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A LINEAR ESTIMATION TO THE NUMBER OF ZEROS FOR ABELIAN INTEGRALS IN A KIND OF QUADRATIC REVERSIBLE CENTERS OF GENUS ONE*

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Abstract In this paper, using the method of Picard-Fuchs equation and Riccati equation, we consider the number of zeros for Abelian integrals in a kind of quadratic reversible centers of genus one under arbitrary polynomial perturbations of degree n, and obtain that the upper bound of the number is $2\left[(n+1)/2\right]+\left[n/2\right]+2$ $(n\geq 1)$, which linearly depends on n.

Keywords Abelian integral, quadratic reversible center, weakened Hilbert's 16th problem, limit cycle.

MSC(2010) 34C07, 34C08, 37G15.

1. Introduction and Main Result

Consider

$$\dot{x}=\frac{H_y(x,y)}{\mu(x,y)}+\varepsilon P(x,y),\ \ \dot{y}=-\frac{H_x(x,y)}{\mu(x,y)}+\varepsilon Q(x,y), \tag{1.1}$$

and

$$\dot{x}=\frac{H_y(x,y)}{\mu(x,y)},\ \ \dot{y}=-\frac{H_x(x,y)}{\mu(x,y)}, \tag{1.2}$$

where ε $(0<\varepsilon\ll 1)$ is a real parameter, $H_y(x,y)/\mu(x,y)$, $H_x(x,y)/\mu(x,y)$, $P(x,y)$, $Q(x,y)$ are all polynomials of x and y, with $\max\{\deg(P(x,y)),\deg(Q(x,y))\}=n$ and $\max\left\{\deg\left(H_y(x,y)/\mu(x,y)\right),\deg\left(H_x(x,y)/\mu(x,y)\right)\right\}=m$. We suppose that the system (1.2) is an integrable system, it has at least one center. The function $H(x,y)$ is a first integral with the integrating factor $\mu(x,y)$, that is, we can define a continuous family of periodic orbits

$$\{\Gamma_h\}\subset\left\{(x,y)\in\mathbb{R}^2: H(x,y)=h, h\in\Delta\right\},$$

which are defined on a maximal open interval $\Delta=(h_1,h_2)$. The problem to be studied in this paper is: for any small number ε, how many limit cycles in the system (1.1) can be bifurcated from the family of periodic orbits $\{\Gamma_h\}$. It is well known that in any compact region of the periodic orbits, the number of limit cycles

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of the system (1.1) is no more than the number of isolated zeros for the following Abelian integrals $A(h)$,

$$A(h) = \oint_{\Gamma_h} \mu(x,y)\left[Q(x,y)\,dx - P(x,y)\,dy\right], \quad h \in \Delta. \tag{1.3}$$

A) If the system (1.2) is a Hamiltonian system, i.e., the $\mu(x,y)$ is constant, then the $H(x,y)$ is a polynomial of x and y with $\deg(H(x,y)) = m+1$. Finding the least upper bound $Z(m,n)$ of the number of isolated zeros for Abelian integrals $A(h)$ is an important and difficult problem, where the upper bound $Z(m,n)$ only depends on m, n, and does not depend on the specific forms of $H(x,y)$, $P(x,y)$, and $Q(x,y)$. This problem is called the weakened Hilbert's 16th problem, it is also called the Hilbert-Arnold problem [1], which has been studied diffusely, such as, for the following system

$$\dot{x} = y, \quad \dot{y} = -g(x) + \varepsilon f(x) y, \tag{1.4}$$

Atabaigi [2] showed that the upper bound of the number of zeros is 3 when $g(x) = x^3(x-1)^2$ and $f(x) = \alpha + \beta x + \gamma x^3$; Moghimi et al. [16] showed that the upper bound is 2 when $g(x) = e^{-x}(1-e^{-x})$ and $f(x) = \alpha+\beta x+\gamma x^2$. For the system (1.1), Rebollo-Perdomo et al. [18] showed the upper bound when $H(x,y) = y(x^2y-1)$; for other specially planar systems, researchers obtain plentiful important results [3, 5, 12, 15, 20], and more specific situations can be found in the books [4, 7], the review article [13], and the references therein.

