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ON THE NUMBER OF LIMIT CYCLES FOR A QUINTIC LIÉNARD SYSTEM UNDER POLYNOMIAL PERTURBATIONS*

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Abstract In this paper, we mainly study the number of limit cycles for a quintic Liénard system under polynomial perturbations. In some cases, we give new estimations for the lower bound of the maximal number of limit cycles.

Keywords Limit cycle, bifurcation, Liénard system, Melnikov function.

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1. Introduction

In 1900, Hilbert proposed the well-known 23 mathematical problems in the Second International Congress of Mathematics [14]. The second part of the 16th problem is to find the maximal number of limit cycles, denoted by $H(n)$, and their relative positions for planar polynomial differential systems with degree n. There are many works on the lower bound of $H(n)$. For example, for the newest results of the lower bound of $H(2)$ and $H(3)$, see [6, 23] and [17, 19] respectively. More detailed introduction on the Hilbert's 16th problem and its related study can be found in [16, 18], etc.

Consider the following Liénard system [20],

$$\dot{x} = y, \quad \dot{y} = -g(x) - \varepsilon f(x)y, \tag{1.1}$$

where $\varepsilon > 0$ is sufficiently small, $f(x)$ and $g(x)$ are polynomials in x with $\deg g = m$ and $\deg f = n$. To find the maximal number of limit cycles for system (1.1) is called the Hilbert's 16th problem for Liénard system.

Let $H(n, m)$ denote the maximal number of limit cycles for system (1.1). For general m and n, there are many results on the lower bound of $H(n, m)$. Blows & Lloyd [5], Han [8] and Christopher and Lynch [7] all studied the number of small-amplitude limit cycles for Liénard system. For $m = 1$, Blows and Lloyd [5] obtained $H(n, 1) \geq [\frac{n}{2}]$. For $m = 2$, Han [8] got the Hopf cyclicity of the Liénard system which shows that $H(n, 2) \geq \left[\frac{2n+1}{3}\right]$. For $m = 2, 3$ and 4, Christopher and Lynch [7] got

$$H(n, 2) \geq \left[\frac{2n+1}{3}\right], \text{ for } n \geq 1, \ H(n, 3) \geq 2\left[\frac{3(n+2)}{8}\right], \text{ for } 1 < n \leq 50.$$

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For arbitrary n and m, by using the averaging theory Llibre et al. [21] proved that

$$H(n,m) \geq \left[\frac{n+m-1}{2}\right], \quad n,m \geq 1.$$

In 2013, Han and Romanovski [10] gave a new method to find the lower bound of $H(n,m)$ for general integers n and m. Later, Xiong and Han [27] improved the existing results for the lower bounds of $H(n,m)$ obtained in [10]. For example, the result for $n \geq m,\ m \geq 7$ in [27] is as follows

$$\begin{aligned} H(n,m) \geq & \tfrac{1}{2}(\tfrac{1}{\ln 2}\ln(m+2)-2)n + \tfrac{1}{3\ln 2}(m+2)\ln(m+2) \\ & -(\tfrac{5}{6}+\tfrac{\ln 3}{3\ln 2})(m+2) + \tfrac{1}{\ln 2}\ln(m+2) + \tfrac{5}{2} - \tfrac{\ln 3}{\ln 2}. \end{aligned}$$

Atabaigi and Zangeneh [2] studied system (1.1) with $g(x) = x(x+1)^2(x-\frac{2}{3})$ and $n = 3$, and showed that this system can undergo degenerated Hopf bifurcation and Poincaré bifurcation, which emerge at most three limit cycles for ε sufficiently small. For the lower bounds of $H(n,5)$, Kazemi et al. [15] got $H(4,5) \geq 3$. Xu and Li [28] proved that $H(2,5) \geq 3$, $H(4,5) \geq 5$, $H(6,5) \geq 10$, $H(8,5) \geq 10$. Xu and Li [29] considered a centrally symmetric Liénard system and obtained $H(10,5) \geq 11$. For $n \geq 5$, Han and Romanovski [10] obtained

$$H(n,6) \geq H(n,5) \geq 2\left[\frac{n-1}{3}\right] + \left[\frac{n-1}{2}\right], \ n \geq 5.$$

Later, Xiong and Han [27] gave new estimations of $H(n,5)$ and $H(n,6)$ with

$$H(n,5) \geq 2\left[\tfrac{n-1}{3}\right] + \left[\tfrac{n}{2}\right] + 2,\ n \geq 5,\ H(n,6) \geq 2\left[\tfrac{n-1}{3}\right] + \left[\tfrac{n-1}{2}\right] + 3,\ n \geq 6. \tag{1.2}$$

For $m = 6$, Asheghi et al. [1] obtained $H(5,6) \geq 9$, $H(6,6) \geq 10$, $H(7,6) \geq 11$. Bakhshalizadeh et al. [3] gave the phase portraits for system $(1.1)|_{\varepsilon=0}$ with $\deg g(x) = 6$ in the case that this system has at least two singular points and a unique center.

For $m = 7$, Sun [24] proved that

$$H(6,7) \geq 11,\ H(8,7) \geq 13,\ H(10,7) \geq 17,\ H(12,7) \geq 17,\ H(14,7) \geq 20. \tag{1.3}$$

Bakhshalizadeh et al. [4] studied the number of limit cycles near an eye-figure loop in a kind of Liénard system and also gave the lower bounds of $H(n,7)$ for $n = 4,6,8,10$. The lower bounds given in [4] are smaller than the ones in [7] and [24]. Moghimi et al. [22] studied the number of limit cycles bifurcated from a Liénard system with a double homoclinic loop of cuspidal type surrounded by a heteroclinic loop. In 2018, Yang and Ding [30] gave new lower bounds of $H(2n,7)$ for $n = 2$ and $4 \leq n \leq 20$.

In 2014, Xiong [26] considered the following system

$$\dot{x} = y, \quad \dot{y} = -\alpha(x-a)(x-b)(x-\alpha_1)(x-\beta)(x-\alpha_2) + \varepsilon q(x,y,\delta), \tag{1.4}$$

where $\alpha > 0$ or $\alpha < 0$, and $q(x,y,\delta) = \sum_{i=0}^{n} a_i x^i y$. When $\varepsilon = 0$, the author obtained that the number of different phase portraits of system (1.4) is 40 if this system has at least a family of periodic orbits.

Especially, for $\alpha = 1$, $a = b = 0$, system (1.4) becomes

$$\dot{x} = y, \quad \dot{y} = -x^2(x-\alpha_1)(x-\beta)(x-\alpha_2) + \varepsilon q(x,y,\delta). \tag{1.5}$$

Further, suppose

$$0 < \alpha_1 < \beta < \alpha_2, \ \alpha_2 = \frac{\beta(2\beta - 3\alpha_1)}{3\beta - 5\alpha_1}. \tag{1.6}$$

Under (1.6), Xiong [26] considered the number of limit cycles of system (1.5) for $1 \le n \le 9$.

In this paper, we also suppose (1.6) holds. For $10 \le n \le 20$ we get the number of limit cycles for system (1.5), which give new estimations for the lower bounds of $H(n,5)$. For $1 \le n \le 9$, we give a further study to (1.5). The main results are as follows:

Theorem 1.1. *For the lower bound of $H(n,5)$ with $1 \le n \le 20$, we have*

$$H(1,5) \ge 1, \ H(2,5) \ge 3, \ H(3,5) \ge 5, \ H(4,5) \ge 9,$$

$$H(n,5) \ge n+4, n = 5,6,7,8, \quad H(9,5) \ge 14,$$

$$H(n,5) \ge n+7-\left[\tfrac{n+1}{6}\right], \ n = 10,11,12, \quad H(n,5) \ge n+8-\left[\tfrac{n+1}{6}\right], \ 13 \le n \le 20.$$

2. Preliminaries

To show our results completely, some similar steps with [26] may appear in the following.

When $\varepsilon = 0$, system (1.5) is a Hamiltonian system and has the following Hamiltonian function

$$H(x,y) = \frac{1}{2}y^2 - \frac{1}{3}\alpha_1\alpha_2\beta x^3 + \frac{1}{4}(\alpha_1\alpha_2 + \alpha_1\beta + \alpha_2\beta)x^4 - \frac{1}{5}(\alpha_1 + \alpha_2 + \beta)x^5 + \frac{1}{6}x^6, \tag{2.1}$$

where $0 < \alpha_1 < \beta < \alpha_2$ and $\alpha_2 = \frac{\beta(2\beta-3\alpha_1)}{3\beta-5\alpha_1}$.

By [26] we know that system (1.5) has a compound loop, denoted by L_0, passing through a nilpotent cusp $(0,0)$ and a hyperbolic saddle $(\beta,0)$, and surrounding two elementary centers $(\alpha_i,0), i = 1,2$. See Figure.1.

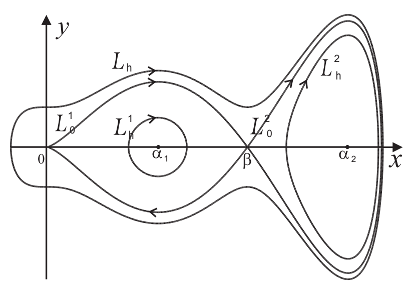

Figure 1. The phase portrait of system $(1.5)|_{\varepsilon=0}$ under (1.6) (see [26]).

