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DYNAMICS OF A STOCHASTIC CHEMOSTAT COMPETITION MODEL WITH PLASMID-BEARING AND PLASMID-FREE ORGANISMS

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Abstract In this paper, we consider a chemostat model of competition between plasmid-bearing and plasmid-free organisms, perturbed by white noise. Firstly, we prove the existence and uniqueness of the global positive solution. Then by constructing suitable Lyapunov functions, we establish sufficient conditions for the existence of a unique ergodic stationary distribution. Furthermore, conditions for extinction of plasmid-bearing organisms are obtained. Theoretical analysis indicates that large noise intensity σ_2^2 is detrimental to the survival of plasmid-bearing organisms and is not conducive to the commercial production of genetically altered organisms. Finally, numerical simulations are presented to illustrate the results.

Keywords Stochastic chemostat model, plasmid-bearing, plasmid-free, stationary distribution.

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1. Introduction

The chemostat, a continuous culture device mainly used for various theoretical studies related to the growth rate of microorganisms, plays an important role in waste treatment and fermentation processes [20]. It has the advantage that the parameters are readily measurable, and thus the mathematics is tractable [3]. Many types of chemostat models have been investigated extensively in the literature (see [1,2,12,17,19,26,38] as well as there references). Especially, competitive chemostat models have been studied by many researchers (see e.g. [18,21,22,27,31]).

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Genetically altered organisms are frequently used to produce desired products. It has been widely used in agriculture, medicine, environmental protection and other fields. The alteration is accomplished by the insertion of a recombinant DNA into the cell in the form of a plasmid [5]. The plasmid-free organism is unencumbered by the added metabolic load the plasmid imposes, and thus may be a better competitor than plasmid-bearing organism. Moreover, the plasmid can be lost in the reproduction, resulting in a plasmid-free organism [6]. Since commercial production can take place on a scale of many generations, it is possible for the plasmid-free organism to take over the culture. The study of chemostat models for the competition between plasmid-bearing and plasmid-free organisms has received considerable attention (see e.g. [8, 23, 24, 29, 32–35] and the references therein). For a chemostat with plasmid-bearing, plasmid-free organisms and periodically pulsed substrate, Xiang and Song [29] showed there exists a asymptotically stable two microorganisms extinction periodic solution. They also established sufficient conditions for the extinction of plasmid-bearing organism and permanence of the other microorganism. Using standard techniques of bifurcation theory, Shi et al. [24] proved the existence of positive periodic solution for a chemostat model with plasmid-bearing, plasmid-free organisms competition and impulsive effect. In [23], Stephanopoulis and Lapidus proposed the following chemostat competition model with Monod response functions

$$\begin{cases} \dfrac{dS(t)}{dt} = D(S^0 - S(t)) - \dfrac{1}{\gamma}\dfrac{\mu_1 S(t)x_1(t)}{K_1 + S(t)} - \dfrac{1}{\gamma}\dfrac{\mu_2 S(t)x_2(t)}{K_2 + S(t)}, \\ \dfrac{dx_1(t)}{dt} = \left(\dfrac{\mu_1 S(t)}{K_1 + S(t)}(1-q) - D\right)x_1(t), \\ \dfrac{dx_2(t)}{dt} = \left(\dfrac{\mu_2 S(t)}{K_2 + S(t)} - D\right)x_2(t) + \dfrac{q\mu_1 S(t)x_1(t)}{K_1 + S(t)}, \end{cases} \tag{1.1}$$

where $S(t)$, $x_1(t)$ and $x_2(t)$ stand for the concentrations of nutrient, plasmid-bearing and plasmid-free organisms at time t, respectively. S_0 is the original input concentration of nutrient and D is the common dilution rate. γ represents the yield constant. μ_1 and μ_2 are the maximum growth rates of plasmid-bearing and plasmid-free organisms, respectively. K_1 and K_2 are the corresponding half-saturation constants. q is the probability that a plasmid is lost in reproduction. Hus et al. [5] studied this model and provided a global analysis of the asymptotic behavior.

However, in reality, chemostat systems are inevitably subject to environmental noise. To reveal the effect of white noise on the continuous culture of microorganisms, some authors have investigated the dynamics of stochastic chemostat systems (see e.g. [9, 25, 28, 30, 36, 37]). For example, for a classical chemostat model in the stochastic environment, Zhao and Yuan [36] derived sharp conditions for the existence of stationary distribution by using the property of Feller process and concluded that noises have negative effects on persistence of the microorganism. Sun et al. [25] considered a stochastic two-species Monod competition chemostat model, which is subject to environmental noise. They analyzed the asymptotic behavior of the solutions. Zhang and Jiang [37] discovered sufficient conditions which guarantee that the principle of competitive exclusion holds for a stochastic chemostat model with Holling type II functional response. Inspired by the relevant works, we assume the environmental noise is proportional to the variables and consider the following stochastic chemostat model with plasmid-bearing, plasmid-free organisms competition. For the sake of simplicity, we use S, x_1 and x_2 to denote $S(t)$, $x_1(t)$ and $x_2(t)$, respectively.

$$\begin{cases} dS = \left[D(S^0 - S) - \dfrac{1}{\gamma}\dfrac{\mu_1 S x_1}{K_1 + S} - \dfrac{1}{\gamma}\dfrac{\mu_2 S x_2}{K_2 + S}\right] dt + \beta_1 S dW_1(t), \\ dx_1 = \left[\left(\dfrac{\mu_1 S}{K_1 + S}(1 - q) - D\right) x_1\right] dt + \beta_2 x_1 dW_2(t), \\ dx_2 = \left[\left(\dfrac{\mu_2 S}{K_2 + S} - D\right) x_2 + \dfrac{q\mu_1 S x_1}{K_1 + S}\right] dt + \beta_3 x_2 dW_3(t), \end{cases} \tag{1.2}$$

where the same notations are used as in (1.1). $W_i(t), i = 1, 2, 3$ are standard one-dimensional independent Brownian motion defined on a complete probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathscr{F}_0$ contains all $\mathbb{P}-$null sets). $\beta_i^2 > 0$ represents the intensity of $W_i(t)$.

Stationary distribution can enrich the dynamical behavior of the stochastic chemostat system. It not only means random weak stability, but also provides a better description of persistence [14], which gives us a deeper understanding of how environmental noise affects the steady state for persistence. To the authors best knowledge, there are few studies on stationary distribution of the stochastic chemostat model with plasmid-bearing, plasmid-free organisms competition in the existing literature. In this paper, we attempt to do some work in this area. Our main effort is to construct suitable Lyapunov functions and find a bounded domain so that the diffusion operator is negative outside the domain.

The organization of the paper is as follows. In the next section, we analyze model (1.2) and give a lemma, which is necessary for later discussion. For the equivalent system (2.1) of model (1.2), we prove the existence and uniqueness of the global positive solution in Section 3. In Section 4, sufficient conditions for the existence of a unique ergodic stationary distribution are established. In Section 5, we obtain conditions for extinction of plasmid-bearing organisms. In Section 6, numerical simulations are carried out to support the theoretical results and we make a further discussion.

