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ON THE NUMBER OF LIMIT CYCLES BY PERTURBING A PIECEWISE SMOOTH HAMILTON SYSTEM WITH TWO STRAIGHT LINES OF SEPARATION*

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Abstract This paper deals with the problem of limit cycle bifurcations for a piecewise smooth Hamilton system with two straight lines of separation. By analyzing the obtained first order Melnikov function, we give upper and lower bounds of the number of limit cycles bifurcating from the period annulus between the origin and the generalized homoclinic loop. It is found that the first order Melnikov function is more complicated than in the case with one straight line of separation and more limit cycles can be bifurcated.

Keywords Piecewise smooth Hamilton system, limit cycle, generalized homoclinic loop, Melnikov function.

MSC(2010) 34C07, 34C05.

1. Introduction and main results

Since the non-smooth phenomena exist widely in various practical applications and natural fields, such as automatic control, neural network, electrical engineering, economics, ecosystem, a big interest has appeared for studying bifurcation theory, especially bifurcation of limit cycles for planar piecewise smooth differential systems.

As pointed out by Kukucka [13], it is usually a nontrivial task to extend the bifurcation theory of smooth differential systems to non-smooth differential systems. So in recent years, the bifurcation of limit cycles for non-smooth differential systems with a straight line of separation has been investigated intensively and many innovative methods have been established. The Melnikov function method was extended to piecewise smooth differential system in [9, 17]. In [17], Liu and Han derived the first order Melnikov function for planar piecewise smooth Hamilton systems which can be used to study the number of limit cycles for these systems. By using the Melnikov function method, Liang, Han and Romanovski [15] studied the number of limit cycles by perturbing a piecewise smooth linear Hamilton system with a

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generalized homoclinic loop around the origin, which takes the form

$$\begin{cases} \dot{x} = -y, \\ \dot{y} = 1 - x, \end{cases} \quad x \geq 0, \qquad \begin{cases} \dot{x} = -y, \\ \dot{y} = x, \end{cases} \quad x < 0. \tag{1.1}$$

For more results about this method, one can see [1, 6, 16, 23, 24, 26–28] and the references therein. Another important method called the averaging method which can be used to detect limit cycles for non-smooth differential systems is developed in [8, 19, 20]. More results on this topic can be found in [2, 3, 5, 7, 14, 18, 21]. Recently, Yang and Zhao [29] extended the Picard-Fuchs method to study the limit cycle bifurcations for piecewise smooth differential systems with a straight line of separation.

However, as far as we know, there are a few papers for estimating the number of limit cycles of piecewise smooth differential systems with two straight lines of separation. In [11, 22], Wang, Han and Constantinescu considered the general form of a piecewise near-Hamiltonian system with two straight lines of separation

$$\begin{cases} \dot{x} = H^1_y(x, y) + \varepsilon f^1(x, y), \\ \dot{y} = -H^1_x(x, y) + \varepsilon g^1(x, y), \end{cases} \quad x > 0, \ y > 0, \tag{1.2}$$

$$\begin{cases} \dot{x} = H^2_y(x, y) + \varepsilon f^2(x, y), \\ \dot{y} = -H^2_x(x, y) + \varepsilon g^2(x, y), \end{cases} \quad x < 0, \ y > 0, \tag{1.3}$$

$$\begin{cases} \dot{x} = H^3_y(x, y) + \varepsilon f^3(x, y), \\ \dot{y} = -H^3_x(x, y) + \varepsilon g^3(x, y), \end{cases} \quad x < 0, \ y < 0 \tag{1.4}$$

and

$$\begin{cases} \dot{x} = H^4_y(x, y) + \varepsilon f^4(x, y), \\ \dot{y} = -H^4_x(x, y) + \varepsilon g^4(x, y), \end{cases} \quad x > 0, \ y < 0. \tag{1.5}$$

or

$$\begin{cases} \dot{x} = H_y(x, y) + \varepsilon f(x, y), \\ \dot{y} = -H_x(x, y) + \varepsilon g(x, y), \end{cases} \tag{1.6}$$

where $0 < |\varepsilon| \ll 1$, $H^i(x, y), f^i(x, y), g^i(x, y) \in C^\infty$, $i = 1, 2, 3, 4$,

$$H(x, y) = \begin{cases} H^1(x, y), \ x > 0, \ y > 0, \\ H^2(x, y), \ x < 0, \ y > 0, \\ H^3(x, y), \ x < 0, \ y < 0, \\ H^4(x, y), \ x > 0, \ y < 0, \end{cases}$$

$$f(x, y) = \begin{cases} f^1(x, y), \ x > 0, \ y > 0, \\ f^2(x, y), \ x < 0, \ y > 0, \\ f^3(x, y), \ x < 0, \ y < 0, \\ f^4(x, y), \ x > 0, \ y < 0, \end{cases} \qquad g(x, y) = \begin{cases} g^1(x, y), \ x > 0, \ y > 0, \\ g^2(x, y), \ x < 0, \ y > 0, \\ g^3(x, y), \ x < 0, \ y < 0, \\ g^4(x, y), \ x > 0, \ y < 0. \end{cases}$$

In order that system (1.6) has a family of periodic orbits near the origin for $\varepsilon = 0$, the following two assumptions are necessary, see [17, 22].

Assumption (I). There exist an interval Σ, and four points $A = (a(h), 0)$, $B = (0, b(h))$, $C = (c(h), 0)$ and $D = (0, d(h))$ such that for all $h \in \Sigma$

$$H^1(A) = H^1(B) = h, \ H^2(B) = H^2(C), \ H^3(C) = H^3(D), \ H^4(D) = H^4(A)$$

with $a(h) \neq c(h)$ and $b(h) \neq d(h)$.

Assumption (II). The system $(1.2)|_{\varepsilon=0}$ has an orbital arc L_h^1 starting from A and ending at B defined by $H^1(x,y) = h, h \in \Sigma, x > 0, y > 0$; the system $(1.3)|_{\varepsilon=0}$ has an orbital arc L_h^2 starting from B and ending at C defined by $H^2(x,y) = H^2(B),\ x < 0, y > 0$; the system $(1.4)|_{\varepsilon=0}$ has an orbital arc L_h^3 starting from C and ending at D defined by $H^3(x,y) = H^3(C),\ x < 0, y < 0$, and the system $(1.5)|_{\varepsilon=0}$ has an orbital arc L_h^4 starting from D and ending at A defined by $H^4(x,y) = H^4(D),\ x > 0, y < 0$. Thus, $L_h = L_h^1 \cup L_h^2 \cup L_h^3 \cup L_h^4$ is a periodic orbit of $(1.6)|_{\varepsilon=0}$ surrounding the origin for $h \in \Sigma$.

