INTERACTIONS OF DELTA SHOCK WAVES FOR A CLASS OF NONSTRICTLY HYPERBOLIC SYSTEM OF CONSERVATION LAWS*

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Abstract In this paper, we study the perturbed Riemann problem for a class of nonstrictly hyperbolic system of conservation laws, and focuse on the interactions of delta shock waves with the shock waves and the rarefaction waves. The global solutions are constructed completely with the method of splitting delta function. In solutions, we find a new kind of nonclassical wave, which is called delta contact discontinuity with Dirac delta function in both components. It is quite different from the previous ones on which only one state variable contains the Dirac delta function. Moreover, by letting perturbed parameter ε tend to zero, we analyze the stability of Riemann solutions.

Keywords Nonstrictly hyperbolic system, wave interaction, delta shock waves, delta contact discontinuity, split delta function.

MSC(2010) 35L65, 35L67.

1. Introduction

In the past over two decades, the investigation for delta shock waves has been an increasingly active topic on the study of conservation laws. As a generalization of an ordinary shock wave, the delta shock wave is a new kind of discontinuity, on which at least one of the state variables may develop an extreme concentration in the form of a weighted Dirac delta function with the discontinuity as its support. Physically, people often use the delta shock wave to represent the process of concentration of the mass [2], or express the galaxies in the universe [22, 30].

For delta shock waves, there are numerous excellent papers for various hyperbolic systems. In 1977, Korchinski first [13] used generalized delta-functions to construct Riemann solution and called it overcompressive singular shock. Afterwards,

more and more scholars started to concern about the existence and uniqueness of nonclassical solutions involving delta functions developing in the state variables. For instance, Keyfitz and Kranzer [12, 14] introduced the singular shock wave. Le Floch [8] introduced the measure solutions. Tan and Zhang [27] independently discovered the delta shock wave. Sheng and Zhang [24] discussed the Riemann problem for the zero-pressure flow. Huang and Wang [11] proved the uniqueness of the weak solution when the initial data are a Radon measure for the zero-pressure flow. Yang [31] solved the Riemann problem for the generalized zero-pressure flow. See also the results in [1, 4, 6, 20, 28] and the references cited therein.

However, it is observed that these investigations on delta shock waves mentioned above have mostly been focused on the case that only one state variable develops the Dirac delta function and the others have bounded variations. Different from these ones, the theory of delta shock waves with Dirac delta functions developing in both state variables has been established by Yang and Zhang [32] for the following class of nonstrictly hyperbolic system of conservation laws

$$
\begin{cases}
u_t + (\phi(u, v)u)_x = 0, \\
v_t + (\phi(u, v)v)_x = 0,
\end{cases}
\tag{1.1}
$$

where $\phi(u, v)$ satisfies the following assumption:

($\mathbf{H_1}$) $\phi = \phi(r)$ is a given smooth function of $r = au + bv$ satisfying $a^2 + b^2 \neq 0$, a and b are constants.

The Riemann problem for system (1.1) with initial data

$$
(u, v)(0, x) = (u_\pm, v_\pm) \quad (\pm x > 0) \tag{1.2}
$$

was constructively solved. In solutions, a special kind of delta shock waves on which both state variables simultaneously contain the Dirac delta functions was found. Moreover, when $\phi(u, v)$ is a smooth function satisfying $\phi(u, v) = \phi(\alpha u, \alpha v)$, $\alpha > 0$ is a constant, this kind of delta shock waves also appears in [33]. More on the theory of delta shock waves with Dirac delta functions developing in multiple state variables, see also [17–19, 35], etc.

When the system (1.1) satisfies assumption ($\mathbf{H_1}$), many important systems can be obtained with assignment for $\phi(r)$, a and b. For example, if $\phi(au + bv) = \frac{1}{2}u$, it is reduced to the system investigated in [13]; while taking $\phi(au + bv) = u$, it becomes one dimensional transport equations studied in [23, 28]. Moreover, if $\phi(au + bv) = 1 + \frac{1}{1-u+v}$ or $1 + \frac{1}{1+v}$, it corresponds to the nonlinear chromatography equation investigated in [5, 9, 34]. Letting $\phi(au + bv) = \frac{1}{1-u+v}$ or $\frac{1}{1+v}$, it is just another form of nonlinear chromatography system considered in [25, 26, 29]. For these special models, the Riemann solutions are constructed completely. The delta shock waves occur in these systems, but they are different. Only one state variable contains the Dirac delta function in [13, 23, 25, 26, 28, 34], while both state variables simultaneously develop Dirac delta functions in [5, 9, 29]. In addition, there are still many other examples, and we will not list them one by one.

For these specific systems, besides the investigation of delta-shock solution, the interaction of delta shock waves is also a very interesting topic. It has significance in the general mathematical theory, numerical calculation and practical applications of quasi-linear hyperbolic equations. For system (1.1), when $\phi(au + bv) = u$, Shen and Sun [23] discussed the interactions of the delta shock waves with the shock

waves and rarefaction waves when the initial data are three piece constant states. When $\phi(au + bv) = 1 + \frac{1}{1-u+v}$ and $1 + \frac{1}{1+v}$, Guo et al. [9] and Zhang [34] consider the perturbed Riemann problem. When $\phi(au + bv) = \frac{1}{1+v}$, Sun [26] studied the Riemann problem with initial data of three piece constant states and constructed the global structures of solutions completely.

Motivated by [9, 23, 26, 34], in which they studied the interaction of delta shock waves and elementary waves by splitting delta function, we in this paper aim to studying the interactions of delta shock waves for the generalized system (1.1). To this end, we consider (1.1) with the initial data of three piecewise constant states as follows

$$(u,v)(0,x) = \begin{cases} (u_-, v_-), & -\infty < x < -\varepsilon, \\ (u_m, v_m), & -\varepsilon < x < \varepsilon, \\ (u_+, v_+), & \varepsilon < x < +\infty, \end{cases} \tag{1.3}$$

where u_i, v_i ($i = \pm, m$) are arbitrary constants and ε is arbitrarily small. Obviously, the initial data (1.3) is a local perturbation of the Riemann initial data (1.2).

As delta shock waves interact with the other elementary waves, it will give rise to the product of $\delta(x)$ and $H(x)$. So we use the method of splitting delta function along a regular curve in R_+^2, which is proposed by Nedeljkov and Oberguggenberger [15, 16], to study the Riemann problem (1.1) and (1.3). Benefited from this method, the product of the piecewise smooth function and discontinuity along such a curve makes sense. This method has been applied to the various systems to investigate the interactions of delta shock waves, such as [9, 10, 21, 23, 26, 34, 36]. Specially, when delta shock wave interact with the rarefaction wave, using the wavefront tracking algorithm [3], we approximate the rarefaction wave by a set of small non-admissible shocks.

After attentive analysis of interaction between delta shock wave and elementary wave, we find that the delta contact discontinuity appears in solutions, on which both state variables simultaneously contain the Dirac delta functions. Generally speaking, the delta contact discontinuity is a kind of nonclassical wave with at least one Dirac delta function supported on it. The delta function propagates along the line of the contact discontinuity and thereby the propagation speed and strength do not change as it travels in space. This kind of discontinuity also appears in [16, 23, 26, 34], etc. In particular, the delta contact discontinuity with both state variables simultaneously containing the Dirac delta functions was also found in [9] for the nonlinear chromatography equations. However, the system in [9] is specific and can be included in (1.1).

As dealing with the Riemann problem (1.1) and (1.3), the expression of the Riemann solution can not be concretely and explicitly formulated, here we use some new ideas and skills to obtain the existence and uniqueness of Riemann solutions qualitatively and abstractly. Moreover, it can be established that the solutions of the perturbed Riemann problem converge to nothing but the corresponding Riemann solutions as $\varepsilon \to 0$, from which the stability of the Riemann solutions with respect to this local small perturbations of the Riemann initial data is obtained.