2. Model analysis and preliminaries

The variables in system (1.2) may be rescaled by S^0. Let

$$\begin{aligned} &s = \frac{S}{S^0},\ x = \frac{x_1}{\gamma S^0},\ y = \frac{x_2}{\gamma S^0},\ \tau = Dt, \\ &\sigma_j = \beta_j\sqrt{\frac{1}{D}},\ B_j(\tau) = \frac{W_j(\frac{\tau}{D})}{\sqrt{\frac{1}{D}}},\ j = 1, 2, 3. \end{aligned}$$

Then system (1.2) is transformed into the following equations (replacing τ with t)

$$\begin{cases} ds = \left[1 - s - \dfrac{m_1 sx}{a_1 + s} - \dfrac{m_2 sy}{a_2 + s}\right] dt + \sigma_1 s dB_1(t), \\ dx = \left[\left(\dfrac{m_1 s}{a_1 + s}(1-q) - 1\right) x\right] dt + \sigma_2 x dB_2(t), \\ dy = \left[\left(\dfrac{m_2 s}{a_2 + s} - 1\right) y + \dfrac{q m_1 sx}{a_1 + s}\right] dt + \sigma_3 y dB_3(t), \end{cases} \tag{2.1}$$

where $m_i = \frac{\mu_i}{D}$, $a_i = \frac{K_i}{S^0}$, $i = 1, 2$. So we can find out the dynamical properties of system (1.2) by studying above model. The corresponding deterministic system to (2.1) is

$$\begin{cases} ds = \left[1 - s - \dfrac{m_1 sx}{a_1 + s} - \dfrac{m_2 sy}{a_2 + s}\right] dt, \\ dx = \left[\left(\dfrac{m_1 s}{a_1 + s}(1-q) - 1\right) x\right] dt, \\ dy = \left[\left(\dfrac{m_2 s}{a_2 + s} - 1\right) y + \dfrac{q m_1 sx}{a_1 + s}\right] dt. \end{cases} \tag{2.2}$$

This model has three equilibria $E_1 : (1, 0, 0)$, $E_2 : (\bar{s}, 0, \bar{y})$, where $\bar{s}$ and $\bar{y}$ satisfy $\frac{m_2 \bar{s}}{a_2 + \bar{s}} = 1$ and $\bar{y} = 1 - \bar{s}$, $E_c : (s^*, x^*, y^*)$, where s^*, x^* and y^* satisfy $1 - s^* - \frac{m_1 s^* x^*}{a_1 + s^*} - \frac{m_2 s^* y^*}{a_2 + s^*} = 0$, $\left(\frac{m_1 s^*}{a_1 + s^*}(1-q) - 1\right) x^* = 0$ and $\left(\frac{m_2 s^*}{a_2 + s^*} - 1\right) y^* + \frac{q m_1 s^* x^*}{a_1 + s^*} = 0$. About the properties of these three equilibria, the reader can refer to Table 1 in Ref. [5].

Next, we present a lemma [11] which gives a criterion for the existence of a unique ergodic stationary distribution. Let $X(t)$ be a homogeneous Markov process in R^l (R^l represents euclidean l-space) satisfying the stochastic equation

$$dX(t) = h(X)dt + \sum_{m=1}^{k} g_m(X) dB_m(t).$$

The diffusion matrix is

$$A(x) = (a_{ij}(x)), \quad a_{ij}(x) = \sum_{m=1}^{k} g_m^{(i)}(x) g_m^{(j)}(x).$$

Lemma 2.1. *Assume there exists a bounded open domain $G \subset R^l$ with regular boundary Γ, which has the following properties*

(A1) In the domain G and some neighborhood thereof, the smallest eigenvalue of the diffusion matrix $A(x)$ is bounded away from zero;

(A2) There exists a non-negative $\mathcal{C}^2$-function V such that LV is negative for any $R^l \setminus G$.

Then the Markov process $X(t)$ has a unique stationary distribution $\pi(\cdot)$. Let $f(x)$ be a function integrable with respect to the measure π. For all $x \in R^l$, the following formula holds

$$\mathbb{P}\left\{\lim_{t\to\infty} \frac{1}{t} \int_0^t f(X(s))ds = \int_{R^l} f(x)\pi(dx)\right\} = 1.$$

For simplicity, we denote

$$\mathbb{R}^d_+=\{x\in\mathbb{R}^d : x_i>0 \text{ for all } 1\leq i\leq d\}, \overline{\mathbb{R}}^d_+=\{x\in\mathbb{R}^d : x_i\geq 0 \text{ for all } 1\leq i\leq d\}.$$

If $f(t)$ is an integrable function on $[0,\infty)$, define $\langle f\rangle_t=\frac{1}{t}\int_0^t f(s)ds$.

3. Existence and uniqueness of positive solution

To research the dynamical behavior of a chemostat model, the first concern is whether the solution is positive and global. In this section, we shall show that system (2.1) has a unique global positive solution for any given initial value by making use of the Lyapunov function method as mentioned in [15].

Theorem 3.1. *For any given initial value* $(s(0),x(0),y(0))\in\mathbb{R}^3_+$, *system* (2.1) *has a unique solution* $(s(t),x(t),y(t))\in\mathbb{R}^3_+$ *for all* $t\geq 0$ *almost surely (a.s.).*

Proof. Since the coefficients of model (2.1) satisfy the local Lipschitz condition, for any given initial value $(s(0),x(0),y(0))\in\mathbb{R}^3_+$, there exists a unique local solution $(s(t),x(t),y(t))$ on $t\in[0,\tau_e)$, where τ_e denotes the explosion time.

Next we show this solution is global. we only need to prove $\tau_e=\infty$ a.s. Let $k_0\geq 1$ be sufficiently large such that $s(0)$, $x(0)$ and $y(0)$ all lie within the interval $[\frac{1}{k_0},k_0]$. For each integer $k\geq k_0$, define the stopping time

$$\tau_k=\inf\left\{t\in[0,\tau_e):\min\{s(t),x(t),y(t)\}\leq\frac{1}{k} \text{ or } \max\{s(t),x(t),y(t)\}\geq k\right\},$$

where throughout this paper, we set $\inf\emptyset=\infty$ (as usual $\emptyset$ denotes the empty set). Obviously, τ_k is increasing as $k\to\infty$. Set $\tau_\infty=\lim_{k\to+\infty}\tau_k$, whence $\tau_\infty\leq\tau_e$ a.s. If we can show that $\tau_\infty=\infty$ a.s., then $\tau_e=\infty$ a.s. and $(s(t),x(t),y(t))\in\mathbb{R}^3_+$ a.s. for all $t\geq 0$. In other words, in order to complete the proof, we need to show $\tau_\infty=\infty$ a.s. If this statement is false, then there is a pair of constants $T>0$ and $\epsilon\in(0,1)$ such that

$$\mathbb{P}\{\tau_\infty\leq T\}>\epsilon.$$

So there is an integer $k_1\geq k_0$ such that

$$\mathbb{P}\{\tau_k\leq T\}\geq\epsilon \text{ for all } k\geq k_1. \tag{3.1}$$

Construct a non-negative $\mathcal{C}^2$-function $\tilde{V}:\mathbb{R}^3_+\to\overline{\mathbb{R}}_+$ by