B) If the system (1.2) is an integrable non-Hamiltonian system, then the $\mu(x,y)$ is not constant. When $H(x,y)$ or $\mu(x,y)$ are not polynomials, the research work of the associated Abelian integrals $A(h)$ becomes much more difficult. Thus, researchers consider this problem by starting from the simplest case, namely m is low. For the specific case of $m=2$, people conjecture that the upper bound $Z(2,n)$ of the number of zeros for associated Abelian integrals $A(h)$ linearly depends on n. For the system (1.1), Novikov et al. [17] showed that the upper bound of the number of zeros is $7n/4+9$ when $H(x,y) = x^2y(1-x-y)$ and $\mu(x,y) = x$; Sui et al. [19] showed the upper bound when $H(x,y) = x^2+y^2$ and $\mu(x,y) = 1/(x^2+1)^m$. Unfortunately, this conjecture is still far from being solved. For quadratic reversible centers of genus one, in reference [6], Gautier et al. showed that there are essentially 22 types in the classification, divided into $(r1)$–$(r22)$ specifically. For the linear dependence of the upper bound of the number of zeros, case $(r1)$ was studied in [21]; case $(r2)$ is a Hamiltonian system; cases $(r3)$–$(r6)$ were studied in [14]; cases $(r9)$, $(r13)$, $(r17)$, and $(r19)$ were studied in [11]; cases $(r11)$, $(r16)$, $(r18)$, and $(r20)$ were studied in [10]; cases $(r12)$ and $(r21)$ were studied in [9]; case $(r10)$ was studied in [8]. All of these upper bounds linearly depend on n. In this paper, we consider the case $(r22)$, and obtain that the upper bound is $2\left[(n+1)/2\right] + [n/2] + 2$ $(n \geq 1)$. Our result shows that the upper bound linearly depends on n.

The quadratic reversible type (Q_3^R) has the form as follows:

$$\dot{z} = -iz + az^2 + 2\left|z\right|^2 + b\bar{z}^2, \quad z = x+yi,$$

or

$$\dot{x} = (a+b+2)x^2 - (a+b-2)y^2 + y, \quad \dot{y} = -x\left[1-2(a-b)y\right].$$

Let $\tilde{x} = 1-2(a-b)y$, $\tilde{y} = x$, $d\tau = -2(a-b)dt$. Using x, y, t instead of $\tilde{x}$, $\tilde{y}$, τ, thus we obtain a new system

$$\dot{x} = -xy, \quad \dot{y} = -\frac{a+b+2}{2(a-b)}y^2 + \frac{a+b-2}{8(a-b)^3}x^2 - \frac{b-1}{2(a-b)^3}x - \frac{a-3b+2}{8(a-b)^3}. \tag{1.5}$$

From the system (1.5), when $(a, b) = (-2, 0)$, we can get the case $(r22)$ as follows:

$$(r22) \qquad \dot{x} = -xy, \qquad \dot{y} = \frac{1}{2^4}x^2 - \frac{1}{2^4}x. \tag{1.6}$$

The system (1.6) is an integrable non-Hamiltonian quadratic system. It has a center $(1, 0)$, a family of periodic orbits $\{\Gamma_h\}$ $(-1/2^5 < h < 0)$, an integral curve $x = 0$ (see Figure 1), and a first integral as follows:

$$H(x, y) = \frac{1}{2}y^2 + \frac{1}{2^5}x^2 - \frac{1}{2^4}x = h, \quad h \in \left(-\frac{1}{2^5}, 0\right), \tag{1.7}$$

with an integrating factor $\mu(x, y) = 1/x$.

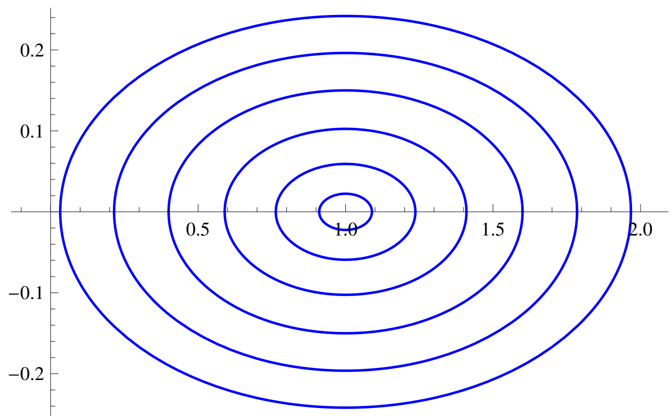

Figure 1. The periodic orbits of the system $(r22)$

In this paper, our main result is the following theorem.