It is not hard to find that the solutions we obtained presents generality, because they contain some exact analytic solutions that are given in papers [9, 23, 26, 34]. That is, this paper to some extent extends their results and proofs. Therefore, we

get more abroad and more common results, which is main objective and innovation of this paper.

This paper is arranged as follows. In Section 2, the Riemann solutions of (1.1) with two piecewise constant states is reviewed. Section 3 discusses the interactions of the delta shock waves and classical waves. The Riemann solutions of (1.1), (1.3) are constructed globally. By taking $\varepsilon \to 0$, the stability of Riemann solutions of (1.1), (1.3) is analyzed.

2. Riemann solutions

In this section, we briefly review the Riemann solutions of (1.1) and (1.2) under the condition

$$\phi_r > 0, \quad (r\phi)_{rr} > 0, \quad \phi(0) = 0. \tag{2.1}$$

The detailed study can be found in [32].

System (1.1) has two eigenvalues $\lambda_1 = \phi$ and $\lambda_2 = \phi + r\phi_r$ with corresponding right eigenvectors $\vec{r}_1 = (b, -a)^T$ and $\vec{r}_2 = (u, v)^T$. Thus it is non-strictly hyperbolic and the set of umbilical points, on which the strictly hyperbolicity fails, is $\sum = \{(u, v) | \lambda_1 = \lambda_2\} = \{(u, v) | r\phi_r = 0\}$. Noticing $\nabla\lambda_1 \cdot \vec{r}_1 \equiv 0$ and $\nabla\lambda_2 \cdot \vec{r}_2 = r(r\phi)_{rr}$, so λ_1 is always linearly degenerate, λ_2 is genuinely nonlinear if $r(r\phi)_{rr} \neq 0$ and linearly degenerate if $r(r\phi)_{rr} = 0$.

Besides the constant state solution, the self-similar waves $(u, v)(\xi)(\xi = x/t)$ of the first family are contact discontinuities

$$J: \quad \xi = \phi(r_-) = \phi(r_+), \quad (r_- = r_+),$$

and those of the second family are rarefaction waves

$$R: \begin{cases} \xi = \phi + r\phi_r, \\ \dfrac{u}{v} = \dfrac{u_-}{v_-}, \quad r_- < r, \end{cases}$$

or shock waves

$$S: \begin{cases} \xi = \sigma = \dfrac{r_+\phi(r_+) - r_-\phi(r_-)}{r_+ - r_-}, \\ \dfrac{u_+}{v_+} = \dfrac{u_-}{v_-}, \quad 0 < r < r_- \quad \text{or} \quad r < r_- < 0. \end{cases}$$

For the case $r_- \geqslant 0 \geqslant r_+$, the delta shock wave appears. In order to define the measure solutions, a two-dimensional weighted delta function $\beta(s)\delta_S$ supported on a smooth curve S parameterized as $t = t(s), x = x(s)(c \leqslant s \leqslant d)$ can be introduced as

$$\left\langle \beta(t(s))\delta_S, \varphi(t(s), x(s)) \right\rangle = \int_c^d \beta(t(s))\varphi(t(s), x(s)) \mathrm{d}s \tag{2.2}$$

for all test functions $\varphi \in C_0^\infty((-\infty, +\infty) \times [0, +\infty))$.

With this definition, we introduce a delta-shock solution to construct the solution of (1.1), which can be expressed as

$$u = U(x,t) + b\beta(t)\delta_s, \qquad v = V(x,t) - a\beta(t)\delta_s, \qquad (2.3)$$

where $S = \{(\sigma t, t) : 0 \leqslant t < \infty\}$,

$$U(x,t) = u_- + [u]H(x - \sigma t), \qquad V(x,t) = v_- + [v]H(x - \sigma t),$$

$$\beta(t) = \frac{1}{b}(\sigma[u] - [u\phi(r)])t, \qquad \phi(r)|_{x=\sigma t} = \sigma, \qquad (2.4)$$

in which $[p] = p_+ - p_-$ denotes the jump of function p across the discontinuity, σ the velocity of the delta shock wave, and $H(x)$ the Heaviside function.

As shown in [32], the solution (u,v) constructed above satisfy

$$\langle u, \varphi_t \rangle + \langle \phi u, \varphi_x \rangle = 0,$$

$$\langle v, \varphi_t \rangle + \langle \phi v, \varphi_x \rangle = 0 \qquad (2.5)$$

for all test functions $\varphi \in C_0^\infty\big((-\infty, +\infty) \times [0, +\infty)\big)$, where

$$\langle u, \varphi \rangle = \int_0^{+\infty} \int_{-\infty}^{+\infty} U\varphi \,\mathrm{d}x\mathrm{d}t + \langle b\beta\delta_S, \varphi \rangle,$$

$$\langle \phi u, \varphi \rangle = \int_0^{+\infty} \int_{-\infty}^{+\infty} \phi(aU + bV)U\varphi \,\mathrm{d}x\mathrm{d}t + \langle \sigma b\beta\delta_S, \varphi \rangle,$$

and v has the similar integral identities as above.

Then, a unique delta-shock solution of (1.1) can be constructed as

$$(u,v)(t,x) = \begin{cases} (u_-, v_-)(t,x), & x < x(t), \\ \big(b\beta(t), -a\beta(t)\big)\delta\big(x - x(t)\big), & x = x(t), \\ (u_+, v_+)(t,x), & x > x(t), \end{cases} \qquad (2.6)$$

in which $x(t)$, $\beta(t)$ and σ satisfy the following generalized Rankine-Hugoniot relation

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = \sigma, \\ b\dfrac{\mathrm{d}\beta(t)}{\mathrm{d}t} = \sigma[u] - [u\phi(r)], \\ -a\dfrac{\mathrm{d}\beta(t)}{\mathrm{d}t} = \sigma[v] - [v\phi(r)], \end{cases} \qquad (2.7)$$

and

$$\phi(r)\big|_{x=x(t)} = \sigma. \qquad (2.8)$$

Moreover, the delta shock wave should satisfy the entropy condition:

$$\lambda_2(r_1) \geqslant \lambda_1(r_1) \geqslant \sigma \geqslant \lambda_1(r_2) \geqslant \lambda_2(r_2), \qquad (2.9)$$

which means that all characteristic curves run into the delta shock curve from both sides. Overcompressibility reflects the fact that delta shock wave should arise only from local concentrations of the quantities u and v due to the conservation law.

Under the entropy condition (2.9), solving the generalized Rankine-Hugoniot relation (2.7) and (2.8) with the initial data $x(0) = 0$ and $w(0) = 0$ yields that

$$\begin{cases} \sigma = \frac{r_+\phi(r_+) - r_-\phi(r_-)}{r_+ - r_-}, \\[2mm] x = \sigma t, \\[2mm] \beta(t) = \frac{\phi(r_+) - \phi(r_-)}{r_+ - r_-}(u_+ v_- - u_- v_+)\, t, \\[2mm] \phi(r)|_{x = \sigma t} = \sigma. \end{cases} \quad (2.10)$$

For the convenience of the discussion in the next section, we briefly review the concept of left- and right-hand side delta functions, further details of which can be found in [15, 16].

Let $\overline{R_2^+}$ be divided into two disjoint open sets Ω_1 and Ω_2 with piecewise smooth boundary curve Γ, that is, $\Omega_1 \cap \Omega_2 = 0$ and $\overline{\Omega_1} \cup \overline{\Omega_2} = \overline{R_2^+}$. Let $C(\Omega_i)$ and $M(\Omega_i)$ be the space of bounded and continuous real-valued functions equipped with the L^∞-norm and the space of measures on Ω_i $(i = 1, 2)$, respectively. Suppose $C_\Gamma = (C(\Omega_1), C(\Omega_2))$ and $M_\Gamma = (M(\Omega_1), M(\Omega_2))$, then the product of $G = (G_1, G_2) \in C_\Gamma$ and $D = (D_1, D_2) \in M_\Gamma$, can be defined as an element $GD = (G_1 D_1, G_2 D_2) \in M_\Gamma$, in which $G_i D_i$ can be defined as the usual product of a continuous function and a measure. It is obvious that the above-defined product makes sense.