$$\begin{aligned}\tilde{V}(s,x,y)=&(s-1-\log s)+(x-1-\log x)+(y-1-\log y)+1\\ :=&\tilde{W}+1,\end{aligned}$$

where $\tilde{W}=(s-1-\log s)+(x-1-\log x)+(y-1-\log y)$. Making use of Itô's formula, we obtain

$$d\tilde{V}(s,x,y)=L\tilde{V}(s,x,y)dt+\sigma_1(s-1)dB_1(t)+\sigma_2(x-1)dB_2(t)+\sigma_3(y-1)dB_3(t),$$

where

$$L\tilde{V}(s,x,y)=\left(1-\frac{1}{s}\right)\left(1-s-\frac{m_1sx}{a_1+s}-\frac{m_2sy}{a_2+s}\right)+\frac{\sigma_1^2}{2}$$

$$
\begin{aligned}
&+(x-1)\left(\frac{m_1 s}{a_1+s}(1-q)-1\right)+\frac{\sigma_2^2}{2}\\
&+\left(1-\frac{1}{y}\right)\left[\left(\frac{m_2 s}{a_2+s}-1\right)y+\frac{q m_1 s x}{a_1+s}\right]+\frac{\sigma_3^2}{2}\\
\leq&4+m_1 q+\frac{m_1}{a_1}x+\frac{m_2}{a_2}y+\frac{\sigma_1^2}{2}+\frac{\sigma_2^2}{2}+\frac{\sigma_3^2}{2}\\
:=&C_1+\frac{m_1}{a_1}x+\frac{m_2}{a_2}y,
\end{aligned}
$$

in which $C_1=4+m_1q+\frac{\sigma_1^2}{2}+\frac{\sigma_2^2}{2}+\frac{\sigma_3^2}{2}$. By the inequalities $x\leq 2(x-1-\log x)+2\log 2\leq 2(\tilde{W}+\log 2)$ and $y\leq 2(y-2-\log y)+2\log 2\leq 2(\tilde{W}+\log 2)$, we get

$$
\begin{aligned}
L\tilde{V}(s,x,y)\leq&C_1+\left(\frac{m_1}{a_1}+\frac{m_2}{a_2}\right)2(\tilde{W}+\log 2))\\
=&C_1+2\left(\frac{m_1}{a_1}+\frac{m_2}{a_2}\right)\log 2+2\left(\frac{m_1}{a_1}+\frac{m_2}{a_2}\right)\tilde{W}\\
\leq&\max\left\{C_1+2\left(\frac{m_1}{a_1}+\frac{m_2}{a_2}\right)\log 2,2\left(\frac{m_1}{a_1}+\frac{m_2}{a_2}\right)\right\}(1+\tilde{W})\\
:=&C_2\tilde{V},
\end{aligned}
$$

where $C_2=\max\left\{C_1+2\left(\frac{m_1}{a_1}+\frac{m_2}{a_2}\right)\log 2,2\left(\frac{m_1}{a_1}+\frac{m_2}{a_2}\right)\right\}$. Then

$$
d\tilde{V}(s,x,y)\leq C_2\tilde{V}(s,x,y)dt+\sigma_1(s-1)dB_1(t)+\sigma_2(x-1)dB_2(t)+\sigma_3(y-1)dB_3(t).
$$

Integrating and taking the expectation yield

$$
\begin{aligned}
\mathbb{E}\tilde{V}(s(\tau_k\wedge T),x(\tau_k\wedge T),y(\tau_k\wedge T))\leq&\tilde{V}(s(0),x(0),y(0))\\
&+C_2\int_0^{\tau_k\wedge T}\mathbb{E}\tilde{V}(s(t),x(t),y(t))dt.
\end{aligned}
$$

By Growrall inequality, we have

$$
\begin{aligned}
\mathbb{E}\tilde{V}(s(\tau_k\wedge T),x(\tau_k\wedge T),y(\tau_k\wedge T))\leq&\tilde{V}(s(0),x(0),y(0))e^{C_2(\tau_k\wedge T)}\\
\leq&\tilde{V}(s(0),x(0),y(0))e^{C_2T}.
\end{aligned}
\tag{3.2}
$$

Set $\Omega_k=\{\tau_k\leq T\}$ for $k\geq k_1$ and according to (3.1), we have $\mathbb{P}(\Omega_k)\geq\epsilon$. Note that for every $\omega\in\Omega_k$, there is $s(\tau_k,\omega)$, $x(\tau_k,\omega)$ or $y(\tau_k,\omega)$ equals either k or $\frac{1}{k}$. So $\tilde{V}(s(\tau_k,\omega),x(\tau_k,\omega),y(\tau_k,\omega))$ is no less than either

$$
k-1-\log k \text{ or } \frac{1}{k}-1-\log\frac{1}{k}=\frac{1}{k}-1+\log k.
$$

Hence

$$
\tilde{V}(s(\tau_k,\omega),x(\tau_k,\omega),y(\tau_m,\omega))\geq(k-1-\log k)\wedge\left(\frac{1}{k}-1+\log k\right).
$$

By (3.2), one can see that

$$
\tilde{V}(s(0),x(0),y(0))e^{C_2T}\geq\mathbb{E}[I_{\Omega_m(\omega)}\tilde{V}(s(\tau_k,\omega),x(\tau_k,\omega),y(\tau_k,\omega))]
$$

$$\geq \epsilon \left[(k-1-\log k) \wedge \left(\frac{1}{k} - 1 + \log k \right) \right],$$

where I_{Ω_k} represents the indicator function of Ω_k. Here $k \to \infty$ leads to the contradiction $\infty > +\infty$, so we must have $\tau_\infty = \infty$ a.s. This completes the proof. $\square$

4. Existence of ergodic stationary distribution

In this section, for system (2.1), we establish sufficient conditions for the existence of a unique ergodic stationary distribution, which implies the plasmid-bearing and plasmid-free organisms can coexist in the chemostat.

Define

$$\bar{\lambda} := \frac{\lambda}{2} - l\bar{s}\left(\bar{s} + \frac{\bar{y}}{2}\right)\sigma_1^2 - \frac{\sigma_2^2}{2} - l\frac{(a_2+\bar{s})(\bar{s}+\bar{y})\bar{y}}{2a_2}\sigma_3^2,$$

where $\lambda = \frac{m_1 \bar{s}}{a_1+\bar{s}}(1-q) - 1$, $l = \frac{m_1^2(1-q)^2}{2\lambda(a_1+\bar{s})^2(1-\sigma_1^2)}$.

Theorem 4.1. *Assume $\bar{\lambda} > 0$ and $1 - \sigma_1^2 > 0$, then system* (2.1) *admits a unique stationary distribution and it has the ergodic property.*

Proof. In order to prove Theorem 4.1, it suffices to verify conditions (A1) and (A2) of Lemma 2.1. The diffusion matrix of system (2.1) is given by

$$A(s,x,y) = \begin{pmatrix} \sigma_1^2 s^2 & 0 & 0 \\ 0 & \sigma_2^2 x^2 & 0 \\ 0 & 0 & \sigma_3^2 y^2 \end{pmatrix},$$

which is positive definite for any compact subset of $\mathbb{R}_+^3$. Clearly, (A1) in Lemma 2.1 holds.