Theorem 1.1. *If $P(x, y)$ and $Q(x, y)$ are any polynomials of x and y, then the upper bound of the number of zeros of Abelian integrals $A(h)$ for the system $(r22)$ depends linearly on n. Concretely, the upper bound is $2\,[(n+1)/2] + [n/2] + 2$ for $n \geq 1$; and the upper bound is 0 for $n = 0$.*

The rest part of this paper is structured as follows. In Section 2, we seek a simple expression of Abelian integrals $A(h)$, prove Proposition 2.1. In Section 3, we study relations among functions $J_m(h)$ and their derivatives $J'_m(h)$ for $m = 0, 1$; relation between $J_0(h)$ and $J_1(h)$, obtain two Picard-Fuchs equations and a variable coefficient first order linear ordinary differential equation. In Section 4, we consider relation between $J(h)$ and $J_1(h)$, obtain a variable coefficient first order linear ordinary differential equation. Finally, we prove Theorem 1.1 using the method of Picard-Fuchs equation and Riccati equation. In Section 5, we give a short conclusion.

2. Simple Expression of Abelian Integrals $A(h)$

In this section, we give a simple expression of Abelian integrals $A(h)$, obtaining Proposition 2.1.

We suppose $P(x,y) = \sum_{0\le i+j\le n} a_{i,j}x^iy^j$ and $Q(x,y) = \sum_{0\le i+j\le n} b_{i,j}x^iy^j$. From (1.3), the Abelian integrals $A(h)$ in Theorem 1.1 have the form

$$A(h) = \oint_{\Gamma_h} x^{-1}\left(\sum_{0\le i+j\le n} b_{i,j}x^iy^j dx - \sum_{0\le i+j\le n} a_{i,j}x^iy^j dy\right),\ h\in\left(-\frac{1}{2^5},0\right),$$

where x^{-1} is an integrating factor.

For conciseness, we introduce functions $I_{i,j}(h)$ as follows:

$$I_{i,j}(h) = \oint_{\Gamma_h} x^{i-1}y^j dx,$$

where $i=-1,0,1,\cdots,n-1,n$; $j=0,1,2,\cdots,n,n+1$, and $0\le i+j\le n$. When $j=1$, we write $I_{i,1}(h)$ as $J_i(h)$.

Note that

$$\oint_{\Gamma_h} x^{i-1}y^j dy = \frac{\oint_{\Gamma_h} x^{i-1}dy^{j+1}}{j+1} = \frac{1-i}{j+1}\oint_{\Gamma_h} x^{i-1-1}y^{j+1}dx = \frac{1-i}{j+1}I_{i-1,j+1}(h).$$

Thus, $A(h)$ can be written as

$$A(h) = \sum_{\substack{0\le i\le n,\\ 0\le j\le n,\\ 0\le i+j\le n}} b_{i,j}I_{i,j}(h) + \sum_{\substack{0\le i\le n,\\ 0\le j\le n,\\ 0\le i+j\le n}} a_{i,j}\frac{i-1}{j+1}I_{i-1,j+1}(h) = \sum_{\substack{-1\le i\le n,\\ 0\le j\le n+1,\\ 0\le i+j\le n}} \tilde{b}_{i,j}I_{i,j}(h), \tag{2.1}$$

where $\tilde{b}_{i,j} = b_{i,j} + i/ja_{i+1,j-1}$.

The following Proposition 2.1 gives a simple expression of Abelian integrals $A(h)$.

Proposition 2.1. *The Abelian integrals $A(h)$ can be expressed as*

$$A(h) = \begin{cases} \dfrac{1}{h}\left[\alpha(h)J_0(h) + \beta(h)J_1(h)\right], & (n\ge 1),\\ \gamma(h)J_{-1}(h), & (n=0), \end{cases} \tag{2.2}$$

where $\alpha(h)$, $\beta(h)$, and $\gamma(h)$ are polynomials of h with $\deg(\alpha(h)) \le [(n+1)/2]$, $\deg(\beta(h)) \le [n/2]$, for $n\ge 1$; $\deg(\gamma(h)) = 0$, for $n=0$.

