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WEAK GALERKIN FINITE ELEMENT METHODS COMBINED WITH CRANK-NICOLSON SCHEME FOR PARABOLIC INTERFACE PROBLEMS

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Abstract This article is devoted to the a priori error estimates of the fully discrete Crank-Nicolson approximation for the linear parabolic interface problem via weak Galerkin finite element methods (WG-FEM). All the finite element functions are discontinuous for which the usual gradient operator is implemented as distributions in properly defined spaces. Optimal order error estimates in both $L^\infty(H^1)$ and $L^\infty(L^2)$ norms are established for lowest order WG finite element space $(\mathcal{P}_k(K),\ \mathcal{P}_{k-1}(\partial K),\ \left[\mathcal{P}_{k-1}(K)\right]^2)$. Finally, we give numerical examples to verify the theoretical results.

Keywords Parabolic, Interface, Finite element method, Weak Galerkin method, Optimal error estimates, Low regularity, Crank-Nicolson.

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1. Introduction

Let Ω be a convex polygonal domain in $\mathbb{R}^2$ with boundary $\partial\Omega$ and $\Omega_1 \subset \Omega$ be an open domain with Lipschitz boundary $\Gamma = \partial\Omega_1$. Let $\Omega_2 = \Omega \backslash \Omega_1$ be an another open domain contained in Ω with boundary $\Gamma \cup \partial\Omega$. In $\Omega = \Omega_1 \cup \Gamma \cup \Omega_2$, we consider following parabolic interface problem

$$u_t - \nabla \cdot (\beta \nabla u) = f \quad \text{in } \Omega \times (0, T] \tag{1.1}$$

with initial and Dirichlet boundary conditions

$$u(x, 0) = u_0(x) \ \text{ in } \Omega; \ \ u = 0 \quad \text{on } \partial\Omega \times (0, T] \tag{1.2}$$

and interface conditions

$$[u] = \psi, \ \ \left[\beta \frac{\partial u}{\partial \eta}\right] = \phi \quad \text{along } \Gamma \times (0, T]. \tag{1.3}$$

Here η is the outward pointing unit normal to Ω_1 and $[v]$ denotes the jump of a quantity v across the interface Γ i.e., $[v](x) = v_1(x) - v_2(x),\ x \in \Gamma$, where $v_i(x) = v(x)|_{\Omega_i},\ i = 1, 2$. The coefficient function $\beta(x)$ is assumed to be positive and piecewise constant across Γ, i.e., $\beta(x) = \beta_k$ for $x \in \Omega_k,\ k = 1, 2$. Across the

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interface Γ, the source function $f : \Omega \times (0, T] \to \mathbb{R}$ can be singular. We assume that f is sufficiently smooth locally. Jump functions $\psi : \Gamma \times (0, T] \to \mathbb{R}$ and $\phi : \Gamma \times (0, T] \to \mathbb{R}$ are given, and $T < \infty$.

Medium heterogeneities makes the construction of stable and accurate numerical schemes for differential equations more challenging. The past few decades have witnessed intensive research activity in interface problems via finite element algorithms. For recent literature, one may refer to [2, 5–8] and the references therein. The objective of the present work is to propose and analyze weak Galerkin finite element method (WG-FEM) for parabolic interface problems. The WG-FEM introduced in [9] refers to the numerical techniques for partial differential equations where the differential operators are approximated by weak forms. In [10], a WG-FEM was developed for the second order elliptic equation in mixed form. The resulting WG mixed finite element schemes turned out to be applicable for general finite element partitions consisting of shape regular polytopes, and the stabilization idea opened a new door for weak Galerkin method. Recently, WG-FEM have been applied to interface problems [4, 5, 8]. The WG algorithm in [8] allows the use of finite element partitions consisting of general polytopal meshes and assume that grid line exactly follows the actual interface. Optimal order error estimate in H^1 norm is established for WG finite element space $(\mathcal{P}_k(K),\ \mathcal{P}_k(\partial K),\ [\mathcal{P}_{k-1}(K)]^2)$. Here, K is any polygonal domain with boundary ∂K and $\mathcal{P}_k(K)$ ($\mathcal{P}_k(\partial K)$) is a set of polynomials on K (∂K) with degree no more than $k \geq 1$. Then the work of [8] has been extended to elliptic and parabolic interface problems in [4, 5] for lowest order WG finite element space $(\mathcal{P}_k(K),\ \mathcal{P}_{k-1}(\partial K),\ [\mathcal{P}_{k-1}(K)]^2)$. The time discretization in [5] is based on the backward Euler approximation. In this paper, we study WG-FEM with Crank-Nicolson scheme for solving parabolic interface problems. Optimal order error estimates in $L^\infty(L^2)$ norm is established for the fully discrete scheme. WG-FEM with second-order accuracy in time for parabolic problems without interface can be found in [11].