Under assumptions (I) and (II), $\{L_h \,|\, h \in \Sigma\}$ is a family of periodic orbits of system $(1.6)|_{\varepsilon=0}$ and each L_h is piecewise smooth. Without loss of generality, suppose that L_h has an anticlockwise orientation, as shown in Fig. 1. From [22],

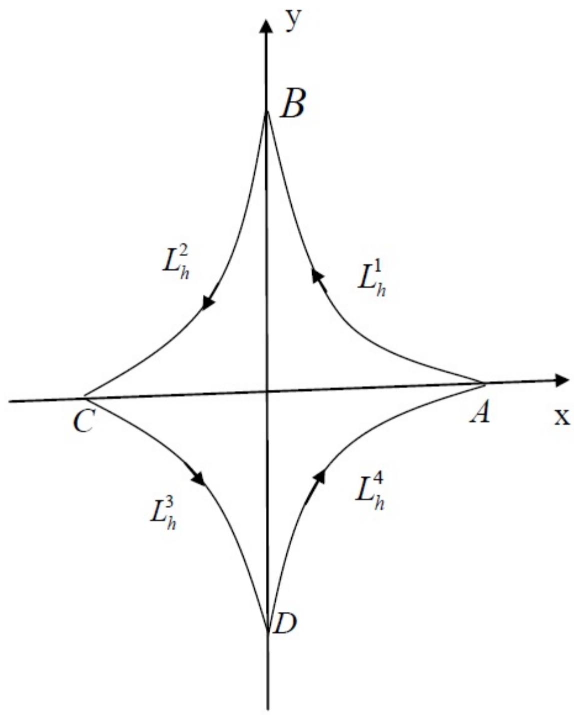

Figure 1. The closed orbits of system $(1.6)|_{\varepsilon=0}$.

the first order Melnikov function $M(h)$ of system (1.6) has the following form

$$\begin{aligned} M(h) =& \frac{H_x^1(A)H_y^2(B)H_x^3(C)H_y^4(D)}{H_x^4(A)H_y^1(B)H_x^2(C)H_y^3(D)} \int_{\widehat{AB}} g^1(x,y)dx - f^1(x,y)dy \\ &+ \frac{H_x^1(A)H_x^3(C)H_y^4(D)}{H_x^4(A)H_x^2(C)H_y^3(D)} \int_{\widehat{BC}} g^2(x,y)dx - f^2(x,y)dy \\ &+ \frac{H_x^1(A)H_y^4(D)}{H_x^4(A)H_y^3(D)} \int_{\widehat{CD}} g^3(x,y)dx - f^3(x,y)dy \\ &+ \frac{H_x^1(A)}{H_x^4(A)} \int_{\widehat{DA}} g^4(x,y)dx - f^4(x,y)dy,\ h \in \Sigma. \end{aligned} \tag{1.7}$$

Further, we know from [11,22] that if $M(h)$ has at most k zeros in h on the interval Σ, then (1.6) has at most k limit cycles bifurcated from the period annulus $\cup_{h\in\Sigma}L_h$. In [12], by using the averaging method of first order, Itikawa et al. obtained the upper bounds of the number of limit cycles bifurcating from the periodic orbits of two kind of isochronous systems, when they are perturbed inside the discontinuous quadratic and cubic polynomials differential systems with two straight lines of separation, respectively. In [25], Xiong investigated the limit cycle bifurcation in perturbations of non-smooth Hamiltonian systems with 4 switching lines via multiple parameters.

In the present paper, motivated by the above references, we will study the number of limit cycles for a piecewise smooth Hamilton system with a generalized homoclinic loop under the perturbations of piecewise polynomials of degree n. More precisely, we consider the following piecewise smooth near-Hamilton system with two straight lines of separation

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\begin{cases}\begin{pmatrix}-y+\varepsilon f^1(x,y)\\ 1-x+\varepsilon g^1(x,y)\end{pmatrix}, & x>0,\ y>0,\\ \begin{pmatrix}-y+\varepsilon f^2(x,y)\\ x+\varepsilon g^2(x,y)\end{pmatrix}, & x<0,\ y>0,\\ \begin{pmatrix}-y+\varepsilon f^3(x,y)\\ x+\varepsilon g^3(x,y)\end{pmatrix}, & x<0,\ y<0,\\ \begin{pmatrix}-y+\varepsilon f^4(x,y)\\ 1-x+\varepsilon g^4(x,y)\end{pmatrix}, & x>0,\ y<0,\end{cases} \tag{1.8}$$

where

$$f^k(x,y)=\sum_{i+j=0}^{n}a_{i,j}^k x^i y^j,\quad g^k(x,y)=\sum_{i+j=0}^{n}b_{i,j}^k x^i y^j,\ k=1,2,3,4.$$

A first integral of system $(1.8)|_{\varepsilon=0}$ is

$$\begin{aligned}H^1(x,y)&=\frac{1}{2}[(x-1)^2-y^2]=\frac{h}{2},\ x>0,y>0,\\ H^2(x,y)&=-\frac{1}{2}[x^2+y^2]=\frac{h-1}{2},\ x<0,y>0,\\ H^3(x,y)&=-\frac{1}{2}[x^2+y^2]=\frac{h-1}{2},\ x<0,y<0,\\ H^4(x,y)&=\frac{1}{2}[(x-1)^2-y^2]=\frac{h}{2},\ x>0,y<0\end{aligned} \tag{1.9}$$

with $h\in(0,1)$. When $\varepsilon=0$, (1.8) has a family of piecewise smooth periodic orbits as follows

$$\begin{aligned}L_h=&\{(x,y)|H^1(x,y)=\frac{h}{2},x>0,y>0\}\\ &\cup\{(x,y)|H^2(x,y)=\frac{h-1}{2},x<0,y>0\}\\ &\cup\{(x,y)|H^3(x,y)=\frac{h-1}{2},x<0,y<0\}\end{aligned}$$

$$\cup\{(x,y)|H^4(x,y)=\frac{h}{2},x>0,y<0\}$$
$$:=L_h^1\cup L_h^2\cup L_h^3\cup L_h^4$$

with $h\in(0,1)$. If $h\to 1$, L_h approaches the origin which is an elementary center of parabolic-focus type, see [4,10]. If $h\to 0$, L_h approaches the generalized homoclinic loop L_0 with a saddle $S(1,0)$, see Fig. 2.

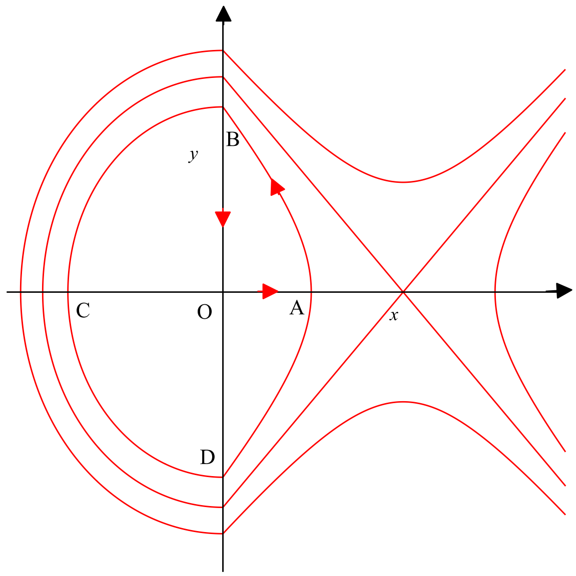

Figure 2. The phase portrait of system $(1.8)|_{\varepsilon=0}$.

Applying the above first order Melnikov function (1.7), we obtain the upper and lower bounds of the number of limit cycles which bifurcate from the period annulus of system $(1.8)|_{\varepsilon=0}$. Our main results are as follows:

Theorem 1.1. *The first order Melnikov function of system (1.8) is*

$$M(h)=P_{[\frac{n}{2}]}(h)\sqrt{1-h}+P_{n+1}(\sqrt{h})+P_{n+1}(\sqrt{1-h})+P_{[\frac{n-1}{2}]}(h)\Theta(h),\quad(1.10)$$

where $h\in(0,1)$, $P_l(u)$ is a polynomial of u with degree l and

$$\Theta(h)=\int_0^{\sqrt{1-h}}\sqrt{h+y^2}dy.\quad(1.11)$$

Theorem 1.2. *Consider system (1.8) with $|\varepsilon|$ small enough. Using the first order Melnikov function (1.7), the number of limit cycles bifurcating from the period annulus is no more than $2n+5[\frac{n-1}{2}]+15$, if $n\geq 2$; 4 if $n=1$, and at least $2n+1$ if $n\geq 1$.*

Corollary 1.1. *If $f^1(x,y)=f^4(x,y)$, $g^1(x,y)=g^4(x,y)$, $f^2(x,y)=f^3(x,y)$ and $g^2(x,y)=g^3(x,y)$, that is, the straight line of separation is $x=0$, then, using the first order Melnikov function (1.7), the number of limit cycles bifurcating from the period annulus is no more than $2[\frac{n}{2}]+3[\frac{n-1}{2}]+3$ (resp. $5[\frac{n-1}{2}]+4$) if n is an even (resp. odd) number.*

Remark 1.1. If $f^1(x,y) = f^4(x,y)$, $g^1(x,y) = g^4(x,y)$, $f^2(x,y) = f^3(x,y)$ and $g^2(x,y) = g^3(x,y)$, then (1.8) becomes (1.1) for $\varepsilon = 0$. The lower bound of the number of limit cycles bifurcating from the period annulus between the origin and the generalized homoclinic loop is $n + [\frac{n+1}{2}]$, see Theorem 1.1 in [15]. The upper bound of the number of limit cycles also can be found in [15], but here we provide an alternative proof which is much more digestible.

