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A FEW EQUIVALENT STATEMENTS OF A HILBERT-TYPE INTEGRAL INEQUALITY WITH THE RIEMANN-ZETA FUNCTION

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Abstract By using the way of real analysis and the weight functions, a few equivalent statements of a Hilbert-type integral inequality in the whole plane with the internal variables and the best possible constant factor related to the Riemann-zeta function is proved. The operator expression and some particular cases are considered.

Keywords Hilbert-type integral inequality, weight function, internal variable, equivalent statement, operator, Riemann-zeta function.

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1. Introduction

Assuming that $0 < \int_0^\infty f^2(x)dx < \infty$ and $0 < \int_0^\infty g^2(y)dy < \infty$, we have the following Hilbert's integral inequality (see [3]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \pi \left(\int_0^\infty f^2(x)dx \int_0^\infty g^2(y)dy\right)^{\frac{1}{2}}, \tag{1.1}$$

where, the constant factor π is the best possible. Recently, by applying the weight functions, a lot of extensions of (1.1) were given by Yang's two books (see [26, 27]). Some new Hilbert-type inequalities and applications were provided by [12, 13, 15, 22, 23, 28–30, 32] and [1].

In 2007, Yang [33] gave a Hilbert-type integral inequality in the whole plane with the exponent function as the internal variables as follows:

$$\int_{-\infty}^\infty \int_{-\infty}^\infty \frac{f(x)g(y)}{(1+e^{x+y})^\lambda}dxdy$$
$$< B(\frac{\lambda}{2}, \frac{\lambda}{2}) \left(\int_{-\infty}^\infty e^{-\lambda x} f^2(x)dx \int_{-\infty}^\infty e^{-\lambda y} g^2(y)dy\right)^{\frac{1}{2}}, \tag{1.2}$$

where, the constant factor $B(\frac{\lambda}{2}, \frac{\lambda}{2})$ is the best possible ($\lambda > 0$, $B(u, v)$ is the beta function). He et al. [4–7, 14, 16, 17, 19, 24, 25, 34, 39, 40] and [2] proved some new Hilbert-type integral inequalities in the whole plane.

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In 2017, Hong [8] gave two equivalent statements between a Hilbert-type inequalities with the homogenous kernel and a few parameters. Some authors continued to discuss this topic to another type of integral inequalities (cf. [9, 18, 20, 35, 37] and [38]).

In this paper, by using the way of real analysis and the weight functions, a few equivalent statements of a Hilbert-type integral inequality in the whole plane with the internal variables and the best possible constant factor related to the Riemann-zeta function is proved in Theorem 3.1 and Theorem 3.2. The operator expressions and some particular cases are considered in Theorem 4.1 and Remark 4.1.

2. An example and a few lemmas

Example 2.1. We set $h(u) := 1 - \tanh(\gamma u)$ $(u > 0)$, where, $\tanh(u) = \frac{e^u - e^{-u}}{e^u + e^{-u}}$ is the hyperbolic tangent function. For $\gamma > 0, \sigma > 1$, we find

$$\begin{aligned} k(\sigma) &:= \int_0^\infty h(u)u^{\sigma-1}du = \int_0^\infty (1 - \tanh(\gamma u))u^{\sigma-1}du \\ &= \int_0^\infty (1 - \frac{e^{\gamma u} - e^{-\gamma u}}{e^{\gamma u} + e^{-\gamma u}})u^{\sigma-1}du = \int_0^\infty \frac{2e^{-2\gamma u}u^{\sigma-1}}{1 + e^{-2\gamma u}}du \\ &= 2\int_0^\infty \sum_{k=0}^\infty (-1)^k e^{-2(k+1)\gamma u}u^{\sigma-1}du \\ &= 2\int_0^\infty \sum_{k=0}^\infty [e^{-2(2k+1)\gamma u} - e^{-2(2k+2)\gamma u}]u^{\sigma-1}du. \end{aligned}$$

By Lebesgue term by term theorem (cf. [10]), setting $u = 2(k+1)\gamma u$, it follows that

$$\begin{aligned} k(\sigma) &= 2\sum_{k=0}^\infty \int_0^\infty [e^{-2(2k+1)\gamma u} - e^{-2(2k+2)\gamma u}]u^{\sigma-1}du \\ &= 2\sum_{k=1}^\infty (-1)^{k-1} \int_0^\infty e^{-2k\gamma u}u^{\sigma-1}du \\ &= \frac{1}{2^{\sigma-1}\gamma^\sigma} \sum_{k=1}^\infty \frac{(-1)^{k-1}}{k^\sigma} \int_0^\infty e^{-v}v^{\sigma-1}dv \\ &= \frac{\Gamma(\sigma)}{2^{\sigma-1}\gamma^\sigma} \left[\sum_{k=1}^\infty \frac{1}{k^\sigma} - \sum_{k=1}^\infty \frac{2}{(2k)^\sigma} \right] \\ &= \frac{1 - 2^{1-\sigma}}{2^{\sigma-1}\gamma^\sigma}\Gamma(\sigma)\zeta(\sigma) \in \mathbf{R}_+, \end{aligned}$$

where, $\Gamma(s) := \int_0^\infty e^{-v}v^{s-1}dv$ $(\operatorname{Re} s > 0)$ is the gamma function and $\zeta(s) := \sum_{k=1}^\infty \frac{1}{k^s}(\operatorname{Re} s > 1)$ is the Riemann-zeta function) (cf. [21]).

For $\mathbf{R} = (-\infty, \infty), \mathbf{R}_+ = (0, \infty), \delta \in \{-1, 1\}, \alpha, \beta \in (-1, 1)$, we set

$$\begin{aligned} &x_\alpha := |x| + \alpha x, y_\beta := |y| + \beta y \ (x, y \in \mathbf{R}), \\ &E_\delta := \{t \in \mathbf{R}; |t|^\delta \geq 1\}, E_{-\delta} = \{t \in \mathbf{R}; |t|^\delta \leq 1\}. \end{aligned} \tag{2.1}$$

Lemma 2.1. *If $\delta \in \{-1, 1\}$, $\gamma = \alpha, \beta$, then for $c > 0$, we have*

$$\int_{E_\delta} t_\gamma^{-c\delta-1} dt = \frac{1}{c}\left[\frac{1}{(1+\gamma)^{c\delta+1}} + \frac{1}{(1-\gamma)^{c\delta+1}}\right] \in \mathbf{R}_+; \tag{2.2}$$

for $c \leq 0$, it follows that

$$\int_{E_\delta} t_\gamma^{-c\delta-1} dt = \infty. \tag{2.3}$$

Proof. Setting $E_\delta^+ := \{t \in \mathbf{R}_+; t^\delta \geq 1\}, E_\delta^- := \{-t \in \mathbf{R}_+; (-t)^\delta \geq 1\}$, we find $E_\delta = E_\delta^+ \cup E_\delta^-$ and

$$\begin{aligned}\int_{E_\delta} t_\gamma^{-c\delta-1} dt &= \int_{E_\delta^+} [(1+\gamma)t]^{-c\delta-1} dt + \int_{E_\delta^-} [(1-\gamma)(-t)]^{-c\delta-1} dt \\ &= \left[\frac{1}{(1+\gamma)^{c\delta+1}} + \frac{1}{(1-\gamma)^{c\delta+1}}\right] \int_{E_\delta^+} t^{-c\delta-1} dt.\end{aligned}$$

Setting $u = t^\delta$ (or $t = u^{\frac{1}{\delta}}$), we find

$$\int_{E_\delta^+} t^{-c\delta-1} dt = \frac{1}{|\delta|}\int_1^\infty u^{\frac{1}{\delta}(-c\delta-1)} u^{\frac{1}{\delta}-1} du = \int_1^\infty u^{-c-1} du.$$

Hence, for $c > 0$, (2.1) follows and for $c \leq 0$, $\int_{E_\delta} t_\gamma^{-c\delta-1} dt = \infty$.