2. Preliminaries and weak Galerkin discretization

Throughout the work, we will follow the usual notation for Sobolev spaces and norms (*cf.* [1]). For a Lebesgue measurable set $\mathcal{M} \subset \mathbb{R}^2$ and $m > 0$, we denote the Hilbertian Sobolev space $H^m(\mathcal{M})$ with the norm $\|.\|_{m,\mathcal{M}}$. For $m = 0$, $L^2(\mathcal{M})$ is a Hilbert space equipped with norm $\|.\|_{\mathcal{M}}$. For simplicity of notation, we skip the subscript $\mathcal{M}$ while defining norm whenever $\mathcal{M} = \Omega$. We also define the standard Bôchner spaces $L^2(J; \mathcal{B})$, where $\mathcal{B}$ is a real Banach space with norm $\|.\|_{\mathcal{B}}$ and $J = [0, T]$, consisting of all measurable functions $\phi : J \to \mathcal{B}$ for which

$$\|\phi\|_{L^\infty(J;\mathcal{B})} := \operatorname*{ess\ sup}_{t\in[0,T]} \|\phi(t)\|_{\mathcal{B}} < \infty.$$

We denote by $H^m(J; \mathcal{B})$, $0 \leq m < \infty$, the space of all measurable functions $\phi : J \to \mathcal{B}$ for which

$$\|u\|_{H^m(J;\mathcal{B})} = \left(\sum_{j=0}^{m} \int_0^T \left\| \frac{\partial^j u(t)}{\partial t^j} \right\|_{\mathcal{B}}^2 dt \right)^{\frac{1}{2}} < \infty.$$

For $m = 0$, we write $L^2(J; \mathcal{B}) = H^0(J; \mathcal{B})$. When no risk of confusion exists, we shall write $L^2(\mathcal{B})$ for $L^2(J; \mathcal{B})$, $L^\infty(\mathcal{B})$ for $L^\infty(J; \mathcal{B})$ and $H^m(\mathcal{B})$ for $H^m(J; \mathcal{B})$.

Let $\mathcal{T}_h$ be a partition of the domain Ω consisting of polygons in two dimension satisfying a set of conditions specified in [8,10]. Denote by $\mathcal{E}_h$ the set of all edges in $\mathcal{T}_h$ and let $\mathcal{E}_h^0 = \mathcal{E}_h \backslash \partial\Omega$ be the set of all interior edges. Let Γ_h be the subset of $\mathcal{E}_h$ of all edges on Γ. For every element $K \in \mathcal{T}_h$, we denote by h_K its diameter and mesh size $h = \max_{K\in\mathcal{T}_h}$ for $\mathcal{T}_h$. Note that

$$\begin{aligned}\mathcal{T}_h \; &= \{K \in \mathcal{T}_h : \; K \nsubseteq \Omega_2 \text{ or } \partial K \cap \Gamma = \emptyset\} \\ &\quad \cup \{K \in \mathcal{T}_h : \; K \subseteq \Omega_2 \text{ and } \partial K \cap \Gamma \neq \emptyset\} \\ &=: \mathcal{T}_1 \cup \mathcal{T}_2. \end{aligned} \tag{2.1}$$

Note that $\mathcal{T}_1$ contains all elements in Ω_1 and non-interface elements in Ω_2.