The layout of the rest of this paper is as follows. The algebraic structure of the first order Melnikov function $M(h)$ and the proof of Theorem 1.1 will be given in section 2. The Theorem 1.2 and Corollary 1.3 will be proved in sections 3 and 4.

2. Algebraic structure of $M(h)$

In the following, we will obtain the algebraic structure of $M(h)$ of system (1.8). Noting that (1.7) and

$$\frac{H_x^1(A)H_y^2(B)H_x^3(C)H_y^4(D)}{H_x^4(A)H_y^1(B)H_x^2(C)H_y^3(D)} = \frac{H_x^1(A)H_x^3(C)H_y^4(D)}{H_x^4(A)H_x^2(C)H_y^3(D)} = \frac{H_x^1(A)H_y^4(D)}{H_x^4(A)H_y^3(D)} = \frac{H_x^1(A)}{H_x^4(A)} = 1,$$

we obtain that the first Melnikov function $M(h)$ has the form

$$\begin{aligned} M(h) = &\int_{L_h^1} g^1(x,y)dx - f^1(x,y)dy + \int_{L_h^2} g^2(x,y)dx - f^2(x,y)dy \\ &+ \int_{L_h^3} g^3(x,y)dx - f^3(x,y)dy + \int_{L_h^4} g^4(x,y)dx - f^4(x,y)dy. \end{aligned} \tag{2.1}$$

For $h \in (0,1)$ and $i, j \in \mathbb{N}$, we denote

$$\begin{aligned} I_{i,j}(h) &= \int_{L_h^1} x^i y^j dy, \quad J_{i,j}(h) = \int_{L_h^2} x^i y^j dy, \\ \hat{J}_{i,j}(h) &= \int_{L_h^3} x^i y^j dy, \quad \hat{I}_{i,j}(h) = \int_{L_h^4} x^i y^j dy. \end{aligned}$$

It is easy to check that

$$\hat{I}_{i,j}(h) = (-1)^j I_{i,j}(h), \quad \hat{J}_{i,j}(h) = (-1)^j J_{i,j}(h). \tag{2.2}$$

Lemma 2.1. *For $h \in (0,1)$ and $n \geq 2$,*

$$\begin{aligned} M(h) = &\alpha_1(h)I_{0,0}(h) + \beta_1(h)I_{1,0}(h) + \gamma_1(h)I_{0,1}(h) + \delta_1(h)I_{1,1}(h) \\ &+ \alpha_2(h)J_{0,0}(h) + \beta_2(h)J_{1,0}(h) + \gamma_2(h)J_{0,1}(h) + \delta_2(h)J_{1,1}(h) \\ &+ \varphi_{n+1}(\sqrt{h}) + \psi_{n+1}(\sqrt{1-h}), \end{aligned} \tag{2.3}$$

where $\varphi_{n+1}(u)$ is a polynomial in u of degree $n+1$, $\psi_{n+1}(u)$ is a polynomial in u of degree $n+1$ without constant term, and $\alpha_k(h)$, $\beta_k(h)$, $\gamma_k(h)$ and $\delta_k(h)$ are polynomials of h with

$$\deg \alpha_k(h) \leq [\frac{n}{2}], \ \deg \beta_k(h), \deg \gamma_k(h) \leq [\frac{n-1}{2}], \ \deg \delta_k(h) \leq [\frac{n-2}{2}], \ k = 1, 2.$$

Proof. For the sake of clearness, we split the proof into two steps.

(1) *We first assert that*

$$M(h) = \sum_{i+j=0}^{n} \tau_{i,j} I_{i,j}(h) + \sum_{i+j=0}^{n} \sigma_{i,j} J_{i,j}(h) + \varphi_{n+1}(\sqrt{h}) + \psi_{n+1}(\sqrt{1-h}), \quad (2.4)$$

where $\tau_{i,j}$ and $\sigma_{i,j}$ are arbitrary real constants, $\varphi_{n+1}(u)$ is a polynomial in u of degree $n+1$ and $\psi_{n+1}(u)$ is a polynomial in u of degree $n+1$ without constant term.

In fact, Let Ω be the interior of $L_h^1 \cup \overrightarrow{BO} \cup \overrightarrow{OA}$, see Fig. 1. Using the Green's Formula, we have for $i \geq 0$ and $j \geq 1$

$$\begin{aligned} \int_{L_h^1} x^i y^j \, dx &= \oint_{L_h^1 \cup \overrightarrow{BO} \cup \overrightarrow{OA}} x^i y^j \, dx = j \iint_{\Omega} x^i y^{j-1} dx dy \\ &= -\frac{j}{i+1} \oint_{L_h^1 \overrightarrow{BO} \cup \overrightarrow{OA}} x^{i+1} y^{j-1} dy \\ &= -\frac{j}{i+1} I_{i+1,j-1}(h). \end{aligned} \quad (2.5)$$

In a similar way, we have for $i \geq 0$ and $j \geq 1$

$$\begin{aligned} \int_{L_h^2} x^i y^j \, dx &= -\frac{j}{i+1} J_{i+1,j-1}(h), \\ \int_{L_h^3} x^i y^j \, dx &= -\frac{j}{i+1} \hat{J}_{i+1,j-1}(h), \\ \int_{L_h^4} x^i y^j \, dx &= -\frac{j}{i+1} \hat{I}_{i+1,j-1}(h). \end{aligned} \quad (2.6)$$

On the other hand, we have for $i \geq 0$ and $j = 0$

$$\begin{aligned} \int_{L_h^1} x^i dx &= \oint_{L_h^1 \cup \overrightarrow{BO} \cup \overrightarrow{OA}} x^i dx - \int_{\overrightarrow{BO}} x^i dx - \int_{\overrightarrow{OA}} x^i dx = -\int_{\overrightarrow{OA}} x^i dx, \\ \int_{L_h^2} x^i dx &= \oint_{L_h^2 \cup \overrightarrow{CO} \cup \overrightarrow{OB}} x^i dx - \int_{\overrightarrow{CO}} x^i dx - \int_{\overrightarrow{OB}} x^i dx = \int_{\overrightarrow{OC}} x^i dx, \\ \int_{L_h^3} x^i dx &= \oint_{L_h^3 \cup \overrightarrow{DO} \cup \overrightarrow{OC}} x^i dx - \int_{\overrightarrow{DO}} x^i dx - \int_{\overrightarrow{OC}} x^i dx = -\int_{\overrightarrow{OC}} x^i dx, \\ \int_{L_h^4} x^i dx &= \oint_{L_h^4 \cup \overrightarrow{AO} \cup \overrightarrow{OD}} x^i dx - \int_{\overrightarrow{AO}} x^i dx - \int_{\overrightarrow{OD}} x^i dx = \int_{\overrightarrow{OA}} x^i dx. \end{aligned} \quad (2.7)$$

