g(.), g_1, g_2) >_{\widetilde{\mathcal{H}}} =< f(.), g(.) >_{W_2^2} \tag{2.8}$$

and corresponding norm

$$\|(f(.), f_1, f_2)\|_{\widetilde{\mathcal{H}}} = \|f(.)\|_{W_2^2}. \tag{2.9}$$

It can be verify easily that, all axioms of inner product are satisfied.

Lemma 2.1. *The inner-product space $\widetilde{\mathcal{H}}$ is a Hilbert space.*

Proof. Let $F_n = (f_n(.), f_{1n}, f_{2n}) \in \widetilde{\mathcal{H}}, n = 1, 2, ...$ be any Cauchy sequence with respect to the norm (2.9). Then by (2.9) the sequence $(f_n(.))$, which consist of the first components of (F_n), will be a Cauchy sequence in the Hilbert space $W_2^2(a,c) \oplus W_2^2(c,b)$. Thus, there exists $f = f(.) \in W_2^2(a,c) \oplus W_2^2(c,b)$ such that $\|f_n - f\|^2_{W_2^2(a,c)\oplus W_2^2(c,b)} \to 0$. By virtue of the fact that, the embeddings $W_2^2(a,c) \subset C[a,c]$ and $W_2^2(c,b) \subset C[c,b]$ are continuous, the sequences $\Psi_3(f_n)$ and $\Psi_4(f_n)$ are converges to $\Psi_3(f)$ and $\Psi_4(f)$, accordingly. Thus, $\Psi_3(f) = \Psi_4(f) = 0$ for all n by (2.7). Now, defining $F = (f(.), B'_a(f), -B'_b(f)) \in \widetilde{\mathcal{H}}$ we see that $\|F_n - F\|_{\widetilde{\mathcal{H}}} \to 0$ as $n \to \infty$, so, the arbitrary Cauchy sequence in $\widetilde{\mathcal{H}}$ is convergent. The proof is complete. □

3. Topological isomorphism and coercive solvability. The resolvent operator.

Let us construct the operator $\widetilde{\mathcal{L}} : \mathcal{H} \to \mathcal{H}$ with the domain

$$\begin{aligned} dom(\widetilde{\mathcal{L}}) := \Big\{ & F = (f(x), f_1, f_2) : f(x), f'(x) \ \in AC_{loc}(a,c) \cap AC_{loc}(c,b),\\ & \text{with the a finite limits } f(c \mp 0) \text{ and } f'(c \mp 0);\ \Psi F \in L_2(a,c) \oplus L_2(c,b);\\ & \Psi_3(f) = \Psi_4(f) = 0;\ f_1 = B'_a(f),\ f_2 = -B'_b(f) \Big\}\end{aligned}$$

and action low

$$\widetilde{\mathcal{L}}(f(x), B'_a(f), -B'_b(f)) = (\Psi f, B_a(f), B_b(f)).$$

Obviously, the operator $\widetilde{\mathcal{L}}$ is well-defined in the Hilbert space $\mathcal{H}$. Then the problem (1.1)–(1.5) acquires the operator equation form

$$\widetilde{\mathcal{L}}F = \mu F, \quad F = (f(x), B'_a(f), -B'_b(f)) \in dom(\widetilde{\mathcal{L}})$$

in the Hilbert space $\mathcal{H}$.

Remark 3.1. The eigenvalues μ_k of the problem (1.1)–(1.5) and those of the operator $\widetilde{\mathcal{L}}$ coincide and there exist a correspondence

$$y_k(x) \leftrightarrow Y_k = (y_k(x), B'_a(y_k), -B'_b(y_k))$$

between eigenfunctions $y_k(x)$ of the problem (1.1) – (1.5) and eigenelements Y_k of the operator $\widetilde{\mathcal{L}}$.

Lemma 3.1. *The linear operator $\widetilde{\mathcal{L}}$ is densely defined on $\mathcal{H}$, i.e. $\overline{dom(\widetilde{\mathcal{L}})} = \mathcal{H}$.*

Proof. Suppose that, $G_0 = (g_0(.), g_1, g_2) \in \mathcal{H}$ is orthogonal to all $F = (f(.), B'_a(f), -B'_b(f)) \in D(\widetilde{\mathcal{L}})$, i.e.

$$\begin{aligned} < F, G_0 >_{\mathcal{H}} = \frac{\Delta_{34}}{p^-} \int_a^{c-} f(x)\overline{g_0(x)}dx + \frac{\Delta_{12}}{p^+} \int_{c+}^b f(x)\overline{g_0(x)}dx \\ + \frac{\Delta_{34}}{\theta_1} B'_a(f)\overline{g_1} - \frac{\Delta_{12}}{\theta_2} B'_b(f)\overline{g_2} = 0 \end{aligned} \tag{3.1}$$