Now we are in the position to validate the condition (A2) of Lemma 2.1. We need to show there is a non-negative $\mathcal{C}^2$-function V and a bounded domain $D_\varepsilon \subset \mathbb{R}_+^3$ such that LV is negative for any $(s,x,y) \in \mathbb{R}_+^3 \setminus D_\varepsilon$.

Define a $\mathcal{C}^2$-function $V_1 : \mathbb{R}_+^3 \to \mathbb{R}$ as follows

$$\begin{aligned} V_1(s,x,y) =& -\log x + l\left[\frac{(s-\bar{s})^2}{2} + \bar{y}\left(s - \bar{s} - \bar{s}\log\frac{s}{\bar{s}}\right) + \frac{(a_2+\bar{s})(\bar{s}+\bar{y})}{a_2}\left(y - \bar{y} - \bar{y}\log\frac{y}{\bar{y}}\right)\right] \\ :=& -\log x + l\left[U_1 + \bar{y}U_2 + \frac{(a_2+\bar{s})(\bar{s}+\bar{y})}{a_2}U_3\right] \\ :=& -\log x + lW, \end{aligned}$$

where $U_1 = \frac{(s-\bar{s})^2}{2}$, $U_2 = s - \bar{s} - \bar{s}\log\frac{s}{\bar{s}}$, $U_3 = y - \bar{y} - \bar{y}\log\frac{y}{\bar{y}}$, $W = U_1 + \bar{y}U_2 + \frac{(a_2+\bar{s})(\bar{s}+\bar{y})}{a_2}U_3$. By Itô's formula, the basic inequality $(a+b)^2 \leq 2(a^2+b^2)$, $\bar{s} + \frac{m_2\bar{s}\bar{y}}{a_2+\bar{s}} = 1$ and $\frac{m_2\bar{s}}{a_2+\bar{s}} = 1$, we get

$$\begin{aligned} LU_1 =& (s-\bar{s})\left(1 - s - \frac{m_1 sx}{a_1+s} - \frac{m_2 sy}{a_2+s}\right) + \frac{\sigma_1^2}{2}s^2 \\ =& (s-\bar{s})\left(\bar{s} - s + \frac{m_2\bar{s}\bar{y}}{a_2+\bar{s}} - \frac{m_2 sy}{a_2+s} - \frac{m_1 sx}{a_1+s}\right) + \frac{\sigma_1^2}{2}(s - \bar{s} + \bar{s})^2 \end{aligned}$$

$$
\begin{aligned}
&\leq -(s-\bar{s})^2 + m_2(s-\bar{s})\left(\frac{\bar{s}\bar{y}}{a_2+\bar{s}} - \frac{sy}{a_2+s}\right) + m_1\bar{s}x + \sigma_1^2[(s-\bar{s})^2+\bar{s}^2]\\
&= -(1-\sigma_1^2)(s-\bar{s})^2 + m_2(s-\bar{s})\left(\frac{\bar{s}\bar{y}}{a_2+\bar{s}} - \frac{sy}{a_2+s} + \frac{\bar{s}y}{a_2+s} - \frac{\bar{s}y}{a_2+s}\right)\\
&\quad + m_1\bar{s}x + \bar{s}^2\sigma_1^2\\
&\leq -(1-\sigma_1^2)(s-\bar{s})^2 + m_2(s-\bar{s})\left(\frac{\bar{s}\bar{y}}{a_2+\bar{s}} - \frac{\bar{s}y}{a_2+s}\right) + m_1\bar{s}x + \bar{s}^2\sigma_1^2\\
&= -(1-\sigma_1^2)(s-\bar{s})^2 + m_2\bar{s}(s-\bar{s})\left(\frac{\bar{y}}{a_2+\bar{s}} - \frac{y}{a_2+s} + \frac{\bar{y}}{a_2+s} - \frac{\bar{y}}{a_2+s}\right)\\
&\quad + m_1\bar{s}x + \bar{s}^2\sigma_1^2\\
&= -(1-\sigma_1^2)(a-\bar{a})^2 + \frac{m_2\bar{s}}{a_2+\bar{s}}\frac{\bar{y}(s-\bar{s})^2}{a_2+s} - m_2\bar{s}\frac{(s-\bar{s})(y-\bar{y})}{a_2+s} + m_1\bar{s}x + \bar{s}^2\sigma_1^2\\
&\leq -(1-\sigma_1^2)(s-\bar{s})^2 + \frac{\bar{y}(s-\bar{s})^2}{s} - m_2\bar{s}\frac{(s-\bar{s})(y-\bar{y})}{a_2+s} + m_1\bar{s}x + \bar{s}^2\sigma_1^2, \qquad (4.1)
\end{aligned}
$$

$$
\begin{aligned}
LU_2 =&\frac{s-\bar{s}}{s}\left(1 - s - \frac{m_1sx}{a_1+s} - \frac{m_2sy}{a_2+s}\right) + \frac{\bar{s}}{2}\sigma_1^2\\
=&\frac{s-\bar{s}}{s}\left(\bar{s} - s + \frac{m_2\bar{s}\bar{y}}{a_2+\bar{s}} - \frac{m_2sy}{a_2+s} - \frac{m_1sx}{a_1+s}\right) + \frac{\bar{s}}{2}\sigma_1^2\\
=&-\frac{(s-\bar{s})^2}{s} - \frac{m_2(s-\bar{s})^2\bar{y}}{(a_2+\bar{s})(a_2+s)s} - \frac{m_2(s-\bar{s})(y-\bar{y})}{a_2+s} - \frac{m_1(s-\bar{s})x}{a_1+s} + \frac{\bar{s}}{2}\sigma_1^2\\
\leq&-\frac{(s-\bar{s})^2}{s} - \frac{m_2(s-\bar{s})(y-\bar{y})}{a_2+s} + \frac{m_1\bar{s}}{a_1}x + \frac{\bar{s}}{2}\sigma_1^2, \qquad (4.2)
\end{aligned}
$$

$$
\begin{aligned}
LU_3 =&\frac{y-\bar{y}}{y}\left[\left(\frac{m_2s}{a_2+s} - 1\right)y + \frac{qm_1sx}{a_1+s}\right] + \frac{\bar{y}}{2}\sigma_3^2\\
=&(y-\bar{y})\left(\frac{m_2s}{a_2+s} - \frac{m_2\bar{s}}{a_2+\bar{s}}\right) + \frac{qm_1sx}{a_1+s} - \frac{qm_1\bar{y}sx}{(a_1+s)y} + \frac{\bar{y}}{2}\sigma_3^2\\
\leq&\frac{m_2a_2}{a_2+\bar{s}}\frac{(s-\bar{s})(y-\bar{y})}{a_2+s} + qm_1x + \frac{\bar{y}}{2}\sigma_3^2. \qquad (4.3)
\end{aligned}
$$

Let $h = \bar{s}\left(1+\frac{\bar{y}}{a_1}\right) + \frac{q(a_2+\bar{s})(\bar{s}+\bar{y})}{a_2}$. From (4.1)-(4.3), it then follows that

$$
LW \leq -(1-\sigma_1^2)(s-\bar{s})^2 + m_1hx + \bar{s}\left(\bar{s}+\frac{\bar{y}}{2}\right)\sigma_1^2 + \frac{(a_2+\bar{s})(\bar{s}+\bar{y})\bar{y}}{2a_2}\sigma_3^2. \qquad (4.4)
$$