From (2.1), (2.2) and (2.5)-(2.7), we obtain

$$\begin{aligned} M(h) = &\sum_{i=0}^{n} b_{i,0}^1 \int_{L_h^1} x^i dx - \sum_{\substack{i+j=1, \\ i \geq 0, j \geq 1}}^{n} \frac{j}{i+1} b_{i,j}^1 I_{i+1,j-1}(h) - \sum_{i+j=0}^{n} a_{i,j}^1 I_{i,j}(h) \\ &+ \sum_{i=0}^{n} b_{i,0}^2 \int_{L_h^2} x^i dx - \sum_{\substack{i+j=1, \\ i \geq 0, j \geq 1}}^{n} \frac{j}{i+1} b_{i,j}^2 J_{i+1,j-1}(h) - \sum_{i+j=0}^{n} a_{i,j}^2 J_{i,j}(h) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=0}^{n} b_{i,0}^{3}\int_{L_h^3} x^i dx - \sum_{\substack{i+j=1,\\ i\geq 0, j\geq 1}}^{n} \frac{j}{i+1} b_{i,j}^{3}\hat{J}_{i+1,j-1}(h) - \sum_{i+j=0}^{n} a_{i,j}^{3}\hat{J}_{i,j}(h)\\
&+\sum_{i=0}^{n} b_{i,0}^{4}\int_{L_h^4} x^i dx - \sum_{\substack{i+j=1,\\ i\geq 0, j\geq 1}}^{n} \frac{j}{i+1} b_{i,j}^{4}\hat{I}_{i+1,j-1}(h) - \sum_{i+j=0}^{n} a_{i,j}^{4}\hat{I}_{i,j}(h)\\
=&\sum_{i+j=0}^{n}\tau_{i,j}I_{i,j}(h) + \sum_{i+j=0}^{n}\sigma_{i,j}J_{i,j}(h)\\
&+\sum_{i=0}^{n}(b_{i,0}^{4}-b_{i,0}^{1})\int_{\overrightarrow{OA}} x^i dx + \sum_{i=0}^{n}(b_{i,0}^{2}-b_{i,0}^{3})\int_{\overrightarrow{OC}} x^i dx,
\end{aligned}
\tag{2.8}
$$

where

$$
\begin{aligned}
\tau_{0,j} &= -a_{0,j}^{1} - a_{0,j}^{4}, \quad \sigma_{0,j} = -a_{0,j}^{1} - a_{0,j}^{4},\ j\geq 0,\\
\tau_{i,j} &= -[a_{i,j}^{1} + (-1)^j a_{i,j}^{4}] - \frac{j+1}{i}[b_{i,j}^{1} + (-1)^j b_{i,j}^{4}],\ i\geq 1, j\geq 0,\\
\sigma_{i,j} &= -[a_{i,j}^{1} + (-1)^j a_{i,j}^{4}] - \frac{j+1}{i}[b_{i,j}^{1} + (-1)^j b_{i,j}^{4}],\ i\geq 1, j\geq 0.
\end{aligned}
$$

Thus, $\tau_{i,j}$ and $\sigma_{i,j}$ can be chosen arbitrarily. By direct computation, we have

$$
\begin{aligned}
\sum_{i=0}^{n}(b_{i,0}^{4}-b_{i,0}^{1})\int_{\overrightarrow{OA}} x^i dx =& \sum_{i=0}^{n}(b_{i,0}^{4}-b_{i,0}^{1})\int_{0}^{1-\sqrt{h}} x^i dx\\
=&\sum_{i=0}^{n}\frac{b_{i,0}^{4}-b_{i,0}^{1}}{i+1}(1-\sqrt{h})^{i+1}\\
:=&\varphi_{n+1}(\sqrt{h}),
\end{aligned}
\tag{2.9}
$$

$$
\begin{aligned}
\sum_{i=0}^{n}(b_{i,0}^{2}-b_{i,0}^{3})\int_{\overrightarrow{OC}} x^i dx =& \sum_{i=0}^{n}(b_{i,0}^{2}-b_{i,0}^{3})\int_{0}^{-\sqrt{1-h}} x^i dx\\
=&\sum_{i=0}^{n}\frac{b_{i,0}^{2}-b_{i,0}^{3}}{i+1}(-1)^{i+1}(1-h)^{\frac{i+1}{2}}\\
:=&\psi_{n+1}(\sqrt{1-h}),
\end{aligned}
$$

where $\varphi_{n+1}(u)$ is a polynomial of u with degree $n+1$ and $\psi_{n+1}(u)$ is a polynomial of u with degree $n+1$ without constant term. Substituting (2.9) into (2.8) gives (2.4).

(2) *We assert that*

$$
\begin{aligned}
\sum_{i+j=0}^{n}\tau_{i,j}I_{i,j}(h) &= \alpha_1(h)I_{0,0}(h)+\beta_1(h)I_{1,0}(h)+\gamma_1(h)I_{0,1}(h)+\delta_1(h)I_{1,1}(h),\\
\sum_{i+j=0}^{n}\sigma_{i,j}J_{i,j}(h) &= \alpha_2(h)J_{0,0}(h)+\beta_2(h)J_{1,0}(h)+\gamma_2(h)J_{0,1}(h)+\delta_2(h)J_{1,1}(h),
\end{aligned}
\tag{2.10}
$$

where $\alpha_k(h)$, $\beta_k(h)$, $\gamma_k(h)$ and $\delta_k(h)$ are polynomials of h with

$$
\deg\alpha_k(h)\leq[\frac{n}{2}],\ \deg\beta_k(h),\deg\gamma_k(h)\leq[\frac{n-1}{2}],\ \deg\delta_k(h)\leq[\frac{n-2}{2}],\ k=1,2.
$$

Without loss of generality, we only prove the first equality in (2.10), and the other one can be shown in a similar way. Differentiating $H^1(x,y)=\frac{h}{2}$ defined in (1.9) with respect to y, we have

$$-y+x\frac{\partial x}{\partial y}-\frac{\partial x}{\partial y}=0. \tag{2.11}$$

Multiplying (2.11) by $x^iy^{j-1}dy$, integrating over L_h^1 and noting that (2.5), we have

$$I_{i,j}(h)=\frac{j-1}{i+1}I_{i+1,j-2}(h)-\frac{j-1}{i+2}I_{i+2,j-2}(h),\quad i\geq 0,\ j\geq 2. \tag{2.12}$$

On the other hand, multiplying $H^1(x,y)=\frac{h}{2}$ defined in (1.9) by $x^{i-2}y^jdy$ and integrating over L_h^1 yield

$$I_{i,j}(h)=(h-1)I_{i-2,j}(h)+I_{i-2,j+2}(h)+2I_{i-1,j}(h),\quad i\geq 0,\ j\geq 0. \tag{2.13}$$

From (2.12) and (2.13), we have

$$I_{i,j}(h)=-\frac{j-1}{i+j+1}\big[(h-1)I_{i,j-2}(h)+\frac{i}{i+1}I_{i+1,j-2}(h)\big],\quad i\geq 0,\ j\geq 2 \tag{2.14}$$

and

$$I_{i,j}(h)=\frac{i}{i+j+1}\big[(h-1)I_{i-2,j}(h)+\frac{2i+j-1}{i-1}I_{i-1,j}(h)\big],\ i\geq 2,\ j\geq 0. \tag{2.15}$$