The lemma is proved. □

In what follows, we assume that $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \delta \in \{-1, 1\}, \alpha, \beta \in (-1, 1), \gamma > 0$, $\sigma > 1$, $\sigma_1 \in \mathbf{R}, h(u) = 1 - \tanh(\gamma u), k(\sigma) = \frac{1-2^{1-\sigma}}{2^{\sigma-1}\gamma^\sigma}\Gamma(\sigma)\zeta(\sigma)$, and

$$\begin{aligned}K_{\alpha,\beta}(\sigma) &:= \frac{2k(\sigma)}{(1-\alpha^2)^{1/q}(1-\beta^2)^{1/p}} \\ &= \frac{2^{2-\sigma}\left(1 - 2^{1-\sigma}\right)\Gamma(\sigma)\zeta(\sigma)}{(1-\alpha^2)^{1/q}(1-\beta^2)^{1/p}\gamma^\sigma}.\end{aligned} \tag{2.4}$$

For $n \in \mathbf{N} = \{1, 2, \cdots\}, E_{-1} = [-1, 1]$, $x \in E_{-\delta}$, we define the following expressions:

$$\begin{aligned}I^{(-)}(x) &:= \int_{-1}^0 [1 - \tanh(\frac{\gamma y_\beta}{x_\alpha^\delta})] y_\beta^{\sigma+\frac{1}{qn}-1} dy, \\ I^{(+)}(x) &:= \int_0^1 [1 - \tanh(\frac{\gamma y_\beta}{x_\alpha^\delta})] y_\beta^{\sigma+\frac{1}{qn}-1} dy, \\ I(x) &:= \int_{E_{-1}} [1 - \tanh(\frac{\gamma y_\beta}{x_\alpha^\delta})] y_\beta^{\sigma+\frac{1}{qn}-1} dy = I^{(-)}(x) + I^{(+)}(x).\end{aligned}$$

Since $y_\beta = |y| + \beta y = (sgn(y) + \beta) y$, where,

$$sgn(y) := \begin{cases} -1, y < 0, \\ 0, y = 0, \\ 1, y > 0, \end{cases}$$

$$x_\alpha^{-\delta} = (1+\alpha\cdot sgn(x))^{-\delta}|x|^{-\delta} \geq \min_{\delta\in\{-1,1\}}(1\pm|\alpha|)^{-\delta} \ (x\in E_{-\delta}),$$

and $1-|\alpha| \leq (1+|\alpha|)^{-1} \leq 1+|\alpha| \leq (1-|\alpha|)^{-1}$, we have

$$(1\pm\beta)x_\alpha^{-\delta} \geq m_{\alpha,\beta} := (1-|\beta|)(1-|\alpha|) > 0 \ (x\in E_{-\delta}). \tag{2.5}$$

For fixed $x\in E_{-\delta}$, setting $u = x_\alpha^{-\delta}y_\beta$, we find

$$\begin{aligned}
I^{(-)}(x) &= \frac{x_\alpha^{\delta(\sigma+\frac{1}{qn})}}{1-\beta}\int_0^{(1-\beta)x_\alpha^{-\delta}}(1-\tanh(\gamma u))u^{\sigma+\frac{1}{qn}-1}du \\
&\geq \frac{x_\alpha^{\delta(\sigma+\frac{1}{qn})}}{1-\beta}\int_0^{m_{\alpha,\beta}}(1-\tanh(\gamma u))u^{\sigma+\frac{1}{qn}-1}du, \\
I^{(+)}(x) &= \frac{x_\alpha^{\delta(\sigma+\frac{1}{qn})}}{1+\beta}\int_0^{(1+\beta)x_\alpha^{-\delta}}(1-\tanh(\gamma u))u^{\sigma+\frac{1}{qn}-1}du \\
&\geq \frac{x_\alpha^{\delta(\sigma+\frac{1}{qn})}}{1+\beta}\int_0^{m_{\alpha,\beta}}(1-\tanh(\gamma u))u^{\sigma+\frac{1}{qn}-1}du, \\
I(x) &= x_\alpha^{\delta(\sigma+\frac{1}{qn})}\left[\frac{1}{1-\beta}\int_0^{(1-\beta)x_\alpha^{-\delta}}(1-\tanh(\gamma u))u^{\sigma+\frac{1}{qn}-1}du\right. \\
&\quad\left. + \frac{1}{1+\beta}\int_0^{(1+\beta)x_\alpha^{-\delta}}(1-\tanh(\gamma u))u^{\sigma+\frac{1}{qn}-1}du\right] \\
&\geq \frac{2x_\alpha^{\delta(\sigma+\frac{1}{qn})}}{1-\beta^2}\int_0^{m_{\alpha,\beta}}(1-\tanh(\gamma u))u^{\sigma+\frac{1}{qn}-1}du.
\end{aligned} \tag{2.6}$$

For $n\in\mathbf{N}$, $x\in E_\delta$, we define the following expressions:

$$\begin{aligned}
J^{(-)}(x) &:= \int_{-\infty}^{-1}[1-\tanh(\frac{\gamma y_\beta}{x_\alpha^\delta})]y_\beta^{\sigma-\frac{1}{qn}-1}dy, \\
J^{(+)}(x) &:= \int_1^{\infty}[1-\tanh(\frac{\gamma y_\beta}{x_\alpha^\delta})]y_\beta^{\sigma-\frac{1}{qn}-1}dy, \\
J(x) &:= \int_{E_1}[1-\tanh(\frac{\gamma y_\beta}{x_\alpha^\delta})]y_\beta^{\sigma-\frac{1}{qn}-1}dy = J^{(-)}(x)+J^{(+)}(x).
\end{aligned}$$

Since for $x\in E_\delta$,

$$x_\alpha^{-\delta} = (1+\alpha\cdot sgn(x))^{-\delta}|x|^{-\delta} \leq \max_{\delta\in\{-1,1\}}\{(1\pm|\alpha|)^{-\delta}\} = (1-|\alpha|)^{-1} \ ,$$

we have

$$M_{\alpha,\beta} := (1+|\beta|)(1-|\alpha|)^{-1} \geq (1\pm\beta)x_\alpha^{-\delta} \ (x\in E_\delta). \tag{2.7}$$