Let K be any polygonal domain with interior K^0 and boundary ∂K. A weak function on the region K refers to a pair of scalar-valued functions $v = \{v_0, v_b\}$ such that $v_0 \in L^2(K)$ and $v_b \in H^{\frac{1}{2}}(\partial K)$. For a given $k \geq 1$, let V_h be WG finite element space associated with $\mathcal{T}_h$ defined as follows

$$V_h = \{v = \{v_0, v_b\} : \; v_0|_{K^0} \in \mathcal{P}_k(K), \; v_b|_e \in \mathcal{P}_{k-1}(e), \; e \in \partial K, \; K \in \mathcal{T}_h\} \tag{2.2}$$

and $V_h^0 = \{v = \{v_0, v_b\} \in V_h : \; v_b = 0 \;\text{ on } \partial\Omega\}$.

For each $v \in V_h$, the weak gradient of it, denoted by ∇_w, is defined as the unique polynomial $(\nabla_w v) \in [\mathcal{P}_{k-1}(K)]^2$ that satisfies the following equation

$$(\nabla_w v, \phi)_K = -\int_K v_0(\nabla\cdot\phi)dK + \int_{\partial K} v_b(\phi\cdot\mathbf{n})ds \;\; \forall\phi \in [\mathcal{P}_{k-1}(K)]^2, \tag{2.3}$$

where $\mathbf{n}$ is the outward normal to ∂K.

The usual L^2-inner product can be written locally on each element as follows

$$(\nabla_w v, \nabla_w w) = \sum_{K\in\mathcal{T}_h} (\nabla_w v, \nabla_w w)_K. \tag{2.4}$$

For each element $K \in \mathcal{T}_h$, denote by Q_0 the usual L^2 projection operator from $L^2(K)$ onto $\mathcal{P}_k(K)$ and by Q_b the L^2 projection from $L^2(e)$ onto $\mathcal{P}_{k-1}(e)$ for any $e \in \mathcal{E}_h$. We shall combine Q_0 with Q_b by writing $Q_h = \{Q_0, Q_b\}$. We recall following crucial approximation properties for local projections Q_h.

Lemma 2.1 (Lemma 3.4, [8])**.** *Let $\mathcal{T}_h$ be a finite element partition of Ω satisfying the shape regularity assumption as specified in [10]. Then, for $u \in H^{k+1}(\Omega_i)$ with $i = 1, \; 2$, we have*

$$\sum_{K\in\mathcal{T}_h} \Big(\|u - Q_0 u\|_K^2 + h_K^2 \|\nabla(u - Q_0 u)\|_K^2\Big) \leq C h^{2(k+1)} \sum_{i=1}^{2} \|u\|_{k+1,\Omega_i}^2.$$

Let $e \subset \Gamma$ be shared by two elements $K_1 \subset \Omega_1$ and $K_2 \subset \Omega_2$. We define four forms

$$\langle \psi, \beta\nabla_w v\cdot\eta\rangle_\Gamma = \sum_{e\in\Gamma_h} \langle \psi, \beta_2 \nabla_w (v|_{K_2})\cdot\eta\rangle_e,$$

$$h^{-1}\langle \psi, v_0 - v_b\rangle_\Gamma = \sum_{e\in\Gamma_h} h^{-1}\langle \psi, v_0|_{K_2} - v_b\rangle_e,$$

$$h^{-1}\langle\psi, Q_b v_0 - v_b\rangle_\Gamma = \sum_{e\in\Gamma_h} h^{-1}\langle\psi, Q_b(v_0|_{K_2}) - v_b\rangle_e,$$

$$\langle\phi, v_b\rangle_\Gamma = \sum_{e\in\Gamma_h}\langle\phi, v_b\rangle_e.$$

Here, $\langle\cdot,\cdot\rangle_e$ denotes the L^2 inner product on $e \in \mathcal{E}_h$.