We will prove the conclusion by induction on n. It could be noticed that $n=2$ corresponds to $(i,j)=(0,2)$ and $(2,0)$ and $n=3$ corresponds to $(i,j)=(0,3),(1,2),(2,1)$ and $(3,0)$. Hence, in view of (2.14) and (2.15), we have for $n=2,3$

$$\begin{cases} I_{0,2}(h)=-\frac{1}{3}(h-1)I_{0,0}(h),\\ I_{2,0}(h)=\frac{2}{3}(h-1)I_{0,0}(h)+2I_{1,0}(h),\\ I_{0,3}(h)=-\frac{1}{2}(h-1)I_{0,1}(h),\\ I_{1,2}(h)=-\frac{1}{4}(h-1)I_{1,0}(h)-\frac{1}{8}I_{2,0}(h),\\ I_{2,1}(h)=\frac{1}{2}(h-1)I_{0,1}(h)+2I_{1,1}(h),\\ I_{3,0}(h)=\frac{3}{4}(h-1)I_{1,0}(h)+\frac{15}{8}I_{2,0}(h), \end{cases} \tag{2.16}$$

which yield the conclusion for $n=2,3$. Now assume that the result holds for all $i+j\leq n-1$ ($n\geq 5$). Then, for $i+j=n$, taking $(i,j)=(0,n),(1,n-1),(2,n-2),\cdots,(n-2,2)$ in (2.14) and $(i,j)=(n-1,1),(n,0)$ in (2.15), respectively, we obtain

$$\begin{pmatrix} I_{0,n}(h)\\ I_{1,n-1}(h)\\ I_{2,n-2}(h)\\ \vdots\\ I_{n-2,2}(h)\\ I_{n-1,1}(h)\\ I_{n,0}(h) \end{pmatrix}=-\frac{1}{n+1}\begin{pmatrix} (n-1)(h-1)I_{0,n-2}(h)\\ (n-2)\big((h-1)I_{1,n-3}(h)+\frac{1}{2}I_{2,n-3}(h)\big)\\ (n-3)\big((h-1)I_{2,n-4}(h)+\frac{2}{3}I_{3,n-4}(h)\big)\\ \vdots\\ (h-1)I_{n-2,0}(h)+\frac{n-2}{n-1}I_{n-1,0}(h)\\ (1-n)\big((h-1)I_{n-3,1}(h)+\frac{2n-2}{n-2}I_{n-2,1}(h)\big)\\ -n\big((h-1)I_{n-2,0}(h)+\frac{2n-1}{n-1}I_{n-1,0}(h)\big) \end{pmatrix}, \tag{2.17}$$

which implies the first equality in (2.10) holds for $i+j=n$.

Now we discuss the degree of polynomials $\alpha_1(h)$, $\beta_1(h)$, $\gamma_1(h)$ and $\delta_1(h)$ defined in (2.10). In view of (2.17), we have for $(i,j)=(0,n)$

$$\begin{aligned} I_{0,n}(h)=&h\big[\alpha^{(n-2)}(h)I_{0,0}(h)+\beta^{(n-2)}(h)I_{1,0}(h)+\gamma^{(n-2)}(h)I_{0,1}(h)+\delta^{(n-2)}(h)I_{1,1}(h)\big]\\ :=&\alpha^{(n)}(h)I_{0,0}(h)+\beta^{(n)}(h)I_{1,0}(h)+\gamma^{(n)}(h)I_{0,1}(h)+\delta^{(n)}(h)I_{1,1}(h), \end{aligned}$$

where $\alpha^{(n-2)}(h),\beta^{(n-2)}(h)$, $\gamma^{(n-2)}(h)$ and $\delta^{(n-2)}(h)$ are polynomials of h satisfying

$$\begin{aligned} &\deg\alpha^{(n-2)}(h)\le[\frac{n-2}{2}],\ \deg\delta^{(n-2)}(h)\le[\frac{n-4}{2}],\\ &\deg\beta^{(n-2)}(h),\deg\gamma^{(n-2)}(h)\le[\frac{n-3}{2}]. \end{aligned}$$

It is easy to check that

$$\deg\alpha^{(n)}(h)\le[\frac{n}{2}],\ \deg\beta^{(n)}(h),\deg\gamma^{(n)}(h)\le[\frac{n-1}{2}],\ \deg\delta^{(n)}(h)\le[\frac{n-2}{2}].$$

If $(i,j)=(1,n-1),(2,n-2),\cdots,(n,0)$, then, by (2.17), we have

$$\begin{aligned} I_{i,j}(h)=&\alpha^{(n-1)}(h)I_{0,0}(h)+\beta^{(n-1)}(h)I_{1,0}(h)+\gamma^{(n-1)}(h)I_{0,1}(h)+\delta^{(n-1)}(h)I_{1,1}(h)\\ &+h\big[\alpha^{(n-2)}(h)I_{0,0}(h)+\beta^{(n-2)}(h)I_{1,0}(h)+\gamma^{(n-2)}(h)I_{0,1}(h)+\delta^{(n-2)}(h)I_{1,1}(h)\big]\\ :=&\alpha^{(n)}(h)I_{0,0}(h)+\beta^{(n)}(h)I_{1,0}(h)+\gamma^{(n)}(h)I_{0,1}(h)+\delta^{(n)}(h)I_{1,1}(h), \end{aligned}$$

where $\alpha^{(n-s)}(h),\beta^{(n-s)}(h)$, $\gamma^{(n-s)}(h)$ and $\delta^{(n-s)}(h)$ are polynomials of h satisfying

$$\begin{aligned} &\deg\alpha^{(n-s)}(h)\le[\frac{n-s}{2}],\ \deg\delta^{(n-s)}(h)\le[\frac{n-2-s}{2}],\\ &\deg\beta^{(n-s)}(h),\deg\gamma^{(n-s)}(h)\le[\frac{n-1-s}{2}],\ s=1,2. \end{aligned}$$

Hence, we have

$$\deg\alpha^{(n)}(h)\le[\frac{n}{2}],\ \deg\beta^{(n)}(h),\deg\gamma^{(n)}(h)\le[\frac{n-1}{2}],\ \deg\delta^{(n)}(h)\le[\frac{n-2}{2}].$$

To sum up, substituting (2.10) into (2.4), we obtain (2.3). The proof is completed. □

Proof of the Theorem 1.1. By some straightforward calculations, we have

$$\begin{aligned} &I_{0,0}(h)=\sqrt{1-h}, &&I_{1,0}(h)=\sqrt{1-h}-\Theta(h),\\ &I_{0,1}(h)=\frac{1}{2}(1-h), &&I_{1,1}(h)=\frac{1}{2}(1-h)-\frac{1}{3}(1-h^{\frac{3}{2}}),\\ &J_{0,0}(h)=-\sqrt{1-h}, &&J_{1,0}(h)=\frac{\pi}{4}(1-h),\\ &J_{0,1}(h)=-\frac{1}{2}(1-h), &&J_{1,1}(h)=\frac{1}{3}(1-h)^{\frac{3}{2}}, \end{aligned}\tag{2.18}$$

where $\Theta(h)$ is defined in (1.11). Substituting (2.18) into (2.3) gives (1.10). This completes the proof of Theorem 1.1. □

3. Proof of the Theorem 1.2

In the following, we denote by $P_l(h)$ the polynomial of h with degree no more than l and by $\#\{\phi(h)=0, h\in(\varrho_1,\varrho_2)\}$ the number of zeros of $\phi(h)$ on the open interval (ϱ_1,ϱ_2), taking into account the multiplicity.