Every measure on $\overline{\Omega_i}$ can be considered as a measure on $\overline{R_2^+}$ with support in $\overline{\Omega_i}(i = 1, 2)$. Based on this, we can obtain the mapping $m : M_\Gamma \to M(\overline{R_2^+})$ by taking $m(D) = D_1 + D_2$. Similarly, we have $m(GD) = G_1 D_1 + G_2 D_2$.

The solution concept used in this paper can be described as follows: perform all nonlinear operations on functions in the space C_Γ, carry out the multiplication and composition in the space M_Γ and then take the mapping $m : M_\Gamma \to M(\overline{R_2^+})$ before differentiation in the space of distributions, and require that the equation is satisfied in the weak sense of distribution.

Now we rewrite the solution (2.6) by employing the above definition of the split delta function. At this moment, the delta-shock solution of (1.1) and (1.2) can be expressed as

$$\begin{cases} u(x, t) = u_- + (u_+ - u_-)H + b\big(\beta_-(t)D^- + \beta_+(t)D^+\big), \\ v(x, t) = v_- + (v_+ - v_-)H - a\big(\beta_-(t)D^- + \beta_+(t)D^+\big), \end{cases} \quad (2.11)$$

where H is the Heaviside function, $\beta(t)D = \beta_-(t)D^- + \beta_+(t)D^+$ is a split delta function, and all of them are supported by the line $x = \sigma t$. What is more, D^- is the delta measure on the set $\overline{R_+^2} \cap \{(x, t) | x \leqslant \sigma t\}$, and D^+ on the set $\overline{R_+^2} \cap \{(x, t) | x \geqslant \sigma t\}$, $\beta_-(t)$ and $\beta_+(t)$ are to be determined.

From (2.11), we can compute that

$$u_t(x, t) = \big(-\sigma(u_+ - u_-) + b\beta'_-(t) + b\beta'_+(t)\big)\delta - b\sigma(\beta_-(t) + \beta_+(t))\delta', \quad (2.12)$$

$$(\phi(r)u)_x = (\phi(r_+)u_+ - \phi(r_-)u_-)\delta + b(\phi(r_-)\beta_-(t) + \phi(r_+)\beta_+(t))\delta'. \quad (2.13)$$

Substituting the above two equations into the first equation of (1.1) yields that

$$- \sigma(u_+ - u_-) + b\beta'_-(t) + b\beta'_+(t) + \phi(r_+)u_+ - \phi(r_-)u_- = 0, \qquad (2.14)$$

$$- b\sigma\big(\beta_-(t) + \beta_+(t)\big) + b\phi(r_-)\beta_-(t) + b\phi(r_+)\beta_+(t) = 0. \qquad (2.15)$$

By calculation, we get

$$\beta_-(t) = \frac{r_-(u_+r_- - u_-r_+)(\phi(r_-) - \phi(r_+))}{b(r_+ - r_-)^2}t, \qquad (2.16)$$

and

$$\beta_+(t) = \frac{r_+(u_+r_- - u_-r_+)(\phi(r_+) - \phi(r_-))}{b(r_+ - r_-)^2}t, \qquad (2.17)$$

in which

$$\frac{1}{b}(u_+r_- - u_-r_+) = \frac{1}{b}(u_+(au_- + bv_-) - u_-(au_+ + bv_+)) = u_+v_- - u_-v_+$$

since $r = au + bv$. Then, we have

$$\beta_-(t) = \frac{r_-(u_+v_- - u_-v_+)(\phi(r_-) - \phi(r_+))}{(r_+ - r_-)^2}t, \qquad (2.18)$$

$$\beta_+(t) = \frac{r_+(u_+v_- - u_-v_+)(\phi(r_+) - \phi(r_-))}{(r_+ - r_-)^2}t. \qquad (2.19)$$

It is easy to see that $\beta(t) = \beta_-(t) + \beta_+(t)$. Similarly, from (2.11) and the second equation of (1.1), the same results can be obtained.

Using classical waves and delta shock waves, we now construct the solutions of the Riemann problem (1.1) and (1.2) as follows:

(a) when $r_+ < r_- < 0$, the solution is $\overleftarrow{S} + J$;

(b) when $r_- < r_+ \leqslant 0$, the solution is $\overleftarrow{R} + J$;

(c) when $r_- < 0 < r_+$, the solution is $\overleftarrow{R} + \overrightarrow{R}$;

(d) when $r_+ > r_- \geqslant 0$, the solution is $J + \overrightarrow{R}$;

(e) when $r_- > r_+ > 0$, the solution is $J + \overrightarrow{S}$.

(f) when $r_- \geqslant 0 \geqslant r_+$, the solution is a delta shock wave.

3. Interactions of delta shock waves

In this section, we investigate the Riemann problem (1.1) and (1.3). Considering that the problem is classical when the delta shock wave does not appear in the interaction process, so in the presented paper, we mainly focus on the interactions of waves involving delta shock waves. In order to cover all the cases completely, we discuss this problem case by case according to the relations among r_-, r_m and r_+.

Case 1. $r_- > 0 = r_m > r_+$.

In this case, two delta shock waves δ_1 and δ_2 will emit from $(-\varepsilon, 0)$ and $(\varepsilon, 0)$ respectively, as shown in Fig. 1, where (i) means (u_i, v_i). The propagating speeds of

them are $\sigma_1 = \phi(r_-)$ and $\sigma_2 = \phi(r_+)$, respectively. It is clear that δ_1 will overtake δ_2 at a finite time because $\sigma_1 > \sigma_2$. The intersection (x_1, t_1) is determined by

$$\begin{cases} x_1 + \varepsilon = \sigma_1 t_1, \\ x_1 - \varepsilon = \sigma_2 t_1. \end{cases} \tag{3.1}$$

By simple calculation, we get

$$(x_1, t_1) = \left(\frac{\phi(r_-) + \phi(r_+)}{\phi(r_-) - \phi(r_+)} \, \varepsilon, \ \frac{2\varepsilon}{\phi(r_-) - \phi(r_+)} \right). \tag{3.2}$$
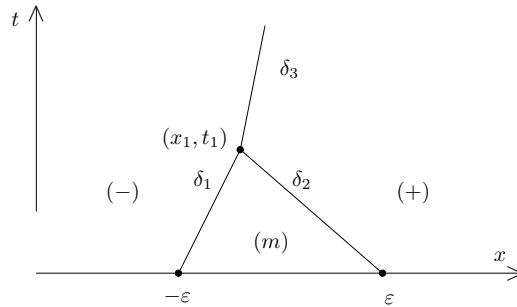

Figure 1. $r_- > 0 = r_m > r_+$

At the intersection (x_1, t_1), a new Riemann problem for (1.1) with initial data

$$u|_{t=t_1} = \begin{cases} u_-, & x < x_1 \\ u_+, & x > x_1 \end{cases} + b\alpha(t_1)\delta(x_1, t_1) \tag{3.3}$$

and

$$v|_{t=t_1} = \begin{cases} v_-, & x < x_1 \\ v_+, & x > x_1 \end{cases} - a\alpha(t_1)\delta(x_1, t_1) \tag{3.4}$$

will be formed, where $b\alpha(t_1)$ and $-a\alpha(t_1)$ denote the strengths of the incoming delta shocks δ_1 and δ_2 on u and v at the time t_1, and $\alpha(t_1)$ can be expressed as

$$\alpha(t_1) = \left(\frac{\phi(r_m) - \phi(r_-)}{r_m - r_-}(u_m v_- - u_- v_m) + \frac{\phi(r_+) - \phi(r_m)}{r_+ - r_m}(u_+ v_m - u_m v_+) \right) t_1. \tag{3.5}$$