For fixed $x\in E_\delta$, setting $u = x_\alpha^{-\delta}y_\beta$, we find

$$\begin{aligned}
J^{(-)}(x) &= \frac{x_\alpha^{\delta(\sigma-\frac{1}{qn})}}{1-\beta}\int_{(1-\beta)x_\alpha^{-\delta}}^{\infty}(1-\tanh(\gamma u))u^{\sigma-\frac{1}{qn}-1}du \\
&\geq \frac{x_\alpha^{\delta(\sigma-\frac{1}{qn})}}{1-\beta}\int_{M_{\alpha,\beta}}^{\infty}(1-\tanh(\gamma u))u^{\sigma-\frac{1}{qn}-1}du,
\end{aligned}$$

$$J^{(+)}(x) = \frac{x_\alpha^{\delta(\sigma-\frac{1}{qn})}}{1+\beta}\int_{(1+\beta)x_\alpha^{-\delta}}^{\infty}(1-\tanh(\gamma u))u^{\sigma-\frac{1}{qn}-1}du$$
$$\geq \frac{x_\alpha^{-\delta(\sigma-\frac{1}{qn})}}{1+\beta}\int_{M_{\alpha,\beta}}^{\infty}\frac{sech(\gamma u)}{e^{\gamma u}}u^{\sigma-\frac{1}{qn}-1}du,$$
$$J(x) = x_\alpha^{\delta(\sigma-\frac{1}{qn})}\left[\frac{1}{1-\beta}\int_{(1-\beta)x_\alpha^{-\delta}}^{\infty}(1-\tanh(\gamma u))u^{\sigma-\frac{1}{qn}-1}du\right.$$
$$\left.+\frac{1}{1+\beta}\int_{(1+\beta)x_\alpha^{-\delta}}^{\infty}(1-\tanh(\gamma u))u^{\sigma-\frac{1}{qn}-1}du\right]$$
$$\geq \frac{2x_\alpha^{\delta(\sigma-\frac{1}{qn})}}{1-\beta^2}\int_{M_{\alpha,\beta}}^{\infty}(1-\tanh(\gamma u))u^{\sigma-\frac{1}{qn}-1}du. \tag{2.8}$$

In view of (2.6) and (2.8), it follows that

Lemma 2.2. *We have the following inequalities:*

$$I_1 := \int_{E_{-\delta}} I(x)x_\alpha^{-\delta(\sigma_1-\frac{1}{pn})-1}dx$$
$$\geq \frac{2}{1-\beta^2}\int_{E_{-\delta}} x_\alpha^{\delta(\sigma-\sigma_1+\frac{1}{n})-1}dx\int_0^{m_{\alpha,\beta}}(1-\tanh(\gamma u))u^{\sigma+\frac{1}{qn}-1}du, \tag{2.9}$$
$$J_1 := \int_{E_\delta} J(x)x_\alpha^{-\delta(\sigma_1+\frac{1}{pn})-1}dx$$
$$\geq \frac{2}{1-\beta^2}\int_{E_\delta} x_\alpha^{-\delta(\sigma_1-\sigma+\frac{1}{n})-1}dx\int_{M_{\alpha,\beta}}^{\infty}(1-\tanh(\gamma u))u^{\sigma-\frac{1}{qn}-1}du. \tag{2.10}$$

We set $\widetilde{I}_1 = (0,a), \widetilde{I}_2 = (b,\infty)$, where, $0 < a \leq m_{\alpha,\beta} \leq M_{\alpha,\beta} \leq b < \infty$.

Lemma 2.3. *If there exists a constant M, such that for any nonnegative measurable functions $f(x)$ and $g(y)$ in $\mathbf{R}$, the following inequality*

$$I := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(1-\tanh(\frac{\gamma y_\beta}{x_\alpha^\delta})\right)f(x)g(y)dxdy$$
$$\leq M\left[\int_{-\infty}^{\infty}x_\alpha^{p(1+\delta\sigma_1)-1}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_{-\infty}^{\infty}y_\beta^{q(1-\sigma)-1}g^q(y)dy\right]^{\frac{1}{q}} \tag{2.11}$$

holds true, then we have $\sigma_1 = \sigma$.

Proof. If $\sigma_1 > \sigma$, then for $n \geq \frac{1}{\sigma_1-\sigma}$ $(n \in \mathbf{N})$, we set functions

$$f_n(x) := \begin{cases} x_\alpha^{-\delta(\sigma_1-\frac{1}{pn})-1}, & x \in E_{-\delta}, \\ 0, & x \in \mathbf{R}\backslash E_{-\delta}, \end{cases}$$
$$g_n(y) := \begin{cases} y_\beta^{\sigma+\frac{1}{qn}-1}, & y \in E_{-1}, \\ 0, & y \in \mathbf{R}\backslash E_{-1}, \end{cases}$$

and by (2.1) and (2.2), we find

$$\begin{aligned}\widetilde{J}_1 &:= \left[\int_{-\infty}^{\infty} x_\alpha^{p(1+\delta\sigma_1)-1} f_n^p(x)dx\right]^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} y_\beta^{q(1-\sigma)-1} g_n^q(y)dy\right]^{\frac{1}{q}}\\ &= \left(\int_{E_{-\delta}} x_\alpha^{\frac{\delta}{n}-1}dx\right)^{\frac{1}{p}} \left(\int_{E_{-1}} y_\beta^{\frac{1}{n}-1}dy\right)^{\frac{1}{q}}\\ &= n\left[\frac{1}{(1+\alpha)^{\frac{-\delta}{n}+1}} + \frac{1}{(1-\alpha)^{\frac{-\delta}{n}+1}}\right]^{\frac{1}{p}}\\ &\quad\times \left[\frac{1}{(1+\beta)^{-\frac{1}{n}+1}} + \frac{1}{(1-\beta)^{-\frac{1}{n}+1}}\right]^{\frac{1}{q}} < \infty.\end{aligned}$$

By (2.9) and (2.11), we have

$$\begin{aligned}&\frac{2}{1-\beta^2}\int_{E_{-\delta}} x_\alpha^{\delta(\sigma-\sigma_1+\frac{1}{n})-1}dx \int_0^{m_{\alpha,\beta}} (1-\tanh(\gamma u))u^{\sigma+\frac{1}{qn}-1}du\\ &\le I_1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(1-\tanh(\frac{\gamma y_\beta}{x_\alpha^\delta})\right) f_n(x)g_n(y)dxdy \le M\widetilde{J}_1 < \infty.\end{aligned}$$

Since for any $n \ge \frac{1}{\sigma_1-\sigma}$ $(n \in \mathbf{N}), \sigma - \sigma_1 + \frac{1}{n} \le 0$, by (2.3), it follows that

$$\int_{E_{-\delta}} x_\alpha^{\delta(\sigma-\sigma_1+\frac{1}{n})-1}dx = \infty.$$

In view of $\int_0^{m_{\alpha,\beta}}(1-\tanh(\gamma u))u^{\sigma+\frac{1}{qn}-1}du > 0$, we find that $\infty \le M\widetilde{J}_1 < \infty$, which is a contradiction.

If $\sigma > \sigma_1$, then for $n \ge \frac{1}{\sigma-\sigma_1}$ $(n \in \mathbf{N})$, we set functions

$$\widetilde{f}_n(x) := \begin{cases} x_\alpha^{-\delta(\sigma_1+\frac{1}{pn})-1}, & x \in E_\delta,\\ 0, & x \in \mathbf{R}\backslash E_\delta,\end{cases}$$

$$\widetilde{g}_n(y) := \begin{cases} y_\beta^{\sigma-\frac{1}{qn}-1}, & y \in E_1,\\ 0, & y \in \mathbf{R}\backslash E_1,\end{cases}$$

and by (2.1) and (2.2), we find

$$\begin{aligned}\widetilde{J}_2 &:= \left[\int_{-\infty}^{\infty} x_\alpha^{p(1-\delta\sigma_1)-1} \widetilde{f}_n^p(x)dx\right]^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} y_\beta^{q(1-\sigma)-1} \widetilde{g}_n^q(y)dy\right]^{\frac{1}{q}}\\ &= \left(\int_{E_\delta} x_\alpha^{\frac{-\delta}{n}-1}dx\right)^{\frac{1}{p}} \left(\int_{E_1} y_\beta^{-\frac{1}{n}-1}dy\right)^{\frac{1}{q}}\\ &= n\left[\frac{1}{(1+\alpha)^{\frac{\delta}{n}+1}} + \frac{1}{(1-\alpha)^{\frac{\delta}{n}+1}}\right]^{\frac{1}{p}}\\ &\quad\times \left[\frac{1}{(1+\beta)^{\frac{1}{n}+1}} + \frac{1}{(1-\beta)^{\frac{1}{n}+1}}\right]^{\frac{1}{q}}.\end{aligned}$$