Proof. Since the periodic orbits Γ_h are symmetric about x-axis, thus, $I_{i,j}(h) = 0$ as j is even, so, we only need to consider the case of j is odd.

From (1.7), we obtain

$$y\frac{\partial y}{\partial x} + 2Cx - 2C = 0, \tag{2.3}$$

where $C = 1/2^5$.

Multiplied the equality (2.3) by $x^iy^{j-2}dx$ and integrated it over Γ_h, we see that

$$\frac{i}{j}I_{i,j}(h) = 2C\left[I_{i+2,j-2}(h) - I_{i+1,j-2}(h)\right], \tag{2.4}$$

where $j = 1,3,5,\cdots,2[n/2]+1$. We restrict $i=-1,0,1,2,\cdots,n-1$, and $0\le i+j\le n$.

(i) For $i=0$, that is, $(i,j)=(0,1),(0,3),\cdots,(0,2[(n+1)/2]-1)$, from (2.4), we obtain

$$I_{2,j-2}(h)=I_{1,j-2}(h). \tag{2.5}$$

From (2.5), let $j=3$, one has

$$J_2(h)=J_1(h). \tag{2.6}$$

(ii) For $i\neq 0$, that is, $(i,j)\neq(0,1),(0,3),\cdots,(0,2[(n+1)/2]-1)$, from (2.4), we have

$$I_{i,j}(h)=\frac{2jC}{i}\left[I_{i+2,j-2}(h)-I_{i+1,j-2}(h)\right], \tag{2.7}$$

which indicates that $I_{i,j}(h)$ can be expressed in terms of $I_{i+2,j-2}(h)$ and $I_{i+1,j-2}(h)$. Then step by step, since j is a positive odd number, we use $(j-1)/2$ times (2.7) and obtain that $I_{i,j}(h)$ can be written as a linear combination of $J_k(h)(k=-1,0,\cdots)$ and $I_{0,j}(h)\,(j=1,3,\cdots,2\,[n/2]-1)$ with the form

$$I_{i,j}(h)=\begin{cases} J_i, \quad (i\neq 0, j=1),\\ \sum_{k=0}^{\frac{j-1}{2}} c_{(i,j),\,i+\frac{j-1}{2}+k}J_{i+\frac{j-1}{2}+k}(h), \quad (i\geq 1, j\geq 3),\\ \sum_{k=0}^{\frac{j-3}{2}} c_{(-1,j),\,1+\frac{j-3}{2}+k}J_{1+\frac{j-3}{2}+k}(h)+d_{(-1,j),0}I_{0,j-2}(h),\\ \qquad\qquad\qquad\qquad \left(i=-1, 3\leq j\leq 2\left[\frac{n}{2}\right]+1\right), \end{cases} \tag{2.8}$$

where $c_{(i,j),i+(j-1)/2+k}$ represents the coefficient obtained when $I_{i,j}(h)$ generates $J_{i+(j-1)/2+k}(h)$, and $d_{(-1,j),0}$ represents the coefficient obtained when $I_{-1,j}(h)$ generates $I_{0,j-2}(h)$, they are all real number.

From (2.1) and (2.8), we have

$$A(h)=A_0(h)+A_1(h)+A_2(h)+A_3(h), \tag{2.9}$$

where

$$\begin{aligned} A_0(h)&=\sum_{k=1}^{[(n+1)/2]}\tilde{b}_{0,2k-1}I_{0,2k-1}(h)+\sum_{k=1}^{[n/2]}\tilde{b}_{-1,2k+1}d_{(-1,2k+1),0}I_{0,2k-1}(h),\\ A_1(h)&=\tilde{b}_{-1,1}J_{-1}(h)+\sum_{k=1}^{n-1}\tilde{b}_{k,1}J_k(h),\\ A_2(h)&=\sum_{\substack{1\leq i\leq n,\\ 4\leq i+j\leq n}}\tilde{b}_{i,j}\sum_{k=0}^{(j-1)/2}c_{(i,j),i+(j-1)/2+k}J_{i+(j-1)/2+k}(h),\\ A_3(h)&=\sum_{\substack{i=-1,\\ 2\leq i+j\leq n}}\tilde{b}_{-1,j}\sum_{k=0}^{(j-3)/2}c_{(-1,j),1+(j-3)/2+k}J_{1+(j-3)/2+k}(h). \end{aligned}$$