The continuous-time weak Galerkin finite element approximation to (1.1)-(1.3) can be obtained by seeking $u_h = \{u_0, u_b\} : [0,T] \to V_h^0$ satisfying both $u_h(0) = Q_h u(0)$ and following equation for any $v = \{v_0, v_b\} \in V_h^0$ (*cf.* [5])

$$(u_{ht}, v_0) + a(u_h, v) = (f, v_0) + \langle\psi, \beta\nabla_w v\cdot\eta\rangle_\Gamma + \langle\phi, v_b\rangle_\Gamma - \frac{1}{h}\langle\psi, Q_b v_0 - v_b\rangle_\Gamma. \quad (2.5)$$

The bilinear map $a(\cdot,\cdot)$ on V_h^0 is given by

$$a(u_h, v) = \sum_{K\in\mathcal{T}_h}(\beta\nabla_w u_h, \nabla_w v)_K + s(u_h, v), \quad (2.6)$$

where the stabilizer $s(\cdot,\cdot)$ is defined as

$$s(v, w) = \sum_{K\in\mathcal{T}_h} h_K^{-1}\langle Q_b v_0 - v_b, Q_b w_0 - w_b\rangle_{\partial K}. \quad (2.7)$$

The finite element space V_h^0 is a normed linear space with respect to a triple-bar norm given by (*cf.* [5])

$$|||w|||^2 = \sum_{K\in\mathcal{T}_h}\|\beta^{\frac{1}{2}}\nabla_w w\|_K^2 + \sum_{K\in\mathcal{T}_h} h_K^{-1}\|Q_b w_0 - w_b\|_{\partial K}^2 = a(w, w). \quad (2.8)$$

We, now, turn our attention to discrete time weak Galerkin procedures. We first divide the interval $[0, T]$ into M equally-spaced subintervals by the following points

$$0 = t^0 < t^1 < \cdots < t^M = T,$$

with $t^n = n\tau$, $\tau = T/M$ be the time step. For a smooth function ξ on $[0, T]$, define $\xi^n = \xi(t^n)$ and

$$\bar\partial\xi^n = \frac{\xi^n - \xi^{n-1}}{\tau}, \quad \hat\xi^n = \frac{\xi^n + \xi^{n-1}}{2}. \quad (2.9)$$

Let $U^n = U_h^n = \{U_0^n, U_b^n\} \in V_h^0$ be the fully discrete approximation of u at $t = t^n$ which we shall define through the following scheme: Given U^{n-1} in V_h^0, we now determine $U^n \in V_h^0$ satisfying

$$\begin{aligned}(\bar\partial U^n, v_0) + a(\hat U^n, v) &= (\hat f^n, v_0) + \langle\hat\psi^n, \beta\nabla_w v\cdot\eta\rangle_\Gamma + \langle\hat\phi^n, v_b\rangle_\Gamma \\ &\quad - h^{-1}\langle\hat\psi^n, Q_b v_0 - v_b\rangle_\Gamma \ \forall v = \{v_0, v_b\} \in V_h^0, \end{aligned} \quad (2.10)$$

with $U^0 = U_h^0 = Q_h u(0)$.

3. Error analysis for the fully discrete scheme

This section deals with the error analysis for the fully discrete scheme. We derive optimal order error bounds in H^1-norm and L^2-norm. The basic idea applied is to use elliptic projection. Let X^* be the collection of all $v \in L^2(\Omega)$ with the property that $v \in \{H^2(\Omega_1) \cup H^2(\Omega_2)\} \cap \{\psi : \psi = 0 \text{ on } \partial\Omega\}$ and $[v] = \psi_v$ and $\left[\beta \frac{\partial v}{\partial \eta}\right] = \phi_v$ along Γ. Define

$$f^* = \begin{cases} -\nabla \cdot (\beta_1 \nabla v) & \text{in } \Omega_1, \\ -\nabla \cdot (\beta_2 \nabla v) & \text{in } \Omega_2. \end{cases}$$