for all $F \in D(\widetilde{\mathcal{L}})$. Denote by $C_0^\infty[a, c) \cup C_0^\infty(c, b]$ the set of infinitely differentiable functions on $[a, c)\cup(c, b]$, each of which vanishes on some neighborhoods of the points $x = a$, $x = c$ and $x = b$. Since $(f(.), 0, 0) \in D(\widetilde{\mathcal{L}})$ for all $f(.) \in C_0^\infty[a, c) \oplus C_0^\infty(c, b]$, we have from (3.1) that $\langle f, g_0\rangle_{\mathcal{H}_1} = 0$ for all $f(.) \in C_0^\infty[a, c) \oplus C_0^\infty(c, b]$, which in turn implies that $\langle f, g_0\rangle_{L_2(a,c)} = \langle f, g_0\rangle_{L_2(c,b)} = 0$ for all $f \in C_0^\infty[a, c) \oplus C_0^\infty(c, b]$. Taking into account that $C_0^\infty[a, c]$ and $C_0^\infty[c, b]$ are dense in $L_2(a, c)$ and $L_2(c, b)$, respectively, we have that the function $g_0(x)$ vanishes on $[a, c) \cup (c, b]$. It is easy to see that, there is an element $\widetilde{F}_0 = (\widetilde{f}_0(.), B'_a(\widetilde{f}_0), -B'_b(\widetilde{f}_0)) \in D(\widetilde{\mathcal{L}})$ such that $B'_a(\widetilde{f}_0) = g_1$ and $B'_b(\widetilde{f}_0) = -g_2$. Putting $F = \widetilde{F}_0$ in (3.1) we have $g_1 = g_2 = 0$. Consequently $(dom(\widetilde{\mathcal{L}}))^\perp = (0, 0, 0)$. The proof is complete. □

Now, consider the nonhomogeneous boundary value-transmission problem

$$\Psi y - \mu y = f(x) \;\; x \in [a, c) \cup (c, b], \tag{3.2}$$

$$\Psi_1(\mu)y = f_1, \;\; \Psi_2(\mu)y = f_2, \;\; \Psi_3 y = \Psi_4 y = 0 \tag{3.3}$$

for $f \in L_2(a, c)\oplus L_2(c, b)$, $f_1, f_2 \in \mathbb{C}$. Denote $Y(x) := (y(x), B'_a(y), -B'_b(y)) \in D(\widetilde{\mathcal{L}})$ and $F = (f(x), f_1, f_2) \in \mathcal{H}$. Then the problem (3.2)-(3.3) reduces to the operator equation

$$(\mu I - \widetilde{\mathcal{L}})Y = F, \;\; F \in \mathcal{H} \tag{3.4}$$

in the Hilbert space $\mathcal{H}$. Throughout in below we shall use the notations

$$G_\delta = \{\mu \in \mathbb{C} \mid \delta < arg\mu < 2\pi - \delta\}, \;\; 0 \le \delta < 2\pi$$

and

$$U_\infty(r) = \{\mu \in \mathbb{C} : \mid \mu \mid > r\}, \;\; r > 0.$$

Theorem 3.1. *Suppose that the linear operator $\mathfrak{B}$ is compact from $W_2^2(a,c)\oplus W_2^2(c,b)$ into $L_2(a,c)\oplus L_2(c,b)$. Then, for any $\delta>0$ there exists $r_\delta>0$ such that for all complex numbers $\mu\in G_\delta\cap U_\infty(r_\delta)$ the operator $\mu I-\widetilde{\mathcal{L}}$ is an topological isomorphism from $\widetilde{\mathcal{H}}$ onto $\mathcal{H}$ and for these μ the coercive estimate*

$$||Y_\mu(.)||_{\widetilde{\mathcal{H}}}+|\mu|\ ||Y_\mu(.)||_{\mathcal{H}}\leq C_\delta\ ||F||_{\mathcal{H}} \tag{3.5}$$

holds for the solution $Y=Y_\mu(.)$ of the equation (3.4) where C_δ is a constant, which depend only of δ.

Proof. From the definitions of $\widetilde{\mathcal{L}},\mathcal{H}$ and $\widetilde{\mathcal{H}}$ it follows immediately that the linear operator $\mu I-\widetilde{\mathcal{L}}$ acts from $\widetilde{\mathcal{H}}$ into $\mathcal{H}$ continuously for all $\mu\in\mathbb{C}$. Following the same procedure as in [23], we obtain that for any $\delta>0$ there exists $r_\delta>0$ such that for all complex numbers $\mu\in G_\delta\cap U_\infty(r_\delta)$ the operator $\Im(\mu):y\to(\mu y-\Psi(\mu)y,\Psi_1(\mu)y,\Psi_2(\mu)y)$ is an isomorphism from $W_2^2(a,c)\oplus W_2^2(c,b)$ onto $(L_2(a,c)\oplus L_2(c,b))\oplus\mathbb{C}^2$ and for these μ the coercive estimate

$$||y||_{W_2^2}+|\mu|(\ ||y||_{L_2}+|B_a^{'}(y)|+|B_b^{'}(y)|)\leq C_\delta(||f||_{L_2}+|f_1|+|f_2|) \tag{3.6}$$

holds for a solution $y(x)$ of the problem (3.2)–(3.3). Consequently, the operator $\widetilde{\mathcal{L}}-\mu I$ is an isomorphism from $\widetilde{\mathcal{H}}$ onto $\mathcal{H}$. The estimate (3.5) follows from (3.6). $\square$

Corollary 3.1. *If the linear operator $\mathfrak{B}$ is compact from $W_2^2(a,c)\oplus W_2^2(c,b)$ into $L_2(a,c)\oplus L_2(c,b)$ then, for any $\delta>0$ there exists $r_\delta>0$ such that for all complex numbers $\mu\in G_\delta\cap U_\infty(r_\delta)$ are regular point of the operator $\widetilde{\mathcal{L}}$ and for the resolvent operator of $\widetilde{\mathcal{L}}$ the estimate*

$$\|(\widetilde{\mathcal{L}}-\mu I)^{-1}\|_{\mathcal{H}\to\mathcal{H}}\leq C_\delta\ |\mu|^{-1} \tag{3.7}$$

holds, where $C_\delta>0$ is a constant which depend only of δ.