The interaction of δ_1 and δ_2 results in a new delta shock wave, denoted by δ_3, which is determined by

$$\begin{cases} u(x,t) = u_- + (u_+ - u_-)H + b\big(\beta_-(t)D^- + \beta_+(t)D^+\big), \\ v(x,t) = v_- + (v_+ - v_-)H - a\big(\beta_-(t)D^- + \beta_+(t)D^+\big), \end{cases} \tag{3.6}$$

where H is the Heaviside function and $\beta(t)D = \beta_-(t)D^- + \beta_+(t)D^+$ is a split delta function. All of them are the functions of $x - x_1 - (t - t_1)\sigma_3$, in which σ_3 is the propagating speed of δ_3. What is more, D^- is the delta measure on the set $\overline{R_+^2} \cap \{(x,t)|x \leqslant x_1 + (t - t_1)\sigma_3\}$, and D^+ is the delta measure on the set $\overline{R_+^2} \cap \{(x,t)|x \geqslant x_1 + (t - t_1)\sigma_3\}$, $\beta_-(t)$, $\beta_+(t)$ and σ_3 are to be determined.

From (3.6), by a simple calculation, we obtain that

$$u_t(x,t) = \big(-\sigma_3(u_+ - u_-) + b\beta'_-(t) + b\beta'_+(t)\big)\delta - b\sigma_3(\beta_-(t) + \beta_-(t))\delta', \quad (3.7)$$

$$(\phi(r)u)_x = (\phi(r_+)u_+ - \phi(r_-)u_-)\delta + b(\phi(r_-)\beta_-(t) + \phi(r_+)\beta_+(t))\delta', \quad (3.8)$$

and

$$v_t(x,t) = \big(-\sigma_3(v_+ - v_-) - a\beta'_-(t) - a\beta'_+(t)\big)\delta + a\sigma_3(\beta_-(t) + \beta_-(t))\delta', \quad (3.9)$$

$$(\phi(r)v)_x = (\phi(r_+)v_+ - \phi(r_-)v_-)\delta - a(\phi(r_-)\beta_-(t) + \phi(r_+)\beta_+(t))\delta'. \quad (3.10)$$

Substituting (3.7)-(3.10) into the system (1.1) and comparing the coefficients of δ and δ', it yields that

$$-\sigma_3(u_+ - u_-) + b\beta'_-(t) + b\beta'_+(t) + \phi(r_+)u_+ - \phi(r_-)u_- = 0, \quad (3.11)$$

$$-b\sigma_3\big(\beta_-(t) + \beta_+(t)\big) + b\phi(r_-)\beta_-(t) + b\phi(r_+)\beta_+(t) = 0, \quad (3.12)$$

and

$$-\sigma_3(v_+ - v_-) - a\beta'_-(t) - a\beta'_+(t) + \phi(r_+)v_+ - \phi(r_-)v_- = 0, \quad (3.13)$$

$$a\sigma_3\big(\beta_-(t) + \beta_+(t)\big) - a\phi(r_-)\beta_-(t) - a\phi(r_+)\beta_+(t) = 0. \quad (3.14)$$

Considering the initial conditions (3.3) and (3.4), from (3.11)-(3.14), it leads to

$$\sigma_3 = \frac{r_+\phi(r_+) - r_-\phi(r_-)}{r_+ - r_-}, \quad (3.15)$$

$$\beta(t) = \beta_-(t) + \beta_+(t) = \frac{\phi(r_+) - \phi(r_-)}{r_+ - r_-}(u_+v_- - u_-v_+)(t - t_1) + \alpha(t_1). \quad (3.16)$$

Obviously, the entropy condition (2.9) is satisfied, this is, the new formed delta shock wave δ_3 is overcompresive. Furthermore, the strengths of δ_3 about u and v are $b\beta(t)$ and $-a\beta(t)$, respectively.

Therefore, the conclusion is that two delta shock waves conclude with each other to form a single delta shock wave. Moreover, it is easy to see that $(x_1, t_1) \to (0,0)$ and $\alpha(t_1) \to 0$ as $\varepsilon \to 0$. In other words, the limit of Riemann solution of (1.1) and (1.3) is just the corresponding one of (1.1) and (1.2) as $\varepsilon \to 0$.

Case 2. $r_- \geqslant 0 > r_+ > r_m$. (When $r_m > r_- \geqslant 0 > r_+$, the interaction situation is similar.)

For this case, we study the interaction of the delta shock wave δ_1 emitting from $(-\varepsilon, 0)$ with the rarefaction wave $\overleftarrow{R}$ followed by a contact discontinuity J emitting from $(\varepsilon, 0)$ (see Fig. 2). The propagating speed of the delta shock wave δ_1 is $\sigma_1 = \frac{r_m\phi(r_m) - r_-\phi(r_-)}{r_m - r_-}$, and that of the wave back in rarefaction wave $\overleftarrow{R}$ is λ_{2m}. Here and after, we use λ_{2i} to denote $(\phi + r\phi_r)|_{r=r_i}$. According to the entropy

condition (2.9), it is obvious to see that $\sigma_1 > \lambda_{2m}$. So δ_1 and $\overleftarrow{R}$ will meet at a finite time. The intersection (x_1, t_1) is determined by

$$\begin{cases} x_1 + \varepsilon = \sigma_1 t_1, \\ x_1 - \varepsilon = \lambda_{2m} t_1. \end{cases} \tag{3.17}$$
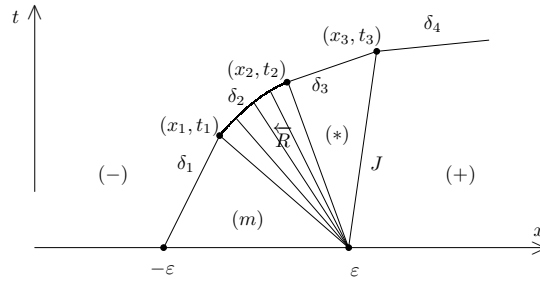

Figure 2. $r_- \geqslant 0 > r_+ > r_m$

The strengths of δ_1 at (x_1, t_1) about u and v are $b\alpha(t_1)$ and $-a\alpha(t_1)$, where

$$\alpha(t_1) = \frac{\phi(r_m) - \phi(r_-)}{r_m - r_-}(u_m v_- - u_- v_m)t_1. \tag{3.18}$$

At the time $t = t_1$, a new delta shock wave generates, denoted by δ_2. Here we use $\Gamma : \{(x(t), t) : t \geqslant t_1\}$ to express the curve of δ_2 with (u_-, v_-) on the left-hand side and $(u(\xi), v(\xi))$ on the right-hand side, where $\lambda_{2m} \leqslant \frac{x-\varepsilon}{t} = \xi = \phi + r\phi_r \leqslant \lambda_{2+}$, $r_m \leqslant r(\xi) = au(\xi) + bv(\xi) \leqslant r_+ \leqslant 0$, $\frac{u(\xi)}{v(\xi)} = \frac{u_m}{v_m}$. A delta shock wave supported on Γ can be constructed as follows:

$$u(x, t) = \begin{cases} u_-, & x < x(t) \\ u(\xi), & x > x(t) \end{cases} + b\big(\beta_-(t)D^- + \beta_+(t)D^+\big), \tag{3.19}$$

$$v(x, t) = \begin{cases} v_-, & x < x(t) \\ v(\xi), & x > x(t) \end{cases} - a\big(\beta_-(t)D^- + \beta_+(t)D^+\big), \tag{3.20}$$

where $\beta(t)D = \beta_-(t)D^- + \beta_+(t)D^+$ is a split delta function supported on Γ and $b\beta(t) = b(\beta_-(t) + \beta_+(t))$, $-a\beta(t) = -a(\beta_-(t) + \beta_+(t))$ are the strengths of δ_2 about u and v at the time t. From (3.19)-(3.20), we can compute that

$$u_t(x, t) = \big(-x'(t)(u(\xi) - u_-) + b\beta'_-(t) + b\beta'_+(t)\big)\delta - bx'(t)(\beta_-(t) + \beta_-(t))\delta',$$
$$(\phi(r)u)_x = (\phi(r(\xi))u(\xi) - \phi(r_-)u_-)\delta + b(\phi(r_-)\beta_-(t) + \phi(r(\xi))\beta_+(t))\delta'.$$