If $n\geq 2$, let $\lambda=\sqrt{1-h}$, $\lambda\in(0,1)$, then we have

$$\Theta(\lambda)=\int_0^\lambda \sqrt{1-\lambda^2+y^2}dy, \tag{3.1}$$

and

$$\widetilde{M}(\lambda)=P_{[\frac{n}{2}]}(\lambda^2)\lambda+P_{n+1}(\sqrt{1-\lambda^2})+P_{n+1}(\lambda)+P_{[\frac{n-1}{2}]}(\lambda^2)\Theta(\lambda), \lambda\in(0,1). \tag{3.2}$$

Let $y=\sqrt{1-\lambda^2}x$, we have

$$\Theta(\lambda)=(1-\lambda^2)\int_0^{\frac{\lambda}{\sqrt{1-\lambda^2}}} \sqrt{1+x^2}dx:=(1-\lambda^2)\Theta_1(\lambda). \tag{3.3}$$

Hence, $\widetilde{M}(\lambda)$ can be written as

$$\begin{aligned}\widetilde{M}(\lambda)=&P_{[\frac{n}{2}]}(\lambda^2)\lambda+P_{n+1}(\sqrt{1-\lambda^2})+P_{n+1}(\lambda)+P_{[\frac{n-1}{2}]}(\lambda^2)(1-\lambda^2)\Theta_1(\lambda)\\ :=&P_{n+1}(\lambda)+P_{n+1}(\sqrt{1-\lambda^2})+P_{[\frac{n-1}{2}]+1}(\lambda^2)\Theta_1(\lambda).\end{aligned} \tag{3.4}$$

Clearly, $M(h)$ and $\widetilde{M}(\lambda)$ have the same number of zeros on $(0,1)$. Suppose that $\Sigma=(0,1)\setminus\{\lambda\in(0,1)|P_{[\frac{n-1}{2}]+1}(\lambda^2)=0\}$. By direct computation, we obtain for $h\in\Sigma$

$$\begin{aligned}M_1(\lambda)=&\frac{d}{d\lambda}\Big(\frac{\widetilde{M}(\lambda)}{P_{[\frac{n-1}{2}]+1}(\lambda^2)}\Big)\\ =&\frac{1}{(1-\lambda^2)^2}+\Big(\frac{P_{n+1}(\lambda)+P_{n+1}(\sqrt{1-\lambda^2})}{P_{[\frac{n-1}{2}]+1}(\lambda^2)}\Big)'\\ =&\frac{P_{n+2[\frac{n-1}{2}]+6}(\lambda)\sqrt{1-\lambda^2}+P_{2[\frac{n-1}{2}]+7}(\lambda)P_n(\sqrt{1-\lambda^2})+P_{2[\frac{n-1}{2}]+5}(\lambda)P_{n+2}(\sqrt{1-\lambda^2})}{P^2_{[\frac{n-1}{2}]+1}(\lambda^2)(1-\lambda^2)^2}\\ =&\begin{cases}\frac{P_{[\frac{n+2}{2}]}(\lambda^2)+P_{n+2[\frac{n-1}{2}]+6}(\lambda)\sqrt{1-\lambda^2}}{P^2_{[\frac{n-1}{2}]+1}(\lambda^2)(1-\lambda^2)^2}, & n \text{ even},\\ \frac{P_{[\frac{n+1}{2}]}(\lambda^2)+P_{n+2[\frac{n-1}{2}]+6}(\lambda)\sqrt{1-\lambda^2}}{P^2_{[\frac{n-1}{2}]+1}(\lambda^2)(1-\lambda^2)^2}, & n \text{ odd}.\end{cases}\end{aligned} \tag{3.5}$$

Next we will estimate the upper bound for the number of zeros of $\widetilde{M}(\lambda)$ on (0,1). When n is an even number, we have

$$M_1(\lambda)=\frac{P_{[\frac{n+2}{2}]}(\lambda^2)+P_{n+2[\frac{n-1}{2}]+6}(\lambda)\sqrt{1-\lambda^2}}{P^2_{[\frac{n-1}{2}]+1}(\lambda^2)(1-\lambda^2)^2}.$$

Let $P_{[\frac{n+2}{2}]}(\lambda^2)+P_{n+2[\frac{n-1}{2}]+6}(\lambda)\sqrt{1-\lambda^2}=0$. That is,

$$P_{[\frac{n+2}{2}]}(\lambda^2)=-P_{n+2[\frac{n-1}{2}]+6}(\lambda)\sqrt{1-\lambda^2}.$$

By squaring the above equation, we can deduce that the function $P_{[\frac{n+2}{2}]}(\lambda^2)+P_{n+2[\frac{n-1}{2}]+6}(\lambda)\sqrt{1-\lambda^2}$ has at most $2n+4[\frac{n-1}{2}]+14$ zeros on $(0,1)$. Therefore,

$$\begin{aligned}\#\{M(h)=0,h\in(0,1)\}=&\#\{\widetilde{M}(\lambda)=0,\lambda\in(0,1)\}\\ \leq&2n+5[\frac{n-1}{2}]+15.\end{aligned}$$

When n is an odd number, following the lines of the discussion above, we have

$$\begin{aligned}\#\{M(h)=0,h\in(0,1)\}=&\#\{\widetilde{M}(\lambda)=0,\lambda\in(0,1)\}\\ \leq&2n+5[\frac{n-1}{2}]+15.\end{aligned}$$

If $n=1$, then we have

$$M(h)=a_0+a_1\sqrt{h}+a_2h+a_3\sqrt{1-h}+a_4\Theta(h), \tag{3.6}$$

where

$$\begin{aligned}a_0=&\frac{1}{2}b^2_{1,0}-\frac{1}{2}b^3_{1,0}-\frac{1}{2}a^1_{0,1}+\frac{1}{2}a^2_{0,1}-\frac{1}{2}a^3_{0,1}+\frac{1}{2}a^4_{0,1}-a^2_{1,0}-a^3_{1,0}\\ &-b^2_{0,1}-b^3_{0,1}-b^1_{0,0}+b^4_{0,0}+\frac{1}{2}b^4_{1,0}-\frac{1}{2}b^1_{1,0},\\ a_1=&b^1_{0,0}-b^4_{0,0}-b^4_{1,0}+b^1_{1,0},\\ a_2=&\frac{1}{2}b^3_{1,0}-\frac{1}{2}b^2_{1,0}+\frac{1}{2}a^1_{0,1}-\frac{1}{2}a^2_{0,1}+\frac{1}{2}a^3_{0,1}-\frac{1}{2}a^4_{0,1}+a^2_{1,0}+a^3_{1,0}\\ &+b^2_{0,1}+b^3_{0,1}+\frac{1}{2}b^4_{1,0}-\frac{1}{2}b^1_{1,0},\\ a_3=&b^3_{0,0}-b^2_{0,0}+a^2_{0,0}-a^1_{0,0}+a^3_{0,0}-a^4_{0,0}-a^4_{1,0}-a^1_{1,0}-b^1_{0,1}-b^4_{0,1},\\ a_4=&a^4_{1,0}+a^1_{1,0}+b^1_{0,1}+b^4_{0,1}.\end{aligned}$$

Similar to (3.4), equality (3.6) becomes

$$\widetilde{M}(\lambda)=a_0+a_2+a_1\sqrt{1-\lambda^2}+a_3\lambda-a_2\lambda^2+a_4(1-\lambda^2)\int_0^{\frac{\lambda}{\sqrt{1-\lambda^2}}}\sqrt{1+x^2}dx.$$

Similar to the proof of the case $n\geq 2$, we obtain that $M(h)$ has at most 4 zeros on (0,1).