Corollary 3.2. *The resolvent operator $R(\mu,\widetilde{\mathcal{L}})=(\widetilde{\mathcal{L}}-\mu I)^{-1}$ acted boundedly from $\mathcal{H}$ into $\widetilde{\mathcal{H}}$.*

Theorem 3.2. *Let the conditions of the Theorem 3.1 be satisfied. Then for any $\delta>0$ there exists $r_\delta>0$ such that for all complex numbers $\mu\in G_\delta\cap U_\infty(r_\delta)$ the resolvent operator $R(\mu,\widetilde{\mathcal{L}})$ from $\mathcal{H}$ into $\mathcal{H}$ is compact.*

Proof. At first show that the embedding $\widetilde{\mathcal{H}}\subset\mathcal{H}$ is compact. For this, let $F_n=(f_n(.),B_a'(f_n),-B_b'(f_n)),n=1,2,...$ be any bounded sequence in $\widetilde{\mathcal{H}}$. Then the sequence $(f_n(.))$ consisting of the first components of (F_n) will be bounded in the direct sum space $W_2^2(a,c)\oplus W_2^2(c,b)$. Since the embeddings $W_2^2(a,c)\subset L_2[a,c]$ and $W_2^2(c,b)\subset L_2[c,b]$ are compact, the sequence $(f_n(.))$ has an convergent subsequence $(f_{n_k}(.))$ in the space $L_2(a,c)\oplus L_2(c,b)$. Let $f_0(.)\in L_2(a,c)\oplus L_2(c,b)$ be limit of this subsequence. Further, since the embeddings $W_2^2(a,c)\subset C[a,c]$ and $W_2^2(c,b)\subset C[c,b]$ are compact, the sequence$(f_{n_k}(.))$ has an convergent subsequence $(f_{n_{k_s}}(.))$ in spaces $C[a,c]$ and $C[c,b]$ respectively. Consequently the numerical sequences $(B_a'(f_{n_{k_s}}))$ and $(B_b'(f_{n_{k_s}}))$ are convergent. Let $f_1,f_2\in\mathbb{C}$ are limits of this numerical sequences respectively. Now defining $F_0=(f_0(.),f_1,f_2)$, we see that $F_0\in\mathcal{H}$ and the subsequence $(F_{n_{k_s}})$ converges to F_0 in the Hilbert space $\mathcal{H}$, so the embedding $\widetilde{\mathcal{H}}\subset\mathcal{H}$ is compact. Further, from Corollary 3.2, follows that the resolvent operator $R(\mu,\widetilde{\mathcal{L}})$ is bounded from $\mathcal{H}$ into $\widetilde{\mathcal{H}}$. Consequently, the resolvent operator $R(\mu,\widetilde{\mathcal{L}})$ is compact from $\mathcal{H}$ into itself. $\square$

4. A pure differential operator associated with the problem

Consider the pure differential part (i.e. without operator $\mathfrak{B}$) of the considered problem (1.1)–(1.5). Let $\pounds$ be linear differential operator in Hilbert space $\mathcal{H}$ with domain $D(\pounds) = D(\widetilde{\pounds})$ and action low

$$\pounds(f(x), B'_a[f], -B'_b[f]) = (\Phi f, B_a[f], B_b[f]).$$

Lemma 4.1. *The operator $\pounds$ is symmetric in $\mathcal{H}$.*

Proof. Let $F = (f(x), B'_a(f), -B'_b(f)), G = (g(x), B'_a(g), -B'_b(g)) \in dom(\pounds)$. By partial integration we have

$$\begin{aligned}\langle \pounds F, G\rangle_{\mathcal{H}} - \langle F, \pounds G\rangle_{\mathcal{H}} =& \Delta_{34}\ W(f,\overline{g};c-) - \Delta_{34}\ W(f,\overline{g};a)\\ &+ \Delta_{12}\ W(f,\overline{g};b) - \Delta_{12}\ W(f,\overline{g};c+)\\ &+ \frac{\Delta_{34}}{\theta_1}(B_a(f)\overline{B'_a(g)} - B'_a(f)\overline{B_a(g)})\\ &+ \frac{\Delta_{12}}{\theta_2}(B'_b(f)\overline{B_b(g)} - B_b(f)\overline{B'_b(g)}) \end{aligned} \tag{4.1}$$

where, as usual, $W(f,\overline{g};x)$ denotes the Wronskians of the functions f and $\overline{g}$. From the definitions of boundary functionals we get that

$$B_a(f)\overline{B'_a(g)} - B'_a(f)\overline{B_a(g)} = \theta_1 W(f,\overline{g};a), \tag{4.2}$$
$$B'_b(f)\overline{B_b(g)} - B_b(f)\overline{B'_b(g)} = -\theta_2 W(f,\overline{g};b). \tag{4.3}$$

By using the transmission conditions (1.4)–(1.5) we derive that

$$W(f,\overline{g};c-) = \frac{\Delta_{12}}{\Delta_{34}}\ W(f,\overline{g};c+). \tag{4.4}$$