By virtue of Young inequality, we obtain

$$
\begin{aligned}
L(-\log x) =& -\frac{m_1s}{a_1+s}(1-q) + 1 + \frac{1}{2}\sigma_2^2\\
=& -\left(\frac{m_1\bar{s}}{a_1+\bar{s}}(1-q) - 1\right) + \left(\frac{m_1\bar{s}}{a_1+\bar{s}} - \frac{m_1s}{a_1+s}\right)(1-q) + \frac{1}{2}\sigma_2^2\\
\leq& -\lambda + \frac{m_1(1-q)|s-\bar{s}|}{a_1+\bar{s}} + \frac{1}{2}\sigma_2^2
\end{aligned}
$$

$$\leq -\frac{\lambda}{2}+\frac{m_1^2(1-q)^2}{2\lambda(a_1+\bar{s})^2}(s-\bar{s})^2+\frac{1}{2}\sigma_2^2.$$

This, together with (4.4), yields

$$LV_1 \leq -\bar{\lambda}+lm_1hx. \tag{4.5}$$

Define a $\mathcal{C}^2$-function $V_2(s)=-\log s$. Applying Itô's formula, we calculate that

$$LV_2=-\frac{1}{s}+1+\frac{m_1x}{a_1+s}+\frac{m_2y}{a_2+s}+\frac{\sigma_1^2}{2}\leq -\frac{1}{s}+1+\frac{m_1x}{a_1}+\frac{m_2y}{a_2}+\frac{\sigma_1^2}{2}. \tag{4.6}$$

Define a $\mathcal{C}^2$-function $V_3(y)=-\log y$. Then

$$LV_3=-\frac{m_2s}{a_2+s}+1-\frac{qm_1sx}{(a_1+s)y}+\frac{\sigma_3^2}{2}\leq 1-\frac{qm_1sx}{(a_1+s)y}+\frac{\sigma_3^2}{2}. \tag{4.7}$$

Define a $\mathcal{C}^2$-function $V_4:\mathbb{R}_+^3\to\mathbb{R}$ in the following form

$$V_4(s,x,y)=\frac{1}{\theta+1}(s+x+y)^{\theta+1},$$

where θ is a constant satisfying $0<\theta<\frac{2}{\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2}$. Using Itô's formula leads to

$$\begin{aligned}LV_4=&(s+x+y)^\theta[1-(s+x+y)]+\frac{\theta}{2}(s+x+y)^{\theta-1}(\sigma_1^2s^2+\sigma_2^2x^2+\sigma_3^2y^2)\\ \leq&(s+x+y)^\theta-\left[1-\frac{\theta}{2}(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2)\right](s+x+y)^{\theta+1}\\ \leq&K_1-\frac{1}{2}\left[1-\frac{\theta}{2}(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2)\right](s^{\theta+1}+x^{\theta+1}+y^{\theta+1}),\end{aligned} \tag{4.8}$$

where

$$K_1=(s+x+y)^\theta-\frac{1}{2}\left[1-\frac{\theta}{2}(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2)\right](s^{\theta+1}+x^{\theta+1}+y^{\theta+1}).$$

Then we define a $\mathcal{C}^2$-function $F:\mathbb{R}_+^3\to\mathbb{R}$ by

$$F(s,x,y)=MV_1+V_2+V_3+V_4,$$

where $M>0$ satisfies

$$-M\bar{\lambda}+K_2\leq -2, \tag{4.9}$$

and

$$\begin{aligned}K_2=\sup_{(s,x,y)\in\mathbb{R}_+^3}\bigg\{&-\frac{1}{4}\left[1-\frac{\theta}{2}(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2)\right](s^{\theta+1}+x^{\theta+1}+y^{\theta+1})\\ &+\frac{m_1x}{a_1}+\frac{m_2y}{a_2}+K_1+2+\frac{\sigma_1^2+\sigma_3^2}{2}\bigg\}.\end{aligned} \tag{4.10}$$

It is easy to check that $\liminf_{k\to\infty,(s,x,y)\in\mathbb{R}_+^3\setminus Q_k}F(s,x,y)=+\infty$, where $Q_k=(\frac{1}{k},k)\times(\frac{1}{k},k)\times(\frac{1}{k},k)$. Furthermore, F is a continuous function. Hence, $F(s,x,y)$ has a minimum point $F(s_0,x_0,y_0)$ in the interior of $\mathbb{R}_+^3$.

According to the above analysis, we construct a non-negative $\mathcal{C}^2$-function $V : \mathbb{R}^3_+ \to \overline{\mathbb{R}}_+$ as follows

$$V(s,x,y) = F(s,x,y) - F(s_0,x_0,y_0).$$

By (4.5)-(4.8) and (4.10), we derive

$$\begin{aligned} LV \leq & - M\bar{\lambda} + Mlm_1hx - \frac{1}{s} - \frac{qm_1sx}{(a_1+s)y} \\ & - \frac{1}{4}\left[1 - \frac{\theta}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right](s^{\theta+1} + x^{\theta+1} + y^{\theta+1}) + K_2. \end{aligned} \tag{4.11}$$

Define a bounded closed domain

$$D_\varepsilon = \left\{\varepsilon \leq s \leq \frac{1}{\varepsilon}, \varepsilon \leq x \leq \frac{1}{\varepsilon}, \varepsilon^3 \leq y \leq \frac{1}{\varepsilon^3}\right\},$$

where $\varepsilon > 0$ is a sufficiently small constant. In the set $\mathbb{R}^3_+ \setminus D_\varepsilon$, we can choose ε sufficiently small such that the following conditions hold

$$-\frac{1}{\varepsilon} + K_3 \leq -1, \tag{4.12}$$

$$Mlm_1h\varepsilon \leq 1, \tag{4.13}$$

$$-\frac{qm_1}{(a_1+\varepsilon)\varepsilon} + K_3 \leq -1, \tag{4.14}$$

$$-\frac{1}{8}\left[1 - \frac{\theta}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right]\frac{1}{\varepsilon^{\theta+1}} + K_4 \leq -1, \tag{4.15}$$

where K_3, K_4 are constants which can be found from (4.16) and (4.17). For convenience, we divide $\mathbb{R}^3_+ \setminus D_\varepsilon$ into six domains,

$$\begin{aligned} & D_1 = \{(s,x,y) \in \mathbb{R}^3_+ : 0 < s < \varepsilon\}, \quad D_2 = \{(s,x,y) \in \mathbb{R}^3_+ : 0 < x < \varepsilon\}, \\ & D_3 = \{(s,x,y) \in \mathbb{R}^3_+ : \varepsilon \leq s, \varepsilon \leq x, 0 < y < \varepsilon^3\}, \quad D_4 = \{(s,x,y) \in \mathbb{R}^3_+ : s > \frac{1}{\varepsilon}\}, \\ & D_5 = \{(s,x,y) \in \mathbb{R}^3_+ : x > \frac{1}{\varepsilon}\}, \quad D_6 = \{(s,x,y) \in \mathbb{R}^3_+ : y > \frac{1}{\varepsilon^3}\}. \end{aligned}$$

Next we will show that $LV(s,x,y) \leq -1$ on $\mathbb{R}^3_+ \setminus D_\varepsilon$, which is equivalent to proving it on the above six domains, respectively.