By (2.10) and (2.11), we have

$$\frac{2}{1-\beta^2}\int_{E_\delta} x_\alpha^{-\delta(\sigma_1-\sigma+\frac{1}{n})-1}dx\int_{M_{\alpha,\beta}}^{\infty}(1-\tanh(\gamma u))u^{\sigma-\frac{1}{qn}-1}du$$
$$\leq J_1=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(1-\tanh(\frac{\gamma y_\beta}{x_\alpha^\delta})\right)\widetilde{f}_n(x)\widetilde{g}_n(y)dxdy\leq M\widetilde{J}_2<\infty.$$

Since for $n\geq\frac{1}{\sigma-\sigma_1}$ ($n\in\mathbf{N}$), $\sigma_1-\sigma+\frac{1}{n}\leq 0$, by (2.3), it follows that

$$\int_{E_\delta} x_\alpha^{-\delta(\sigma_1-\sigma+\frac{1}{n})-1}dx=\infty.$$

In view of $\int_{M_{\alpha,\beta}}^{\infty}(1-\tanh(\gamma u))u^{\sigma-\frac{1}{qn}-1}du>0$, we have $\infty\leq M\widetilde{J}_2<\infty$, which is a contradiction.

Hence, we conclude that $\sigma_1=\sigma$.

The lemma is proved. □

For $\sigma_1=\sigma$, we still have

Lemma 2.4. *If there exists a constant M, such that for any nonnegative measurable functions $f(x)$ and $g(y)$ in $\mathbf{R}$, the following inequality*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(1-\tanh(\frac{\gamma y_\beta}{x_\alpha^\delta})\right)f(x)g(y)dxdy$$
$$\leq M\left[\int_{-\infty}^{\infty}x_\alpha^{p(1+\delta\sigma)-1}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_{-\infty}^{\infty}y_\beta^{q(1-\sigma)-1}g^q(y)dy\right]^{\frac{1}{q}}\tag{2.12}$$

holds true, then we have $K_{\alpha,\beta}(\sigma)\leq M$.

Proof. For $\sigma_1=\sigma$, by (2.6), we have

$$I_1=\int_{E_{-\delta}}I(x)x_\alpha^{-\delta(\sigma-\frac{1}{pn})-1}dx=\frac{L^{(-)}+L^{(+)}}{1-\beta}+\frac{K^{(-)}+K^{(+)}}{1+\beta},\tag{2.13}$$

where, we indicate

$$L^{(+)}:=\int_{E_{-\delta}^+}x_\alpha^{\frac{\delta}{n}-1}\int_0^{(1-\beta)x_\alpha^{-\delta}}(1-\tanh(\gamma u))u^{\sigma+\frac{1}{qn}-1}dudx,$$
$$L^{(-)}:=\int_{E_{-\delta}^-}x_\alpha^{\frac{\delta}{n}-1}\int_0^{(1-\beta)x_\alpha^{-\delta}}(1-\tanh(\gamma u))u^{\sigma+\frac{1}{qn}-1}dudx,$$
$$K^{(+)}:=\int_{E_{-\delta}^+}x_\alpha^{\frac{\delta}{n}-1}\int_0^{(1+\beta)x_\alpha^{-\delta}}(1-\tanh(\gamma u))u^{\sigma+\frac{1}{qn}-1}dudx,$$
$$K^{(-)}:=\int_{E_{-\delta}^-}x_\alpha^{\frac{\delta}{n}-1}\int_0^{(1+\beta)x_\alpha^{-\delta}}(1-\tanh(\gamma u))u^{\sigma+\frac{1}{qn}-1}dudx.$$

By Fubini theorem (cf. [10]), we find

$$L^{(+)}=(1+\alpha)^{\frac{\delta}{n}-1}\int_{E_{-\delta}^+}x^{\frac{\delta}{n}-1}\int_0^{\frac{1-\beta}{(1+\alpha)^\delta}x^{-\delta}}(1-\tanh(\gamma u))u^{\sigma+\frac{1}{qn}-1}dudx$$

$$
\begin{aligned}
=&(1+\alpha)^{\frac{\delta}{n}-1}\int_1^\infty y^{-\frac{1}{n}-1}\int_0^{\frac{1-\beta}{(1+\alpha)^\delta}y}(1-\tanh(\gamma u))u^{\sigma+\frac{1}{qn}-1}dudy\\
=&(1+\alpha)^{\frac{\delta}{n}-1}\left[\int_1^\infty y^{-\frac{1}{n}-1}dy\int_0^{\frac{1-\beta}{(1+\alpha)^\delta}}(1-\tanh(\gamma u))u^{\sigma+\frac{1}{qn}-1}du\right.\\
&\left.+\int_1^\infty y^{-\frac{1}{n}-1}\int_{\frac{1-\beta}{(1+\alpha)^\delta}}^{\frac{1-\beta}{(1+\alpha)^\delta}y}(1-\tanh(\gamma u))u^{\sigma+\frac{1}{qn}-1}dudy\right]\\
=&(1+\alpha)^{\frac{\delta}{n}-1}\left\{n\int_0^{\frac{1-\beta}{(1+\alpha)^\delta}}(1-\tanh(\gamma u))u^{\sigma+\frac{1}{qn}-1}du\right.\\
&\left.+\int_{(1-\beta)(1+\alpha)^{-\delta}}^\infty\left[\int_{\frac{(1+\alpha)^\delta u}{1-\beta}}^\infty y^{-\frac{1}{n}-1}dy\right](1-\tanh(\gamma u))u^{\sigma+\frac{1}{qn}-1}du\right\}\\
=&\frac{n}{(1+\alpha)^{\frac{-\delta}{n}+1}}\left[\int_0^{\frac{1-\beta}{(1+\alpha)^\delta}}(1-\tanh(\gamma u))u^{\sigma+\frac{1}{qn}-1}du\right.\\
&\left.+(1-\beta)^{\frac{1}{n}}(1+\alpha)^{\frac{-\delta}{n}}\int_{\frac{1-\beta}{(1+\alpha)^\delta}}^\infty(1-\tanh(\gamma u))u^{\sigma-\frac{1}{pn}-1}du\right],\\
L^{(-)}=&(1-\alpha)^{\frac{\delta}{n}-1}\int_{F_\delta^-}(-x)^{\frac{\delta}{n}-1}\int_0^{\frac{1-\beta}{(1+\alpha)^\delta}(-x)^{-\delta}}(1-\tanh(\gamma u))u^{\sigma+\frac{1}{qn}-1}dudx\\
=&(1-\alpha)^{\frac{\delta}{n}-1}\int_1^\infty y^{-\frac{1}{n}-1}\int_0^{\frac{1-\beta}{(1+\alpha)^\delta}y}(1-\tanh(\gamma u))u^{\sigma+\frac{1}{qn}-1}dudy\\
=&(1-\alpha)^{\frac{\delta}{n}-1}\int_1^\infty y^{-\frac{1}{n}-1}dy\int_0^{\frac{1-\beta}{(1+\alpha)^\delta}}(1-\tanh(\gamma u))u^{\sigma+\frac{1}{qn}-1}du\\
&+(1-\alpha)^{\frac{\delta}{n}-1}\int_1^\infty y^{-\frac{1}{n}-1}\int_{\frac{1-\beta}{(1+\alpha)^\delta}}^{\frac{1-\beta}{(1+\alpha)^\delta}y}(1-\tanh(\gamma u))u^{\sigma+\frac{1}{qn}-1}dudy\\
=&\frac{n}{(1-\alpha)^{\frac{-\delta}{n}+1}}\left[\int_0^{\frac{1-\beta}{(1+\alpha)^\delta}}(1-\tanh(\gamma u))u^{\sigma+\frac{1}{qn}-1}du\right.\\
&\left.+(1-\beta)^{\frac{1}{n}}(1-\alpha)^{\frac{-\delta}{n}}\int_{\frac{1-\beta}{(1+\alpha)^\delta}}^\infty(1-\tanh(\gamma u))u^{\sigma-\frac{1}{pn}-1}du\right],\\
K^{(+)}=&(1+\alpha)^{\frac{\delta}{n}-1}\int_{F_\delta^+}x^{\frac{\delta}{n}-1}\int_0^{\frac{1+\beta}{(1+\alpha)^\delta}x^{-\delta}}(1-\tanh(\gamma u))u^{\sigma+\frac{1}{qn}-1}dudx\\
=&(1+\alpha)^{\frac{\delta}{n}-1}\int_1^\infty y^{-\frac{1}{n}-1}\int_0^{\frac{1+\beta}{(1+\alpha)^\delta}y}(1-\tanh(\gamma u))u^{\sigma+\frac{1}{qn}-1}dudy\\
=&(1+\alpha)^{\frac{\delta}{n}-1}\left[\int_1^\infty y^{-\frac{1}{n}-1}dy\int_0^{\frac{1+\beta}{(1+\alpha)^\delta}}(1-\tanh(\gamma u))u^{\sigma+\frac{1}{qn}-1}du\right.\\
&\left.+\int_1^\infty y^{-\frac{1}{n}-1}\int_{\frac{1+\beta}{(1+\alpha)^\delta}}^{\frac{1+\beta}{(1+\alpha)^\delta}y}(1-\tanh(\gamma u))u^{\sigma+\frac{1}{qn}-1}dudy\right]
\end{aligned}
$$