Substituting the above two equations into the first equation of (1.1) yields

$$-x'(t)(u(\xi) - u_-) + b\beta'_-(t) + b\beta'_+(t) + \phi(r(\xi))u(\xi) - \phi(r_-)u_- = 0, \tag{3.21}$$

$$- bx'(t)\big(\beta_-(t) + \beta_+(t)\big) + b\phi(r_-)\beta_-(t) + b\phi(r(\xi))\beta_+(t) = 0. \tag{3.22}$$

Similarly, combining (3.19), (3.20) and the second equation of (1.1), we have

$$- x'(t)(v(\xi) - v_-) - a\beta'_-(t) - a\beta'_+(t) + \phi(r(\xi))v(\xi) - \phi(r_-)v_- = 0, \tag{3.23}$$

$$ax'(t)\big(\beta_-(t) + \beta_+(t)\big) - a\phi(r_-)\beta_-(t) - a\phi(r(\xi))\beta_+(t) = 0. \tag{3.24}$$

From (3.21) and (3.23), the propagating speed of δ_2 is

$$\sigma_2 = x'(t) = \frac{\phi(r(\xi))r(\xi) - \phi(r_-)r_-}{r(\xi) - r_-} \triangleq f(r(\xi)). \tag{3.25}$$

Functions $f(r(\xi))$ and $\frac{\partial f}{\partial x} = f'(r)r'(\xi)\frac{1}{t}$ are continuous, so the solution of ordinary differential equation (3.25) with the initial condition $x_1 = x(t_1)$ is existent and unique. On one hand, it's easy to know that $f'(r) = \frac{\phi'(r)r(r - r_-) + r_-(\phi(r_-) - \phi(r))}{(r - r_-)^2} > 0$ from (3.25). On the other hand, $r(\xi)$ increases when δ_2 propagates forwards. So the propagating speed of δ_2 accelerate gradually and the curve Γ is no long a straight line.

In addition, from (3.21), we can get

$$\beta(t) = \beta_-(t) + \beta_+(t) = \alpha(t_1) + \frac{1}{b}\int_{t_1}^{t}(\sigma_2[u] - [u\phi])\mathrm{d}t \tag{3.26}$$

for $t \geqslant t_1$. Furthermore, we have $\beta_-(t) = \frac{\phi(r(\xi)) - x'(t)}{\phi(r(\xi)) - \phi(r_-)}\beta(t)$ and $\beta_+(t) = \frac{\phi(r_-) - x'(t)}{\phi(r_-) - \phi(r(\xi))}\beta(t)$ from (3.22) and (3.26). So the strengths of δ_2 about u and v are $b\beta(t)$ and $-a\beta(t)$. Entropy condition is obviously satisfied.

The delta shock wave δ_2 will penetrate over the whole rarefaction wave $\overleftarrow{R}$ in a finite time because $r_- \geqslant 0 > r_+$. The interaction point of δ_2 and the head of the rarefaction wave $\overleftarrow{R}$ is denoted by (x_2, t_2), which is determined by

$$\begin{cases} x_2 = x(t_2), \\ x_2 - \varepsilon = \lambda_{2+}t_2. \end{cases} \tag{3.27}$$

The strengths of δ_2 about u and v at (x_2, t_2) can be calculated by (3.26). After the time t_2, δ_2 will propagate with an invariant speed $\sigma_3 = \frac{r_*\phi(r_*) - r_-\phi(r_-)}{r_* - r_-}$, and be denoted by δ_3, where $r_* = r_+$. The strengths of it about u and v are $b\gamma(t)$ and $-a\gamma(t)$, where

$$\gamma(t) = \frac{\phi(r_*) - \phi(r_-)}{r_* - r_-}(u_*v_- - u_-v_*)(t - t_2) + \beta(t_2).$$

Because $r_* = r_+$ and $\frac{u_m}{v_m} = \frac{u_*}{v_*}$, the above formula can be also written as

$$\gamma(t) = \frac{r_+(\phi(r_+) - \phi(r_-))}{r_m(r_+ - r_-)}(u_mv_- - u_-v_m)(t - t_2) + \beta(t_2).$$

Since the speed of contact discontinuity J is $\phi(r_+)$, δ_3 must meet J at some point (x_3, t_3). At this moment, a new initial value problem is formed. The intersection

point (x_3, t_3) is determined by the equations

$$
\begin{cases}
\dfrac{x_3 - \varepsilon}{t_3} = \phi(r_+), \\[2mm]
x_3 - x_2 = \dfrac{r_+\phi(r_+) - r_-\phi(r_-)}{r_+ - r_-}(t_3 - t_2).
\end{cases}
\tag{3.28}
$$

After the interaction of δ_3 and J, a new delta shock wave is formed and denoted by δ_4, whose velocity is $\sigma_4 = \frac{r_+\phi(r_+)-r_-\phi(r_-)}{r_+-r_-}$. The strengths of δ_4 about u and v can be obtained from the following formula

$$
\eta(t) = \frac{\phi(r_+) - \phi(r_-)}{r_+ - r_-}(u_+v_- - u_-v_+)(t - t_3) + \gamma(t_3).
$$

Obviously, the entropy condition is hold.

From (3.17), (3.27) and (3.28), we can get that (x_1, t_1), (x_2, t_2) and (x_3, t_3) all tend to $(0, 0)$ as $\varepsilon \to 0$. Furthermore, we also have $\lim\limits_{\varepsilon \to 0} \alpha(t_1) = \lim\limits_{\varepsilon \to 0} \beta(t_2) = \lim\limits_{\varepsilon \to 0} \gamma(t_3) = 0$. It is easy to know that the limit of the solution of (1.1) and (1.3) is just the delta shock wave, which is the corresponding Riemann solution of (1.1) and (1.2).

Case 3. $r_- \geqslant 0 > r_m > r_+$. (If $r_- > r_m > 0 \geqslant r_+$, the structure of the solution is similar.)

In this case, a delta shock wave δ_1 emits from $(-\varepsilon, 0)$ and a shock wave $\overleftarrow{S}$ followed by a contact discontinuity J emits from $(\varepsilon, 0)$ (see Fig. 3). The propagating speed of δ_1 is $\sigma_1 = \frac{r_m\phi(r_m)-r_-\phi(r_-)}{r_m-r_-}$ and that of $\overleftarrow{S}$ is $\tau = \frac{r_+\phi(r_+)-r_m\phi(r_m)}{r_+-r_m}$. So δ_1 and $\overleftarrow{S}$ will overtake in a finite time. The intersection (x_1, t_1) is determined by

$$
\begin{cases}
x_1 + \varepsilon = \sigma_1 t_1, \\
x_1 - \varepsilon = \tau t_1,
\end{cases}
\tag{3.29}
$$

which yields that

$$
(x_1, t_1) = \left(\frac{\varepsilon(\sigma_1 + \tau)}{\sigma_1 - \tau}, \ \frac{2\varepsilon}{\sigma_1 - \tau} \right).
\tag{3.30}
$$