Finally, substituting (4.2), (4.3) and (4.4) in (1.2) we have

$$\langle \pounds F, G\rangle_{\mathcal{H}} = \langle F, \pounds G\rangle_{\mathcal{H}} \text{ for every } F, G \in dom(\pounds),$$

so the operator $\pounds$ is symmetric in $\mathcal{H}$. The proof is complete. □

Corollary 4.1. *The eigenvalues of $\pounds$ are real.*

Theorem 4.1. *$\pounds$ is self-adjoint linear operator in the Hilbert space $\mathcal{H}$.*

Proof. Since $\pounds$ is symmetric, it is enough to prove that $D(\pounds^*) = D(\pounds)$. Let $F \in D(\pounds^*)$. Then for all $G \in D(\pounds)$

$$\langle \pounds G, F\rangle_{\mathcal{H}} = \langle G, \pounds^* F\rangle_{\mathcal{H}}.$$

Let μ_0 be any regular value of $\pounds$ such that $Im\mu_0 \neq 0$. Then

$$\langle(\mu_0 I - \pounds)G, F\rangle_{\mathcal{H}} = \langle G, (\overline{\mu_0} I - \pounds^*)F\rangle_{\mathcal{H}}, \quad \forall\ G \in D(\pounds). \tag{4.5}$$

Since μ_0 is a regular point of $\pounds$ the operator $\overline{\mu_0} I - \pounds$ has the inverse $(\overline{\mu_0} I - \pounds)^{-1}$ which is defined on whole Hilbert space $\mathcal{H}$. Then defining $F_0 \in D(\pounds)$ as

$$F_0 = (\overline{\mu_0} I - \pounds)^{-1}(\overline{\mu_0} F - \pounds^* F),$$

we have
$$(\overline{\mu_0}I - \pounds)F_0 = \overline{\mu_0}F - \pounds^* F.$$
By using the last equalities and applying the previous Theorem we have

$$\begin{aligned}\langle(\mu_0 I - \pounds)G, F\rangle_{\mathcal{H}} &= \langle G, (\overline{\mu_0}I - \pounds^*)F\rangle_{\mathcal{H}} = \langle G, (\overline{\mu_0}I - \pounds)F_0\rangle_{\mathcal{H}} \\ &= \langle G, \overline{\mu_0}F_0\rangle_{\mathcal{H}} - \langle G, \pounds F_0\rangle_{\mathcal{H}} = \langle \mu_0 G, F_0\rangle_{\mathcal{H}} - \langle \pounds G, F_0\rangle_{\mathcal{H}} \\ &= \langle(\mu_0 I - \pounds)G, F_0\rangle_{\mathcal{H}}\end{aligned}$$

for an arbitrary $G \in D(\pounds)$. Consequently, for an arbitrary element $G \in D(\pounds)$ the equality
$$\langle(\mu_0 I - \pounds)G, F - F_0\rangle_{\mathcal{H}} = 0,$$
holds. Since μ_0 is a regular point of $\pounds$, we can put $G = R(\mu_0, \pounds)(F - F_0)$ in the last equality. Then we have $\|F - F_0\|_{\mathcal{H}} = 0$, namely, $F = F_0$. Thus we find that $F \in D(\pounds)$ i.e. $D(\pounds^*) = D(\pounds)$. The proof is complete. □

Let $\mu = s^2$. Following the same procedure as in [24] we have the next Theorem.

Theorem 4.2. *The operator $\pounds$ has precisely denumerable many real eigenvalues, whose behavior may be expressed by two sequence $\{s_{n,1}\}_1^\infty$ and $\{s_{n,2}\}_1^\infty$ with following asymptotics as $n \to \infty$.*
i) *If $\alpha'_{21} \neq 0$ and $\alpha'_{11} \neq 0$, then*

$$s_{n,1} = \frac{\sqrt{p_1}(n-3)\pi}{(c-a)} + O\left(\frac{1}{n}\right), \; s_{n,2} = \frac{\sqrt{p_2}n\pi}{(b-c)} + O\left(\frac{1}{n}\right). \tag{4.6}$$

ii) *If $\alpha'_{21} \neq 0$ and $\alpha'_{11} = 0$, then*

$$s_{n,1} = \frac{\sqrt{p_1}(2n+1)\pi}{2(c-a)} + O\left(\frac{1}{n}\right), \; s_{n,2} = \frac{\sqrt{p_2}(n-2)\pi}{(b-c)} + O\left(\frac{1}{n}\right). \tag{4.7}$$

iii) *If $\alpha'_{21} = 0$ and $\alpha'_{11} \neq 0$, then*

$$s_{n,1} = \frac{\sqrt{p_1}(n-2)\pi}{(c-a)} + O\left(\frac{1}{n}\right), \; s_{n,2} = \frac{\sqrt{p_2}(2n+1)\pi}{2(b-c)} + O\left(\frac{1}{n}\right). \tag{4.8}$$

iv) *If $\alpha'_{21} = 0$ and $\alpha'_{11} = 0$, then*

$$s_{n,1} = \frac{\sqrt{p_1}(2n-3)\pi}{2(c-a)} + O\left(\frac{1}{n}\right), \; s_{n,2} = \frac{\sqrt{p_2}(2n+1)\pi}{2(b-c)} + O\left(\frac{1}{n}\right). \tag{4.9}$$

5. The structure of the spectrum and asymptotic behaviour of the eigenvalues in complex plane

Let us present here some needed basic definitions about spectrum of the general linear operators in Hilbert space(see, for example, [24]).