For $A_0(h)$, we can get

$$A_0(h)=\sum_{k=1}^{[(n+1)/2]}e_{0,2k-1}I_{0,2k-1}(h), \tag{2.10}$$

where $e_{0,2k-1} = \tilde{b}_{0,2k-1} + \tilde{b}_{-1,2k+1}d_{(-1,2k+1),0}$ $(k = 1, 2, \cdots, n/2)$, for n is even; $e_{0,2k-1} = \tilde{b}_{0,2k-1} + \tilde{b}_{-1,2k+1}d_{(-1,2k+1),0}$ $(k = 1, 2, \cdots, (n-1)/2)$, $e_{0,2k-1} = \tilde{b}_{0,2k-1}$ $(k = (n+1)/2)$, for n is odd.

For $A_2(h)$, the maximum number of $i + (j-1)/2 + k$ for $k = 0, 1, \cdots, (j-1)/2$ is $i + (j-1)/2 + (j-1)/2 = i + j - 1 = n - 1$, and the minimum number is $1 + (3-1)/2 + 0 = 2$, that is, $\{i + (j-1)/2 + k\} = \{2, 3, 4, \cdots, n-2, n-1\}$.

For $A_3(h)$, the maximum number of $1 + (j-3)/2 + k$ for $k = 0, 1, \cdots, (j-3)/2$ is $1 + (j-3)/2 + (j-3)/2 = j - 2 = 2[n/2] - 1 \leq n - 1$, and the minimum number is $1 + (3-3)/2 + 0 = 1$, that is, $\{i + (j-3)/2 + k\} = \{1, 2, 3, \cdots, 2[n/2] - 1\}$.

Suppose that $A_4(h) := A_1(h) + A_2(h) + A_3(h)$, so $A_4(h)$ is a linear combination of $J_{-1}(h)$, $J_1(h)$, $J_2(h)$, $\cdots$, $J_{n-1}(h)$. Thus, we have that

$$A_4(h) = \tilde{b}_{-1,1}J_{-1}(h) + \sum_{k=1}^{n-1} e_k J_k(h), \tag{2.11}$$

where $e_k \in \mathbb{R}$ $(k = 1, 2, 3, \cdots, n-1)$.

From (2.9), we have

$$A(h) = A_0(h) + A_4(h). \tag{2.12}$$

Again, from (1.7), we have

$$\frac{1}{2}y^2 + Cx^2 - 2Cx = h. \tag{2.13}$$

Multiplied the equality (2.13) by $x^{i-1}y^{j-2}dx$ and integrated it over Γ_h, we see that

$$\frac{1}{2}I_{i,j}(h) = hI_{i,j-2}(h) - CI_{i+2,j-2}(h) + 2CI_{i+1,j-2}(h). \tag{2.14}$$

(i) For $i \neq 0$, let $j = 3$, by the equality (2.7), the equality (2.14) becomes

$$(i+3)CJ_{i+2}(h) = ihJ_i(h) + (2i+3)CJ_{i+1}(h), (i \neq 0). \tag{2.15}$$

A) For $i \geq 3$, we can rewrite equality (2.15) as

$$hJ_i(h) = \frac{i-2}{(i+1)C}h^2 J_{i-2}(h) + \frac{2i-1}{i+1}hJ_{i-1}(h),$$

which indicates that $hJ_i(h)$ can be expressed in terms of $h^2 J_{i-2}(h)$ and $hJ_{i-1}(h)$. Then step by step, moreover $J_2(h) = J_1(h)$, we obtain that $hJ_i(h)$ can be written as a linear combination of $J_0(h)$ and $J_1(h)$ with polynomial coefficients of h,

$$hJ_i(h) = \alpha_i(h)J_0(h) + \beta_i(h)J_1(h),$$

where $\alpha_i(h)$ and $\beta_i(h)$ are polynomials of h with $\alpha_i(h) = 0$, $\deg(\beta_i(h)) \leq [(i+1)/2]$.

B) For $i = 2$, by the equality (2.6), one has

$$hJ_2(h) = hJ_1(h).$$

C) For $i = 1$, $hJ_1(h)$ can also be a linear combination of $J_0(h)$ and $J_1(h)$ as $hJ_1(h) = hJ_1(h)$.