Next we will give the lower bound for the number of zeros of $M(h)$. For the sake of simplicity, we choose

$$\begin{aligned}&f^1(x,y)=\sum_{i=0}^{n}a^1_{i,0}x^i,\ g^k(x,y)=\sum_{i=0}^{n}b^k_{i,0}x^i,\ k=1,2,\\ &f^i(x,y)=g^j(x,y)=0,\ i=2,3,4,j=3,4.\end{aligned}$$

Hence,

$$
\begin{aligned}
M(h) &= \sum_{i=0}^{n} b_{i,0}^{1} \int_{L_h^1} x^i dx + \sum_{i=0}^{n} b_{i,0}^{2} \int_{L_h^2} x^i dx - \sum_{i=0}^{n} a_{i,0}^{1} I_{i,0}(h) \\
&= \sum_{i=0}^{n} (-1)^{i+1} \frac{b_{i,0}^{2}}{i+1} (1-h)^{\frac{i+1}{2}} - \sum_{i=0}^{n} \frac{b_{i,0}^{1}}{i+1} (1-\sqrt{h})^{i+1} - \sum_{i=0}^{n} a_{i,0}^{1} I_{i,0}(h).
\end{aligned} \tag{3.7}
$$

From (1.9), $H^1(x,y) = \frac{1}{2}[(x-1)^2 - y^2] = \frac{h}{2}$ is equivalent to

$$
2x - x^2 = 1 - h - y^2 \triangleq z. \tag{3.8}
$$

Consider (3.8) in x near $x = 0$. There is a unique C^∞ solution

$$
x = p(z) = \sum_{j=1}^{\infty} \mu_j z^j,
$$

where $\mu_1 = \frac{1}{2},\ \mu_2 = \frac{1}{8},\ \mu_3 = \frac{1}{16}, \cdots$. Then, we have

$$
\sum_{i=0}^{n} a_{i,0}^{1} x^i = \sum_{i=0}^{n} a_{i,0}^{1} \Big(\sum_{j=1}^{\infty} \mu_j z^j \Big)^i = \sum_{i=0}^{\infty} \rho_i z^i,
$$

where

$$
\begin{aligned}
&\rho_0 = a_{0,0}^{1},\ \rho_1 = \frac{1}{2} a_{1,0}^{1},\ \rho_2 = \frac{1}{4} a_{2,0}^{1} + \frac{1}{8} a_{1,0}^{1}, \cdots, \\
&\rho_n = \frac{1}{2^n} a_{n,0}^{1} + L(a_{1,0}^{1}, a_{2,0}^{1}, \cdots, a_{n-1,0}^{1}), \cdots,
\end{aligned}
$$

$L(\cdot)$ denotes a linear combination of $a_{1,0}^{1}, a_{2,0}^{1}, \cdots, a_{n-1,0}^{1}$. Thus,

$$
\begin{aligned}
\sum_{i=0}^{n} a_{i,0}^{1} I_{i,0}(h) &= \int_{L_h^1} \sum_{i=0}^{n} a_{i,0}^{1} x^i dy = \sum_{i=0}^{\infty} \rho_i \int_{L_h^1} z^i dy \\
&= \sum_{i=0}^{\infty} \rho_i \int_0^{\sqrt{1-h}} (1-h-y^2)^i dy.
\end{aligned}
$$

Let $y = \sqrt{1-h}u$, we have

$$
\begin{aligned}
\sum_{i=0}^{n} a_{i,0}^{1} I_{i,0}(h) &= \sum_{i=0}^{\infty} \rho_i \int_0^1 (1-u^2)^i du (1-h)^{i+\frac{1}{2}} \\
&\triangleq \sum_{i=0}^{\infty} \rho_i A_i (1-h)^{i+\frac{1}{2}},
\end{aligned} \tag{3.9}
$$

where $A_i = \int_0^1 (1-u^2)^i du$. The expansion of $1 - \sqrt{h}$ for $0 < 1 - h \ll 1$ is

$$
1 - \sqrt{h} = \sum_{j=1}^{\infty} \sigma_j (1-h)^j,
$$

where $\sigma_1 = \frac{1}{2}$, $\sigma_2 = \frac{1}{8}$, $\sigma_3 = \frac{1}{16}$, $\cdots$. Thus,

$$\begin{aligned}\sum_{i=0}^{n} \frac{b_{i,0}^1}{i+1}(1-\sqrt{h})^{i+1} &= \sum_{i=0}^{n} \frac{b_{i,0}^1}{i+1}\Big(\sum_{j=1}^{\infty} \sigma_j (1-h)^j\Big)^{i+1} \\ &= \sum_{i=1}^{\infty} \nu_i (1-h)^i,\end{aligned} \tag{3.10}$$

where

$$\begin{aligned}&\nu_1 = \frac{1}{2}b_{0,0}^1,\ \nu_2 = \frac{1}{8}b_{1,0}^1 + \frac{1}{8}b_{0,0}^1,\ \nu_3 = \frac{1}{24}b_{2,0}^1 + \frac{1}{16}b_{1,0}^1 + \frac{1}{16}b_{0,0}^1, \cdots, \\ &\nu_{n+1} = \frac{1}{2^{n+1}(n+1)}b_{n,0}^1 + L(b_{0,0}^1, b_{1,0}^1, \cdots, b_{n-1,0}^1), \cdots .\end{aligned}$$

Then inserting (3.9) and (3.10) into (3.7) gives

$$\begin{aligned}M(h) &= \sum_{i=0}^{n} (-1)^{i+1} \frac{b_{i,0}^2}{i+1}(1-h)^{\frac{i+1}{2}} - \sum_{i=0}^{\infty} \rho_i A_i (1-h)^{i+\frac{1}{2}} - \sum_{i=1}^{\infty} \nu_i (1-h)^i \\ &\triangleq \sum_{i=0}^{2n+1} \xi_i (1-h)^{\frac{i+1}{2}} + \cdots ,\end{aligned} \tag{3.11}$$

where

$$\begin{aligned}\xi_0 &= -b_{0,0}^2 - a_{0,0}^1, \\ \xi_1 &= \frac{1}{2}b_{1,0}^2 - \frac{1}{2}b_{0,0}^1, \\ \xi_2 &= -\frac{1}{3}b_{2,0}^2 - \frac{A_1}{2}a_{1,0}^1, \\ \xi_3 &= \frac{1}{4}b_{3,0}^2 - \frac{1}{8}b_{1,0}^1 - \frac{1}{8}b_{0,0}^1, \\ &\ \vdots \\ \xi_n &= \begin{cases} -\frac{1}{n+1}b_{n,0}^2 - \frac{A_{\frac{n}{2}}}{2^{\frac{n}{2}}}a_{\frac{n}{2},0}^1 + L(a_{1,0}^1, \cdots, a_{\frac{n-2}{2},0}^1), & n \text{ even}, \\ \frac{1}{n+1}b_{n,0}^2 - \frac{1}{(n+1)2^{\frac{n-1}{2}}}b_{\frac{n-1}{2},0}^1 - L(b_{0,0}^1, \cdots, b_{\frac{n-3}{2},0}^1), & n \text{ odd}, \end{cases} \\ \xi_{n+1} &= \begin{cases} -\frac{1}{(n+2)2^{\frac{n}{2}}}b_{\frac{n}{2},0}^1 - L(b_{0,0}^1, \cdots, b_{\frac{n-2}{2},0}^1), & n \text{ even}, \\ -\frac{A_{\frac{n+1}{2}}}{2^{\frac{n+1}{2}}}a_{\frac{n+1}{2},0}^1 + L(a_{1,0}^1, \cdots, a_{\frac{n-1}{2},0}^1), & n \text{ odd}, \end{cases} \\ \xi_{n+2} &= \begin{cases} -\frac{A_{\frac{n+2}{2}}}{2^{\frac{n+2}{2}}}a_{\frac{n+2}{2},0}^1 + L(a_{1,0}^1, \cdots, a_{\frac{n}{2},0}^1), & n \text{ even}, \\ -\frac{1}{(n+3)2^{\frac{n+1}{2}}}b_{\frac{n+1}{2},0}^1 - L(b_{0,0}^1, \cdots, b_{\frac{n-1}{2},0}^1), & n \text{ odd}, \end{cases} \\ &\ \vdots \\ \xi_{2n-1} &= -\frac{1}{n2^n}b_{n-1,0}^1 - L(b_{0,0}^1, \cdots, b_{n-2,0}^1), \\ \xi_{2n} &= -\frac{A_n}{2^n}a_{n,0}^1 + L(a_{1,0}^1, \cdots, a_{n-1,0}^1),\end{aligned}$$

$$\xi_{2n+1} = -\frac{1}{(n+1)2^{n+1}} b^1_{n,0} - L(b^1_{0,0}, \cdots, b^1_{n-1,0}),$$

$$\vdots$$

Therefore, if n is an even number, then we have

$$\frac{\partial(\xi_0, \xi_1, \cdots, \xi_n, \xi_{n+1}, \xi_{n+2}, \cdots, \xi_{2n}, \xi_{2n+1})}{\partial(b^2_{0,0}, b^2_{1,0}, \cdots, b^2_{n,0}, b^1_{\frac{n}{2},0}, a^1_{[\frac{n+2}{2}],0}, \cdots, a^1_{n,0}, b^1_{n,0})} =$$

$$\begin{pmatrix} -1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{2} & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -\frac{1}{n+1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & -\frac{1}{(n+2)2^{\frac{n}{2}}} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & -\frac{A_{\frac{n+2}{2}}}{2^{\frac{n+2}{2}}} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & * & \cdots & -\frac{A_n}{2^n} & 0 \\ 0 & 0 & \cdots & 0 & * & 0 & \cdots & 0 & -\frac{1}{(n+1)2^{n+1}} \end{pmatrix}.$$