Let A be densely defined closed operator in complex Hilbert space E. The point μ of the complex plane is called a regular point of an operator A in E, if the operator $A - \mu I$ is invertible(i.e has a bounded inverse operator $(A - \mu I)^{-1}$ which defined on whole E). In this case the operator $R(\mu, A) = (A - \mu I)^{-1}$ is called the resolvent of the operator A.The complement of the set of regular points $\rho(A)$ to the entire complex plane is called the spectrum $\sigma(A)$ of the operator(obviously, all eigenvalues belongs to the spectrum).

Let μ_0 be eigenvalue of A. The linear manifold

$$N_{\mu_0} = \cup_{n=1}^{\infty}\{f \in E : f \in D(A^n), (A - \mu_0 I)^n f = 0\}$$

is called a root lineal corresponding to eigenvalue μ_0. The dimension of the lineal N_{μ_0} is called an algebraic multiplicity of the eigenvalue μ_0. The spectrum $\sigma(A)$ of the operator A is called discrete if $\sigma(A)$ consist of isolated eigenvalues with finite algebraic multiplicities and infinity is the only possible limit point of $\sigma(A)$.

Theorem 5.1. *The spectrum of the problem* (1.1)–(1.5) *is discrete.*

Proof. Since the operator $\widetilde{\mathcal{L}}$ has a compact resolvent in the Hilbert space $\mathcal{H}$, by virtue of well-known Theorem of Functional Analysis [[13], chapter III, section 6)] the spectrum of $\widetilde{\mathcal{L}}$ is discrete. □

Now, let us denote by $\widetilde{\mathfrak{B}}$ the linear operator in the Hilbert space $\mathcal{H}$ with domain $D(\widetilde{\mathfrak{B}}) = D(\widetilde{\mathcal{L}})$ and action low

$$\widetilde{\mathfrak{B}}Y = \Big(\mathfrak{B}y\mid_x, 0, 0\Big) \tag{5.1}$$

for $Y = (y(x), y_1, y_2) \in D(\widetilde{\mathfrak{B}})$. Then, the main problem can be written in the operator-equation form as

$$(\mathcal{L} + \widetilde{\mathfrak{B}})F = \mu F, \quad F \in D(\widetilde{\mathcal{L}}). \tag{5.2}$$

Let A be an linear operator with discrete spectrum and let S be a subset of complex plane $\mathbb{C}$. In below we shall denote by $N(r, S, A)$ the sum of the algebraic multiplicities of all the eigenvalues of A contained in $S \cap \{\mu \in \mathbb{C} : |\mu| < r\}$.

Definition 5.1. Let A_1 be any closed linear operator having at least one regular point. A linear operator A_2 is said to be A_1-compact (or compact with respect to A_1) if $D(A_2) \supseteq D(A_1)$ and if for some regular point $\mu_0 \in \rho(A_1)$ the operator $A_2R(\mu_0, A_1) = A_2(A_1 - \mu_0 I)^{-1}$ is compact (see, for example, [16]).

The next theorem follows immediately from the [17]

Theorem 5.2. *Let $\Im$ be a self-adjoint linear operator in Hilbert space with discrete spectrum and let T be $\Im$-compact operator. Then if $\Im$ has a precisely denumerable many positive eigenvalues and*

$$N\left(r(1+\varepsilon), R^+, \Im\right) \sim N(r, R^+, \Im) \text{ as } r \to \infty,\ \varepsilon \to 0 \tag{5.3}$$

then for any δ $(0 < \delta < \frac{\pi}{2})$

$$N\left(r, G_\delta, \Im + T\right) \sim N(r, R^+, \Im) \text{ as } r \to \infty \tag{5.4}$$

where $R^+ = (0, \infty)$, $\quad G_\delta := \{\mu \in \mathbb{C} \mid \delta < arg\mu < 2\pi - \delta\}$ and $f(\mu) \sim g(\mu)$ as $r \to \infty$ is the abbreviation for $\lim\limits_{r\to\infty} f(r)/g(r) = 1$.

Theorem 5.3. *Let the linear operator $\mathfrak{B}$ be $\mathcal{L}-$ compact operator in the Hilbert space $\mathcal{H}$. Then the spectrum of $\widetilde{\mathcal{L}} = \mathcal{L} + \widetilde{\mathfrak{B}}$ is discrete and consist of precisely denumerable many eigenvalues. For any arbitrary small $\alpha > 0$ all eigenvalues of*

$\widetilde{\pounds}$ with the possible exception of a finite number lie in the sector $\psi_\alpha = \{\mu \in \mathbb{C} : |arg\mu| < \alpha\}$ of angular 2α and for the sequence of eigenvalues $\{\mu_{n,\alpha}\}_1^\infty, n \geq 0$, belonging to the sector ψ_α, which when listen according to nondecreasing modulus and repeated according to algebraic multiplicity the asymptotic expression

$$|\mu_{n,\alpha}| = \big(\frac{\sqrt{p_1p_2}}{\sqrt{p_1}(b-c)+\sqrt{p_2}(c-a)}\big)^2\pi^2n^2 + o(n^2) \tag{5.5}$$

is valid, where the expression $a_n = o(n^2)$ is the abbreviation for $\lim\limits_{n\to\infty}\frac{a_n}{n^2} = 0$.