It is obvious that $H(0,0) = H(\beta,0) = 0$. Denote the heteroclinic loop $H(x,y) = 0, 0 \le x \le \beta$ by L_0^1 and the homoclinic loop $H(x,y) = 0, \beta \le x \le 1$ by L_0^2, and let

$$h_{\alpha_1} \equiv H(\alpha_1, 0), \quad h_{\alpha_2} \equiv H(\alpha_2, 0).$$

Then there are three families of periodic orbits

$$L_h : H(x,y) = h,\ h > 0,\quad L_h^i : H(x,y) = h,\ h_{\alpha_i} < h < 0,\ i = 1,2,$$

and, correspondingly, three Melnikov functions

$$\begin{aligned} M(h,\delta) &= \oint_{L_h} q(x,y,\delta)\mathrm{d}x,\quad h > 0,\\ M_i(h,\delta) &= \oint_{L_h^i} q(x,y,\delta)\mathrm{d}x,\quad h_{\alpha_i} < h < 0,\quad i = 1,2. \end{aligned} \tag{2.2}$$

We know that there exists a transformation of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/k_i & 0 \\ 0 & 1/\bar{k}_i \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + S_i,$$

and a time scaling $t \to \tau_i = k_i \bar{k}_i t (k_i > 0, \bar{k}_i > 0)$, where $S_1 = (\beta, 0)^T,\ S_2 = (\alpha_1, 0)^T,\ S_3 = (\alpha_2, 0)^T$, such that system (1.5) becomes

$$\frac{\mathrm{d}u}{\mathrm{d}\tau_i} = H_v^{(i)},\ \frac{\mathrm{d}v}{\mathrm{d}\tau_i} = -H_u^{(i)} + \varepsilon q_i(u,v,\delta),$$

where

$$\begin{aligned} H^{(1)}(u,v) &= \tfrac{1}{2}(v^2 - u^2) + \sum_{i=3}^{6} \hat{h}_{i0} u^i,\ q_1(u,v,\delta) = \sum_{i=0}^{n} \hat{b}_{i1} u^i v,\\ H^{(2)}(u,v) &= h_{\alpha_1} + \tfrac{1}{2}(v^2 + u^2) + \sum_{i=3}^{6} \bar{h}_{i0} u^i,\ q_2(u,v,\delta) = \sum_{i=0}^{n} \bar{b}_{i1} u^i v,\\ H^{(3)}(u,v) &= h_{\alpha_2} + \tfrac{1}{2}(v^2 + u^2) + \sum_{i=3}^{6} \widetilde{h}_{i0} u^i,\ q_3(u,v,\delta) = \sum_{i=0}^{n} \widetilde{b}_{i1} u^i v. \end{aligned}$$

By [9, 11, 13, 25, 26], we have the following Lemma.

Lemma 2.1. *For the first order Melnikov functions $M_1(h,\delta)$, $M_2(h,\delta)$ and $M(h,\delta)$ in* (2.2)*, one has*

$$\begin{aligned} M_1(h,\delta) &= c_0(\delta) + B_{00}c_2(\delta)|h|^{\frac{5}{6}} + c_3(\delta)h\ln|h| + c_5(\delta)h + B_{10}c_6(\delta)|h|^{\frac{7}{6}}\\ &\qquad - \tfrac{1}{11}B_{00}c_7(\delta)|h|^{\frac{11}{6}} + c_8(\delta)h^2\ln|h| + O(h^2),\ 0 < -h \ll 1,\\ M_2(h,\delta) &= c_1(\delta) + c_3(\delta)h\ln|h| + c_4(\delta)h + c_8(\delta)h^2\ln|h| + O(h^2),\ 0 < -h \ll 1,\\ M_1(h,\delta) &= \sum_{i\geq 0} b_{l,i}(\delta)(h - h_{\alpha_1})^{i+1},\ 0 < h - h_{\alpha_1} \ll 1,\\ M_2(h,\delta) &= \sum_{k\geq 0} b_{r,k}(\delta)(h - h_{\alpha_2})^{k+1},\ 0 < h - h_{\alpha_2} \ll 1, \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} M(h,\delta) &= \bar{c}_0(\delta) + B_{00}^* c_2(\delta)h^{\frac{5}{6}} + 2c_3(\delta)h\ln h + \bar{c}_5(\delta)h + B_{10}^* c_6(\delta)h^{\frac{7}{6}}\\ &\qquad + \tfrac{1}{11}B_{00}^* c_7(\delta)h^{\frac{11}{6}} + 2c_8(\delta)h^2\ln h + O(h^2),\ 0 < h \ll 1, \end{aligned} \tag{2.4}$$

where $B^*_{00} < 0$, $B^*_{10} < 0$, $B_{00} > 0$, $B_{10} > 0$, *and*

$$\begin{aligned}
&c_0(\delta) = \oint_{L_0^1} q(x,y,\delta)\mathrm{d}x, \quad c_1(\delta) = \oint_{L_0^2} q(x,y,\delta)\mathrm{d}x, \quad \bar{c}_0(\delta) = c_0(\delta) + c_1(\delta),\\
&c_2(\delta) = 2\sqrt{2}\, h_{30}^{-\frac{1}{3}} a_0, h_{30} = -\tfrac{1}{3}\alpha_1\alpha_2\beta, \quad c_3(\delta) = -\hat{b}_{01},\\
&c_4(\delta) = \oint_{L_0^2} [q_y(x,y,\delta) - q_y(\beta,0,\delta)]\mathrm{d}t + \mu_1 c_3(\delta),\\
&c_5(\delta) = \oint_{L_0^1} q_y(x,y,\delta)\mathrm{d}t, \ \mathit{if}\ c_2(\delta) = c_3(\delta) = 0,\\
&c_6(\delta) = \tilde{r}_{10}, \quad c_7(\delta) = 2\tilde{r}_{30} + 9\tilde{r}_{01}, \ c_8(\delta) = \tfrac{1}{2}(\hat{b}_{21} + 3\hat{b}_{11}\hat{h}_{30}) + \mu_2 c_3(\delta),\\
&\bar{c}_5(\delta) = c_4(\delta) + c_5(\delta), \ \mathit{if}\ c_2(\delta) = c_3(\delta) = 0,\\
&\quad b_{l,0} = 2\pi\bar{b}_{01}, \ b_{r,0} = 2\pi\widetilde{b}_{01},
\end{aligned} \tag{2.5}$$

where μ_1, μ_2 *are some constants,* $\widetilde{r}_{10}$, $\widetilde{r}_{30}$ *and* $\widetilde{r}_{01}$ *can be obtained by [13]. The coefficients* $b_{l,i}$ *and* $b_{r,k}$ *can be obtained by the programs in [12].*

By using Lemma 2.1 we can prove

Theorem 2.1. *Consider system* (1.5). *If there exists a parameter* δ_0 *such that*

$$\begin{aligned}
c_0(\delta_0) = \cdots = c_7(\delta_0) = 0, \quad c_8(\delta_0) \neq 0,\\
b_{l,0}(\delta_0) = \cdots = b_{l,i-1}(\delta_0) = 0, \quad b_{l,i}(\delta_0) \neq 0,\\
b_{r,0}(\delta_0) = \cdots = b_{r,k-1}(\delta_0) = 0, \quad b_{r,k}(\delta_0) \neq 0,
\end{aligned} \tag{2.6}$$

and

$$\mathrm{rank}\frac{\partial(c_0,\ldots,c_7,b_{l,0},\ldots,b_{l,i-1},b_{r,0},\ldots,b_{r,k-1})}{\partial\delta}\bigg|_{\delta=\delta_0} = 8 + i + k, \tag{2.7}$$

then we have

(i) if $b_{l,i}(\delta_0)b_{r,k}(\delta_0) > 0$, $c_8(\delta_0)b_{l,i}(\delta_0) > 0$, *there exist* $14+i+k+2$ *limit cycles for some* (ε,δ) *near* $(0,\delta_0)$, *where* 14 *limit cycles are near the compound loop, i limit cycles are near the center* $(\alpha_1,0)$ *and k limit cycles are near the center* $(\alpha_2,0)$*;*

(ii) if $b_{l,i}(\delta_0)b_{r,k}(\delta_0) < 0$, *there exist* $14+i+k+1$ *limit cycles for some* (ε,δ) *near* $(0,\delta_0)$*;*

(iii) if $b_{l,i}(\delta_0)b_{r,k}(\delta_0) > 0$, $c_8(\delta_0)b_{l,i}(\delta_0) < 0$, *there exist* $14+i+k$ *limit cycles for some* (ε,δ) *near* $(0,\delta_0)$.