Obviously, the rank of the above matrix is $2n+2$, which means that ξ_i, $0 \le i \le 2n+1$ can be chosen as free parameters such that $0 < |\xi_0| \ll |\xi_1| \ll \cdots \ll |\xi_{2n}| \ll |\xi_{2n+1}| \ll 1$, and $\xi_i\xi_{i+1} < 0$, $0 \le i \le 2n$. Hence, $M(h)$ defined by (3.6) has $2n+1$ simple zeros in $(0,1)$ near $h = 1$. Therefore, system (1.8) can have $2n+1$ limit cycles for $0 < h < 1$.

If n is an odd number, we can prove that system (1.8) has $2n+1$ limit cycles for $0 < h < 1$ in a similar way. This ends the proof of Theorem 1.2. □

4. Proof of the Corollary 1.1

If $f^1(x,y) = f^4(x,y)$, $g^1(x,y) = g^4(x,y)$, $f^2(x,y) = f^3(x,y)$ and $g^2(x,y) = g^3(x,y)$, that is, the straight line of separation is $x = 0$, then system (1.8) becomes

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} -y + \varepsilon f^1(x,y) \\ 1 - x + \varepsilon g^1(x,y) \end{pmatrix}, & x > 0, \\ \begin{pmatrix} -y + \varepsilon f^2(x,y) \\ x + \varepsilon g^2(x,y) \end{pmatrix}, & x < 0. \end{cases} \tag{4.1}$$

From Theorem 1.1 in [9, 17], we know that the number of limit cycles bifurcating from the annulus period of system $(4.1)|_{\varepsilon=0}$ is controlled by the following first order Melnikov function of system (4.1)

$$M(h) = \int_{L_h^+} g^1(x,y)dx - f^1(x,y)dy + \int_{L_h^-} g^2(x,y)dx - f^2(x,y)dy, \tag{4.2}$$

where $L_h^+ = L_h^1 \cup L_h^4$ and $L_h^- = L_h^2 \cup L_h^3$; see Fig. 3. Similar to (2.5), we have for

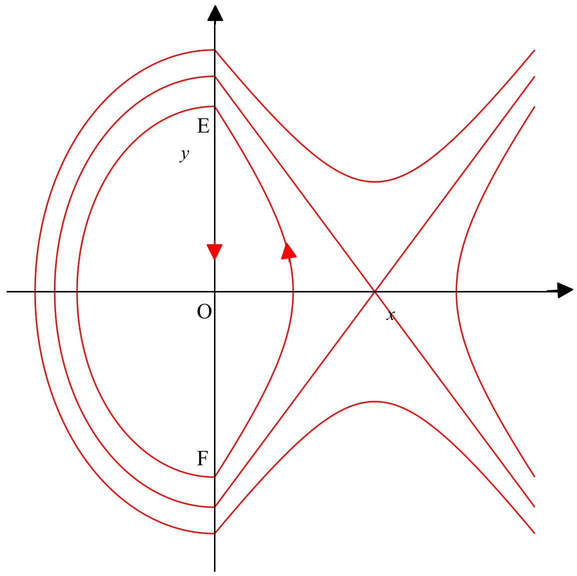

Figure 3. The phase portrait of system $(4.1)|_{\varepsilon=0}$ with a separatrix line $x = 0$.

$i \geq 0$ and $j \geq 0$

$$\int_{L_h^+} x^i y^j \, dx = -\frac{j}{i+1}\Phi_{i+1,j-1}(h),$$
$$\int_{L_h^-} x^i y^j \, dx = -\frac{j}{i+1}\Psi_{i+1,j-1}(h),$$

where

$$\Phi_{i,j}(h) = \int_{L_h^+} x^i y^j \, dy, \quad \Psi_{i,j}(h) = \int_{L_h^-} x^i y^j \, dy.$$

Since $L_h^\pm$ are symmetric with respect to the x-axis, $\Phi_{i,2j+1}(h) = \Psi_{i,2j+1}(h) = 0$. Therefore, the Melnikov function in (4.2) can be written as

$$\begin{aligned} M(h) = & -\sum_{\substack{i+j=1,\\ i\geq 0, j\geq 1}}^{n} \frac{j}{i+1} b_{i,j}^1 \Phi_{i+1,j-1}(h) - \sum_{i+j=0}^{n} a_{i,j}^1 \Phi_{i,j}(h) \\ & -\sum_{\substack{i+j=1,\\ i\geq 0, j\geq 1}}^{n} \frac{j}{i+1} b_{i,j}^2 \Psi_{i+1,j-1}(h) - \sum_{i+j=0}^{n} a_{i,j}^2 \Psi_{i,j}(h) \\ = & \sum_{i+j=0}^{n} \tau_{i,j} \Phi_{i,j}(h) + \sum_{i+j=0}^{n} \sigma_{i,j} \Psi_{i,j}(h). \end{aligned}$$

Following the processes of the analysis of Lemma 2.1, we can obtain the algebraic structure of $M(h)$.

Lemma 4.1. *For $h \in (0,1)$,*

$$M(h) = \alpha_1(h)\Phi_{0,0}(h) + \beta_1(h)\Phi_{1,0}(h) + \alpha_2(h)\Psi_{0,0}(h) + \beta_2(h)\Psi_{1,0}(h), \quad (4.3)$$

where $\alpha_k(h)$ and $\beta_k(h)$ are polynomials of h with

$$\deg \alpha_k(h) \le [\frac{n}{2}], \quad \deg \beta_k(h) \le [\frac{n-1}{2}], \ k = 1, 2.$$

Proof of the Corollary 1.3. By direct computation, we have

$$\begin{aligned} \Phi_{0,0}(h) &= 2\sqrt{1-h}, \quad \Phi_{1,0}(h) = 2\sqrt{1-h} - 2\Theta(h), \\ \Psi_{0,0}(h) &= -2\sqrt{1-h}, \quad \Psi_{1,0}(h) = \frac{\pi}{2}(1-h), \end{aligned} \quad (4.4)$$

where $\Theta(h)$ is defined in (1.11). Substituting (4.4) into (4.3) implies

$$M(h) = P_{[\frac{n}{2}]}(h)\sqrt{1-h} + P_{[\frac{n-1}{2}]+1}(h) + P_{[\frac{n-1}{2}]}(h)\Theta(h), \ h \in (0,1).$$

Following the lines of the proof of Theorem 1.2, we obtain that $M(h)$ has at most $2[\frac{n}{2}] + 3[\frac{n-1}{2}] + 3$ (resp. $5[\frac{n-1}{2}] + 4$) zeros if n is an even (resp. odd) number. This completes the proof of Corollary 1.3. □

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