Clearly $f^* \in L^2(\Omega)$. Define $R_h : X^* \to V_h^0$ by

$$\begin{aligned} a(R_h v, w) = (f^*, w_0) + \langle \psi_v, \beta \nabla_w w \cdot \eta \rangle_\Gamma + \langle \phi_v, w_b \rangle_\Gamma \\ -h^{-1} \langle \psi_v, Q_b w_0 - w_b \rangle_\Gamma \quad \forall w = \{w_0, w_b\} \in V_h^0, \ v \in X^*. \end{aligned} \tag{3.1}$$

Further, in view of (3.1), this definition may be expressed by saying that $R_h v$ is the weak Galerkin finite element solution of the elliptic interface problem with exact solution $v \in H_0^1(\Omega)$ (*cf.* [4,8])

$$\begin{aligned} &-\nabla \cdot (\beta(x) \nabla v) = f^* \quad \text{in } \Omega, \\ &v = 0 \quad \text{on } \partial\Omega, \\ &[v] = \psi_v, \quad \left[\beta \frac{\partial u}{\partial \eta}\right] = \phi_v \quad \text{along } \Gamma. \end{aligned} \tag{3.2}$$

The error estimate for R_h, as shows in the following lemma, should be applied.

Lemma 3.1 ([4,8])**.** *Let R_h be defined by (3.1). Assume that the exact solution of problem (3.2) is so regular that $v \in H^{k+1}(\Omega_i)$, $i = 1, 2$. Then there exists a constant $C > 0$ such that*

$$\begin{aligned} &|||R_h v - Q_h v||| \le C h^k (\|v\|_{k+1,\Omega_1} + \|v\|_{k+1,\Omega_2}), \\ &\|R_h v - Q_h v\|_{L^2(\Omega)} \le C h^{k+1} (\|v\|_{k+1,\Omega_1} + \|v\|_{k+1,\Omega_2}). \end{aligned}$$

For fully discrete error estimates, we now split the errors at $t = t^n$ as follows

$$u^n - U^n = u^n - Q_h u^n + Q_h u^n - U^n.$$

We denote our error as

$$e^n = U^n - Q_h u^n = \{e_0^n, e_b^n\}.$$

Using ρ and θ, error e^n can be further separated as

$$e^n = \theta^n + \rho^n, \tag{3.3}$$

where $\theta^n = U^n - R_h u^n$ and $\rho^n = R_h u^n - Q_h u^n$.

From the definition (3.1), for all $w = \{w_0, w_b\} \in V_h^0$, it is easy to notice that

$$\begin{aligned} a\left(\frac{R_h u^n + R_h u^{n-1}}{\tau}, w\right) = \sum_{i=1}^{2} (-\nabla \cdot (\beta \nabla \hat{u}^n), w_0)_{\Omega_i} + \langle \hat{\psi}^n, \beta \nabla_w w \cdot \eta \rangle_\Gamma \\ + \langle \hat{\phi}^n, w_b \rangle_\Gamma - h^{-1} \langle \hat{\psi}^n, Q_b w_0 - w_b \rangle_\Gamma. \end{aligned} \tag{3.4}$$

Above equation and (1.1) leads to following error equation for θ^n

$$\begin{aligned}(\bar{\partial}\theta^n, v_0) + a(\hat{\theta}^n, v) &= (\hat{f}^n, v_0) + \sum_{i=1}^{2}(\nabla\cdot(\beta\nabla\hat{u}^n), v_0)_{\Omega_i} - (\bar{\partial}R_h u^n, v_0)\\ &= (\hat{u}_t^n, v_0) - (\bar{\partial}R_h u^n, v_0)\\ &:= -(w^n, v_0)\ \forall v = \{v_0, v_b\} \in V_h^0, \end{aligned} \tag{3.5}$$

where $w^n = \bar{\partial}R_h u^n - \hat{u}_t^n$. For simplicity of the exposition, we write $w^n = w_1^n + w_2^n$, where $w_1^n = \bar{\partial}R_h u^n - \bar{\partial}u^n$ and $w_2^n = \bar{\partial}u^n - \hat{u}_t^n$. Now, setting $v = \hat{\theta}^n$ in (3.5), we have

$$(\bar{\partial}\theta^n, \hat{\theta}^n) + a(\hat{\theta}^n, \hat{\theta}^n) = -(w^n, \hat{\theta}^n). \tag{3.6}$$