Proof. We first prove (i). Without loss of generality, suppose $c_8(\delta_0) > 0$, $b_{l,i}(\delta_0) > 0$ and $b_{r,k}(\delta_0) > 0$. By (2.6), we know that $M_1(h,\delta_0)$ and $M_2(h,\delta_0)$ can be written as

$$\begin{aligned}
&M_1(h,\delta_0) = c_8(\delta_0)h^2\ln|h| + O(h^2) < 0, \ 0 < -h \ll 1,\\
&M_1(h,\delta_0) = b_{l,i}(\delta_0)(h-h_{\alpha_1})^{i+1} + O((h-h_{\alpha_1})^{i+2}) > 0, \ 0 < h - h_{\alpha_1} \ll 1,\\
&M_2(h,\delta_0) = c_8(\delta_0)h^2\ln|h| + O(h^2) < 0, \ 0 < -h \ll 1,\\
&M_2(h,\delta_0) = b_{r,k}(\delta_0)(h-h_{\alpha_2})^{k+1} + O((h-h_{\alpha_2})^{k+2}) > 0, \ 0 < h - h_{\alpha_2} \ll 1,
\end{aligned}$$

which show that there exist $h_0 \in (h_{\alpha_1},0)$ and $h_0^* \in (h_{\alpha_2},0)$ such that

$$\begin{aligned}
M_1(h_0,\delta_0) = 0, \ M_1(h_0-\varepsilon_0,\delta_0)M_1(h_0+\varepsilon_0,\delta_0) < 0,\\
M_2(h_0^*,\delta_0) = 0, \ M_2(h_0^*-\varepsilon_0,\delta_0)M_2(h_0^*+\varepsilon_0,\delta_0) < 0
\end{aligned}$$

if ε_0 is sufficiently small.

By (2.7), the coefficients $c_0, \ldots, c_7, b_{l,0}, \ldots, b_{l,i-1}, b_{r,0}, \ldots, b_{r,k-1}$ can be taken as free parameters. Now change $b_{l,0}, \ldots, b_{l,i-1}$ and $b_{r,0}, \ldots, b_{r,k-1}$ such that

$$b_{l,i-1}b_{l,i}(\delta_0) < 0, \quad b_{r,k-1}b_{r,k}(\delta_0) < 0,$$

and

$$|b_{l,0}| \ll |b_{l,1}| \ll ... \ll |b_{l,i-1}| \ll |b_{l,i}(\delta_0)|, \quad b_{l,j}b_{l,j+1} < 0, \quad j = 0, 1, 2, ..., i-2,$$

$$|b_{r,0}| \ll |b_{r,1}| \ll ... \ll |b_{r,k-1}| \ll |b_{r,k}(\delta_0)|, \quad b_{r,j}b_{r,j+1} < 0, \quad j = 0, 1, 2, ..., k-2,$$

which ensure that $M_1(h,\delta)$ has i simple zeros for $0 < h - h_{\alpha_1} \ll 1$ and $M_2(h,\delta)$ has k simple zeros for $0 < h - h_{\alpha_2} \ll 1$.

Then change $c_0, c_1, \ldots, c_7$ such that

$$0 < |c_0| + |c_1| \ll c_2 \ll -c_3 \ll |c_4| + |c_5| \ll -c_6 \ll -c_7 \ll c_8(\delta_0) \tag{2.8}$$

with $c_0 < 0, c_1 > 0, c_0 + c_1 > 0, c_4 < 0, c_5 < 0$, which ensures that for h near $h = 0$ $M_1(h,\delta)$ has 6 simple zeros, $M_2(h,\delta)$ has 3 simple zeros and $M(h,\delta)$ has 5 simple zeros.

At the same time, $M_1(h,\delta)$ has a simple zero near h_0 and $M_2(h,\delta)$ has a simple zero near h_0^*.

Let $0 < \varepsilon \ll \min\{-c_0, c_1, |b_{l,0}|, |b_{r,0}|\}$. Similar to the proof of Theorem 2.3 in [11], it can be seen that system (1.5) has $14+i+k+2$ limit cycles for some (ε,δ) near $(0,\delta_0)$.

By using the same way, we can prove (ii) and (iii). The proof is completed. □

3. The proof of Theorem 1.1

In this section, we consider system (1.5) with $\alpha_1 = \frac{5}{9}, \beta = 1, \alpha_2 = \frac{3}{2}$, i.e.

$$\dot{x} = y, \ \dot{y} = -x^2(x - \frac{5}{9})(x-1)(x - \frac{3}{2}) + \varepsilon \sum_{i=0}^{n} a_i x^i y. \tag{3.1}$$

When $\varepsilon = 0$, system (3.1) becomes

$$\dot{x} = y, \quad \dot{y} = -x^2(x - \frac{5}{9})(x-1)(x - \frac{3}{2}), \tag{3.2}$$

which is a Hamiltonian system and has a nilpotent cusp $(0,0)$, a hyperbolic saddle $(1,0)$ and two elementary centers $(\frac{5}{9},0)$, $(\frac{3}{2},0)$. The Hamiltonian function of system (3.2) is

$$H(x,y) = \frac{1}{2}y^2 - \frac{5}{18}x^3 + \frac{13}{18}x^4 - \frac{11}{18}x^5 + \frac{1}{6}x^6.$$

It is easy to get that

$$h_{\frac{5}{9}} \equiv H(\frac{5}{9},0) = -\frac{10000}{1594323}, \quad h_{\frac{3}{2}} \equiv H(\frac{3}{2},0) = -\frac{3}{128}.$$

Next, we give the coefficients in the expansions of $M(h,\delta)$, $M_1(h,\delta)$, $M_2(h,\delta)$ in (2.3) and (2.4), and the proof of Theorem 1.1 according to the value of n.

(1) $n=20$. For each coefficient $c_i(\delta)$ with $i=0,1,\cdots,5$, according to [26], we directly have

$$c_0(\delta)=\sum_{i=0}^{20}a_iI_i,\quad c_1(\delta)=\sum_{i=0}^{20}a_iJ_i,\quad c_2(\delta)=-\tfrac{2\sqrt{2}}{5}\cdot 5^{\frac{2}{3}}\cdot 18^{\frac{1}{3}}a_0,$$
$$c_3(\delta)=-\tfrac{3\sqrt{2}}{2}\sum_{i=0}^{20}a_i,\quad c_4(\delta)|_{c_2(\delta)=c_3(\delta)=0}=\sum_{i=2}^{20}a_iM_i,\quad c_5(\delta)|_{c_2(\delta)=c_3(\delta)=0}=\sum_{i=2}^{20}a_iN_i,$$

where

$$\begin{aligned}
I_i&=2\int_0^1 x^i\sqrt{\tfrac{5}{9}x^3-\tfrac{13}{9}x^4+\tfrac{11}{9}x^5-\tfrac{1}{3}x^6}\,\mathrm{d}x,\quad i=0,...,20,\\
J_i&=2\int_1^{\frac{5}{3}} x^i\sqrt{\tfrac{5}{9}x^3-\tfrac{13}{9}x^4+\tfrac{11}{9}x^5-\tfrac{1}{3}x^6}\,\mathrm{d}x,\quad i=0,...,20,\\
M_i&=2\int_1^{\frac{5}{3}}\frac{x^i-x}{\sqrt{\frac{5}{9}x^3-\frac{13}{9}x^4+\frac{11}{9}x^5-\frac{1}{3}x^6}}\,\mathrm{d}x,\ i=2,...,20,\\
N_i&=2\int_0^1\frac{x^i-x}{\sqrt{\frac{5}{9}x^3-\frac{13}{9}x^4+\frac{11}{9}x^5-\frac{1}{3}x^6}}\,\mathrm{d}x,\ i=2,...,20,
\end{aligned}$$

with

$$\begin{aligned}
I_0&=\tfrac{299\sqrt{2}}{2592}-\tfrac{125\sqrt{3}}{31104}(\pi+2\arcsin\tfrac{1}{5}),\quad\cdots,\quad I_{20}=\xi_1-\xi_2(\pi+2\arcsin\tfrac{1}{5}),\\
J_0&=\tfrac{299\sqrt{2}}{2592}+\tfrac{125\sqrt{3}}{31104}(\pi-2\arcsin\tfrac{1}{5}),\quad\cdots,\quad J_{20}=\xi_1+\xi_2(\pi-2\arcsin\tfrac{1}{5}),\\
M_2&=\sqrt{3}\,(\pi-2\arcsin\tfrac{1}{5}),\quad\cdots,\quad M_{20}=\xi_3+\xi_4\,(\pi-2\arcsin\tfrac{1}{5}),\\
N_2&=-\sqrt{3}(\pi+2\arcsin\tfrac{1}{5}),\quad\cdots,\quad N_{20}=\xi_3-\xi_4\,(\pi+2\arcsin\tfrac{1}{5}),
\end{aligned}$$

and

$$\begin{aligned}
\xi_1&=\tfrac{940278120762887578126216390627 1\sqrt{2}}{3508859567615530458501926092 8},\ \xi_2=\tfrac{454398809075355529785156 25\sqrt{3}}{7360819185499152234577 92},\\
\xi_3&=\tfrac{179343402179570126237526 73\sqrt{2}}{121006747446801334272 0},\qquad \xi_4=\tfrac{758112807714775765072 7\sqrt{3}}{221861110674043699 2}.
\end{aligned}$$

For the coefficients $c_6(\delta),c_7(\delta)$ and $c_8(\delta)$, by (2.5) and [13], [11], we have

$$\begin{aligned}
c_6(\delta)&=12\sqrt{2}\left(\tfrac{1}{5}\right)^{\frac{5}{3}}18^{\frac{2}{3}}\left(\tfrac{13}{9}a_0+\tfrac{5}{6}a_1\right),\\
c_7(\delta)&=\tfrac{16\sqrt{2}}{25}5^{\frac{2}{3}}18^{\frac{1}{3}}\left(\tfrac{542}{9}a_0+\tfrac{853}{25}a_1+\tfrac{78}{5}a_2+\tfrac{9}{2}a_3\right),\\
c_8(\delta)&=\tfrac{27}{4}\sqrt{2}\big(-\tfrac{9}{8}a_1-\tfrac{7}{4}a_2-\tfrac{15}{8}a_3-\tfrac{3}{2}a_4-\tfrac{5}{8}a_5+\tfrac{3}{4}a_6+\tfrac{21}{8}a_7+5a_8+\tfrac{63}{8}a_9\\
&\quad+\tfrac{45}{4}a_{10}+\tfrac{121}{8}a_{11}+\tfrac{39}{2}a_{12}+\tfrac{195}{8}a_{13}+\tfrac{119}{4}a_{14}+\tfrac{285}{8}a_{15}+42a_{16}+\tfrac{391}{8}a_{17}\\
&\quad+\tfrac{225}{4}a_{18}+\tfrac{513}{8}a_{19}+\tfrac{145}{2}a_{20}\big)+\mu_2c_3(\delta),
\end{aligned}$$

where μ_2 is a constant.