$$
\begin{aligned}
&=\frac{n}{(1+\alpha)^{\frac{-\delta}{n}+1}}\left[\int_0^{\frac{1+\beta}{(1+\alpha)^\delta}}(1-\tanh(\gamma u))u^{\sigma+\frac{1}{qn}-1}du\right.\\
&\left.+(1+\beta)^{\frac{1}{n}}(1+\alpha)^{\frac{-\delta}{n}}\int_{\frac{1+\beta}{(1+\alpha)^\delta}}^{\infty}(1-\tanh(\gamma u))u^{\sigma-\frac{1}{pn}-1}du\right],\\
K^{(-)}&=(1-\alpha)^{\frac{\delta}{n}-1}\int_{E_{-\delta}^-}(-x)^{\frac{\delta}{n}-1}\int_0^{\frac{(1+\beta)(-x)^{-\delta}}{(1-\alpha)^\delta}}(1-\tanh(\gamma u))u^{\sigma+\frac{1}{qn}-1}dudx\\
&=(1-\alpha)^{\frac{\delta}{n}-1}\int_1^\infty y^{-\frac{1}{n}-1}\int_0^{\frac{1+\beta}{(1-\alpha)^\delta}y}(1-\tanh(\gamma u))u^{\sigma+\frac{1}{qn}-1}dudy\\
&=(1-\alpha)^{\frac{\delta}{n}-1}\left[\int_1^\infty y^{-\frac{1}{n}-1}dy\int_0^{\frac{1+\beta}{(1-\alpha)^\delta}}(1-\tanh(\gamma u))u^{\sigma+\frac{1}{qn}-1}du\right.\\
&\left.+\int_1^\infty y^{-\frac{1}{n}-1}\int_{\frac{1+\beta}{(1-\alpha)^\delta}}^{\frac{1+\beta}{(1-\alpha)^\delta}y}(1-\tanh(\gamma u))u^{\sigma+\frac{1}{qn}-1}dudy\right]\\
&=\frac{n}{(1-\alpha)^{\frac{-\delta}{n}+1}}\left[\int_0^{\frac{1+\beta}{(1-\alpha)^\delta}}(1-\tanh(\gamma u))u^{\sigma+\frac{1}{qn}-1}du\right.\\
&\left.+(1+\beta)^{\frac{1}{n}}(1-\alpha)^{\frac{-\delta}{n}}\int_{\frac{1+\beta}{(1-\alpha)^\delta}}^{\infty}(1-\tanh(\gamma u))u^{\sigma-\frac{1}{pn}-1}du\right].
\end{aligned}
$$

By (2.12) and (2.13) (for $f=f_n, g=g_n$), we have

$$\frac{1}{n}I_1=\frac{1}{n}\left(\frac{L^{(-)}+L^{(+)}}{1-\beta}+\frac{K^{(-)}+K^{(+)}}{1+\beta}\right)\le\frac{1}{n}M\widetilde{J}_1.$$

For $n\to\infty$, by Fatou lemma (cf. [10]) and the presented results, we find

$$\frac{2}{1-\beta^2}\cdot\frac{2k(\sigma)}{1-\alpha^2}\le M(\frac{2}{1-\alpha^2})^{\frac{1}{p}}(\frac{2}{1-\beta^2})^{\frac{1}{q}},$$

namely, $K_{\alpha,\beta}(\sigma)=\frac{2k(\sigma)}{(1-\alpha^2)^{1/q}(1-\beta^2)^{1/p}}\le M$.

The lemma is proved. □

Lemma 2.5. *We define the following weight functions:*

$$\omega_\delta(\sigma,y):=y_\beta^\sigma\int_{-\infty}^{\infty}\left(1-\tanh(\frac{\gamma y_\beta}{x_\alpha^\delta})\right)x_\alpha^{-\delta\sigma-1}dx\ (y\in\mathbf{R}),\tag{2.14}$$

$$\varpi_\delta(\sigma,x):=x_\alpha^{-\delta\sigma}\int_{-\infty}^{\infty}\left(1-\tanh(\frac{\gamma y_\beta}{x_\alpha^\delta})\right)y_\beta^{\sigma-1}dy\ (x\in\mathbf{R}).\tag{2.15}$$

Then we have

$$\frac{1-\alpha^2}{2}\omega_\delta(\sigma,y)=\frac{1-\beta^2}{2}\varpi_\delta(\sigma,x)=k(\sigma)\ (x,y\in\mathbf{R}\backslash\{0\}).\tag{2.16}$$

Proof. For fixed $y\in(-\infty,0)\cup(0,\infty)$, setting $u=\frac{y_\beta}{x_\alpha^\delta}$, we find

$$\omega_\delta(\sigma,y)=y_\beta^\sigma\int_{-\infty}^{0}\left(1-\tanh(\frac{\gamma y_\beta}{x_\alpha^\delta})\right)x_\alpha^{-\delta\sigma-1}dx$$

$$+ y_\beta^\sigma \int_0^\infty \left(1 - \tanh(\frac{\gamma y_\beta}{x_\alpha^\delta})\right) x_\alpha^{-\delta\sigma-1} dx$$
$$= \frac{2}{1-\alpha^2} \int_0^\infty (1 - \tanh(\gamma u)) u^{\sigma-1} du = \frac{2k(\sigma)}{1-\alpha^2};$$

for fixed $x \in (-\infty, 0) \cup (0, \infty)$, setting $u = \frac{y_\beta}{x_\alpha^\delta}$, it follows that

$$\varpi_\delta(\sigma, x) = x_\alpha^{-\delta\sigma} \int_{-\infty}^0 \left(1 - \tanh(\frac{\gamma y_\beta}{x_\alpha^\delta})\right) y_\beta^{\sigma-1} dy$$
$$+ x_\alpha^{-\delta\sigma} \int_0^\infty \left(1 - \tanh(\frac{\gamma y_\beta}{x_\alpha^\delta})\right) y_\beta^{\sigma-1} dy$$
$$= \frac{2}{1-\beta^2} \int_0^\infty (1 - \tanh(\gamma u)) u^{\sigma-1} du = \frac{2k(\sigma)}{1-\beta^2}.$$

Hence, we have (2.16).

