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THIRD ORDER BOUNDARY VALUE PROBLEM WITH FINITE SPECTRUM ON TIME SCALES*

Jijun Ao[1,†] and Nana Liu[1]

Abstract The eigenvalues of a class of third order boundary value problem on time scales is investigated. It is shown that this kind of third order boundary value problem has finite number of eigenvalues, and the same results on time scales are previously known only for even order cases. It can be illustrated that the number of eigenvalues depend on the partition of the time scale and the order of the equation.

Keywords Third order boundary value problems, time scales, eigenvalues, finite spectrum.

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1. Introduction

We consider the third order boundary value problem (BVP) on time scales which is governed by the following equation

$$(py^{\Delta\Delta})^{\Delta} + qy^{\sigma} = \lambda wy^{\sigma}, \quad t \in \mathbb{T}, \tag{1.1}$$

together with the boundary condition (BC)

$$AY(a) + BY(e) = 0, \quad Y = (y \;\; y^{\Delta} \;\; py^{\Delta\Delta})^T, \quad A,\ B \in M_3(\mathbb{C}), \tag{1.2}$$

where σ denotes the forward-jump operator, Δ denotes the delta derivative, $M_3(\mathbb{C})$ denotes the set of square matrices of order 3 over the complex numbers $\mathbb{C}$, and λ is the spectral parameter. The coefficients satisfy the conditions

$$r = \frac{1}{p},\ q,\ w \in C_{prd}(\mathbb{T}), \tag{1.3}$$

where C_{prd} denotes the set of all prd-continuous functions on $\mathbb{T}$ [6].

Let $y_1 = y, \;\; y_2 = y^{\Delta}, \;\; y_3 = py^{\Delta\Delta}$, then ones have the system representation of (1.1) as follows:

$$y_1^{\Delta} = y_2,\ y_2^{\Delta} = ry_3,\ y_3^{\Delta} = (\lambda w - q)y_1^{\sigma},\ on\ \ \mathbb{T}. \tag{1.4}$$

†The corresponding author. Email address:george_ao78@sohu.com(J. Ao)
1College of Sciences, Inner Mongolia University of Technology, Hohhot, Inner Mongolia, 010051, China
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According to the classical Sturm-Liouville theory, the spectrum of a regular or singular, self-adjoint Sturm-Liouville problem (SLP) is unbounded and therefore infinite [22]. In 1964, Atkinson in his well known book [7] suggested that if the coefficients of SLP satisfies some special conditions, the problem may have finite eigenvalues. In 2001 Kong et al. confirmed the rationality of Atkinson's judgement [12]. They demonstrated that a certain class of SLP has finite number of eigenvalues. Recently, Ao et al. generalized the finite spectrum results to the fourth order BVPs [3, 4], and even the 2nth order BVPs [5].

It is well known, the odd order BVPs have wide applications, especially the first order and the third order differential equations. There are many achievements on the odd order BVPs. In 2002, Everitt and Poulkou discussed about kramer analytic kernels and first-order BVPs [9]. In [2], Ao discussed two classes of third order BVPs with finite spectrum. For more achievements on third-order BVPs can be found in [10, 13, 16–21].

With the development of the long research history, people hope to find a new theoretical framework which can combine discrete and continuous equations. In this case, Stefan Hilger, a German mathematician, proposed the concept of time scales in his doctoral dissertation in 1988. Then the studies about time scales have been made in last three decades. In 1999, Agarwal et al. discussed the existence and characteristics of the eigenvalues of the second-order SLP on time scales with $p = 1$ [1]. Then Kong generalized their conclusions to the general SLP with the separated BCs and discussed the dependence of eigenvalues on the problem in [11]. The corresponding research achievements on time scales can be found in [8, 14, 15]. In 2018, Ao and Wang discussed the eigenvalues of SLPs with distribution potentials on time scales [6].

However, these results are only restricted into even order cases, and there is no such finite spectrum results for odd order problems on time scales. In this paper, a class of third order BVP with finite spectrum on time scales is studied. Following the method of [2], by partitioning the bounded time scale such that the coefficients of the equation satisfying certain conditions, we construct a class of third order BVP which has exactly finite number of eigenvalues.

The paper is organized as follows: Following this Introduction, Section 2 presents the main results of the finite spectrum of the third order BVP on time scales. The proofs of the main results are given in Section 3.