Next, we give the coefficients $b_{l,i}$ and $b_{r,k}$ in the expansions of $M_i(h,\delta)(i=1,2)$ for $0<h-h_{\alpha_i}\ll 1$.

According to [26], take a transformation $\bar{u}=\frac{5\sqrt{34}}{81}(x-\frac{5}{9})$, $\bar{v}=y$ and a time scaling $t\to\frac{5\sqrt{34}}{81}t$. Then system (3.1) is equivalent to the following system

$$\begin{aligned}
\dot{\bar{u}}&=\bar{v},\\
\dot{\bar{v}}&=-\left(\tfrac{81}{5\sqrt{34}}\right)^2\bar{u}\left(\tfrac{81}{5\sqrt{34}}\bar{u}+\tfrac{5}{9}\right)^2\left(\tfrac{81}{5\sqrt{34}}\bar{u}-\tfrac{4}{9}\right)\left(\tfrac{81}{5\sqrt{34}}\bar{u}-\tfrac{17}{18}\right)\\
&\quad+\tfrac{81}{5\sqrt{34}}\varepsilon\sum_{i=0}^{20}a_i\left(\tfrac{81}{5\sqrt{34}}\bar{u}+\tfrac{5}{9}\right)^i\bar{v}.
\end{aligned}$$

When $\varepsilon = 0$, the above system has the following Hamiltonian function

$$\bar{H}(\bar{u},\bar{v}) = \tfrac{1}{2}(\bar{v}^2+\bar{u}^2) + \tfrac{2673\sqrt{34}}{57800}\bar{u}^3 - \tfrac{17537553}{1445000}\,\bar{u}^4 - \tfrac{387420489}{245650000}\sqrt{34}\,\bar{u}^5 + \tfrac{94143178827}{1228250000}\,\bar{u}^6.$$

Then, by the maple programs in [12], we get

$$\begin{aligned}
b_{l,0}(\delta) &= \tfrac{81}{85}\sqrt{34}\,\pi\sum_{i=0}^{20} a_i\left(\tfrac{5}{9}\right)^i,\\
b_{l,1}(\delta) &= \sqrt{34}\,\pi\big(\tfrac{588262140279}{33408400000}\,a_0 + \tfrac{58207138407}{6681680000}\,a_1 + \tfrac{2117438091}{267267200}\,a_2 + \tfrac{436037499}{53453440}\,a_3\\
&\quad + \tfrac{414456183}{53453440}\,a_4 + \tfrac{71927271}{10690688}\,a_5 + \tfrac{58080915}{10690688}\,a_6 + \tfrac{44415375}{10690688}\,a_7 + \tfrac{97740625}{32072064}\,a_8\\
&\quad + \tfrac{624618125}{288648576}\,a_9 + \tfrac{3889365625}{2597837184}\,a_{10} + \tfrac{23711703125}{23380534656}\,a_{11} + \tfrac{142050390625}{210424811904}\,a_{12}\\
&\quad + \tfrac{838548828125}{1893823307136}\,a_{13} + \tfrac{4888416015625}{17044409764224}\,a_{14} + \tfrac{28191376953125}{153399687878016}\,a_{15}\\
&\quad + \tfrac{161058056640625}{1380597190902144}\,a_{16} + \tfrac{912569580078125}{12425374718119296}\,a_{17} + \tfrac{5133111572265625}{111828372463073664}\,a_{18}\\
&\quad + \tfrac{28686212158203125}{1006455352167662976}\,a_{19} + \tfrac{159381011962890625}{9058098169508966784}\,a_{20}\big),\\
&\cdots,
\end{aligned}$$

where $b_{l,0}$ can also be easily got by (2.5).

To get the coefficients $b_{r,k}$ in (2.3), we take a transformation of the form $\widetilde{u} = \frac{\sqrt{17}}{4}\left(x - \frac{3}{2}\right)$, $\widetilde{v} = y$ and a time scaling $t \to \frac{\sqrt{17}}{4}t$. Then system (3.1) can be transformed into the following system

$$\begin{aligned}
\dot{\widetilde{u}} &= \widetilde{v},\\
\dot{\widetilde{v}} &= -\tfrac{16}{17}\,\widetilde{u}\left(\tfrac{4}{\sqrt{17}}\widetilde{u}+\tfrac{3}{2}\right)^2\left(\tfrac{4}{\sqrt{17}}\widetilde{u}+\tfrac{17}{18}\right)\left(\tfrac{4}{\sqrt{17}}\widetilde{u}+\tfrac{1}{2}\right) + \tfrac{4}{\sqrt{17}}\varepsilon\sum_{i=0}^{20} a_i\left(\tfrac{4}{\sqrt{17}}\widetilde{u}+\tfrac{3}{2}\right)^i\widetilde{v}.
\end{aligned}$$

When $\varepsilon = 0$, the above system has the following Hamiltonian function

$$\widetilde{H}(\widetilde{u},\widetilde{v}) = \frac{1}{2}(\widetilde{v}^2+\widetilde{u}^2) + \frac{896\sqrt{17}}{2601}\widetilde{u}^3 + \frac{4064}{2601}\,\widetilde{u}^4 + \frac{8192\sqrt{17}}{44217}\widetilde{u}^5 + \frac{2048}{14739}\,\widetilde{u}^6.$$

Then, by the maple programs in [12], we get

$$\begin{aligned}
b_{r,0}(\delta) &= \tfrac{8}{\sqrt{17}}\,\pi\sum_{i=0}^{20} a_i\left(\tfrac{3}{2}\right)^i,\\
b_{r,1}(\delta) &= \sqrt{17}\,\pi\big(\tfrac{5540992}{2255067}\,a_0 + \tfrac{2039360}{751689}\,a_1 + \tfrac{709600}{250563}\,a_2 + \tfrac{227504}{83521}\,a_3 + \tfrac{191928}{83521}\,a_4\\
&\quad + \tfrac{126324}{83521}\,a_5 + \tfrac{40770}{83521}\,a_6 - \tfrac{21465}{83521}\,a_7 + \tfrac{109107}{167042}\,a_8 + \tfrac{2111913}{334084}\,a_9 + \tfrac{15481773}{668168}\,a_{10}\\
&\quad + \tfrac{85260195}{1336336}\,a_{11} + \tfrac{406355535}{2672672}\,a_{12} + \tfrac{1773182421}{5345344}\,a_{13} + \tfrac{7289067609}{10690688}\,a_{14}\\
&\quad + \tfrac{28697282559}{21381376}\,a_{15} + \tfrac{109346642955}{42762752}\,a_{16} + \tfrac{406097982945}{85525504}\,a_{17} + \tfrac{1477349115813}{171051008}\,a_{18}\\
&\quad + \tfrac{5283855862587}{342102016}\,a_{19} + \tfrac{18630922175847}{684204032}\,a_{20}\big),\\
&\cdots.
\end{aligned}$$

For $\delta = (a_0, a_1, \cdots, a_{19}, a_{20})$, we can find

$$\delta_0 = (a_0^*, a_1^*, \cdots, a_4^*, a_5, a_6^*, \cdots, a_{10}^*, a_{11}, a_{12}^*, \cdots, a_{16}^*, a_{17}, a_{18}^*, a_{19}^*, a_{20})$$

such that

$$c_0(\delta_0) = c_1(\delta_0) = \cdots = c_7(\delta_0) = 0,$$
$$b_{l,0}(\delta_0) = b_{l,1}(\delta_0) = \cdots = b_{l,6}(\delta_0) = 0,\ b_{r,0}(\delta_0) = b_{r,1}(\delta_0) = 0,$$
$$\mathrm{rank}\frac{\partial(c_0,\ldots,c_7,b_{l,0},\ldots,b_{l,6},b_{r,0},b_{r,1})}{\partial\delta}\bigg|_{\delta=\delta_0} = 17,$$