D) For $i < 0$, from the equality (2.15), we obtain

$$hJ_i(h) = -\frac{2i+3}{i}CJ_{i+1}(h) + \frac{i+3}{i}CJ_{i+2}(h). \tag{2.16}$$

From the equality (2.16), let $i = -1$, we have

$$hJ_{-1}(h) = CJ_0(h) - 2CJ_1(h).$$

As a consequence, all $hJ_i(h)$ $(i = -1, 1, 2, \cdots, n-1)$ can be written as a linear combination of $J_0(h)$ and $J_1(h)$ with polynomial coefficients of h,

$$hJ_i(h) = \alpha_i(h)J_0(h) + \beta_i(h)J_1(h), \tag{2.17}$$

where $\alpha_i(h)$ and $\beta_i(h)$ are polynomials of h with $\alpha_i(h) = 0$, $\deg(\beta_i(h)) \le [(i+1)/2]$, for $i \ge 1$; and $\deg(\alpha_i(h)) = 0$, $\deg(\beta_i(h)) = 0$, for $i = -1$.

Substituting these formulae into $hA_4(h)$, we obtain

$$hA_4(h) = \tilde{\alpha}_1(h)J_0(h) + \tilde{\beta}_1(h)J_1(h), \tag{2.18}$$

where $\tilde{\alpha}_1(h)$, $\tilde{\beta}_1(h)$ are polynomials of h with $\deg(\tilde{\alpha}_1(h)) = 0$, $\deg(\tilde{\beta}_1(h)) \le [n/2]$, for $n \ge 2$; and $\deg(\tilde{\alpha}_1(h)) = 0$, $\deg(\tilde{\beta}_1(h)) = 0$, for $n = 0, 1$.
(ii) For $i = 0$, from (2.14) and (2.5), we have

$$hI_{0,j}(h) = \begin{cases} hJ_0(h), (j = 1), \\ 2h^2 I_{0,j-2}(h) + 2ChI_{1,j-2}(h), \left(3 \le j \le 2\left[\frac{n+1}{2}\right] - 1\right). \end{cases} \tag{2.19}$$

From equalities (2.8) and (2.17), we get

$$hI_{1,j-2}(h) = \tilde{\alpha}_2(h)J_0(h) + \tilde{\beta}_2(h)J_1(h), (j \ge 3), \tag{2.20}$$

where $\tilde{\alpha}_2(h)$, $\tilde{\beta}_2(h)$ are polynomials of h with $\tilde{\alpha}_2(h) = 0$ and $\deg(\tilde{\beta}_2(h)) \le (j-1)/2$.

From equalities (2.19) and (2.20), and then step by step, we have

$$hI_{0,j}(h) = \tilde{\alpha}_3(h)J_0(h) + \tilde{\beta}_3(h)J_1(h), \tag{2.21}$$

where $\tilde{\alpha}_3(h)$, $\tilde{\beta}_3(h)$ are polynomials of h with $\deg(\tilde{\alpha}_3(h)) \le (j+1)/2$, $\deg(\tilde{\beta}_3(h)) \le (j-1)/2$ for $j \ge 3$; and $\deg(\tilde{\alpha}_3(h)) = 1$, $\tilde{\beta}_3(h) = 0$ for $j = 1$.

From (2.10) and (2.21), we obtain

$$\begin{aligned} hA_0(h) &= e_{0,1}hJ_0(h) + e_{0,3}hI_{0,3}(h) + \cdots + e_{0,2[\frac{n+1}{2}]-1}hI_{0,2[\frac{n+1}{2}]-1}(h) \\ &= \tilde{\alpha}_4(h)J_0(h) + \tilde{\beta}_4(h)J_1(h), \end{aligned} \tag{2.22}$$

where $\tilde{\alpha}_4(h)$, $\tilde{\beta}_4(h)$ are polynomials of h with $\deg(\tilde{\alpha}_4(h)) \le [(n+1)/2]$, $\deg(\tilde{\beta}_4(h)) \le [(n-1)/2]$ for $n \ge 3$; $\deg(\tilde{\alpha}_4(h)) = 1$, $\tilde{\beta}_4(h) = 0$ for $n = 1, 2$.