A new Riemann problem is formed at (x_1, t_1) with the following initial data

$$
u(x, t) = \begin{cases}
u_-, & x < x(t) \\
u_*, & x > x(t)
\end{cases} + b\alpha(t_1)\delta(x_1, t_1),
\tag{3.31}
$$

$$
v(x, t) = \begin{cases}
v_-, & x < x(t) \\
v_*, & x > x(t)
\end{cases} - a\alpha(t_1)\delta(x_1, t_1),
\tag{3.32}
$$

where $(u_*, v_*) = (\frac{au_+ + bv_+}{au_m + bv_m}u_m, \frac{au_+ + bv_+}{au_m + bv_m}v_m)$ is the intermediate state between $\overleftarrow{S}$ and J, and $\alpha(t_1)$ has the same expression as (3.18).
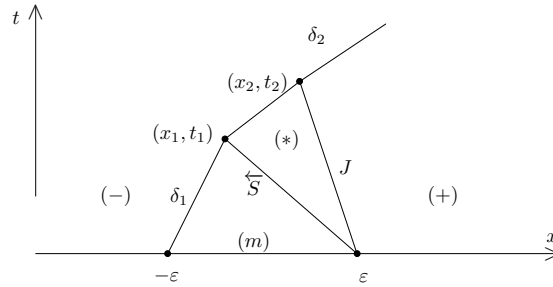
Figure 3. $r_- \geqslant 0 > r_m > r_+$

We solve the new initial data problem as we have done before. Then, a new delta shock wave δ_2 appears at (x_1, t_1) expressed by

$$u(x,t) = \begin{cases} u_-, & x - x_1 < (t - t_1)\sigma_2 \\ u_*, & x - x_1 > (t - t_1)\sigma_2 \end{cases} + b(\beta_-(t)D^- + \beta_+(t)D^+), \qquad (3.33)$$

$$v(x,t) = \begin{cases} v_-, & x - x_1 < (t - t_1)\sigma_2 \\ v_*, & x - x_1 > (t - t_1)\sigma_2 \end{cases} - a(\beta_-(t)D^- - \beta_+(t)D^+), \qquad (3.34)$$

where the propagating speed of δ_2 is $\sigma_2 = \frac{r_*\phi(r_*) - r_-\phi(r_-)}{r_* - r_-}$ with $r_+ = r_*$, and $\beta(t) = \beta_-(t) + \beta_+(t) = \frac{\phi(r_*) - \phi(r_-)}{r_* - r_-}(u_*v_- - u_-v_*)(t - t_1) + \alpha(t_1)$.

After the time t_1, the situation is completely similar to Case 2. That is to say, the delta shock wave δ_2 will pass through J with the same speed. The only difference lies in that the strengths about u and v varies due to the difference choice of (u_*, v_*) and (u_+, v_+). So, the delta shock wave is wholly denoted by δ_2 when $t > t_1$, and the strengths about u and v can be calculated by

$$\gamma(t) = \frac{\phi(r_+) - \phi(r_-)}{r_+ - r_-}(u_+v_- - u_-v_+)(t - t_2) + \beta(t_2).$$

Similarly to the analysis in Case 2, the limits of the solution to (1.1) and (1.3) as $\varepsilon \to 0$ is just the corresponding ones of (1.1) and (1.2).

Case 4. $r_- \geqslant 0 > r_m$ and $r_+ > 0 > r_m$. (If $r_m > 0 > r_-$ and $r_m > 0 \geqslant r_+$, the analysis and computation is similar.)

In this case, there are a delta shock wave δ_1 and two rarefaction waves $\overleftarrow{R}$ and $\overrightarrow{R}$ near $t = 0$ on (x,t)-plane (see Fig. 4). The propagating speed of δ_1 is $\sigma_1 = \frac{r_m\phi(r_m) - r_-\phi(r_-)}{r_m - r_-}$ and that of the wave back of the rarefaction wave $\overleftarrow{R}$ is $\lambda_{2m} = (\phi + r\phi_r)|_{r=r_m}$. Moreover, the strengths of δ_1 about u and v are $b\alpha(t)$ and $-a\alpha(t)$ with $\alpha(t) = \frac{\phi(r_m) - \phi(r_-)}{r_m - r_-}(u_mv_- - u_-v_m)t$.

Similar to Case 2, a new delta shock wave will be generated after δ_1 meets the rarefaction waves $\overleftarrow{R}$ at (x_1, t_1), which is determined by (3.17). The delta

shock wave will cross the whole rarefaction wave $\overleftarrow{R}$ and be denoted by δ_2, whose curve is expressed by $\Gamma : \{(x(t), t) : t \geqslant t_1\}$, with (u_-, v_-) on the left-hand side and $(u(\xi), v(\xi))$ on the right-hand side. Here $\lambda_{2m} \leqslant \frac{x-\varepsilon}{t} = \xi = \phi + r\phi_r \leqslant 0$, $r_m \leqslant r(\xi) = au(\xi) + bv(\xi) \leqslant 0$, $\frac{u(\xi)}{v(\xi)} = \frac{u_m}{v_m}$. The speed and strengths about u and v of δ_2 are determined by (3.25) and (3.26).

The interaction of δ_2 and the head of the rarefaction wave $\overrightarrow{R}$ begins at (x_2, t_2), which can be calculated by

$$\begin{cases} x_2 = x(t_2), \\ x_2 = \varepsilon. \end{cases} \tag{3.35}$$

At the time $t = t_2$, we again obtain a new local Riemann problem with the initial data

$$u(x, t) = \begin{cases} u_-, & x < \varepsilon \\ u(\xi), & x > \varepsilon \end{cases} + b\beta(t_2)\delta(x_2, t_2), \tag{3.36}$$

$$v(x, t) = \begin{cases} v_-, & x < \varepsilon \\ v(\xi), & x > \varepsilon \end{cases} - a\beta(t_2)\delta(x_2, t_2), \tag{3.37}$$

where $0 \leqslant \frac{x-\varepsilon}{t} = \xi = \phi + r\phi_r \leqslant \lambda_{2+}$, $0 \leqslant r(\xi) = au(\xi) + bv(\xi) \leqslant r_+$, $\frac{u(\xi)}{v(\xi)} = \frac{u_+}{v_+}$, and $\beta(t_2)$ can be calculated by (3.26).

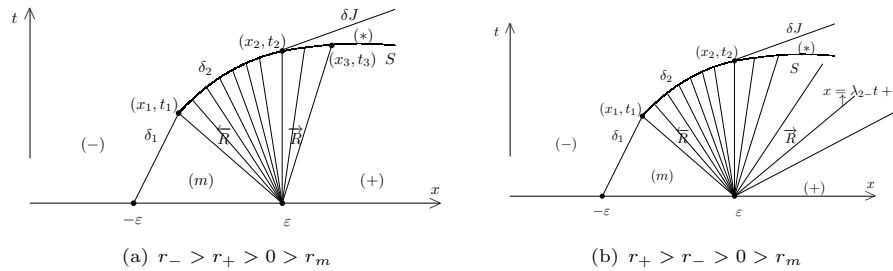(a) $r_- > r_+ > 0 > r_m$ (b) $r_+ > r_- > 0 > r_m$

Figure 4.

In order to deal with the above initial value problem, we assume that the rarefaction wave $\overrightarrow{R}$ is approximated by a set of non-physical shock waves(a method proposed in [3]). Now the value $(u(\xi), v(\xi))$ in $\overrightarrow{R}$ is determined by the right state (u_+, v_+) with $\frac{u(\xi)}{v(\xi)} = \frac{u_+}{v_+}$.