2. Finite spectrum of the problem

Lemma 2.1. *Equation* (1.1) *is equivalent to the following form*

$$Y^{\Delta} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \frac{1}{p(t)} \\ \lambda w(t) - q(t) & 0 & 0 \end{pmatrix} \begin{pmatrix} y_1^{\sigma} \\ y_2 \\ y_3 \end{pmatrix}, \qquad Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix},$$

or

$$Y^{\Delta} = A(t)Y, \tag{2.1}$$

where

$$A(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \frac{1}{p(t)} \\ \lambda w(t) - q(t) & \mu(t)(\lambda w(t) - q(t)) & 0 \end{pmatrix}.$$

Proof. It can be proved by direct calculation. □

Lemma 2.2. *$\forall t_0 \in [a,b]_{\mathbb{T}}$, $A \in C_{rd}$ is $n \times n$ functional matrix, and $\forall t \in [a,t_0]_{\mathbb{T}}$, $I + \mu(t)A(t)$ is invertible matrix, then the initial value problem*

$$Y^{\Delta} = A(t)Y, \quad Y(t_0) = Y_0, \quad Y_0 \in \mathbb{C}^n,$$

has unique solution $Y \in C^1_{rd}$. It can be assumed that $\Phi(t,\lambda) = [\phi_{ij}(t,\lambda)]$, $t \in [a,b]_{\mathbb{T}}$ is the fundamental matrix solution of equation (2.1) satisfied with the initial condition $\Phi(a,\lambda) = I$, where

$$\Phi(t,\lambda) = \begin{pmatrix} \varphi_1(t,\lambda) & \varphi_2(t,\lambda) & \varphi_3(t,\lambda) \\ (\varphi_1^{\Delta})(t,\lambda) & (\varphi_2^{\Delta})(t,\lambda) & (\varphi_3^{\Delta})(t,\lambda) \\ (p\varphi_1^{\Delta\Delta})(t,\lambda) & (p\varphi_2^{\Delta\Delta})(t,\lambda) & (p\varphi_3^{\Delta\Delta})(t,\lambda) \end{pmatrix}.$$

Proof. See [8]. □

Lemma 2.3. *Let (1.3) holds and $\Phi(t,\lambda) = \phi_{ij}(t,\lambda), t \in [a,b]_{\mathbb{T}}$ is the fundamental matrix solution of equation (2.1) determined by the initial condition $\Phi(a,\lambda) = I$, then $\lambda \in \mathbb{C}$ is the eigenvalue of BVP (1.1), (1.2) if and only if $\Delta(\lambda) = 0$. Furthermore, the characteristic function $\Delta(\lambda) = det[A + B\Phi(b,\lambda)]$ can be written as*

$$\Delta(\lambda) = \mathbf{det}(A) + \mathbf{det}(B) + \sum_{i=1}^{3}\sum_{j=1}^{3} c_{ij}\phi_{ij} + \sum_{1\le i,j,k,l\le 3,\ j\neq l} d_{ijkl}\phi_{ij}\phi_{kl}, \tag{2.2}$$

where $c_{ij}, 1 \le i,j \le 3$, $d_{ijkl}, 1 \le i,j,k,l \le 3, j \neq l$ are some constants depending on the matrices A and B.

Proof. This follows from a tedious but straightforward computation. □

The third order problem (1.1) and (1.2), or equivalently (1.4) and (1.2), is said to be degenerate if in (2.2) either $\Delta(\lambda) \equiv 0$ for all $\lambda \in \mathbb{C}$ or $\Delta(\lambda) \neq 0$ for every $\lambda \in \mathbb{C}$ [2].

Assume (1.1) is defined on $\mathbb{T} = [a,b] \cup \{c\} \cup [d,e]$ with $-\infty < a < b < c < d < e < +\infty$ and there exists a partition of the intervals of the time scale $\mathbb{T}$

$$\begin{aligned} a = a_0 < a_1 < a_2 < \cdots < a_{2m} < a_{2m+1} = b, \\ d = b_0 < b_1 < b_2 < \cdots < b_{2n} < b_{2n+1} = e, \end{aligned} \tag{2.3}$$

for some positive integers m, n, so that

$$
\begin{aligned}
&r(t) = \frac{1}{p(t)} = 0 \ on \ [a_{2k}, a_{2k+1}], \ \int_{a_{2k}}^{a_{2k+1}} w(t)dt \neq 0, \\
&\int_{a_{2k}}^{a_{2k+1}} w(t)tdt \neq 0, \ k = 0, 1, \ldots, m, \\
&q(t) = w(t) = 0 \ on \ [a_{2k+1}, a_{2k+2}], \ \int_{a_{2k+1}}^{a_{2k+2}} r(t)dt \neq 0, \\
&\int_{a_{2k+1}}^{a_{2k+2}} r(t)tdt \neq 0, \ k = 0, 1, \ldots, m-1,
\end{aligned} \tag{2.4}
$$