Since $a(\hat{\theta}^n, \hat{\theta}^n) \geq 0$, we have

$$\frac{1}{2\tau}\big(\|\theta^n\|^2 - \|\theta^{n-1}\|^2\big) \leq \frac{1}{2}\|w^n\|\big(\|\theta^n\| + \|\theta^{n-1}\|\big).$$

which implies

$$\begin{aligned}\|\theta^n\| &\leq \|\theta^{n-1}\| + \tau\|w^n\|\\ &\leq \|\theta^0\| + \tau\sum_{j=1}^{n}\|w_1^j\| + \tau\sum_{j=1}^{n}\|w_2^j\|. \end{aligned} \tag{3.7}$$

The term w_1^j can be expressed as

$$\begin{aligned} w_1^j &= R_h\bar{\partial}u^j - \bar{\partial}u^j = (R_h - I)(\bar{\partial}u^j)\\ &= (R_h - I)\frac{1}{\tau}\int_{t^{j-1}}^{t^j} u_t dt = \frac{1}{\tau}\int_{t^{j-1}}^{t^j}(R_h u_t - u_t)dt. \end{aligned} \tag{3.8}$$

An application of Lemma 3.1 leads to

$$\tau\|w_1^j\| \leq Ch^{k+1}\int_{t^{j-1}}^{t^j}(\|u_t\|_{k+1,\Omega_1} + \|u_t\|_{k+1,\Omega_2})dt.$$

Using above estimate, we have

$$\tau\sum_{j=1}^{n}\|w_1^j\| \leq Ch^{k+1}\int_0^{t^n}(\|u_t\|_{k+1,\Omega_1} + \|u_t\|_{k+1,\Omega_2})dt. \tag{3.9}$$

Now, it remains to determine estimates for w_2^j. To find an estimate for w_2^j, we consider the following two parts separately

$$\begin{aligned}\sigma_1^j &:= \frac{u(t^j) - u(t^{j-1})}{\tau} - u'\Big(t^{j-1} + \frac{\tau}{2}\Big),\\ \sigma_2^j &:= u'\Big(t^{j-1} + \frac{\tau}{2}\Big) - \frac{u'(t^j) + u'(t^{j-1})}{2}.\end{aligned}$$

Using the Taylor's expansion, we obtain

$$\|\sigma_1^j\|_{\Omega_i} + \|\sigma_2^j\|_{\Omega_i} \leq C\tau\int_{t^{j-1}}^{t^j}\|u_{ttt}(s)\|_{\Omega_i}ds,\ i = 1, 2. \tag{3.10}$$

Summing over j from $j = 1$ to $j = n$ in (3.10), we obtain

$$\tau \sum_{j=1}^{n} \|w_2^j\| \leq C\tau^2 \int_0^{t^n} \Big\{ \|u_{ttt}\|_{\Omega_1} + \|u_{ttt}\|_{\Omega_2} \Big\} dt. \tag{3.11}$$

Combining (3.9), (3.11) and (3.7), and further using the fact that $\|\theta^0\| = \|Q_h u(0) - R_h u(0)\| \leq Ch^{k+1}\|u(0)\|_{k+1}$, we obtain

$$\|\theta^n\| \leq C(h^{k+1} + \tau^2)\Big(\|u(0)\|_{k+1} + \sum_{i=1}^{2} \int_0^{t^n} \big(\|u_t\|_{k+1,\Omega_i} + \|u_{ttt}\|_{\Omega_i} \big) dt \Big). \tag{3.12}$$

An application of Lemma 3.1 for ρ^n yields

$$\|\rho^n\| \leq Ch^{k+1} \sum_{i=1}^{2} \|u^n\|_{k+1,\Omega_i}.$$

Again, it is easy to verify that

$$\|u^n\|_{k+1,\Omega_i} \leq \|u(0)\|_{k+1,\Omega_i} + \int_0^{t^n} \|u_t\|_{k+1,\Omega_i} dt.$$