In fact, we can construct the solution of the initial value problem (1.1) with (3.36) and (3.37) in the form:

$$u(x, t) = \begin{cases} u_-, & x < x_1(t) \\ \frac{u_+(au_- + bv_-)}{au_+ + bv_+}, & x_1(t) < x < x_2(t) \\ u(\xi), & x > x_2(t) \end{cases} + b\beta(t_2)\delta(x - x_1(t)), \tag{3.38}$$

$$v(x,t) = \begin{cases} v_-, & x < x_1(t) \\ \frac{v_+(au_-+bv_-)}{au_++bv_+}, & x_1(t) < x < x_2(t) \\ v(\xi), & x > x_2(t) \end{cases} - a\beta(t_2)\delta(x - x_1(t)), \qquad (3.39)$$

where $x_1(t) = x_2 + \phi(r_-)(t - t_2)$, and $x_2(t)$ is the shock wave curve satisfying

$$\begin{cases} \dfrac{\mathrm{d}x_2(t)}{\mathrm{d}t} = \dfrac{(au+bv)\phi(au+bv) - r_-\phi(r_-)}{au+bv - r_-}, \\[2mm] \dfrac{x_2(t) - \varepsilon}{t} = \phi + r\phi_r, \\[2mm] 0 \leqslant \dfrac{x_2(t) - \varepsilon}{t} \leqslant \lambda_2+, \\[2mm] x_2(t_2) = x_2. \end{cases} \qquad (3.40)$$

Now we prove that (3.38) and (3.39) are indeed the weak solution of the initial value problem (1.1) with (3.36) and (3.37). For every $\varphi \in C_0^\infty(R \times R_+)$, the following equations

$$\begin{cases} \langle u_t + (\phi(u,v)u)_x, \ \varphi \rangle = 0, \\ \langle v_t + (\phi(u,v)v)_x, \ \varphi \rangle = 0 \end{cases} \qquad (3.41)$$

are hold if supp $\varphi \bigcap \{(x,t) | x = x_2 + \phi(r_-)(t - t_2), t \geqslant t_2\} = \varnothing$. Otherwise, we prove that (3.38) and (3.39) is still the weak solution of (1.1) with (3.36) and (3.37) near the support of the delta function. Substituting (3.38) and (3.39) into the first equation of (1.1), we have

$$\begin{aligned} u_t + (\phi(r)u)_x = {}& -\phi(r_-)\Big(\frac{u_+(au_-+bv_-)}{au_++bv_+} - u_-\Big)\delta - b\phi(r_-)(\beta_-(t_2) + \beta_+(t_2))\delta' \\ & + \phi(r_-)\Big(\frac{u_+(au_-+bv_-)}{au_++bv_+} - u_-\Big)\delta + b\phi(r_-)(\beta_-(t_2) + \beta_+(t_2))\delta' \\ = {}& 0. \end{aligned} \qquad (3.42)$$

Similarly, Substituting (3.38) and (3.39) into the second equation of (1.1), we obtain

$$\begin{aligned} v_t + (\phi(r)v)_x = {}& -\phi(r_-)\Big(\frac{v_+(au_-+bv_-)}{au_++bv_+} - v_-\Big)\delta - b\phi(r_-)(\beta_-(t_2) + \beta_+(t_2))\delta' \\ & + \phi(r_-)\Big(\frac{v_+(au_-+bv_-)}{au_++bv_+} - v_-\Big)\delta + b\phi(r_-)(\beta_-(t_2) + \beta_+(t_2))\delta' \\ = {}& 0, \end{aligned} \qquad (3.43)$$

which means that (1.1) is also satisfied near the line $x = x_1(t)$ in the weak (or distributional) sense.

For the solution (3.38) and (3.39), the Dirac delta function is now supported on the contact discontinuity line and overcompressibility is lost. As in [16], we introduce the delta contact discontinuity as follows.

Definition 3.1. Consider a region Ω and a curve Γ_1 of slope $\lambda_1 = \phi(r) = \phi(au + bv)$ in Ω. A pair of distributions $(u, v) \in C(\Omega) \times D'(\Omega)$ is called a delta contact discontinuity, if u and v are a sum of a locally integrable functions on Ω and a delta functions on Γ_1 which solves (1.1) in the sense of distribution.

Solving the Riemann problem at (x_2, t_2), we obtain the solution involving a delta contact discontinuity δJ and a shock wave S. In other words, when $t > t_2$, δ_2 decomposes and the state $(u_*, v_*) = (\frac{u_+(au_-+bv_-)}{au_++bv_+}, \frac{v_+(au_-+bv_-)}{au_++bv_+})$ lies between δJ and S. For the delta contact discontinuity, both state variables simultaneously contain the Dirac delta functions.

Then, the delta contact discontinuity δJ will continue to move forwards with a constant speed $\phi(r_-)$, while the strengths about u and v are invariant. However, the over-compressibility is lost.

At the same time, the shock wave S will continue to penetrate the rarefaction wave $\overrightarrow{R}$. Its curve should satisfy the differential equations (3.40). Based on the comparison between the values r_- and r_+, we should divide our discussion into the following three subcases.

Subcase 4.1. $r_+ < r_-$

In this subcase, the curve of S will penetrate over the whole rarefaction wave fan $\overrightarrow{R}$ at (x_3, t_3), the intersection can be calculated by

$$
\begin{cases}
x_3 = x_2(t_3), \\
x_3 - \varepsilon = \lambda_{2+}t_3.
\end{cases}
\tag{3.44}
$$

After the time t_3, the shock wave S propagates with an invariant speed $\frac{r_+\phi(r_+)-r_-\phi(r_-)}{r_+-r_-}$. As $\varepsilon \to 0$, (x_1, t_1), (x_2, t_2) and (x_3, t_3) coincide with each other at the point $(0, 0)$. In addition, we have $\alpha(t_1) \to 0$ and $\beta(t_2) \to 0$. Thus, the limit of the solution of (1.1) and (1.3) is the contact discontinuity $J : x = \phi(r_-)t$ and the shock wave $S : x = \frac{r_+\phi(r_+)-r_-\phi(r_-)}{r_+-r_-}t$, which is exactly the corresponding Riemann solution of (1.1) and (1.2) in this subcase.

Subcase 4.2. $r_+ > r_-$

At this moment, the shock wave curve S can not penetrate the rarefaction wave fan $\overrightarrow{R}$ completely and ultimately has $x = \lambda_{2-}t+\varepsilon$ as its asymptote (see Fig. 4(b)).

Moreover, when $\varepsilon \to 0$, the limit of the solution of (1.1) and (1.3) is the contact discontinuity and the rarefaction wave plus the intermediate state $(u_*, v_*) = (\frac{u_+(au_-+bv_-)}{au_++bv_+}, \frac{v_+(au_-+bv_-)}{au_++bv_+})$ between them. Thus, the limit situation is also true for our assertion in this subcase.

Subcase 4.3. $r_+ = r_-$

For this special subcase, the curve of S has $x = \lambda_{2+}t+\varepsilon$ as its asymptote, which is exactly the wave front of the rarefaction wave $\overrightarrow{R}$. When $\varepsilon \to 0$, the limit is a contact discontinuity $J = \phi(r_-)t$ connecting (u_-, v_-) and (u_+, v_+) directly, and the conclusion is obviously identical with our assertion.

Through the discussions in case 4, we know that delta contact discontinuity is introduced beyond a point at which a delta shock wave loses overcompressibility. Its existence can be justified by two facts. First, a contact discontinuity emerges

in the case when one of the characteristic fields is linearly degenerate. Second, if the conservation law has a delta function as initial data, it propagates along the characteristic lines.

Now we have finished the discussion for all kinds of interactions when the delta shock wave is included. The global solutions for the perturbed initial value problem (1.1) and (1.3) have been constructed. From the results above, it is easy to see that the limits of the perturbed Riemann solutions of (1.1) and (1.3) are exactly the corresponding Riemann solutions of (1.1) and (1.2) as $\varepsilon \to 0$, and the asymptotic behavior of the perturbed Riemann solutions is governed completely by the states $(u_\pm, v_\pm)$, that is, the Riemann solutions of (1.1) and (1.2) are stable with respect to such a local small perturbation.