The lemma is proved. □

3. Main results and some particular cases

Theorem 3.1. *If M is a constant, then the following statements (i), (ii) and (iii) are equivalent:*

(i) for any $f(x) \geq 0$, we have the following inequality:

$$J := \left\{ \int_{-\infty}^\infty y_\beta^{p\sigma-1} \left[\int_{-\infty}^\infty \left(1 - \tanh(\frac{\gamma y_\beta}{x_\alpha^\delta})\right) f(x) dx \right]^p dy \right\}^{\frac{1}{p}}$$
$$\leq M \left[\int_{-\infty}^\infty x_\alpha^{p(1+\delta\sigma_1)-1} f^p(x) dx \right]^{\frac{1}{p}}; \tag{3.1}$$

(ii) for any $f(x), g(y) \geq 0$, we have the following inequality:

$$I = \int_{-\infty}^\infty \int_{-\infty}^\infty \left(1 - \tanh(\frac{\gamma y_\beta}{x_\alpha^\delta})\right) f(x) g(y) dx dy$$
$$\leq M \left[\int_{-\infty}^\infty x_\alpha^{p(1+\delta\sigma_1)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_{-\infty}^\infty y_\beta^{q(1-\sigma)-1} g^q(y) dy \right]^{\frac{1}{q}}; \tag{3.2}$$

(iii) $\sigma_1 = \sigma$, and $K_{\alpha,\beta}(\sigma) \leq M$.

Proof. $(i) => (ii)$. By Hölder's inequality (cf. [11]), we have

$$I = \int_{-\infty}^\infty \left[y_\beta^{\sigma-\frac{1}{p}} \int_{-\infty}^\infty \left(1 - \tanh(\frac{\gamma y_\beta}{x_\alpha^\delta})\right) f(x) dx \right] \left(y_\beta^{-\sigma+\frac{1}{p}} g(y) \right) dy$$
$$\leq J \left[\int_{-\infty}^\infty y_\beta^{q(1-\sigma)-1} g^q(y) dy \right]^{\frac{1}{q}}. \tag{3.3}$$

Then by (3.1), we have (3.2).

$(ii) => (iii)$. By Lemma 2.1, we have $\sigma_1 = \sigma$. Then by Lemma 2.2, we have $K_{\alpha,\beta}(\sigma) \leq M$.

$(iii) => (i)$. By Hölder's inequality with weight (see [11]) and (2.14), we have

$$\left[\int_{-\infty}^{\infty}\left(1-\tanh(\frac{\gamma y_\beta}{x_\alpha^\delta})\right)f(x)dx\right]^p$$
$$=\left\{\int_{-\infty}^{\infty}\left(1-\tanh(\frac{\gamma y_\beta}{x_\alpha^\delta})\right)\left[\frac{y_\beta^{(\sigma-1)/p}}{x_\alpha^{(-\delta\sigma-1)/q}}f(x)\right]\left[\frac{x_\alpha^{(-\delta\sigma-1)/q}}{y_\beta^{(\sigma-1)/p}}\right]dx\right\}^p$$
$$\leq\int_{-\infty}^{\infty}\left(1-\tanh(\frac{\gamma y_\beta}{x_\alpha^\delta})\right)\frac{y_\beta^{\sigma-1}f^p(x)}{x_\alpha^{(-\delta\sigma-1)p/q}}dx$$
$$\times\left[\int_{-\infty}^{\infty}\left(1-\tanh(\frac{\gamma y_\beta}{x_\alpha^\delta})\right)\frac{x^{-\delta\sigma-1}dx}{y_\beta^{(\sigma-1)q/p}}\right]^{p/q}$$
$$=\left[\omega_\delta(\sigma,y)y_\beta^{q(1-\sigma)-1}\right]^{p-1}\int_{-\infty}^{\infty}\left(1-\tanh(\frac{\gamma y_\beta}{x_\alpha^\delta})\right)\frac{y_\beta^{\sigma-1}}{x_\alpha^{(-\delta\sigma-1)p/q}}f^p(x)dx$$
$$=(\frac{2k(\sigma)}{1-\alpha^2})^{p-1}y_\beta^{-p\sigma+1}\int_{-\infty}^{\infty}\left(1-\tanh(\frac{\gamma y_\beta}{x_\alpha^\delta})\right)\frac{y_\beta^{\sigma-1}}{x_\alpha^{(-\delta\sigma-1)p/q}}f^p(x)dx. \quad (3.4)$$

By Fubini theorem, (2.15) and (2.16), we have

$$J\leq(\frac{2k(\sigma)}{1-\alpha^2})^{\frac{1}{q}}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(1-\tanh(\frac{\gamma y_\beta}{x_\alpha^\delta})\right)\frac{y_\beta^{\sigma-1}}{x_\alpha^{(-\delta\sigma-1)p/q}}f^p(x)dxdy\right]^{\frac{1}{p}}$$
$$=(\frac{2k(\sigma)}{1-\alpha^2})^{\frac{1}{q}}\left\{\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\left(1-\tanh(\frac{\gamma y_\beta}{x_\alpha^\delta})\right)\frac{y_\beta^{\sigma-1}dy}{x_\alpha^{(-\delta\sigma-1)p/q}}\right]f^p(x)dx\right\}^{\frac{1}{p}}$$
$$=(\frac{2k(\sigma)}{1-\alpha^2})^{\frac{1}{q}}\left[\int_{-\infty}^{\infty}\varpi_\delta(\sigma,x)x_\delta^{p(1+\delta\sigma)-1}f^p(x)dx\right]^{\frac{1}{p}}$$
$$=K_{\alpha,\beta}(\sigma)\left[\int_{-\infty}^{\infty}x_\delta^{p(1+\delta\sigma)-1}f^p(x)dx\right]^{\frac{1}{p}}.$$

For $K_{\alpha,\beta}(\sigma)\leq M$, we have (3.1) (for $\sigma_1=\sigma$).

Therefore, Statements (i), (ii) and (iii) are equivalent.

The theorem is proved. □

For $\sigma_1=\sigma$, we have

Theorem 3.2. *If M is a constant, then following statements (i), (ii) and (iii) are equivalent:*

(i) for any $f(x)\geq 0$, satisfying $0<\int_{-\infty}^{\infty}x_\alpha^{p(1+\delta\sigma)-1}f^p(x)dx<\infty$, we have the following inequality:

$$\left\{\int_{-\infty}^{\infty}y_\beta^{p\sigma-1}\left[\int_{-\infty}^{\infty}\left(1-\tanh(\frac{\gamma y_\beta}{x_\alpha^\delta})\right)f(x)dx\right]^p dy\right\}^{\frac{1}{p}}$$
$$<M\left[\int_{-\infty}^{\infty}x_\alpha^{p(1+\delta\sigma)-1}f^p(x)dx\right]^{\frac{1}{p}}; \quad (3.5)$$

(ii) for any $f(x)\geq 0$, satisfying $0<\int_{-\infty}^{\infty}x_\alpha^{p(1+\delta\sigma)-1}f^p(x)dx<\infty$, and $g(y)\geq 0$, satisfying $0 < \int_{-\infty}^{\infty} y_\beta^{q(1-\sigma)-1} g^q(y)dy < \infty$, we have the following inequality:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(1-\tanh(\frac{\gamma y_\beta}{x_\alpha^\delta})\right) f(x)g(y)dxdy$$
$$< M\left[\int_{-\infty}^{\infty} x_\alpha^{p(1+\delta\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}}\left[\int_{-\infty}^{\infty} y_\beta^{q(1-\sigma)-1} g^q(y)dy\right]^{\frac{1}{q}}; \qquad (3.6)$$

(iii) $K_{\alpha,\beta}(\sigma) \le M$.