Proof. Denote by $\{\mu_n(\pounds)\}_1^\infty$, $n = 0, 1, ...$ the eigenvalues of $\pounds$, which counted with their algebraic multiplicity and listed as $\mu_1(\pounds) \leq \mu_2(\pounds) \leq ...$ Then from (4.6)-(4.9) it follow that there are real numbers $\xi_1 > 0$ and $\xi_2 > 0$ such that

$$\big(\frac{\sqrt{p_1p_2}}{\sqrt{p_1}(b-c)+\sqrt{p_2}(c-a)}\big)^2\pi^2n^2+\xi_1 n \leq \mu_n(\pounds) \leq \big(\frac{\sqrt{p_1p_2}}{\sqrt{p_1}(b-c)+\sqrt{p_2}(c-a)}\big)^2\pi^2n^2+\xi_2 n. \tag{5.6}$$

From these inequalities it follows that

$$N\left(r, R^+, \pounds\right) = \frac{1}{\pi}\big(\frac{(b-c)}{\sqrt{p_2}} + \frac{(c-a)}{\sqrt{p_1}}\big)\sqrt{r} + O\big(\frac{1}{\sqrt{r}}\big),\ r \to \infty.$$

Hence

$$N\left(r(1+\varepsilon), R^+, \pounds\right) \sim N(r, R^+, \pounds),\ \text{for } r \to \infty,\ \varepsilon \to \infty.$$

Applying Theorem 5.2 we have

$$N\left(r, \alpha, \pounds + \widetilde{\mathfrak{B}}\right) \sim N(r, R^+, \pounds),\ r \to \infty. \tag{5.7}$$

Consequently

$$N\left(r, \psi_\alpha, \pounds + \widetilde{\mathfrak{B}}\right) = N(r, R^+, \pounds) + o(N(r, R^+, \pounds)),\ r \to \infty, \tag{5.8}$$

where, as usual, the expression $f(r) = o(g(r)),\ r \to \infty$ is the abbreviation for $\lim\limits_{r\longrightarrow\infty} f(r)/g(r) = 0$. Putting $r = |\mu_{n,\alpha}|$ in the last equality we have the needed asymptotic formula (5.5). The proof is complete. □

Lemma 5.1. *Suppose that the conditions of Theorem 5.3 is hold. Then the spectrum $\sigma(\widetilde{\pounds})$ of the operator $\widetilde{\pounds}$ is discrete and consist of denumerable many eigenvalues $\{\mu_n(\widetilde{\pounds})\}_1^\infty$ which, when arranged in decreasing modulus and counted to their algebraic multiplicity, has the following asymptotic behaviour*

$$Re\mu_n(\widetilde{\pounds}) = \big(\frac{\sqrt{p_1p_2}}{\sqrt{p_1}(b-c)+\sqrt{p_2}(c-a)}\big)^2\pi^2n^2 + o(n^2) \tag{5.9}$$

and

$$|Im\mu_n(\widetilde{\pounds})| = o(n^2)\ as\ n \to \infty. \tag{5.10}$$

Proof. By using the fact that for all $\alpha > 0$, small enough, there are at most finite number eigenvalues of $\widetilde{\pounds}$ outside angle $\psi_\alpha = \{\mu \in \mathbb{C} : |arg\mu| < \alpha\}$, and applying Theorem 5.3 we have the asymptotic formula

$$|\mu_{n,\alpha}(\widetilde{\pounds})| = \big(\frac{\sqrt{p_1p_2}}{\sqrt{p_1}(b-c)+\sqrt{p_2}(c-a)}\big)^2\pi^2n^2 + o(n^2),\ \text{as } n \longrightarrow \infty. \tag{5.11}$$

Again, by the Theorem 3.1 for all $\alpha > 0$, small enough, there is a natural number n_α such that for all $n \geq n_\alpha$ the inequalities

$$Re\mu_n(\widetilde{\mathcal{L}}) > |\mu_n(\widetilde{\mathcal{L}})| \cos\alpha \quad \text{and} \quad |Im\mu_n| < |\mu_n(\widetilde{\mathcal{L}})| \sin\alpha$$

are hold. Taking in view that $\alpha > 0$ is arbitrary real number (small enough) it is easy to see that

$$Re\mu_n(\widetilde{\mathcal{L}}) \sim |\mu_n| \quad \text{and} \quad |Im\mu_n(\widetilde{\mathcal{L}})| = o(|\mu_n(\widetilde{\mathcal{L}})|) \text{ as } n \longrightarrow \infty.$$

Putting (5.11) in these formulas yields the needed equalities (5.9)–(5.10). □

Now, we are ready to prove the main result of this section.

Theorem 5.4. *Let the operator* $\mathfrak{B}$ *acted compactly from* $W_2^2(a,c) \oplus W_2^2(c,b)$ *into* $L_2(a,c). \oplus L_2(c,b)$. *Then, the spectrum of the problem* (1.1)–(1.5) *is discrete and consist of precisely denumerable many eigenvalues* $\mu_n,\ n = 1, 2, ...$ *which, when listed according to decreasing real parts and repeated according to algebraic multiplicity, has the following asymptotic representation:*

$$\mu_n = \Big(\frac{\sqrt{p_1 p_2}}{\sqrt{p_1}(b-c) + \sqrt{p_2}(c-a)}\Big)^2 \pi^2 n^2 + o(n^2). \quad as \quad n \longrightarrow \infty. \tag{5.12}$$