Case 1. If $(s,x,y) \in D_1$, by (4.11) and (4.12), one can derive

$$\begin{aligned} LV \leq & - \frac{1}{s} + Mlm_1hx - \frac{1}{4}\left[1 - \frac{\theta}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right](s^{\theta+1} + x^{\theta+1} + y^{\theta+1}) + K_2 \\ \leq & - \frac{1}{s} + K_3 \leq -\frac{1}{\varepsilon} + K_3 \leq -1, \end{aligned}$$

in which

$$K_3 = \sup_{(s,x,y)\in\mathbb{R}^3_+}\left\{Mlm_1hx - \frac{1}{4}\left[1 - \frac{\theta}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right](s^{\theta+1} + x^{\theta+1} + y^{\theta+1}) + K_2\right\}. \tag{4.16}$$

Therefore,

$$LV(s,x,y) \leq -1 \text{ for any } (s,x,y) \in D_1.$$

Case 2. If $(s,x,y) \in D_2$, it follows from (4.9), (4.11) and (4.13),that

$$LV \leq -M\bar{\lambda} + Mlm_1h\varepsilon + K_2 \leq -2+1 \leq -1.$$

Thus,

$$LV(s,x,y) \leq -1 \text{ for any } (s,x,y) \in D_2.$$

Case 3. If $(s,x,y) \in D_3$, in view of (4.11) and (4.14), one can obtain

$$\begin{aligned} LV \leq & Mlm_1hx - \frac{qm_1sx}{(a_1+s)y} - \frac{1}{4}\left[1-\frac{\theta}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right](s^{\theta+1}+x^{\theta+1}+y^{\theta+1}) + K_2 \\ \leq & -\frac{qm_1}{(a_1+\varepsilon)\varepsilon} + K_3 \leq -1. \end{aligned}$$

Hence,

$$LV(s,x,y) \leq -1 \text{ for any } (s,x,y) \in D_3.$$

Case 4. If $(s,x,y) \in D_4$, by (4.11) and (4.15), we get

$$\begin{aligned} LV \leq & -\frac{1}{8}\left[1-\frac{\theta}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right] s^{\theta+1} + Mlm_1hx \\ & -\frac{1}{8}\left[1-\frac{\theta}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right](s^{\theta+1}+x^{\theta+1}+y^{\theta+1}) + K_2 \\ \leq & -\frac{1}{8}\left[1-\frac{\theta}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right]\frac{1}{\varepsilon^{\theta+1}} + K_4 \leq -1, \end{aligned}$$

in which

$$K_4 = \sup_{(s,x,y)\in\mathbb{R}^3_+}\left\{Mlm_1hx - \frac{1}{8}\left[1-\frac{\theta}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right](s^{\theta+1}+x^{\theta+1}+y^{\theta+1}) + K_2\right\}. \tag{4.17}$$

So,

$$LV(s,x,y) \leq -1 \text{ for any } (s,x,y) \in D_4.$$

Case 5. If $(s,x,y) \in D_5$, from (4.11) and (4.15), it then follows that

$$\begin{aligned} LV \leq & -\frac{1}{8}\left[1-\frac{\theta}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right] x^{\theta+1} + Mlm_1hx \\ & -\frac{1}{8}\left[1-\frac{\theta}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right](s^{\theta+1}+x^{\theta+1}+y^{\theta+1}) + K_2 \\ \leq & -\frac{1}{8}\left[1-\frac{\theta}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right]\frac{1}{\varepsilon^{\theta+1}} + K_4 \leq -1. \end{aligned}$$

Thus,

$$LV(s,x,y) \leq -1 \text{ for any } (s,x,y) \in D_5.$$

Case 6. Similarly, if $(s,x,y) \in D_6$, we have

$$LV \leq -\frac{1}{8}\left[1-\frac{\theta}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)\right] y^{\theta+1} + Mlm_1hx$$

$$-\frac{1}{8}\left[1-\frac{\theta}{2}(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2)\right](s^{\theta+1}+x^{\theta+1}+y^{\theta+1})+K_2$$
$$\leq -\frac{1}{8}\left[1-\frac{\theta}{2}(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2)\right]\frac{1}{\varepsilon^{3(\theta+1)}}+K_4\leq -1.$$

Therefore,

$$LV(s,x,y)\leq -1 \text{ for any } (s,x,y)\in D_6.$$

Based on the above six situations, we can conclude that for a sufficiently small ε,

$$LV(s,x,y)\leq -1 \text{ for any } (s,x,y)\in \mathbb{R}^3_+\setminus D_\varepsilon.$$

The proof is complete. □

5. Extinction

In this section, we investigate the extinction of plasmid-bearing organisms in system (2.1). First of all, we give two useful lemmas.

Lemma 5.1. *For any given initial value $(s(0),x(0),y(0))\in\mathbb{R}^3_+$, the solution $(s(t),x(t),y(t))$ of system (2.1) has the following property*

$$\limsup_{t\to\infty}\frac{\log x(t)}{t}\leq 0,\quad \limsup_{t\to\infty}\frac{\log y(t)}{t}\leq 0\quad a.s.$$

Lemma 5.2. *Let $Z(t)$ be the solution of the stochastic differential equation*

$$dZ(t)=(1-Z(t))dt+\sigma_1 Z(t)dB_1(t),$$

with the initial value $Z(0)=s(0)>0$. Then we have $s(t)\leq Z(t)$ by the comparison theorem [16] and $Z(t)$ converges weakly to the distribution ν, which has the density $\nu(z)=Qz^{-(2+2/\sigma_1^2)}e^{-2/\sigma_1^2 z}, z\in(0,\infty)$, where $Q=(2/\sigma_1^2)^{1+2/\sigma_1^2}\Gamma^{-1}(2/\sigma_1^2+1)$ such that $\int_0^\infty \nu(z)dz=1$ and $\int_0^\infty z\nu(z)dz=1$.

Theorem 5.1. *Let $(s(t),x(t),y(t))$ be the solution of system (2.1) with any given initial value $(s(0),x(0),y(0))\in\mathbb{R}^3_+$. If $m_1(1-q)\int_0^\infty \frac{z\nu(z)}{a_1+z}dz<1+\frac{\sigma_2^2}{2}$ and $m_2\int_0^\infty\frac{z\nu(z)}{a_2+z}dz>1+\frac{\sigma_3^2}{2}$, then*

$$\limsup_{t\to\infty}\frac{\log x(t)}{t}\leq m_1(1-q)\int_0^\infty\frac{z\nu(z)}{a_1+z}dz-\left(1+\frac{\sigma_2^2}{2}\right)<0\quad a.s.,$$

and

$$\liminf_{t\to\infty}\langle y\rangle_t\geq\frac{1}{m_2^2}\left[m_2\int_0^\infty\frac{z\nu(z)}{a_2+z}dz-\left(1+\frac{\sigma_3^2}{2}\right)\right]>0\quad a.s.$$

That is to say, plasmid-bearing organisms will go extinct exponentially with probability one and plasmid-free organisms will survive.