where

$$a_0^* = a_1^* = 0,\ a_2^* = -\tfrac{5}{6}\,a_5 + \tfrac{125}{108}\,a_{11} - \tfrac{419500}{2187}\,a_{17} - \tfrac{668018067664633813498904683 75}{114214616865278252955648}\,a_{20},$$
$$a_3^* = \tfrac{26}{9}\,a_5 - \tfrac{325}{81}\,a_{11} + \tfrac{4362800}{6561}\,a_{17} + \tfrac{17368469759280479150971521 7775}{856609626489586897 16736}\,a_{20},$$
$$a_4^* = -\tfrac{55}{18}\,a_5 + \tfrac{1375}{324}\,a_{11} - \tfrac{4614500}{6561}\,a_{17} - \tfrac{7348199192364567863750646 83375}{3426438505958347588 66944}\,a_{20},$$
$$a_6^* = -\tfrac{455}{54}\,a_{11} + \tfrac{3053960}{2187}\,a_{17} + \tfrac{4863146081430692358493998 75395}{114214616865278252955648}\,a_{20},$$
$$a_7^* = \tfrac{62}{3}\,a_{11} - \tfrac{832288}{243}\,a_{17} - \tfrac{198800852084643464084193174509}{190357694775463754926 08}\,a_{20},$$
$$a_8^* = -\tfrac{79}{3}\,a_{11} + \tfrac{2119867}{486}\,a_{17} + \tfrac{354452115413549628093472400 2717}{266500772685649256896512}\,a_{20},$$
$$a_9^* = \tfrac{995}{54}\,a_{11} - \tfrac{6634565}{2187}\,a_{17} - \tfrac{369796893309775640487560126 1355}{399751159028473885344768}\,a_{20},$$
$$a_{10}^* = -\tfrac{121}{18}\,a_{11} + \tfrac{756712}{729}\,a_{17} + \tfrac{843610903255367788309870860583}{266500772685649256896512}\,a_{20},$$
$$a_{12}^* = -\tfrac{18551}{81}\,a_{17} - \tfrac{29415738719551792156812 82837}{42301709950103056650 24}\,a_{20},$$
$$a_{13}^* = \tfrac{16870}{81}\,a_{17} + \tfrac{1467658857499936369110179 05}{235009499722794759168}\,a_{20},$$
$$a_{14}^* = -\tfrac{10100}{81}\,a_{17} - \tfrac{1329263076946364075054293 85}{365570332902125180928}\,a_{20},$$
$$a_{15}^* = \tfrac{1280}{27}\,a_{17} + \tfrac{7096265898709985471735083 1}{548355499353187771392}\,a_{20},$$
$$a_{16}^* = -\tfrac{187}{18}\,a_{17} - \tfrac{8429544783590998779589625}{365570332902125180928}\,a_{20},$$
$$a_{18}^* = \tfrac{1268363746013732671278 7}{20309462939006954496}\,a_{20},\ a_{19}^* = -\tfrac{203164190920514106025}{3384910489834492416}\,a_{20}.$$

We further get

$$c_8(\delta_0) = \tfrac{8267216014802944\sqrt{2}}{688504393719125271}\,a_{20},\ b_{r,2}(\delta_0) = \tfrac{272409980483463200028 5\sqrt{17}\pi}{2115085497505152832512}\,a_{20},$$
$$b_{l,7}(\delta_0) = -\tfrac{157908910608441147678024116181769859300439 51\sqrt{34}\pi}{936996608594520449578177280000000000000}\,a_{20}, \tag{3.3}$$

which shows that $b_{l,7}(\delta_0)b_{r,2}(\delta_0) < 0$ if $a_{20} \neq 0$.

To find more limit cycles we consider the zero of $M(h,\delta_0)$ for $h > 0$ large. Let $G(x) = -\frac{5}{18}x^3 + \frac{13}{18}x^4 - \frac{11}{18}x^5 + \frac{1}{6}x^6$. Then $H(x,y) = \frac{1}{2}y^2 + G(x)$. And for $h > 0$ the Melnikov function is

$$M(h,\delta) = \oint_{L_h} f(x,\delta)y\mathrm{d}x = 2\int_{x_2(h)}^{x_1(h)} f(x,\delta)\sqrt{2(h-G(x))}\,\mathrm{d}x, \tag{3.4}$$

where

$$f(x,\delta) = \sum_{i=0}^{20} a_i x^i,\quad G(x_i(h)) = h,\quad i = 1,2,\quad x_2(h) < 0,\ x_1(h) > \tfrac{5}{3}.$$

When $h = 1000$, $x_1(1000)$ and $x_2(1000)$ can be obtained by the equation $G(x) = 1000$ with

$$x_2(1000) = -3.699426676...,\ x_1(1000) = 4.925416271....$$

By Yang et al. [12], it is obvious that $a_{6i-1}(i = 1, 2, \cdots)$ has no effect on the zeros of the first order Melnikov function. So we take $a_5 = a_{11} = a_{17} = 0$ in δ_0 and let $\widetilde{\delta}_0 = \delta_0|_{a_5=a_{11}=a_{17}=0}$. It is obvious that

$$c_8(\widetilde{\delta}_0) = c_8(\delta_0),\ b_{l,7}(\widetilde{\delta}_0) = b_{l,7}(\delta_0),\ b_{r,2}(\widetilde{\delta}_0) = b_{r,2}(\delta_0). \tag{3.5}$$

Then, by (2.4), (3.3), (3.4) and (3.5) we obtain

$$\begin{aligned}
&a_{20}M(1000, \widetilde{\delta}_0) = 72027752561243.4077352...\, a_{20}^2 > 0,\\
&a_{20}M(h, \widetilde{\delta}_0) = 2a_{20}c_8(\delta_0)h\ln h + O(h^2) < 0, \quad 0 < h \ll 1,
\end{aligned}$$

if $a_{20} \neq 0$, which shows that there exists $h^{(20)}(0 < h^{(20)} < 1000)$ such that $M(h, \widetilde{\delta}_0)$ has $h^{(20)}$ as its zero with an odd multiplicity.

Hence, by Theorem 2.1 (ii) with $i = 7$ and $k = 2$, system (3.1) has 25 limit cycles for some (ε, δ) near $(0, \widetilde{\delta}_0)$. This proves that $H(20, 5) \geq 25$.

(2) $n = 11$ and $n = 10$.

Similarly, we can find $\delta_0^{(11)}$ such that

$$\begin{gathered}
c_0(\delta_0^{(11)}) = \cdots = c_7(\delta_0^{(11)}) = 0, \quad b_{r,0}(\delta_0^{(11)}) = 0,\\
\operatorname{rank}\frac{\partial(c_0(\delta), \ldots, c_7(\delta), b_{r,0}(\delta))}{\partial\delta}\bigg|_{\delta=\delta_0^{(11)}} = 9,
\end{gathered}$$

with $\delta_0^{(11)} = (a_0^*, a_1^*, \cdots, a_4^*, a_5, a_6^*, \cdots, a_{10}^*, a_{11})$ and

$$\begin{aligned}
&a_0^* = a_1^* = 0,\ a_2^* = -\tfrac{5}{6}\,a_5 - \tfrac{56375}{7168}\,a_{10} - \tfrac{20016125}{387072}\,a_{11},\\
&a_3^* = \tfrac{26}{9}\,a_5 + \tfrac{146575}{5376}\,a_{10} + \tfrac{52041925}{290304}\,a_{11},\ a_4^* = -\tfrac{55}{18}\,a_5 - \tfrac{36625}{1344}\,a_{10} - \tfrac{12986875}{72576}\,a_{11},\\
&a_6^* = \tfrac{364675}{16128}\,a_{10} + \tfrac{41679595}{290304}\,a_{11}, \quad a_7^* = -\tfrac{453515}{16128}\,a_{10} - \tfrac{48875699}{290304}\,a_{11},\\
&a_8^* = \tfrac{1239793}{64512}\,a_{10} + \tfrac{119436265}{1161216}\,a_{11}, \quad a_9^* = -\tfrac{10535}{1536}\,a_{10} - \tfrac{765295}{27648}\,a_{11}.
\end{aligned}$$

Further, we have

$$\begin{gathered}
c_8(\delta_0^{(11)}) = -\tfrac{5\sqrt{2}}{1344}(18a_{10} + 121a_{11}),\\
b_{r,1}(\delta_0^{(11)}) = \tfrac{535\sqrt{17}\pi}{4402048}(18a_{10} + 121a_{11}),\ b_{l,0}(\delta_0^{(11)}) = -\tfrac{15625\sqrt{34}\pi}{1549681956}(18a_{10} + 121a_{11}),
\end{gathered}$$

which shows that $b_{r,1}(\delta_0^{(11)})b_{l,0}(\delta_0^{(11)}) < 0$ if $18a_{10} + 121a_{11} \neq 0$. In this case, we can not find a zero of $M(h, \delta_0^{(11)})$ for $h > 0$ large. Thus, by Theorem 2.1 (ii) with $i = 0$ and $k = 1$, system (3.1) has 16 limit cycles for some (ε, δ) near $(0, \delta_0^{(11)})$ which yields that $H(11, 5) \geq 16$.

Let $a_{11} = 0$. The above proof also shows that $H(10, 5) \geq 16$.

(3) $n = 9$.