Proof. We know from Corollary 3.2 that the resolvent operator $R(\mu, \widetilde{\mathcal{L}})$ acted boundedly from $\mathcal{H}$ to $\widetilde{\mathcal{H}}$. It is clear that, the operator $\widetilde{\mathfrak{B}}$ defined by (5.1) acted compactly from $\widetilde{\mathcal{H}}$ to $\mathcal{H}$ by assumption on $\mathfrak{B}$ and by definition of $\widetilde{\mathcal{H}}$. Hence the operator $\widetilde{\mathfrak{B}}R(\mu, \widetilde{\mathcal{L}})$ is compact in the Hilbert space $\mathcal{H}$, i.e. $\widetilde{\mathfrak{B}}$ is compact with respect to $\widetilde{\mathcal{L}}$. Now it is enough to apply the previous Theorem, to complete the proof. □

6. Discussion of the results

Note that such generalization of Sturm-Liouville problems involving abstract linear operator in the equation has been investigated by us for the first time in literature. It is easy to verify that the pure differential part of the considered problem (1.1)–(1.5) is not self-adjoint in the usual direct sum space $L_2[a,c) \oplus L_2(c,b]$ (i.e. the generated differential operator is not self-adjoint in $L_2([a,c) \oplus L_2(c,b]) \oplus \mathbb{C}^2$). For self-adjoint realization of this problem we develop an own approach. Namely, we define "alternative" inner product in this space "incorporating" with the considered problem. We must emphasize that in our approach the sign of the boundary-transmission determinants $\theta_i (i = 1, 2), \Delta_{12}$ and Δ_{34} play an important role. Indeed, let us consider the following simple case of the problem (1.1)–(1.5) as

$$-y''(x) = \mu y(x) \quad x \in [-1, 0) \cup (0, 1], \tag{6.1}$$

$$y(-1) = 0, \quad (\mu - 1)y'(-1) + \mu y(1) = 0, \tag{6.2}$$

$$y(0-) = y(0+), \quad y'(0-) = -y'(0+). \tag{6.3}$$

for which the condition (2.1) is not hold. By direct calculation we can show that this problem has only one eigenvalue $\mu = 1$ in contract with standard Sturm-Liouville problems which has infinitely many eigenvalues. We find "minimal" conditions (2.1) on coefficients of boundary and transmission conditions under which the pure differential part of the problem (1.1)–(1.5) can be interpreted as self-adjoint eigenvalue problem in some "alternative realization" of usual direct sum space $L_2[a,c) \oplus L_2(c,b] \oplus \mathbb{C}^2$. Even under condition (2.1) the spectral properties of the considered problem is essentially different from the corresponding spectral properties of standard Sturm-Liouville problem. For instance, the eigenvalues of the problem (1.1)–(1.5) are not real numbers in general. But the leading term in asymptotic expansion of eigenvalues is real sequence. Note also that, the second term in asymptotic expansion of eigenvalues appears in the more "weak" form as $o(n^2)$ because of the abstract linear operator $\mathfrak{B}$ in the equation.

References

[1] B. P. Allahverdiev, E. Bairamov and E. Uğurlu, *Eigenparameter dependent Sturm-Liouville problems in boundary conditions with transmission conditions*, J. Funct. Anal., 2013, 401(1), 388–396.

[2] B. P. Allahverdiev, H. Tuna, *Titchmarsh-Weyl theory for dirac systems with transmission conditions*, Mediterr. J. Math., 2018. DOI: 10.1007/s00009-018-1197-6.

[3] W. O. Amrein and V. Georgescu, *Strong asymptotic completeness of wave operators for highly singular potentials*, Helv. Phys. Acta., 1974, 47, 517–533.

[4] N. Aronszajn, *On a problem of Weyl in the theory of singular Sturm-Liouville equations*, Amer. J. Math., 1957, 79(3), 597–610.

[5] P. A. Binding, P. J. Browne And B. A. Watson, *Sturm-Liouville problems with boundary conditions rationallay dependent on the eigenparameter II*, J. Comput. Appl. Math., 2002, 148, 147–169.

[6] G. D. Birkhoff, *Boundary-value and expansion problems of ordinary differential equations*, Trans. Amer. Math. Soc., 1908, 9(4), 373–395.

[7] S. A. Buterin, *On half inverse problem for differential pencils with the spectral parameter in boundary conditions,* Tamkang J.Math., 2011, 42(3), 355–364.

[8] F. Gesztesy and B. Simon, *Inverse spectral analysis with partial information on the potential.II:The case of discrete spectrum*, Trans.Amer.Math.Soc., 2000, 352(6), 2765–2787.

[9] O. R. Hryniv and Y. V. Mykytyuk, *Half-inverse spectral problems for Sturm-Liouville operators with singular potentials*, Inverse Problems, 2004, 20(5), 1423–1444.

[10] M. Kandemir, O. Sh. Mukhtarov and Y.Y. Yakubov, *Irregular boundary value problems with discontinuous coefficients and the eigenvalue parameter*, Mediterr, J. Math., 2009, 6(3), 317–338.

[11] M. Kandemir and O. Sh. Mukhtarov, *Nonlocal Sturm-Liouville problems with integral terms in the boundary conditions*, Electronic Journal of Differential Equations, 2017, 2017(11), 1–12.

[12] T. Kato, *Perturbation of continuous spectra by trace class operators*, Proc. Japan Acad., 1957, 33, 260–264.

[13] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin, New York, 1984.