For $n \ge 1$, we suppose that $J(h) := hA(h)$, from (2.12), (2.18) and (2.22), we have

$$hA(h) = J(h) = \alpha(h)J_0(h) + \beta(h)J_1(h), \tag{2.23}$$

where $\alpha(h)$ and $\beta(h)$ are polynomials of h with $\deg(\alpha(h)) \le [(n+1)/2]$, $\deg(\beta(h)) \le [n/2]$.

For $n = 0$, from (2.1), we obtain $A(h) = \gamma(h)J_{-1}(h)$, where $\gamma(h) = -a_{0,0}$, and $\deg(\gamma(h)) = 0$. □

3. Picard-Fuchs Equations and Riccati Equation

In this section, we give two relations among functions $J_m(h)$ and their derivatives $J_m'(h)$ for $m=0,1$; a relation between $J_0(h)$ and $J_1(h)$, obtaining two Picard-Fuchs equations and a variable coefficient first order linear ordinary differential equation.

The following two lemmas give two relations among functions $J_m(h)$ and their derivatives $J_m'(h)$ for $m=0,1$.

Lemma 3.1. *The functions $J_m(h)$ for $m=0,1$ satisfy the following Picard-Fuchs equation*

$$\begin{pmatrix} J_0(h) \\ J_1(h) \end{pmatrix} = \begin{pmatrix} 2h & 2C \\ 0 & h+C \end{pmatrix} \begin{pmatrix} J_0'(h) \\ J_1'(h) \end{pmatrix}. \tag{3.1}$$

Proof. By (1.7), we have $y^2 = 2h - 2Cx^2 + 4Cx$, $\partial y/\partial h = 1/y$, and $ydy = (2C-2Cx)dx$. Since $J_i(h) = \oint_{\Gamma_h} x^{i-1} ydx$, $J_i'(h) = \oint_{\Gamma_h} x^{i-1}/ydx$. Thus

$$\begin{aligned} J_i(h) &= \oint_{\Gamma_h} \frac{x^{i-1}y^2}{y} dx = \oint_{\Gamma_h} \frac{x^{i-1}\left(2h - 2Cx^2 + 4Cx\right)}{y} dx \\ &= 2hJ_i'(h) - 2CJ_{i+2}'(h) + 4CJ_{i+1}'(h), \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} iJ_i(h) &= \oint_{\Gamma_h} ix^{i-1} ydx = \oint_{\Gamma_h} ydx^i = -\oint_{\Gamma_h} x^i \frac{2C-2Cx}{y} dx \\ &= 2CJ_{i+2}'(h) - 2CJ_{i+1}'(h). \end{aligned} \tag{3.3}$$

From (3.2) and (3.3), we have

$$(i+1)J_i(h) = 2hJ_i'(h) + 2CJ_{i+1}'(h). \tag{3.4}$$

By (3.4), let $i=0,1$ respectively, we obtain

$$J_0(h) = 2hJ_0'(h) + 2CJ_1'(h), \tag{3.5}$$
$$J_1(h) = hJ_1'(h) + CJ_2'(h). \tag{3.6}$$

From (2.6), one has

$$J_2'(h) = J_1'(h). \tag{3.7}$$

From three simultaneous equations (3.5)–(3.7), it follows that

$$J_0(h) = 2hJ_0'(h) + 2CJ_1'(h), \tag{3.8}$$
$$J_1(h) = (h+C)J_1'(h). \tag{3.9}$$

From equalities (3.8) and (3.9), we obtain (3.1). □

Lemma 3.2. *The functions $J_m(h)$ for $m=0,1$ satisfy the following Picard-Fuchs equation*

$$\begin{pmatrix} J_0'(h) \\ J_1'(h) \end{pmatrix} = \frac{1}{B(h)} \begin{pmatrix} h+C & -2C \\ 0 & 2h \end{pmatrix} \begin{pmatrix} J_0(h) \\ J_1(h) \end{pmatrix}, \tag{3.10}$$

where $B(h) = 2h\left(h + 1/2^5\right)$.

Proof. It can be calculated directly from Lemma 3.1. □

Lemma 3.3. *$J_i\left(-1/2^5\right) = 0$ $(i = 0,1)$; $J_i(h) < 0$ $(i = -1,0,1)$, when $h \in \left(-1/2^5, 0\right)$.*

Since $J_i(h) = \oint_{\Gamma_h} x^{i-1} y dx$. The proof only requires some simple calculations, so it is omitted.