Moreover, if statement (iii) holds true, then the constant factor $M = K_{\alpha,\beta}(\sigma)$ in (3.5) and (3.6) is the best possible.

In particular, (1) for $\delta = 1$, we have the following equivalent inequalities with the homogeneous kernel of degree 0 and the best possible constant factor $K_{\alpha,\beta}(\sigma)$:

$$\left\{\int_{-\infty}^{\infty} y_\beta^{p\sigma-1}\left[\int_{-\infty}^{\infty}\left(1-\tanh(\frac{\gamma y_\beta}{x_\alpha})\right) f(x)dx\right]^p dy\right\}^{\frac{1}{p}}$$
$$< K_{\alpha,\beta}(\sigma)\left[\int_{-\infty}^{\infty} x_\alpha^{p(1+\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}}, \qquad (3.7)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(1-\tanh(\frac{\gamma y_\beta}{x_\alpha})\right) f(x)g(y)dxdy$$
$$< K_{\alpha,\beta}(\sigma)\left[\int_{-\infty}^{\infty} x_\alpha^{p(1+\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}}\left[\int_{-\infty}^{\infty} y_\beta^{q(1-\sigma)-1} g^q(y)dy\right]^{\frac{1}{q}}; \qquad (3.8)$$

(2) for $\delta = -1$, we have the following equivalent inequalities with the nonhomogeneous kernel and the best possible constant factor $K_{\alpha,\beta}(\sigma)$:

$$\left\{\int_{-\infty}^{\infty} y_\beta^{p\sigma-1}\left[\int_{-\infty}^{\infty}\left(1-\tanh(\gamma x_\alpha y_\beta)\right) f(x)dx\right]^p dy\right\}^{\frac{1}{p}}$$
$$< K_{\alpha,\beta}(\sigma)\left[\int_{-\infty}^{\infty} x_\alpha^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}}, \qquad (3.9)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(1-\tanh(\gamma x_\alpha y_\beta)\right) f(x)g(y)dxdy$$
$$< K_{\alpha,\beta}(\sigma)\left[\int_{-\infty}^{\infty} x_\alpha^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}}\left[\int_{-\infty}^{\infty} y_\beta^{q(1-\sigma)-1} g^q(y)dy\right]^{\frac{1}{q}}. \qquad (3.10)$$

Proof. For $\sigma_1 = \sigma$ and the assumption of (i), if (3.4) takes the form of equality for a $y \in (-\infty, 0) \cup (0, \infty)$, then (see [11]), there exist constants A and B, such that they are not all zero, and

$$A\frac{y_\beta^{\sigma-1}}{x_\alpha^{(-\delta\sigma-1)p/q}} f^p(x) = B\frac{x^{\delta\sigma-1}}{y_\beta^{(\sigma-1)q/p}} \text{ a.e. in } \mathbf{R}.$$

We suppose that $A \ne 0$ (otherwise $B = A = 0$). Then it follows that

$$x_\delta^{p(1+\delta\sigma)-1} f^p(x) = y_\beta^{q(1-\sigma)}\frac{B}{Ax_\alpha} \text{ a.e. in } \mathbf{R}.$$

Since $\int_{-\infty}^{\infty} x_\alpha^{-1} dx = \infty$, it contradicts the fact that $0 < \int_{-\infty}^{\infty} x_\alpha^{p(1+\delta\sigma)-1} f^p(x) dx < \infty$. Hence, (3.4) takes the form of strict inequality; so does (3.1). Hence, (3.5) and (3.6) are valid.

In view of Theorem 3.1, we still can conclude that statements (i), (ii) and (iii) in Theorem 3.2 are equivalent.

When statement (iii) holds true, namely, $K_{\alpha,\beta}(\sigma) \leq M$, if there exists a constant $M \leq K_{\alpha,\beta}(\sigma)$, such that (3.6) is valid, then we can conclude that the constant factor $M = K_{\alpha,\beta}(\sigma)$ in (3.6) is the best possible.

The constant factor $M = K_{\alpha,\beta}(\sigma)$ in (3.5) is still the best possible. Otherwise, by (3.3) (for $\sigma_1 = \sigma$), we would reach a contradiction that the constant factor $M = K_{\alpha,\beta}(\sigma)$ in (3.6) is not the best possible.

The theorem is proved. □

4. Operator expressions

We set the following functions: $\varphi(x) := x_\alpha^{p(1+\delta\sigma)-1} (x \in \mathbf{R})$, and $\psi(y) := y_\beta^{q(1-\sigma)-1}$, wherefrom, $\psi^{1-p}(y) = y_\beta^{p\sigma-1}$ $(y \in \mathbf{R})$. Define the following real normed linear spaces:

$$L_{p,\varphi}(\mathbf{R}) := \left\{ f : ||f||_{p,\varphi} := \left(\int_{-\infty}^{\infty} \varphi(x)|f(x)|^p dx \right)^{\frac{1}{p}} < \infty \right\},$$

$$L_{q,\psi}(\mathbf{R}) := \left\{ g : ||g||_{q,\psi} = \left(\int_{-\infty}^{\infty} \psi(y)|g(y)|^q dy \right)^{\frac{1}{q}} < \infty \right\},$$

$$L_{p,\psi^{1-p}}(\mathbf{R}) := \left\{ h : ||h||_{p,\psi^{1-p}} = \left(\int_{-\infty}^{\infty} \psi^{1-p}(y)|h(y)|^p dy \right)^{\frac{1}{p}} < \infty \right\}.$$

In view of Theorem 3.2, for $f \in L_{p,\varphi}(\mathbf{R})$, setting

$$h_1(y) := \int_{-\infty}^{\infty} \left(1 - \tanh(\frac{\gamma y_\beta}{x_\alpha^\delta}) \right) f(x) dx \ (y \in \mathbf{R}),$$

by (3.5), we have

$$||h_1||_{p,\psi^{1-p}} = \left[\int_{-\infty}^{\infty} \psi^{1-p}(y) h_1^p(y) dy \right]^{\frac{1}{p}} < M||f||_{p,\varphi} < \infty. \tag{4.1}$$

Definition 4.1. Define a Hilbert-type integral operator with the nonhomogeneous kernel $T : L_{p,\varphi}(\mathbf{R}) \to L_{p,\psi^{1-p}}(\mathbf{R})$ as follows: For any $f \in L_{p,\varphi}(\mathbf{R})$, there exists a unique representation $Tf = h_1 \in L_{p,\psi^{1-p}}(\mathbf{R})$, satisfying for any $y \in \mathbf{R}, Tf(y) = h_1(y)$.