Thus, we have

$$\|\rho^n\| \leq Ch^{k+1}\Big(\|u(0)\|_{k+1} + \sum_{i=1}^{2} \int_0^{t^n} \|u_t\|_{k+1,\Omega_i} dt \Big). \tag{3.13}$$

Combining estimates (3.12) and (3.13) along with Lemma 2.1, we obtain following optimal $L^\infty(L^2)$ norm error estimate

Theorem 3.1. *Let u and U be the solutions of the problem (1.1)-(1.3) and (2.10), respectively. Assume the exact solution $u \in H^1(H^{k+1}(\Omega_i)) \cap H^3(L^2(\Omega_i))$, $i = 1, 2$. Then there exists a constant $C > 0$ such that*

$$\begin{aligned} &\|U^n - u^n\| \\ \leq & C(h^{k+1} + \tau^2)\Big\{ \|u(0)\|_{k+1} + \sum_{i=1}^{2} (\|u\|_{H^1(H^{k+1}(\Omega_i))} + \|u_{ttt}\|_{L^2(L^2(\Omega_i))}) \Big\}. \end{aligned}$$

Standard inverse inequality (Lemma A.4, [10]), together with estimates (3.12) and (3.13), and Lemma 2.1 leads to following $L^\infty(H^1)$ norm error estimate

Theorem 3.2. *Let u and U be the solutions of the problem (1.1)-(1.3) and (2.10), respectively. Assume the exact solution $u \in H^1(H^{k+1}(\Omega_i)) \cap H^3(L^2(\Omega_i))$, $i = 1, 2$. Then there exists a constant $C > 0$ such that*

$$\begin{aligned} &\|U^n - u^n\|_1 \\ \leq & C(h^{k} + \tau^2)\Big\{ \|u(0)\|_{k+1} + \sum_{i=1}^{2} (\|u\|_{H^1(H^{k+1}(\Omega_i))} + \|u_{ttt}\|_{L^2(L^2(\Omega_i))}) \Big\}. \end{aligned}$$

Remark 3.1. The proposed fully discrete finite element scheme can be easily extended for the numerical approximation of the solutions to the following IBVP

$$\begin{cases} \sigma u_t - \nabla.(\beta \nabla u) = f & \text{in } \Omega \times (0, T], \\ u(x, 0) = u_0, \;\; u_t(x, 0) = v_0 & \text{in } \Omega, \\ u(x, t) = 0 & \text{on } \partial\Omega \times (0, T], \end{cases} \tag{3.14}$$

coupled with the jump conditions (1.3). For numerical validation, we refer to numerical Example 4.2.

4. Numerical Example

We present in this section numerical results to validate the theoretical estimates presented in Section 3. For our numerical experiment, we use lowest order weak Galerkin space $(\mathcal{P}_1(K),\ \mathcal{P}_0(\partial K),\ [\mathcal{P}_0(K)]^2)$ based on uniform triangulations of Ω_i, $i = 1, 2$. The nodes of the triangulations of Ω_1 and Ω_2 coincide on the interface Γ. Note that for each iteration, the spatial mesh size becomes half of the previous mesh size. We choose the uniform time step $\tau = \frac{1}{10}h$.

Example 4.1. We consider the two dimensional domain $\Omega = (-1, 1) \times (-1, 1)$ and the interface is taken to be the circle $x^2 + y^2 = \frac{1}{4}$. We select the appearing in (1.1)-(1.3) setting exact solution as

$$u_1(x, y, t) = t(0.25 - x^2 - y^2) \quad \text{in} \;\; \Omega_1 \times (0, 1],$$
$$u_2(x, y, t) = t(0.25 - x^2 - y^2)\sin(\pi x)\sin(\pi y) \quad \text{in} \;\; \Omega_2 \times (0, 1],$$

with $\beta_1 = 10^{-4}$ and $\beta_2 = 1$.