[14] M. V. Keldysh, *On the characteristic values and characteristic functions of certain classes of non-selfadjoint equations*, Dokl. AN SSSR, 1951, 77, 11–14.

[15] A. V. Likov and Y. A. Mikhailov, *The theory of Heat and Mass Transfer*, Qosenergaizdat, Russian, 1963.

[16] A. S. Markus, *Introduction to the Spectral Theory of Polynomial Operator Pencils*, Shtiintsa, Kishinev, 1986.

[17] A. S. Markus and V. I. Matsayev, *Comparison theorems for spectra of linear Operators and spectral asymptotics*, Tr. Mosk. Mat. Obshch., 1982, 45, 133–181.

[18] M. Marletta, A. Shkalikov and C. Tretter, *Pencils of differential operators containing the eigenvalue parameter in the boundary conditions*, Proc. R. Soc. Edinb., Sect. A, Math., 2003, 133(4), 893–917.

[19] O. Martinyuk and V. Pivovarchik, *On the Hochstadt-Lieberman theorem*,Inverse Problems, 2010, 26(3), 035011.

[20] S. S. Mirzoyev, A. R. Aliev and L. A. Rustamova, *Solvability conditions for boundary-value problems for elliptic operator-differential equations with discontinuous coefficient*, Math. Notes, 2012, 92(5), 722–726

[21] O. S. Mukhtarov and K. Aydemir,*Minimization principle and generalized fourier series for discontinuous Sturm-liouville systems in direct sum spaces*, Journal of Applied Analysis and Computation, 2018, 8(5), 1511–1523.

[22] O. S. Mukhtarov, H. Olğar and K. Aydemir,*Resolvent Operator and Spectrum of New Type Boundary Value Problems*, Filomat, 2015, 29(7), 1671–1680.

[23] H. Olğar and F. S. Muhtarov, *The basis property of the system of weak eigenfunctions of a discontinuous Sturm-liouville problem*, Mediterr. J. Math., 2017. DOI:10.1007/s00009-017-0915-9.

[24] H. Olğar, O. Sh. Mukhtarov, and K. Aydemir, *Some properties of eigenvalues and generalized eigenvectors of one boundary value problem*, Filomat 2018, 32(3), 911–920.

[25] E. Penahlı and A. Ercan, *Stability problem of singular Sturm-Liouville equation*, TWMS J. Pure Appl. Math., 2017, 8(2), 148–159.

[26] M. L. Rasulov, *Methods of Contour Integration*, North-Holland Publishing Company, Amsterdam, 1967.

[27] M. Rosenblum, *Perturbation of the continuous spectrum and unitary equivalence*, Pacific J. Math., 1957, 7(1), 997–1010.

[28] L. Sakhnovich, *Half inverse problems on the finite interval*,Inverse Problems, 2001, 17(3), 527–532.

[29] E. Şen, O. S. Mukhtarov, *Spectral properties of discontinuous Sturm-Liouville problems with a finite number of transmission conditions*, Mediterranean Journal of Mathematics, 2016, 13(1), 153–170.

[30] A. A. Shkalikov, *Boundary problems for ordinary differential equations with parameter in the boundary conditions* , Tr. Semin. Im. I.G. Petrovskogo 1983, 9, 190–229 (Russian), translation in J. Sov. Math., 1986, 33(6), 1311–1342.

[31] Y. D. Tamarkin, *On Some General Problems of the Theory of Ordinary Linear Differential Equations and on the Expansion of Arbitrary Functions in Series*, Petrograd, Russian, 1917.

[32] A. N. Tikhonov and A. A. Samarskii, *Equations of Mathematical Physics* , Pergamon, Oxford and New York, 1963.

[33] I. Titeux and Y. Yakubov, *Completeness of root functions for thermal conduction in a strip with piecewise continuous coefficients*, Math. Models Methods Appl. Sc., 1997, 7(7), 1035–1050.

[34] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, North-Holland, Amsterdam, 1978.

[35] I. Trooshin and M. Yamamoto, *Hochstadt-Lieberman type theorem foranonsymmetric system of first-order ordinary differential operators*, Recent Development in Theories Numerics, 2003, 202–211.

[36] E. Uğurlu and E. Bairamov, *Krein's theorem for the dissipative operators with finite impulsive effects* , Numer. Func. Anal. Optim., 2015, 36(2), 256–270.

[37] N. N. Voitovich , B. Z. Katsenelbaum and A. N. Sivov, *Generalized Method of Eigen-vibration in the Theory of Diffraction* , Nakua, Mockow, 1997 (Russian).

[38] A. Wang, J. Sun and A. Zettl, *The classification of self-adjoint boundary conditions: separated, coupled, and mixed*, J. Funct. Anal., 2008, 255(6), 1554–1573.

[39] H. Weyl, *Das Asymptotische Verteilungsgezetz der Eigenverte linearer Partiler Differentialgleihun-gen*, Math. Anal., 1912, 71, 441–479 .

[40] S. Y. Yakubov, *Completeness of Root Functions of Regular Differential Operators*, Longman Scientific and Technical, Newyork, 1994.

[41] C. F. Yang and A. Zettl, *Half inverse problems for quadratic pencils of Sturm-Liouville operators*, Taiwanese J. Math., 2012, 16(5), 1829–1846.

[42] M. Yücel, O. S. Mukhtarov, *A New Treatment of the Decomposition Method for Nonclassical Boundary Value Problems*, Journal of Advanced Physics, 2018, 7(2), 161–166.