In this case, we can find $\delta_0^{(9)} = (a_0^*, a_1^*, \cdots, a_4^*, a_5, a_6^*, a_7^*, a_8^*, a_9)$ with

$$a_0^* = a_1^* = 0,\ a_2^* = -\tfrac{5}{6}a_5 + \tfrac{28825}{4914}a_9,\ a_3^* = \tfrac{26}{9}a_5 - \tfrac{11530}{567}a_9,$$
$$a_4^* = -\tfrac{55}{18}a_5 + \tfrac{148300}{7371}a_9,\ a_6^* = -\tfrac{860}{63}a_9,\ a_7^* = \tfrac{1490}{117}a_9,\ a_8^* = -\tfrac{3131}{546}a_9,$$

such that

$$\operatorname{rank}\tfrac{\partial(c_0(\delta),c_1(\delta),\ldots,c_7(\delta))}{\partial\delta}\Big|_{\delta=\delta_0^{(9)}} = 8,$$
$$c_0(\delta_0^{(9)}) = c_1(\delta_0^{(9)}) = \cdots = c_7(\delta_0^{(9)}) = 0,$$

and

$$c_8(\delta_0^{(9)}) = \frac{48\sqrt{2}}{91}a_9,\ b_{l,0}(\delta_0^{(9)}) = -\frac{800000\,\pi}{7399253043}\sqrt{34}\,a_9,\ b_{r,0}(\delta_0^{(9)}) = -\frac{9\,\pi}{6188}\sqrt{17}\,a_9.$$

On the other hand, we can not find a zero of $M(h, \delta_0^{(9)})$ for $h > 0$ large. Thus, by Theorem 2.1 (iii) with $i = 0$ and $k = 0$, system (3.1) has 14 limit cycles for some (ε, δ) near $(0, \delta_0^{(9)})$, which yields that $H(9, 5) \geq 14$.

(4) $n = 8$.

There exists $\delta_0^{(8)} = (a_0^*, a_1^*, \cdots, a_4^*, a_5, a_6^*, a_7^*, a_8)$ with

$$a_0^* = 0,\ a_1^* = 0,\ a_2^* = -\tfrac{5}{6}a_5 - \tfrac{63755}{5184}a_8,\ a_3^* = \tfrac{26}{9}a_5 + \tfrac{319151}{7776}a_8,$$
$$a_4^* = -\tfrac{55}{18}a_5 - \tfrac{18095}{486}a_8,\ a_6^* = \tfrac{7679}{576}a_8,\ a_7^* = -\tfrac{187}{32}a_8,$$

such that

$$\operatorname{rank}\tfrac{\partial(c_0(\delta),c_1(\delta),\ldots,c_6(\delta))}{\partial\delta}\Big|_{\delta=\delta_0^{(8)}} = 7,$$
$$c_0(\delta_0^{(8)}) = c_1(\delta_0^{(8)}) = \cdots = c_6(\delta_0^{(8)}) = 0,$$

and

$$c_7(\delta_0^{(8)}) = -\tfrac{55\sqrt{2}}{12}5^{\frac{2}{3}}18^{\frac{1}{3}}a_8,\ c_8(\delta_0^{(8)}) = \tfrac{\sqrt{2}}{2}\,a_8,$$
$$b_{l,0}(\delta_0^{(8)}) = \tfrac{5000\sqrt{34}}{9034497}\,\pi a_8,\ b_{r,0}(\delta_0^{(8)}) = -\tfrac{5\sqrt{17}}{4352}\,\pi a_8.$$

In this case, we also can not find a zero of $M(h, \delta_0^{(8)})$. Without loss of generality, suppose $a_8 > 0$. Then, we have

$$M_1(h, \delta_0^{(8)}) = -\tfrac{1}{11}B_{00}c_7(\delta_0^{(8)})|h|^{\frac{11}{6}} + O(h^2\ln|h|) > 0,\ \text{for } 0 < -h \ll 1,$$
$$M_1(h, \delta_0^{(8)}) = b_{l,0}(\delta_0^{(8)})(h - h_{\frac{5}{9}}) + O((h - h_{\frac{5}{9}})^2) > 0,\ \text{for } 0 < h - h_{\frac{5}{9}} \ll 1,$$
$$M_2(h, \delta_0^{(8)}) = c_8(\delta_0^{(8)})h^2\ln|h| + O(h^2) < 0,\ \text{for } 0 < -h \ll 1,$$
$$M_2(h, \delta_0^{(8)}) = b_{r,0}(\delta_0^{(8)})(h - h_{\frac{3}{2}}) + O((h - h_{\frac{3}{2}})^2) < 0,\ \text{for } 0 < h - h_{\frac{3}{2}} \ll 1,$$

which show that we can not find a zero of $M_i(h, \delta_0^{(8)})$ for each $i (i = 1, 2)$. Then, similar to the proof of Theorem 2.1, change the free parameters $c_0, c_1, \cdots, c_6$ such that

$$0 < \varepsilon \ll |c_0| + |c_1| \ll c_2 \ll -c_3 \ll |c_4| + |c_5| \ll -c_6 \ll \min\{-c_7(\delta_0^{(8)}), c_8(\delta_0^{(8)})\}$$

with $c_0<0, c_1>0, c_0+c_1>0, c_4<0, c_5<0$. Then system (3.1) has 12 limit cycles for some (ε,δ) near $(0,\delta_0^{(8)})$, of which 5 limit cycles are near L_0^1, 3 limit cycles are near L_0^2 and 4 limit cycles are near L_0. This proves that $H(8,5)\geq 12$.

(5) $n=7$.

We can find $\delta_0^{(7)}=(a_0^*,a_1^*,\cdots,a_4^*,a_5,a_6^*,a_7)$ with

$$\begin{aligned}
a_0^*&=0,\ a_1^*=\tfrac{14}{55}\rho\left(625\sqrt{3}\pi-1250\sqrt{3}\arcsin\left(\tfrac{1}{5}\right)-2796\sqrt{2}\right)a_7,\\
a_2^*&=-\tfrac{1}{5940}\rho[(11350350\sqrt{3}\pi-38788200\sqrt{2}-22700700\sqrt{3}\arcsin\left(\tfrac{1}{5}\right))a_5\\
&\quad+(261217550\sqrt{3}\arcsin\left(\tfrac{1}{5}\right)+444604596\sqrt{2}-130608775\sqrt{3}\pi)a_7],\\
a_3^*&=\tfrac{1}{8910}\rho[(59021820\sqrt{3}\pi-118043640\sqrt{3}\arcsin\left(\tfrac{1}{5}\right)-201698640\sqrt{2})a_5\\
&\quad+(2188711428\sqrt{2}+1279119110\sqrt{3}\arcsin\left(\tfrac{1}{5}\right)-639559555\sqrt{3}\pi)a_7],\\
a_4^*&=-\tfrac{1}{17820}\rho[(124853850\sqrt{3}\pi-249707700\sqrt{3}\arcsin\left(\tfrac{1}{5}\right)-426670200\sqrt{2})a_5\\
&\quad+(2173180630\sqrt{3}\arcsin\left(\tfrac{1}{5}\right)+3723966948\sqrt{2}-1086590315\sqrt{3}\pi)a_7],\\
a_6^*&=-\tfrac{1}{330}\rho\left(4499885\sqrt{3}\pi-8999770\sqrt{3}\arcsin\left(\tfrac{1}{5}\right)-15419964\sqrt{2}\right)a_7,
\end{aligned}$$

where $\rho=\frac{1}{2293\sqrt{3}\pi-4586\sqrt{3}\arcsin\left(\frac{1}{5}\right)-7836\sqrt{2}}<0$, such that

$$\operatorname{rank}\frac{\partial(c_0(\delta),c_1(\delta),\ldots,c_5(\delta))}{\partial\delta}\bigg|_{\delta=\delta_0^{(7)}}=6,$$
$$c_0(\delta_0^{(7)})=c_1(\delta_0^{(7)})=\cdots=c_4(\delta_0^{(7)})=b_{r,0}(\delta_0^{(7)})=0,$$

and

$$\begin{aligned}
c_5(\delta_0^{(7)})&=-\tfrac{9344\sqrt{6}\pi}{165}\rho a_7=2.135427...a_7,\\
c_8(\delta_0^{(7)})&=\tfrac{3\sqrt{2}}{55}\rho\left(31895\sqrt{3}\pi-63790\sqrt{3}\arcsin\left(\tfrac{1}{5}\right)-106356\sqrt{2}\right)a_7\\
&=-0.3385901...a_7,\\
b_{l,0}(\delta_0^{(7)})&=\tfrac{32\sqrt{34}\pi}{3247695}\rho\left(1155625\sqrt{3}\pi-2311250\sqrt{3}\arcsin\left(\tfrac{1}{5}\right)-3656076\sqrt{2}\right)a_7\\
&=-0.2756545...a_7,\\
b_{r,1}(\delta_0^{(7)})&=\tfrac{1024\sqrt{17}\pi}{162129}\rho\left(8599\sqrt{3}\pi-17198\sqrt{3}\arcsin\left(\tfrac{1}{5}\right)-28932\sqrt{2}\right)a_7\\
&=0.0494921...a_7.
\end{aligned}$$

Without loss of generality, suppose $a_7>0$. It is easy to see that

$$\begin{aligned}
&M(h,\delta_0^{(7)})=c_5(\delta_0^{(7)})h+O(h^{\frac{7}{6}})>0,\ \text{for } 0<h\ll 1,\\
&M(1000,\delta_0^{(7)}|_{a_5=0})=-1139044.03564...a_7<0,
\end{aligned}$$

which shows that there exists $h^*\in(0,1000)$ satisfying $M(h^*,\delta_0^{(7)}|_{a_5=0})=0$. Then, change the free parameters $c_0,c_1,\cdots,c_4$ and $b_{r,0}$ such that

$$0<|c_0|+|c_1|\ll -c_2\ll c_3\ll c_4\ll\min\{c_5(\delta_0^{(7)}),-c_8(\delta_0^{(7)})\},0<-b_{r,0}\ll b_{r,1}(\delta_0^{(7)})$$

with $c_0 > 0, c_1 < 0, c_0 + c_1 < 0$, which ensures that each $M_i(h,\delta)(i = 1,2)$ has three simple zeros near $h = 0$, $M_2(h,\delta)$ has one simple zero near $h = h_{\frac{5}{9}}$, $M(h,\delta)$ has three simple zeros near $h = 0$. At the same time, $M(h,\delta_0^{(7)})$ has a simple zero near h^*. Further, for some (ε,δ) near $(0,\delta_0^{(7)})$ we can get 11 limit cycles of system (3.1). This shows that $H(7,5) \geq 11$.