For the $L^\infty(0, T; L^2(\Omega))$ error with $\tau = \mathrm{O}(h)$, we observe its experimental order of convergence (EOC). For each run i, EOC of a given sequence of $L^\infty(0, T; L^2(\Omega))$ errors $e(i)$ defined on a sequence of meshes of size $h(i)$ by

$$\mathrm{EOC}(e(i)) = \frac{\log\Big(e(i+1)/e(i)\Big)}{\log\Big(h(i+1)/h(i)\Big)}.$$

The convergence behavior of the fully discrete weak Galerkin solutions at final time $T = 1$ with respect to the L^2-norm and H^1-norm are also depicted in Tables 1-2. It is clear from these tables that we have achieved optimal order of convergence in both the norms, which confirm the theoretical prediction as proved in Theorems 3.1-3.2.

Table 1. Numerical results for $L^\infty(L^2)$-norm error

l (runs)	h	Error	EOC
1	$\frac{1}{8}$	8.38744×10^{-2}	-
2	$\frac{1}{16}$	3.05176×10^{-2}	1.45
3	$\frac{1}{32}$	7.04421×10^{-3}	2.11
4	$\frac{1}{64}$	1.84338×10^{-3}	1.93
5	$\frac{1}{128}$	4.40906×10^{-4}	2.01

Table 2. Numerical results for $L^\infty(H^1)$-norm error

l (runs)	h	Error	EOC
1	$\frac{1}{8}$	1.17124	-
2	$\frac{1}{16}$	6.84037×10^{-1}	0.77
3	$\frac{1}{32}$	3.17659×10^{-1}	1.10
4	$\frac{1}{64}$	1.59264×10^{-1}	0.99
5	$\frac{1}{128}$	7.93007×10^{-2}	1.00

Example 4.2. In our second numerical example, we consider the square domain $\Omega = (-1,1)\times(-1,1)$ and the interface is taken to be the ellipse $\{(x,y) : 4x^2+16y^2 = r^2 = 1\}$. We select the data in (3.14) such that the exact solution u is given by

$$u(x,y,t) = \begin{cases} 10^{-1}(1-r^2)t^2\exp(-t) & \text{if } r^2 \le 1, \\ 10^{-2}(1-r^2)\sin(0.25\pi t)\sin(\pi x)\sin(\pi y) & \text{if } r^2 > 1. \end{cases}$$

The second set of physical coefficients borrowed from Dai etc [3] that corresponds to the classical Pennes bio heat transfer model is given by

$$(\sigma,\beta) = \begin{cases} (4.08, 0.0052) & \text{in } \ 4x^2+16y^2 \le 1, \\ (3.06, 0.0021) & \text{in } \ 4x^2+16y^2 > 1. \end{cases}$$

Table 3. Numerical results for $L^\infty(L^2)$-norm error

l (runs)	h	Error	EOC
1	$\frac{1}{4}$	2.3268×10^{-2}	-
2	$\frac{1}{8}$	5.68903×10^{-3}	2.03
3	$\frac{1}{16}$	1.48856×10^{-3}	1.93
4	$\frac{1}{32}$	3.60021×10^{-4}	2.04
5	$\frac{1}{64}$	9.6392×10^{-5}	2.01

Table 4. Numerical results for $L^\infty(H^1)$-norm error

l (runs)	h	Error	EOC
1	$\frac{1}{4}$	2.037×10^{-1}	-
2	$\frac{1}{8}$	1.32786×10^{-1}	0.61
3	$\frac{1}{16}$	6.59948×10^{-2}	1.00
4	$\frac{1}{32}$	3.17777×10^{-2}	1.05
5	$\frac{1}{64}$	1.57923×10^{-2}	1.01

Tables 3-4 represent the numerical solution errors and convergence rates in L^2 and H^1 norms, respectively. In both cases, errors are calculated at time $t = 1$ and clearly demonstrates the second order of convergence in L^2 norm and first order of convergence in H^1 norm.

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