For the relation between $J_0(h)$ and $J_1(h)$, assume that $U(h) := J_0(h)/J_1(h)$, we obtain the following corollary.

Corollary 3.1. *The function $U(h)$ satisfies the following variable coefficient first order linear ordinary differential equation*

$$B(h)U'(h) = (C-h)U(h) - 2C, \tag{3.11}$$

where $B(h) = 2h\left(h + 1/2^5\right)$.

Proof. Using Lemma 3.2, and differentiated both sides of $U(h)$ with respect to h, we obtain (3.11). □

4. The Number of Zeros for Abelian Integrals $A(h)$

In this section, we give a relation between function $J(h)$ and $J_1(h)$, obtaining a variable coefficient first order linear ordinary differential equation. Finally, we prove Theorem 1.1 using the method of Picard-Fuchs equation and Riccati equation.

For the relation between $J(h)$ and $J_1(h)$, assume that $V(h) := J(h)/J_1(h)$, we obtain the following lemma.

Lemma 4.1. *For $n \geq 1$, the function $V(h)$ satisfies the following variable coefficient first order linear ordinary differential equation*

$$B(h)\alpha(h)V'(h) = D(h)V(h) + G(h), \tag{4.1}$$

where $D(h) = B(h)\alpha'(h) + (C-h)\alpha(h)$, $G(h) = B(h)\alpha(h)\beta'(h) - B(h)\alpha'(h)\beta(h) - (C-h)\alpha(h)\beta(h) - 2C\alpha^2(h)$. Thus, $\deg(D(h)) \leq [(n+1)/2] + 1$, and $\deg(G(h)) \leq [(n+1)/2] + [n/2] + 1$.

Proof. Using the equality (2.23) and Corollary 3.1, differentiated both sides of $V(h)$ with respect to h, we obtain (4.1). □

We use $\sharp A(h)$ to denote the number of zeros of Abelian integrals $A(h)$ in Δ, and we need the following lemma.

Lemma 4.2 ([14]). *The smooth functions $W(h)$, $\phi(h)$, $\psi(h)$, $\xi(h)$, and $\eta(h)$ satisfy the following Riccati equation*

$$\eta(h)W'(h) = \phi(h)W^2(h) + \psi(h)W(h) + \xi(h),$$

then

$$\sharp W(h) \leq \sharp\eta(h) + \sharp\xi(h) + 1.$$

Lemma 4.2 is Lemma 5.3 in [14], and the proof can be found in [14], so it is omitted.

Finally, we complete the proof of Theorem 1.1 using the method of Picard-Fuchs equation and Riccati equation.

Proof. Using the equality (2.23), Proposition 2.1, Lemma 4.1 and Lemma 4.2, therefore

$$\sharp A(h) = \sharp J(h) = \sharp V(h) \leq \sharp B(h) + \sharp\alpha(h) + \sharp G(h) + 1.$$

For $n \geq 1$, since $\deg(\alpha(h)) \leq [(n+1)/2]$, $\deg(G(h)) \leq [(n+1)/2] + [n/2] + 1$, noticing that $B(h) = 2h\left(h + 1/2^5\right)$ and there is no zero in $\left(-1/2^5, 0\right)$, we obtain

$$\sharp A(h) \leq \left[\frac{n+1}{2}\right] + \left(\left[\frac{n+1}{2}\right] + \left[\frac{n}{2}\right] + 1\right) + 1 = 2\left[\frac{n+1}{2}\right] + \left[\frac{n}{2}\right] + 2.$$

For $n = 0$, since $A(h) = \gamma(h)J_{-1}(h)$, where $\deg(\gamma(h)) = 0$, $J_{-1}(h) < 0$, we have $\sharp A(h) = 0$. $\square$

5. Conclusion

In this paper, we study the linear estimation to the number of zeros for Abelian integrals in the quadratic reversible system $(r22)$ under arbitrary polynomial perturbations of degree n, according to the method of Picard-Fuchs equation and Riccati equation. At the same time, we prove that the upper bound of the number is $2\left[(n+1)/2\right] + [n/2] + 2$ $(n \geq 1)$. Our result shows that the upper bound depends linearly on n.

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