(6) $n = 6$.

Without loss of generality, suppose $a_6 > 0$. We can find $\delta_0^{(6)} = (a_0^*, \cdots, a_4^*, a_5, a_6)$ with

$$
\begin{aligned}
&a_0^* = 0,\ a_1^* = \tfrac{7\rho_1}{1728}\left(3588\sqrt{2} + 125\sqrt{3}\pi - 250\sqrt{3}\arcsin\left(\tfrac{1}{5}\right)\right)a_6,\\
&a_2^* = -\tfrac{\rho_1}{864}[(-95040\sqrt{2} + 25200\sqrt{3}\pi - 50400\sqrt{3}\arcsin\left(\tfrac{1}{5}\right))a_5\\
&\qquad +(-249564\sqrt{2} + 80165\sqrt{3}\pi - 160330\sqrt{3}\arcsin\left(\tfrac{1}{5}\right))a_6],\\
&a_3^* = \tfrac{\rho_1}{1728}[(-658944\sqrt{2} + 174720\sqrt{3}\pi - 349440\sqrt{3}\arcsin\left(\tfrac{1}{5}\right))a_5\\
&\qquad +(-1742004\sqrt{2} + 492679\sqrt{3}\pi - 985358\sqrt{3}\arcsin\left(\tfrac{1}{5}\right))a_6],\\
&a_4^* = -\tfrac{\rho_1}{216}[\left(-87120\sqrt{2} + 23100\sqrt{3}\pi - 46200\sqrt{3}\arcsin\left(\tfrac{1}{5}\right)\right)a_5\\
&\qquad + \left(-180732\sqrt{2}\,a_6 + 49213\sqrt{3}\pi - 98426\sqrt{3}\arcsin\left(\tfrac{1}{5}\right)\right)a_6],
\end{aligned}
$$

where $\rho_1 = \frac{1}{35\sqrt{3}\pi - 70\sqrt{3}\arcsin\left(\frac{1}{5}\right) - 132\sqrt{2}} < 0$, such that

$$
\begin{gathered}
\operatorname{rank}\frac{\partial(c_0(\delta), c_1(\delta), \ldots, c_4(\delta))}{\partial\delta}\bigg|_{\delta=\delta_0^{(6)}} = 5,\\
c_0(\delta_0^{(6)}) = c_1(\delta_0^{(6)}) = \cdots = c_4(\delta_0^{(6)}) = 0,
\end{gathered}
$$

and

$$
\begin{aligned}
c_5(\delta_0^{(6)}) &= \tfrac{148}{27}\rho_1\sqrt{6}\,\pi a_6 < 0,\\
c_8(\delta_0^{(6)}) &= -\tfrac{\rho_1}{32}\sqrt{2}\left(-3084\sqrt{2} + 1033\sqrt{3}\pi - 2066\sqrt{3}\arcsin\left(\tfrac{1}{5}\right)\right)a_6 > 0,\\
b_{l,0}(\delta_0^{(6)}) &= -\tfrac{\rho_1}{111537}\sqrt{34}\,\pi\left(15125\sqrt{3}\pi - 30250\sqrt{3}\arcsin\left(\tfrac{1}{5}\right) - 20508\sqrt{2}\right)a_6 > 0,\\
b_{r,0}(\delta_0^{(6)}) &= -\tfrac{\rho_1}{9792}\sqrt{17}\,\pi\left(2293\sqrt{3}\pi - 4586\sqrt{3}\arcsin\left(\tfrac{1}{5}\right) - 7836\sqrt{2}\right)a_6 < 0.
\end{aligned}
\tag{3.6}
$$

Here we can not find zeros of $M_i(h,\delta_0^{(6)}), i = 1,2$. Let $\bar{\delta}_0^{(6)} = \delta_0^{(6)}|_{a_5=0}$. We get

$$M(h,\bar{\delta}_0^{(6)}) < 0,\ \text{for } 0 < -h \ll 1,\ \text{and } M(1000,\bar{\delta}_0^{(6)}) = 477110.349...a_6 > 0,$$

which shows that there exists $h^{(6)}$ such that $M(h^{(6)},\bar{\delta}_0^{(6)}) = 0$, where $0 < h^{(6)} < 1000$. Then, by changing the free parameters as follows

$$0 < |c_0| + |c_1| \ll c_2 \ll -c_3 \ll -c_4 \ll \min\{-c_5(\bar{\delta}_0^{(6)}), c_8(\bar{\delta}_0^{(6)})\}$$

with $c_0 < 0, c_1 > 0, c_0 + c_1 > 0$, we get three simple zeros of $M_i(h,\delta)$ near $h = 0$ for each $i(i = 1,2)$ and three simple zeros of $M(h,\delta)$ near $h = 0$. At this moment, $M(h,\bar{\delta}_0^{(6)})$ has a simple zero near $h^{(6)}$. Thus, for some (ε,δ) near $(0,\bar{\delta}_0^{(6)})$, system (3.1) has 10 limit cycles which yields that $H(6,5) \geq 10$.

(7) $n = 4$

There exists $\delta_0^{(4)} = (a_0^*, a_1^*, a_2^*, a_3^*, a_4)$ with

$$a_0^* = 0,\ a_1^* = -\frac{35}{48}a_4,\ a_2^* = \frac{65}{24}a_4,\ a_3^* = -\frac{143}{48}a_4,$$

such that

$$\text{rank}\frac{\partial(c_0(\delta), c_1(\delta), \ldots, c_3(\delta))}{\partial\delta}\bigg|_{\delta=\delta_0^{(4)}} = 4,\ c_0(\delta_0^{(4)}) = c_1(\delta_0^{(4)}) = \cdots = c_3(\delta_0^{(4)}) = 0$$

and

$$\begin{aligned}
b_{l,0}(\delta_0^{(4)}) &= \tfrac{20}{1377}\sqrt{34}\,\pi\, a_4,\ b_{r,0}(\delta_0^{(4)}) = \tfrac{\sqrt{17}}{272}\,\pi\, a_4,\\
c_4(\delta_0^{(4)}) &= -\tfrac{1}{288}\left(132\sqrt{2} - 35\sqrt{3}\pi + 70\sqrt{3}\arcsin\tfrac{1}{5}\right) a_4 = -0.0716687...a_4,\\
c_5(\delta_0^{(4)}) &= -\tfrac{1}{288}\left(35\sqrt{3}\pi + 132\sqrt{2} + 70\sqrt{3}\arcsin\tfrac{1}{5}\right) a_4 = -1.39423...a_4,\\
\bar{c}_5(\delta_0^{(4)}) &= c_4(\delta_0^{(4)}) + c_5(\delta_0^{(4)}) = -\tfrac{1}{72}\left(35\sqrt{3}\arcsin\tfrac{1}{5} + 66\sqrt{2}\right) a_4.
\end{aligned}$$

We also have

$$a_4 M(h, \delta_0^{(4)}) < 0,\ \text{for } 0 < h \ll 1,\ a_4 M(1000, \delta_0^{(4)}) = 35693.662...a_4^2 > 0,$$

if $a_4 \neq 0$. This shows that there exists $h^{(4)}$ such that $M(h^{(4)}, \delta_0^{(4)}) = 0$. By Corollary 2.3(iii) for $k = 3$ in Appendix of Xiong [26], there exist 8 limit cycles of system (3.1) for some (ε, δ) near $(0, \delta_0^{(4)})$. At the same time, $M(h, \delta_0^{(4)})$ has a simple zero near $h^{(4)}$ which leads to a limit cycle. Thus, system (3.1) has 9 limit cycles which shows that $H(4, 5) \geq 9$. By [31], it is obvious that $H(5, 5) \geq 9$.

(8) $n = 3$.

There exists $\delta_0^{(3)} = (a_0^*, a_1^*, a_2^*, a_3)$ with

$$a_0^* = 0,\ a_1^* = \frac{5}{6}a_3,\ a_2^* = -\frac{85}{42}a_3,$$

such that

$$\text{rank}\frac{\partial(c_0(\delta), c_1(\delta), c_2(\delta))}{\partial\delta}\bigg|_{\delta=\delta_0^{(3)}} = 3,\ c_0(\delta_0^{(3)}) = c_1(\delta_0^{(3)}) = c_2(\delta_0^{(3)}) = 0$$

and

$$c_3(\delta_0^{(3)}) = \frac{2\sqrt{2}}{7}a_3,\ b_{l,0}(\delta_0^{(3)}) = \frac{10\sqrt{34}}{1071}\pi a_3,\ b_{r,0}(\delta_0^{(3)}) = \frac{4\sqrt{17}}{119}\pi a_3.$$

By Corollary 2.3 (ii) for $k = 2$ in [26], system (3.1) has 5 limit cycles for some (ε, δ) near $(0, \delta_0^{(3)})$ which yields that $H(3, 5) \geq 5$.