Proof. An application of Itô's formula yields

$$d\log x(t)=\left(\frac{m_1 s}{a_1+s}(1-q)-\left(1+\frac{\sigma_2^2}{2}\right)\right)dt+\sigma_2 dB_2(t).$$

Integrating this from 0 to t and dividing by t on both sides result in

$$\begin{aligned}\frac{\log x(t)}{t} =& m_1(1-q)\left\langle \frac{s}{a_1+s}\right\rangle_t - \left(1+\frac{\sigma_2^2}{2}\right) + \frac{\sigma_2 B_2(t)}{t} + \frac{\log x(0)}{t} \\ \leq& m_1(1-q)\left\langle \frac{Z}{a_1+Z}\right\rangle_t - \left(1+\frac{\sigma_2^2}{2}\right) + \frac{\sigma_2 B_2(t)}{t} + \frac{\log x(0)}{t}. \end{aligned} \tag{5.1}$$

Similarly,

$$d\log y(t) = \left(\frac{m_2 s}{a_2+s} + \frac{q m_1 s x}{(a_1+s)y} - \left(1+\frac{\sigma_3^2}{2}\right)\right)dt + \sigma_3 dB_3(t). \tag{5.2}$$

The strong law of large numbers [10] implies that

$$\lim_{t\to\infty}\frac{\sigma_i B_i(t)}{t} = 0,\ i=2,3 \quad a.s. \tag{5.3}$$

In view of (5.1) and (5.3), we obtain

$$\limsup_{t\to\infty}\frac{\log x(t)}{t} \leq m_1(1-q)\int_0^\infty \frac{z\nu(z)}{a_1+z}dz - \left(1+\frac{\sigma_2^2}{2}\right) < 0 \quad a.s.,$$

which means that

$$\lim_{t\to\infty} x(t) = 0 \quad a.s. \tag{5.4}$$

Using Itô's formula, one can derive

$$\begin{aligned}\frac{\log s(t)}{t} =& \frac{\log s(0)}{t} + \left\langle \frac{1}{s}\right\rangle_t - m_1\left\langle \frac{x}{a_1+s}\right\rangle_t - m_2\left\langle \frac{y}{a_2+s}\right\rangle_t \\ &- \left(1+\frac{\sigma_1^2}{2}\right) + \frac{\sigma_1 B_1(t)}{t}, \end{aligned} \tag{5.5}$$

and

$$\frac{\log Z(t)}{t} = \frac{\log s(0)}{t} + \left\langle \frac{1}{Z}\right\rangle_t - \left(1+\frac{\sigma_1^2}{2}\right) + \frac{\sigma_1 B_1(t)}{t}. \tag{5.6}$$

From (5.5) and (5.6), it follows that

$$\begin{aligned} 0 \geq \frac{\log s(t) - \log Z(t)}{t} &= \left\langle \frac{1}{s} - \frac{1}{Z}\right\rangle_t - m_1\left\langle \frac{x}{a_1+s}\right\rangle_t - m_2\left\langle \frac{y}{a_2+s}\right\rangle_t \\ &\geq \left\langle \frac{1}{s} - \frac{1}{Z}\right\rangle_t - \frac{m_1}{a_1}\langle x\rangle_t - \frac{m_2}{a_2}\langle y\rangle_t. \end{aligned}$$

Thus,

$$\left\langle \frac{1}{s} - \frac{1}{Z}\right\rangle_t \leq \frac{m_1}{a_1}\langle x\rangle_t + \frac{m_2}{a_2}\langle y\rangle_t. \tag{5.7}$$

From (5.2), we get

$$\frac{\log y(t)}{t} = m_2\left\langle \frac{s}{a_2+s}\right\rangle_t + q m_1\left\langle \frac{sx}{(a_1+s)y}\right\rangle_t - \left(1+\frac{\sigma_3^2}{2}\right) + \frac{\sigma_3 B_3(t)}{t} + \frac{\log y(0)}{t}$$

$$
\begin{aligned}
&= m_2 \left\langle \frac{Z}{a_2+Z} \right\rangle_t - m_2 a_2 \left\langle \frac{Z-s}{(a_2+s)(a_2+Z)} \right\rangle_t + q m_1 \left\langle \frac{sx}{(a_1+s)y} \right\rangle_t \\
&\quad - \left(1+\frac{\sigma_3^2}{2}\right) + \frac{\sigma_3 B_3(t)}{t} + \frac{\log y(0)}{t} \\
&\geq m_2 \left\langle \frac{Z}{a_2+Z} \right\rangle_t - m_2 a_2 \left\langle \frac{1}{s} - \frac{1}{Z} \right\rangle_t - \left(1+\frac{\sigma_3^2}{2}\right) + \frac{\sigma_3 B_3(t)}{t} + \frac{\log y(0)}{t} \\
&\geq m_2 \left\langle \frac{Z}{a_2+Z} \right\rangle_t - m_2 a_2 \left(\frac{m_1}{a_1}\langle x\rangle_t + \frac{m_2}{a_2}\langle y\rangle_t\right) - \left(1+\frac{\sigma_3^2}{2}\right) \\
&\quad + \frac{\sigma_3 B_3(t)}{t} + \frac{\log y(0)}{t}.
\end{aligned}
$$

Then it is easy to see that

$$
\begin{aligned}
\langle y\rangle_t =& \frac{1}{m_2^2}\left[m_2 \left\langle \frac{Z}{a_2+Z} \right\rangle_t - \left(1+\frac{\sigma_3^2}{2}\right)\right] - \frac{m_1 a_2}{m_2 a_1}\langle x\rangle_t \\
&+ \frac{1}{m_2^2}\left[\frac{\sigma_3 B_3(t)}{t} + \frac{\log y(0)}{t}\right] - \frac{1}{m_2^2}\frac{\log y(t)}{t}.
\end{aligned}
\tag{5.8}
$$

Taking the inferior limit on both sides of (5.8) and combining with Lemma 5.1, from (5.3) and (5.4), we obtain

$$
\liminf_{t\to\infty}\langle y\rangle_t \geq \frac{1}{m_2^2}\left[m_2 \int_0^\infty \frac{z\nu(z)}{a_2+z}dz - \left(1+\frac{\sigma_3^2}{2}\right)\right] > 0 \quad a.s.
$$

This completes the proof. □

6. Numerical simulations and discussion

This paper is devoted to a chemostat model with plasmid-bearing, plasmid-free organisms competition, which is disturbed by white noise. We first prove the system has a unique global positive solution for any initial value. Then, using the boundary equilibrium point E_2 of system (2.2), we construct appropriate Lyapunov functions and establish sufficient conditions for the existence of stationary distribution. Specifically, if $\bar{\lambda} := \frac{\lambda}{2} - l\bar{s}\left(\bar{s}+\frac{\bar{y}}{2}\right)\sigma_1^2 - \frac{\sigma_2^2}{2} - l\frac{(a_2+\bar{s})(\bar{s}+\bar{y})\bar{y}}{2a_2}\sigma_3^2 > 0$ and $1-\sigma_1^2 > 0$, then system (2.1) admits a unique ergodic stationary distribution. Moreover, we show that if $m_1(1-q)\int_0^\infty \frac{z\nu(z)}{a_1+z}dz < 1+\frac{\sigma_2^2}{2}$ and $m_2\int_0^\infty \frac{z\nu(z)}{a_2+z}dz > 1+\frac{\sigma_3^2}{2}$, then the plasmid-bearing organism will go extinct exponentially with probability one and the plasmid-free organism will survive. This implies large noise intensity σ_2^2 is detrimental to the survival of plasmid-bearing organisms. In this case, the plasmid-free organism will take over the chemostat and exclude the plasmid-bearing organism. The results obtained in the present paper may be of interest in biotechnology. In the commercial production process of genetically altered organisms, in order to avoid capture by the plasmid-free organism, some measures can be taken to reduce the noise intensity so that two microorganisms can coexist to produce the desired products.