and

$$
\begin{aligned}
&r(t) = \frac{1}{p(t)} = 0 \ on \ [b_{2i}, b_{2i+1}], \ \int_{b_{2i}}^{b_{2i+1}} w(t)dt \neq 0, \\
&\int_{b_{2i}}^{b_{2i+1}} w(t)tdt \neq 0, \ i = 0, 1, \ldots, n, \\
&q(t) = w(t) = 0 \ on \ [b_{2i+1}, b_{2i+2}], \ \int_{b_{2i+1}}^{b_{2i+2}} r(t)dt \neq 0, \\
&\int_{b_{2i+1}}^{b_{2i+2}} r(t)tdt \neq 0, \ i = 0, 1, \ldots, n-1.
\end{aligned} \tag{2.5}
$$

To show our main result the following notations are needed. Given (2.3)-(2.5), let

$$
\begin{aligned}
r_k &= \int_{a_{2k+1}}^{a_{2k+2}} r(t)dt, \quad \hat{r}_k = \int_{a_{2k+1}}^{a_{2k+2}} r(t)tdt, \quad k = 0, 1, \ldots, m-1; \\
q_k &= \int_{a_{2k}}^{a_{2k+1}} q(t)dt, \quad \hat{q}_k = \int_{a_{2k}}^{a_{2k+1}} q(t)tdt, \quad k = 0, 1, \ldots, m; \\
w_k &= \int_{a_{2k}}^{a_{2k+1}} w(t)dt, \quad \hat{w}_k = \int_{a_{2k}}^{a_{2k+1}} w(t)tdt, \quad k = 0, 1, \ldots, m; \\
\tilde{r}_i &= \int_{b_{2i+1}}^{b_{2i+2}} r(t)dt, \quad \check{r}_i = \int_{b_{2i+1}}^{b_{2i+2}} r(t)tdt, \quad i = 0, 1, \ldots, n-1; \\
\tilde{q}_i &= \int_{b_{2i}}^{b_{2i+1}} q(t)dt, \quad \check{q}_i = \int_{b_{2i}}^{b_{2i+1}} q(t)tdt, \quad i = 0, 1, \ldots, n; \\
\tilde{w}_i &= \int_{b_{2i}}^{b_{2i+1}} w(t)dt, \quad \check{w}_i = \int_{b_{2i}}^{b_{2i+1}} w(t)tdt, \quad i = 0, 1, \ldots, n.
\end{aligned}
$$

Then we can state the main theorem of this paper.

Theorem 2.1. *Let $m, n \in \mathbb{N}$, and let* (2.3)-(2.5) *hold. Then the third order BVP* (1.1), (1.2) *has at most $2m + 2n + 3$ eigenvalues.*

Proof. This is given in the next section. □

3. Proof of main result

In this section, we will prove Theorem 2.1. Before giving the detailed proof, some additional lemmas are needed.

Lemma 3.1. *Let* (1.3), (2.3)-(2.5) *hold,* $\Phi(t,\lambda) = [\phi_{ij}(t,\lambda)]$ *be the fundamental matrix solution of the system* (1.4) *determined by the initial condition* $\Phi(a,\lambda) = I$ *for each* $\lambda \in \mathbb{C}$. *Let*

$$F_k(t,\lambda,a_k) = \begin{pmatrix} 1 & t-a_k & 0 \\ 0 & 1 & 0 \\ \int_{a_k}^t (\lambda w - q)\Delta x & \int_{a_k}^t (\lambda w - q)(x-a_k)\Delta x & 1 \end{pmatrix}, \quad k = 0, 2, \ldots, 2m;$$

$$F_k(t,\lambda,a_k) = \begin{pmatrix} 1 & t-a_k & \int_{a_k}^t r(t-x)\Delta x \\ 0 & 1 & \int_{a_k}^t r\Delta x \\ 0 & 0 & 1 \end{pmatrix}, \quad k = 1, 3, \ldots, 2m-1.$$