(9) $n = 2$.

There exists $\delta_0^{(2)} = (a_0^*, a_1^*, a_2)$ with

$$a_0^* = \frac{25}{36}a_2,\ a_1^* = -\frac{23}{12}a_2,$$

such that

$$\mathrm{rank}\frac{\partial(c_0(\delta), c_1(\delta))}{\partial\delta}\bigg|_{\delta=\delta_0^{(2)}} = 2,\ c_0(\delta_0^{(2)}) = c_1(\delta_0^{(2)}) = 0$$

and

$$\begin{aligned}
&c_2(\delta_0^{(2)}) = -\tfrac{5\sqrt{2}}{18}18^{\frac{1}{3}}5^{\frac{2}{3}}a_2,\ c_3(\delta_0^{(2)}) = \tfrac{\sqrt{2}}{3}a_2,\\
&b_{l,0}(\delta_0^{(2)}) = -\tfrac{\sqrt{34}}{17}\pi a_2,\ b_{r,0}(\delta_0^{(2)}) = \tfrac{5\sqrt{17}}{153}\pi a_2,\\
&M(1000, \delta_0^{(2)}) = 3702.7782608...a_2.
\end{aligned}$$

By Corollary 2.3 (ii) for $k = 1$ in [26], system (3.1) has 3 limit cycles for some (ε, δ) near $(0, \delta_0^{(2)})$ which yields that $H(2, 5) \geq 3$.

(10) $n = 1$.

There exists $\delta_0^{(1)} = (a_0^*, a_1)$ with

$$a_0^* = -\frac{1}{30}\rho_2\left(116724\sqrt{2} - 15625\sqrt{3}\pi - 31250\sqrt{3}\arcsin\left(\frac{1}{5}\right)\right)a_1,$$

such that

$$\begin{aligned}
c_0(\delta_0^{(1)}) &= 0,\ c_1(\delta_0^{(1)}) = \tfrac{800}{9}\sqrt{6}\rho_2\pi a_1,\\
c_2(\delta_0^{(1)}) &= \tfrac{2^{\frac{1}{2}}5^{\frac{2}{3}}18^{\frac{1}{3}}}{75}\rho_2\left(116724\sqrt{2} - 15625\sqrt{3}\pi - 31250\sqrt{3}\arcsin\left(\tfrac{1}{5}\right)\right)a_1\\
&= 2.32008893...a_1,\\
b_{l,0}(\delta_0^{(1)}) &= -\tfrac{9\sqrt{34}\pi}{850}\rho_2\left(170772\sqrt{2} - 40625\sqrt{3}\pi - 81250\sqrt{3}\arcsin\left(\tfrac{1}{5}\right)\right)a_1\\
&= 0.35513382...a_1,\\
b_{r,0}(\delta_0^{(1)}) &= \tfrac{64\sqrt{17}\pi}{255}\rho_2\left(2796\sqrt{2} + 625\sqrt{3}\pi + 1250\sqrt{3}\arcsin\left(\tfrac{1}{5}\right)\right)a_1\\
&= 5.88095014...a_1,
\end{aligned}$$

and $M(1000, \delta_0^{(1)}) = 55.110058...a_1$, where

$$\rho_2 = \frac{1}{-125\sqrt{3}\pi - 250\sqrt{3}\arcsin\left(\frac{1}{5}\right) + 3588\sqrt{2}} > 0.$$

By Corollary 2.3 (iii) for $k = 0$ in [26], system (3.1) has one limit cycle for some (ε, δ) near $(0, \delta_0^{(1)})$ which yields that $H(1, 5) \geq 1$.

(11) $12 \leq n \leq 19$.

For $12 \leq n \leq 19$, we can find $\delta_0^{(n)}$ such that (2.6) and (2.7) hold for some i and k. By [31] we know that $a_{6i-1}, i \geq 1$ has no effect on the zeros of the Melnikov function. So, similar to the proof for $n = 20$, let $\bar{\delta}_0^{(n)} = \delta_0^{(n)}|_{a_{6i-1}=0, i\geq 1}$. We can get $M(1000, \bar{\delta}_0^{(n)})$ and $M(h, \bar{\delta}_0^{(n)})(0 < -h \ll 1)$ as follows. If $M(1000, \bar{\delta}_0^{(n)})M(h, \bar{\delta}_0^{(n)}) < 0$ for $0 < -h \ll 1$, then there exists $h^{(n)}$ such that $M(h^{(n)}, \bar{\delta}_0^{(n)}) = 0$ which yields a limit cycle outside the compound loop.

For $12 \leq n \leq 19$, the results of $M(1000, \bar{\delta}_0^{(n)})$, $c_8(\bar{\delta}_0^{(n)})$ and the sign of

$$M(1000, \bar{\delta}_0^{(n)})M(h, \bar{\delta}_0^{(n)})$$

for $0 < h \ll 1$, denoted by "Sign", are listed in the following table.

n	$M(1000,\bar{\delta}_0^{(n)})$	$c_8(\bar{\delta}_0^{(n)})$	Sign
12	$0.1326413295... \times 10^{10}\, a_{12}$	$-\frac{1155\sqrt{2}}{27392}a_{12}$	+
13	$0.2736600916... \times 10^{8}\, a_{13}$	$\frac{46923776\sqrt{2}}{3960046827}a_{13}$	-
14	$0.1831658452... \times 10^{11} a_{14}$	$\frac{13324288\sqrt{2}}{87972561}a_{14}$	-
15	$-0.1375283302... \times 10^{9}\, a_{15}$	$-\frac{166789120\sqrt{2}}{85570799301}a_{15}$	-
16	$0.2835166607... \times 10^{12}\, a_{16}$	$\frac{109665665024\sqrt{2}}{1339584728265}a_{16}$	-
18	$0.4495797844... \times 10^{13}\, a_{18}$	$\frac{32923320639422464\sqrt{2}}{1221465343958812953}a_{18}$	-
19	$-0.5005575888... \times 10^{11}\, a_{19}$	$-\frac{28906830169674612736\sqrt{2}}{7262814863307964584224 7}a_{19}$	-

The following table shows the results of $c_8(\bar{\delta}_0^{(n)})$, $b_{l,i}(\bar{\delta}_0^{(n)})$, $b_{r,k}(\bar{\delta}_0^{(n)})$ and the lower bounds of $H(n,5)$ for $n=12,13,14,15,16,18,19$.

n	$c_8(\bar{\delta}_0^{(n)}), b_{l,i}(\bar{\delta}_0^{(n)}), b_{r,k}(\bar{\delta}_0^{(n)})$	The lower bound of $H(n,5)$
12	$c_8 = -\frac{1155\sqrt{2}}{27392}\, a_{12}$ $b_{l,0} = -\frac{2899211875\sqrt{34}\,\pi}{5969374894512}\, a_{12}$ $b_{r,2} = \frac{1409933\sqrt{17}\pi}{71493976}\, a_{12}$	17
13	$c_8 = \frac{46923776\sqrt{2}}{3960046827}\, a_{13}$ $b_{l,3} = -\frac{64645052944752\sqrt{34}\,\pi}{1092660076931225}\, a_{13}$ $b_{r,0} = \frac{1718105\sqrt{17}\pi}{2484735264}\, a_{13}$	19
14	$c_8 = \frac{13324288\sqrt{2}}{87972561}a_{14}$ $b_{l,4} = -\frac{211466643362658885 4551\sqrt{34}\,\pi}{594214715080684 00000}a_{14}$ $b_{r,0} = \frac{8330301019\sqrt{17}\pi}{480447479808}\, a_{14}$	20
15	$c_8 = -\frac{166789120\sqrt{2}}{85570799301}\, a_{15}$ $b_{l,4} = \frac{58007501654167989\sqrt{34}\,\pi}{174894669893905000}\, a_{15}$ $b_{r,1} = -\frac{26849545\sqrt{17}\pi}{8776492236}\, a_{15}$	21
16	$c_8 = \frac{109665665024\sqrt{2}}{1339584728265}\, a_{16}$ $b_{l,5} = -\frac{75949001371481415202966941\sqrt{34}}{118208723563148000000000}\pi\, a_{16}$ $b_{r,1} = \frac{193192824246859\sqrt{17}\pi}{914489841162240}\, a_{16}$	22
18	$c_8 = \frac{32923320639422464\sqrt{2}}{1221465343958812953}\, a_{18}$ $b_{l,6} = -\frac{1458378600285388813351022742800 13\sqrt{34}\,\pi}{150600114781550493740800000 00}\, a_{18}$ $b_{r,1} = \frac{142030050983063483245\sqrt{17}\pi}{121287807244976917708 8}\, a_{18}$	23
19	$c_8 = -\frac{28906830169674612736\sqrt{2}}{72628148633079645842247}\, a_{19}$ $b_{l,6} = \frac{143332026286914762728864018216822 1\sqrt{34}\,\pi}{118009218610552871641189375000 00}\, a_{19}$ $b_{r,2} = -\frac{3593799255949532925 40\sqrt{17}\pi}{128167321117199375015 73}\, a_{19}$	24

From $H(16,5)\geq 22$ and [31], it follows that $H(17,5)\geq 22$.

This ends the proof of Theorem 1.1.

We should note that for the cases of $1\leq n\leq 9$ the results of $b_{l,0}$ and $b_{r,0}$ in our paper are different with ones in (3.7) and (3.8) of [26].

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