In view of (4.1), it follows that $||Tf||_{p,\psi^{1-p}} = ||h_1||_{p,\psi^{1-p}} < M||f||_{p,\varphi}$, and then the operator T is bounded satisfying

$$||T|| = \sup_{f(\neq\theta)\in L_{p,\varphi}(\mathbf{R})} \frac{||Tf||_{p,\psi^{1-p}}}{||f||_{p,\varphi}} \leq M.$$

If we define the formal inner product of Tf and g as follows:

$$(Tf,g):=\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\left(1-\tanh(\frac{\gamma y_{\beta}}{x_{\alpha}^{\delta}})\right)f(x)dx\right]g(y)dy,$$

then we can rewrite Theorem 3.2 as follows:

Theorem 4.1. *If M is a constant, then the following statements (i), (ii) and (iii) are equivalent:*

(i) for any $f(x)\geq 0, f\in L_{p,\varphi}(\mathbf{R})$, $||f||_{p,\varphi}>0$, we have the following inequality:

$$||Tf||_{p,\psi^{1-p}}<M||f||_{p,\varphi}; \tag{4.2}$$

(ii) for any $f(x),g(y)\geq 0, f\in L_{p,\varphi}(\mathbf{R}), g\in L_{q,\psi}(\mathbf{R}), ||f||_{p,\varphi}, ||g||_{q,\psi}>0$, we have the following inequality:

$$(Tf,g)<M||f||_{p,\varphi}||g||_{q,\psi}; \tag{4.3}$$

(iii) $K_{\alpha,\beta}(\sigma)\leq M$.

Moreover, if statement (iii) holds true, then the constant factor $M=K_{\alpha,\beta}(\sigma)$ in (4.2) and (4.3) is the best possible, namely, $||T||=K_{\alpha,\beta}(\sigma)$.

Remark 4.1. (1) In particular, for $\alpha=\beta=0$ in (3.7) and (3.8) we have the following equivalent inequalities with the best possible constant factor $\frac{1-2^{1-\sigma}}{2^{\sigma-2}\gamma^{\sigma}}\Gamma(\sigma)\zeta(\sigma)$:

$$\left\{\int_{-\infty}^{\infty}|y|^{p\sigma-1}\left[\int_{-\infty}^{\infty}\left(1-\tanh(\frac{\gamma|y|}{|x|})\right)f(x)dx\right]^{p}dy\right\}^{\frac{1}{p}}$$
$$<\frac{1-2^{1-\sigma}}{2^{\sigma-2}\gamma^{\sigma}}\Gamma(\sigma)\zeta(\sigma)\left[\int_{-\infty}^{\infty}|x|^{p(1+\sigma)-1}f^{p}(x)dx\right]^{\frac{1}{p}}, \tag{4.4}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(1-\tanh(\frac{\gamma|y|}{|x|})\right)f(x)g(y)dxdy$$
$$<\frac{1-2^{1-\sigma}}{2^{\sigma-2}\gamma^{\sigma}}\Gamma(\sigma)\zeta(\sigma)\left[\int_{-\infty}^{\infty}|x|^{p(1+\sigma)-1}f^{p}(x)dx\right]^{\frac{1}{p}}$$
$$\times\left[\int_{-\infty}^{\infty}|y|^{q(1-\sigma)-1}g^{q}(y)dy\right]^{\frac{1}{q}}. \tag{4.5}$$

If $f(-x)=f(x), g(-y)=g(y)$ $(x,y\in\mathbf{R}_{+})$, then we have the following equivalent inequalities:

$$\left\{\int_{0}^{\infty}y^{p\sigma-1}\left[\int_{0}^{\infty}\left(1-\tanh(\frac{\gamma y}{x})\right)f(x)dx\right]^{p}dy\right\}^{\frac{1}{p}}$$
$$<\frac{1-2^{1-\sigma}}{2^{\sigma-1}\gamma^{\sigma}}\Gamma(\sigma)\zeta(\sigma)\left[\int_{0}^{\infty}x^{p(1+\sigma)-1}f^{p}(x)dx\right]^{\frac{1}{p}}, \tag{4.6}$$

$$\int_{0}^{\infty}\int_{0}^{\infty}\left(1-\tanh(\frac{\gamma y}{x})\right)f(x)g(y)dxdy$$
$$<\frac{1-2^{1-\sigma}}{2^{\sigma-1}\gamma^{\sigma}}\Gamma(\sigma)\zeta(\sigma)\left[\int_{0}^{\infty}x^{p(1+\sigma)-1}f^{p}(x)dx\right]^{\frac{1}{p}}$$

$$\times\left[\int_0^\infty y^{q(1-\sigma)-1}g^q(y)dy\right]^{\frac{1}{q}}, \tag{4.7}$$

where, the constant factor $\frac{1-2^{1-\sigma}}{2^{\sigma-1}\gamma^\sigma}\Gamma(\sigma)\zeta(\sigma)$ is the best possible.

(2) For $\alpha=\beta=0$ in (3.9) and (3.10) we have the following equivalent inequalities with the best possible constant factor $\frac{1-2^{1-\sigma}}{2^{\sigma-2}\gamma^\sigma}\Gamma(\sigma)\zeta(\sigma)$:

$$\begin{aligned}&\left[\int_{-\infty}^\infty |y|^{p\sigma-1}\left(\int_{-\infty}^\infty (1-\tanh(\gamma|xy|))\,f(x)dx\right)^p dy\right]^{\frac{1}{p}}\\ &<\frac{1-2^{1-\sigma}}{2^{\sigma-2}\gamma^\sigma}\Gamma(\sigma)\zeta(\sigma)\left[\int_{-\infty}^\infty |x|^{p(1-\sigma)-1}f^p(x)dx\right]^{\frac{1}{p}},\end{aligned} \tag{4.8}$$

$$\begin{aligned}&\int_{-\infty}^\infty\int_{-\infty}^\infty (1-\tanh(\gamma|xy|))\,f(x)g(y)dxdy\\ &<\frac{1-2^{1-\sigma}}{2^{\sigma-2}\gamma^\sigma}\Gamma(\sigma)\zeta(\sigma)\left[\int_{-\infty}^\infty |x|^{p(1-\sigma)-1}f^p(x)dx\right]^{\frac{1}{p}}\\ &\times\left[\int_{-\infty}^\infty |y|^{q(1-\sigma)-1}g^q(y)dy\right]^{\frac{1}{q}}.\end{aligned} \tag{4.9}$$

If $f(-x)=f(x), g(-y)=g(y)$ $(x,y\in\mathbf{R}_+)$, then we have the following equivalent inequalities:

$$\begin{aligned}&\left[\int_0^\infty y^{p\sigma-1}\left(\int_0^\infty (1-\tanh(\gamma xy))\,f(x)dx\right)^p dy\right]^{\frac{1}{p}}\\ &<\frac{1-2^{1-\sigma}}{2^{\sigma-1}\gamma^\sigma}\Gamma(\sigma)\zeta(\sigma)\left[\int_0^\infty x^{p(1-\sigma)-1}f^p(x)dx\right]^{\frac{1}{p}},\end{aligned} \tag{4.10}$$

$$\begin{aligned}&\int_0^\infty\int_0^\infty (1-\tanh(\gamma xy))\,f(x)g(y)dxdy\\ &<\frac{1-2^{1-\sigma}}{2^{\sigma-1}\gamma^\sigma}\Gamma(\sigma)\zeta(\sigma)\left[\int_0^\infty x^{p(1-\sigma)-1}f^p(x)dx\right]^{\frac{1}{p}}\\ &\times\left[\int_0^\infty y^{q(1-\sigma)-1}g^q(y)dy\right]^{\frac{1}{q}},\end{aligned} \tag{4.11}$$

where, the constant factor $\frac{1-2^{1-\sigma}}{2^{\sigma-1}\gamma^\sigma}\Gamma(\sigma)\zeta(\sigma)$ is the best possible.

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