Then for $1 \le k \le 2m+1$ *we have*

$$\Phi(a_k,\lambda) = F_{k-1}(a_k,\lambda,a_{k-1})\,\Phi(a_{k-1},\lambda).$$

If we let

$$T_0 = F_0(a_1,\lambda,a_0),\ T_k = F_{2k}(a_{2k+1},\lambda,a_{2k})\,F_{2k-1}(a_{2k},\lambda,a_{2k-1}),\ k = 1, 2, \ldots, m,$$

then

$$\Phi(a_1,\lambda) = F_0(a_1,\lambda,a_0) = T_0, \quad \Phi(a_{2k+1},\lambda) = T_k\,\Phi(a_{2k-1},\lambda),\ k = 1, 2, \ldots, m.$$

Hence the following formula can be obtained

$$\Phi(a_{2k+1},\lambda) = T_k T_{k-1} \cdots T_0, \quad k = 0, 1, \ldots, m.$$

Proof. See [2]. □

Lemma 3.2. *Let* (1.3), (2.3)-(2.5) *hold,* $\Psi(t,\lambda) = [\psi_{ij}(t,\lambda)]$ *be the fundamental matrix solution of the system* (1.4) *determined by the initial condition* $\Psi(d,\lambda) = I$ *for each* $\lambda \in \mathbb{C}$. *Let*

$$\tilde{F}_i(t,\lambda,b_i) = \begin{pmatrix} 1 & t-b_i & 0 \\ 0 & 1 & 0 \\ \int_{b_i}^t (\lambda w - q)\Delta x & \int_{b_i}^t (\lambda w - q)(x-b_i)\Delta x & 1 \end{pmatrix}, \quad i = 0, 2, \ldots, 2n;$$

$$\tilde{F}_i(t,\lambda,b_i) = \begin{pmatrix} 1 & t-b_i & \int_{b_i}^t r(t-x)\Delta x \\ 0 & 1 & \int_{b_i}^t r\Delta x \\ 0 & 0 & 1 \end{pmatrix}, \quad i = 1, 3, \ldots, 2n-1.$$

Then for $1 \le i \le 2n+1$ *we have*

$$\Psi(b_i,\lambda) = \tilde{F}_{i-1}(b_i,\lambda,b_{i-1})\,\Psi(b_{i-1},\lambda).$$

Still let

$$\tilde{T}_0 = \tilde{F}_0(b_1,\lambda,b_0),\ \tilde{T}_i = \tilde{F}_{2i}(b_{2i+1},\lambda,b_{2i})\,\tilde{F}_{2i-1}(b_{2i},\lambda,b_{2i-1}),\ i = 1, 2, \ldots, n,$$

then

$$\Psi(b_1,\lambda)=\tilde{F}_0(b_1,\lambda,b_0)=\tilde{T}_0,\quad \Psi(b_{2i+1},\lambda)=\tilde{T}_i\,\Psi(b_{2i-1},\lambda),\ i=1,2,\dots,n.$$

Hence the following formula can be obtained

$$\Psi(b_{2i+1},\lambda)=\tilde{T}_i\tilde{T}_{i-1}\cdots\tilde{T}_0,\quad i=0,1,\dots,n.$$

Proof. The proof is similar to the Lemma 3.1. □

Lemma 3.3. *Let (2.3)-(2.5) hold.* $\Phi(t,\lambda),\Psi(t,\lambda)$ *be defined as in the Lemma 3.1 and Lemma 3.2 respectively, then*

$$\Phi(e,\lambda)=\Psi(e,\lambda)N(\lambda)\Phi(b,\lambda),\tag{3.1}$$

where $N(\lambda)=N_2(\lambda)N_1(\lambda)$, *and*

$$N_1(\lambda)=\begin{pmatrix}1 & c-b & 0\\ 0 & 1 & (c-b)r(c)\\ (c-b)[\lambda w(b)-q(b)] & (c-b)^2[\lambda w(b)-q(b)] & 1\end{pmatrix},$$

$$N_2(\lambda)=\begin{pmatrix}1 & d-c & 0\\ 0 & 1 & (d-c)r(d)\\ (d-c)[\lambda w(c)-q(c)] & (d-c)^2[\lambda w(c)-q(c)] & 1\end{pmatrix}.$$

Proof. From (1.4) and Definition 3 in [6], we know that

$$y^{\Delta}(b)=\frac{y(c)-y(b)}{c-b},\qquad y^{\Delta\Delta}(b)=\frac{y^{\Delta}(c)-y^{\Delta}(b)}{c-b},\tag{3.2}$$

$$(py^{\Delta\Delta})^{\Delta}(b)=\frac{(py^{\Delta\Delta})(c)-(py^{\Delta\Delta})(b)}{c-b},\quad (py^{\Delta\Delta})^{\Delta}(b)=[\lambda w(b)-q(b)]y(c).\tag{3.3}$$