Now we are in the position to present some numerical examples which will support our analytical results. Using Milstein's Higher Order Method mentioned in [7], we get the following discretization transformation of system (2.1).

$$\begin{cases} s_{j+1} = s_j + \left(1 - s_j - \frac{m_1 s_j x_j}{a_1+s_j} - \frac{m_2 s_j y_j}{a_2+s_j}\right)\triangle t + \sigma_1 s_j \sqrt{\triangle t}\omega_{1j} + \frac{\sigma_1^2}{2} s_j(\omega_{1j}^2 - 1)\triangle t, \\ x_{j+1} = x_j + \left(\frac{m_1 s_j}{a_1+s_j}(1-q) - 1\right) x_j \triangle t + \sigma_2 x_j \sqrt{\triangle t}\omega_{2j} + \frac{\sigma_2^2}{2} x_j(\omega_{2j}^2 - 1)\triangle t, \\ y_{j+1} = y_j + \left(\left(\frac{m_2 s_j}{a_2+s_j} - 1\right) y_j + \frac{q m_1 s_j x_j}{a_1+s_j}\right)\triangle t + \sigma_3 y_j \sqrt{\triangle t}\omega_{3j} + \frac{\sigma_3^2}{2} y_j(\omega_{3j}^2 - 1)\triangle t, \end{cases}$$

where the time increment $\triangle t > 0$, $\omega_{ij}, i = 1,2,3$ are the Gaussian random variables which follow the distribution $N(0,1)$. We take initial value $(s(0), x(0), y(0)) = (1, 1, 0.04)$. For the sake of convenience and simplicity, we always keep some parameters fixed as follows:

$$m_1 = 1.9,\ m_2 = 1.6,\ a_1 = 0.18,\ a_2 = 0.3.$$

Example 6.1. Chose environmental noise intensities $\sigma_1^2 = 0.0225,\ \sigma_2^2 = 0.01,\ \sigma_3^2 = 0.01$ and $q = 0.005$. By computation, we get

$$\bar{\lambda} := \frac{\lambda}{2} - l\bar{s}\left(\bar{s} + \frac{\bar{y}}{2}\right)\sigma_1^2 - \frac{\sigma_2^2}{2} - l\frac{(a_2+\bar{s})(\bar{s}+\bar{y})\bar{y}}{2a_2}\sigma_3^2 = 0.0369 > 0,$$

and

$$1 - \sigma_1^2 = 0.9775 > 0.$$

It follows from Theorem 4.1 that system (2.1) admits a unique ergodic stationary distribution. Simulation in Figure 1 can confirm this.

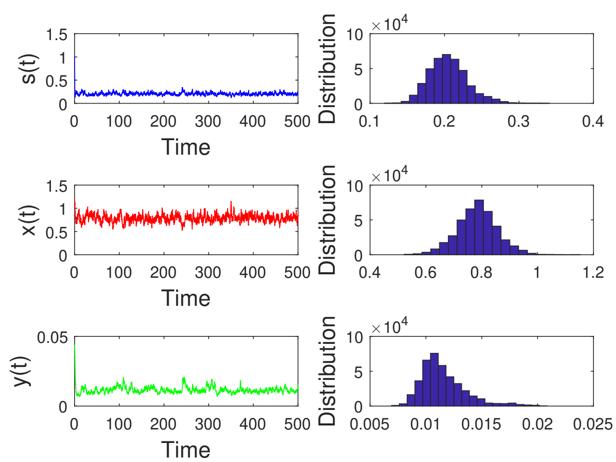

Figure 1. The pictures on the left are the solutions of system (2.1). The pictures on the right are the density functions of system (2.1) with initial value $(s(0), x(0), y(0)) = (1, 1, 0.04)$, $\sigma_1^2 = 0.0225$, $\sigma_2^2 = 0.01$, $\sigma_3^2 = 0.01$, $q = 0.005$. (Color figure online)

Example 6.2. Let the environmental noise intensities $\sigma_1^2 = 0.0225,\ \sigma_2^2 = 0.9025,\ \sigma_3^2 = 0.01$ and $q = 0.5$. Simple calculations show that

$$\limsup_{t\to\infty} \frac{\log x(t)}{t} \le m_1(1-q)\int_0^\infty \frac{z\nu(z)}{a_1+z}dz - \left(1 + \frac{\sigma_2^2}{2}\right) = -0.6473 < 0 \quad a.s.,$$

and

$$\liminf_{t\to\infty}\langle y\rangle_t \geq \frac{1}{m_2^2}\left[m_2\int_0^\infty \frac{z\nu(z)}{a_2+z}dz - \left(1+\frac{\sigma_3^2}{2}\right)\right] = 0.2233 > 0 \quad a.s.$$

By Theorem 5.1, we can conclude that plasmid-bearing organisms will become extinct and plasmid-free organisms will survive, which is supported by Figure 2.

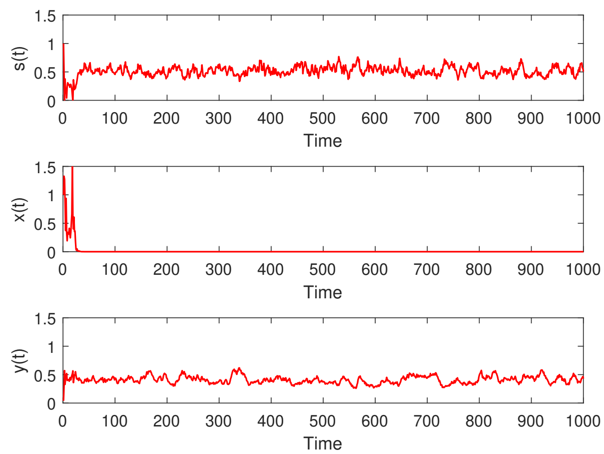

Figure 2. The solutions of system (2.1) with initial value $(s(0), x(0), y(0)) = (1, 1, 0.04)$, $\sigma_1^2 = 0.0225$, $\sigma_2^2 = 0.9025$, $\sigma_3^2 = 0.01$, $q = 0.5$. (Color figure online)

Some interesting topics deserve further consideration. Notice that some scholars [4,13] have studied the dynamics of stochastic chemostat models with pulsed input. Next, we will investigate the effects of impulsive perturbations on system (1.2) and find the optimal period of impulsive input. In addition, the chemostat is inevitably affected by temperature, humidity or illumination. At the micro level, the system continuously experiences a transition from one state to another. It is interesting to study model (1.2) with regime switching.

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