Remark It is noticed that, as pointed in [32], the theory does not work in some typical systems. For instance, the following equations of geometrical optics

$$\begin{cases} u_t + \left(\dfrac{u^2}{\sqrt{u^2 + v^2}} \right)_x = 0, \\ v_t + \left(\dfrac{uv}{\sqrt{u^2 + v^2}} \right)_x = 0, \end{cases} \tag{3.45}$$

which were proposed by Engquist and Runborg [7] in 1996. To our knowledge, the interactions of delta shock waves for the above geometrical optics (3.45) have not been discussed so far. Therefore, it is nature to consider the interaction of delta shock waves for the system (1.1) with the assumption that $\phi(u, v)$ is a given smooth function satisfying $\phi(u, v) = \phi(\alpha u, \alpha v)$, $\alpha > 0$ is a constant.

Obviously, under this assumption, one can find that the system (3.45) is the very prototype of (1.1) by taking $\phi(u, v) = \frac{u}{\sqrt{u^2+v^2}}$. We leave them for our future study.

References

[1] Y. Brenier, *Solutions with concentration to the Riemann problem for one-dimensional Chaplygin gas dynamics*, J. Math. Fluid Mech., 2005, 7, S326–S331.

[2] Y. Brenier and E. Grenier, *Sticky particles and scalar conservation laws*, SIAM J. Numer. Anal., 1998, 35(6), 2317–2328.

[3] A. Bressan, *Hyperbolic systems of conservation laws: the one-dimensional Cauchy problem*, Oxford Lecture Ser. Math. Appl., Vol. 20. Oxford University Press, Oxford, 2000.

[4] G. Chen and H. Liu, *Formation of delta-shocks and vacuum states in the vanishing pressure limit of solutions to the isentropic Euler equations*, SIAM J. Math. Anal., 2003, 34(4), 925–938.

[5] H. Cheng and H. Yang, *Delta shock waves in chromatography equations*, J. Math. Anal. Appl., 2011, 380(2), 475–485.

[6] V. G. Danilov and D. Mitrovic, *Delta shock wave formation in the case of triangular hyperbolic system of conservation laws*, J. Differential Equations, 2008, 245(12), 3704–3734.

[7] B. Engquist and O. Runborg, *Multiphase computations in geometrical optics*, J. Comput. Appl. Math., 1996, 74, 175–192.

[8] P. Le Floch, *An existence and uniqueness result for two nonstrictly hyperbolic systems*, IMA Vol. Math. Appl., Vol. 27. Springer-Verlag, Berlin/New York, 1990.

[9] L. Guo, L. Pan and G. Yin, *The perturbed Riemann problem and delta contact discontinuity in chromatography equations*, Nonlinear Analysis, 2014, 106, 110–123.

[10] L. Guo, Y. Zhang and G. Yin, *Interactions of delta shock waves for the Chaplygin gas equations with split delta functions*, J. Math. Anal. Appl., 2014, 410(1), 190–201.

[11] F. Huang and Z Wang, *Well-posedness for pressureless flow*, Commun. Math. Phys., 2001, 222(1), 117–146.

[12] B. L. Keyfitz and H. C. Kranzer, *A viscosity approximation to a system of conservation laws with no classical Riemann solution*, Lecture Notes in Math., Vol. 1402. Springer-Verlag, Berlin-New York, 1989.

[13] D. J. Korchinski, *Solution of a Riemann problem for a 2×2 system of conservation laws possessing no classical weak solution*, thesis, Adelphi University, 1977.

[14] H. C. Kranzer and B. L. Keyfitz, *A strictly hyperbolic system of conservation laws admitting singular shock*, IMA Vol. Math. Appl., Vol. 27. Springer-Verlag, Berlin-New York, 1990.

[15] M. Nedeljkov, *Delta and singular delta locus for one dimensional systems of conservation laws*, Math. Methods Appl. Sci., 2004, 27(8), 931–955.

[16] M. Nedeljkov and M. Oberguggenberger, *Interactions of delta shock waves in a strictly hyperbolic system of conservation laws*, J. Math. Anal. Appl., 2008, 344(2), 1143–1157.

[17] B. Nilsson, O. S. Rozanova and V. M. Shelkovich, *Mass, momentum and energy conservation laws in zero-pressure gas dynamics and δ-shocks*, II, Appl. Anal., 2011, 90(5), 831–842.

[18] B. Nilsson and V. M. Shelkovich, *Mass, momentum and energy conservation laws in zero-pressure gas dynamics and delta-shocks*, Appl. Anal., 2011, 90(11), 1677–1689.

[19] Y. Pang, *Delta shock wave in the compressible Euler equations for a Chaplygin gas*, J. Math. Anal. Appl., 2017, 448(1), 245–261.

[20] E. Y. Panov and V. M. Shelkovich, *$\delta'-$shock waves as a new type of solutions to systems of conservation laws*, J. Differential Equations, 2006, 73(1), 49-86.

[21] A. Sen, T. R. Sekhar and V. D. Sharma, *Wave interactions and stability of the Riemann solution for a strictly hyperbolic system of conservation laws*, Q. Appl. Math., 2017, 75(3), 539–554.

[22] S. F. Shandarin and Y. B. Zeldovich, *The large-scale structure of the universe: turbulence, intermittency, structure in a selfgravitating medium*, Rev. Modern Phys., 1989, 61(2), 185–220.

[23] C. Shen and M. Sun, *Stability of the Riemann solutions for a nonstrictly hyperbolic system of conservation laws*, Nonlinear Analysis, 2010, 73(10), 3284–3294.

[24] W. Sheng and T. Zhang, *The Riemann problem for transportation equation in gas dynamics*, Mem. Am. Math. Soc., 1999, 137, 1–77.

[25] M. Sun, *Delta shock waves for the chromatography equations as self-similar viscosity limits*, Q. Appl. Math., 2011, 69(3), 425–443.

[26] M. Sun, *Interactions of delta shock waves for the chromatography equations*, Appl. Math. Lett., 2013, 26(6), 631–637.

[27] D. Tan and T. Zhang, *Two-dimensional Riemann problem for a hyperbolic system of nonlinear conservation laws, I. Four-J cases, II. Initial data involving some rarefaction waves*, J. Differential Equations, 1994, 111(2), 203–253.

[28] D. Tan, T. Zhang and Y. Zheng, *Delta shock waves as limits of vanishing viscosity for hyperbolic systems of conversation laws*, J. Differential Equations, 1994, 112(1), 1–32.

[29] G. Wang, *One-dimensional nonlinear chromatography system and delta-shock waves*, Z. Angew. Math. Phys., 2013, 64(5), 1451–1469.

[30] E. Weinan, Yu. G. Rykov and Ya. G. Sinai, *Generalized variational principles, global weak solutions and behavior with random initial data for systems of conservation laws arising in ashesion particle dynamics*, Comm. Math. Phys., 1996, 177(2), 349–380.

[31] H. Yang, *Riemann problems for a class of coupled hyperbolic systems of conservation laws*, J. Differential Equations, 1999, 159, 447–484.

[32] H. Yang and Y. Zhang, *New developments of delta shock waves and its applications in systems of conservation laws*, J. Differential Equations, 2012, 252(11), 5951–5993.

[33] H. Yang and Y. Zhang, *Delta shock waves with Dirac delta function in both components for systems of conservation laws*, J. Differential Equations, 2014, 257(12), 4369–4402.

[34] Q. Zhang, *Interactions of delta shock waves and stability of Riemann solutions for nonlinear chromatography equations*, Z. Angew. Math. Phys., 2016, 67(1), 1451–1469.

[35] Y. Zhang and Y. Zhang, *The Riemann problem and interaction of waves in two-dimensional steady zero-pressure adiabatic flow*, Internat. J. Non-Linear Mech., 2018, 104, 100–108.

[36] Y. Zhang and Y. Zhang and J. Wang, *Interaction of delta shock waves for the Chaplygin Euler equations of compressible fluid flow with split delta functions*, Math. Meth. Appl. Sci., 2018, 41(3), 7678–7697.