Calculate from (3.2), (3.3) we have that

$$Y(c)=N_1(\lambda)Y(b),$$

where

$$N_1(\lambda)=\begin{pmatrix}1 & c-b & 0\\ 0 & 1 & (c-b)r(c)\\ (c-b)[\lambda w(b)-q(b)] & (c-b)^2[\lambda w(b)-q(b)] & 1\end{pmatrix}.$$

Similarly, we have

$$Y(d)=N_2(\lambda)Y(c),$$

where

$$N_2(\lambda) = \begin{pmatrix} 1 & d-c & 0 \\ 0 & 1 & (d-c)r(d) \\ (d-c)[\lambda w(c) - q(c)] & (d-c)^2[\lambda w(c) - q(c)] & 1 \end{pmatrix}.$$

Also because

$$Y(b) = \Phi(b,\lambda)Y(a), \qquad Y(e) = \Psi(e,\lambda)Y(d),$$

then $Y(e) = \Psi(e,\lambda)N(\lambda)\Phi(b,\lambda)Y(a)$, and $Y(e) = \Phi(e,\lambda)Y(a)$.

From

$$det(I + \mu(b)A(b)) \neq 0,$$

and Lemma 2.2 we have

$$\Phi(e) = \Psi(e,\lambda)N(\lambda)\Phi(b,\lambda).$$

The proof is completed. □

Corollary 3.1. *Let* $N(\lambda) = \begin{pmatrix} 1 & d-b & n_{13} \\ n_{21}(\lambda) & n_{22}(\lambda) & n_{23} \\ n_{31}(\lambda) & n_{32}(\lambda) & n_{33}(\lambda) \end{pmatrix}$, *for the fundamental matrix* Φ *one has*

$$\begin{aligned} \phi_{ij}(e,\lambda) &= H_{ij}\lambda^{m+n+1} + \tilde{\phi}_{ij}(\lambda), \quad i,j = 1,2, \quad or \quad i = j = 3; \\ \phi_{ij}(e,\lambda) &= H_{ij}\lambda^{m+n+2} + \tilde{\phi}_{ij}(\lambda), \quad i = 3, \quad j = 1,2; \\ \phi_{ij}(e,\lambda) &= H_{ij}\lambda^{m+n} + \tilde{\phi}_{ij}(\lambda), \quad i = 1,2, \quad j = 3, \end{aligned} \tag{3.4}$$

where H_{ij} *are some constants related to* $r_k, \hat{r}_k, k = 0,1,\ldots,m-1,\quad w_k, \hat{w}_k, k = 0,1\ldots,m, \tilde{r}_i, \check{r}_i, i = 0,1,\ldots,n-1,\quad \tilde{w}_i, \check{w}_i, i = 0,1,\ldots,n$ *and the end points* $b,\ c,\ d,\ e,\ \tilde{\phi}_{ij}(\lambda)$ *are functions of* λ*, in which the degrees of* λ *are smaller then* $m+n+1, m+n+2,$ *or* $m+n$ *respectively.*

Now the proof of Theorem 2.1 can be stated as follow.

Proof of Theorem 2.1. Since $\Delta(\lambda) = \mathbf{det}[A + B\Phi(e,\lambda)]$, where $\Phi(e,\lambda) = [\phi_{ij}(e,\lambda)]$. From Lemma 2.3 and Corollary 3.1 we know that the characteristic function $\Delta(\lambda)$ is a polynomial of λ and has the form of (2.2). We denote the maximum of degree of λ in $\phi_{ij}(e,\lambda)$ by $d_{ij},\ 1 \leq i,j \leq 3$, by Corollary 3.1 the maximum of degree of λ in the matrix $\Phi(e,\lambda)$ can be written as the following matrix

$$(d_{ij}) = \begin{pmatrix} m+n+1 & m+n+1 & m+n \\ m+n+1 & m+n+1 & m+n \\ m+n+2 & m+n+2 & m+n+1 \end{pmatrix}. \tag{3.5}$$

In term of (2.2) and (3.5), we conclude that the maximum of the degree of λ in $\Delta(\lambda)$ is $2m+2n+3$. Thus from the Fundamental Theorem of Algebra, $\Delta(\lambda)$ has at most $2m+2n+3$ roots. □

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