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APPROXIMATE CONTROLLABILITY OF SECOND-ORDER SEMILINEAR EVOLUTION SYSTEMS WITH STATE-DEPENDENT INFINITE DELAY*

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Abstract In this article, we study the problem of approximate controllability for a class of semilinear second-order control systems with state-dependent delay. We establish some sufficient conditions for approximate controllability for this kind of systems by constructing fundamental solutions and using the resolvent condition and techniques on cosine family of linear operators. Particularly, theory of fractional power operators for cosine families is also applied to discuss the problem so that the obtained results can be applied to the systems involving derivatives of spatial variables. To illustrate the applications of the obtained results, two examples are presented in the end.

Keywords Second-order evolution equation, approximate controllability, cosine operator, fundamental solution, fractional power operator.

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1. Introduction

The concept of controllability has played a central role throughout the history of modern control theory. Moreover, approximately controllable systems are more prevalent and very often approximate controllability is completely adequate in applications. Therefore, various approximate controllability problems for different kinds of non-linear controlled systems represented by evolution equations have been extensively investigated in literature in these years, see [4, 10, 25, 36], for example.

We are concerned with in this paper the approximate controllability of systems governed by a second-order semilinear functional evolution equation with state-dependent delay of the form

$$\begin{cases} \dfrac{d^2}{dt^2}y(t) = -Ay(t) + L\left(y_t\right) + F(t, y_{\rho(t,y_t)}) + Bu(t), \quad t \in [0,T], \\ y_0 = \varphi \in \mathscr{B}, \quad y'(0) = x^0 \in X, \quad t \in (-\infty, 0], \end{cases} \tag{1.1}$$

where the state $y(\cdot)$ and the control function $u(\cdot)$ take values respectively in Hilbert spaces X and U. The histories $y_t : (-\infty, 0] \to X$, given in the usual way by

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$y_t(\theta) = y(t+\theta)$ for $\theta \in (-\infty, 0]$, belong to the phase space $\mathscr{B}$ defined axiomatically. Here $y'(0)$ denotes the right derivative of $y(\cdot)$ at zero. We assume that $-A : D(-A) \subset X \to X$ is the infinitesimal generator of a strongly continuous cosine family $(C(t))_{t\in\mathbb{R}}$ on X and $L : \mathscr{B} \to X$ and $B : U \to X$ are bounded linear operators. $F : [0,T] \times \mathscr{B}_\alpha \to X$ is a nonlinear continuous and uniformly bounded function to be specified later, and $\rho : [0,T] \times \mathscr{B}_\alpha \to (-\infty, T]$ is also a continuous function.

Eq. (1.1) is the abstract form of some semilinear second order partial functional differential equations (PFDEs) with state-dependent delay (for example delayed wave equations), which describe many physical, chemical and biological problems. An important way of studying semilinear second order PFDEs is to transfer them into this kind of abstract second order evolution equations. Indeed, in many cases it is more advantageous to treat the second-order abstract differential equations directly than converting them to the first order systems. The basic theory of abstract second-order evolution equations governed by generator of strongly continuous cosine family was initiated and developed respectively by Fattorini [12,13], Travis and Webb [44,45]. In the past decades, by means of the theory of cosine and sine operator families, existence, uniqueness, regularity, periodicity, stability and controllability of solutions for this kind of evolution equations have been much studied by various authors, see [3, 5, 7, 17, 26, 32, 37]. Particularly, the approximate controllability of second-order semilinear system with delay has also been discussed in some papers, see [11, 22, 31, 40, 42]. Henríquez and Hernández M [22] studied the approximate controllability of the following second-order neutral functional differential systems with infinite delay

$$\begin{cases} \dfrac{d}{dt}[x'(t) - g(t, x_t)] = Ax(t) + Bu(t) + f(t, x_t), \quad t \in [0, \tau], \\ x_0 = \varphi \in \mathscr{B}, \qquad x'(0) = w \in X, \end{cases} \tag{1.2}$$

where A, B, are defined as above. The authors established the approximate controllability results for Systems (1.2) under some assumptions of range type (also see Naito [36]). It seems that generally it is difficult in applications to verify such range conditions for evolution systems with delay.

On the other hand, Numerous models that arise in applications are properly described using functional (partial) differential equations with state-dependent delay. We refer to the survey paper of Hartung et al. [16], for many examples. For this reason, the theory of functional differential equations with state-dependent delay has become, in recent years, an attractive area of research. We mention the works [1, 6, 8, 21, 28, 33] and the references therein for information on recent results in this area. The problem of existence of solutions for the first-order and second-order partial functional differential equations with state-dependent delay have been studied in [17–20, 29, 30], while qualitative properties of solutions of these equations with state-dependent delay have been investigated in [2, 9, 24].

There is no doubt that the control problems are also very interesting and challenging topics for functional evolution systems with state-dependent delay. Actually, some work on the approximate controllability of first and second order functional differential equations with state-dependent delay can be found in [11, 14, 39–41, 43, 47]. In [14] the authors considered the approximate controllability of systems represented in the following semilinear neutral functional differential equations with state-dependent delay

$$\begin{cases} \dfrac{d}{dt}[x(t)+F(t,x_t)] = -Ax(t)+Bu(t)+G(t,x_{\rho(t,x_t)}), \quad t\in[0,T], \\ x_0=\phi\in\mathscr{B}_\alpha. \end{cases} \tag{1.3}$$

By concerning analytic semigroup theory and the so-called resolvent condition introduced in [4], sufficient conditions of approximate controllability for System (1.3) were formulated and proved there. While in [40], Sakthivel et al. obtained sufficient conditions for the approximate controllability of second-order functional differential equation with state-dependent delay described by the equation

$$\begin{cases} x''(t) = Ax(t)+Bu(t)+f(t,x_{\rho(t,x_t)}), \quad t\in J=[0,b], \\ x_0=\phi\in\mathscr{B}, \qquad x'(0)=\xi_0\in X. \end{cases} \tag{1.4}$$

In their paper, the authors also used the resolvent condition to obtain the sufficient conditions for the approximate controllability of (1.4). Note that in recent years this resolvent condition, which is equivalent to the linear system is approximately controllable, was extensively employed to investigate the approximate controllability of many deterministic or stochastic evolution systems with (without) delay since it is easier to be verified in applications, see [14,31,34,42] among others. The outstanding feature of this technique is that the nonlinear functions in the considered systems should commonly satisfy the condition of uniform boundedness.

Our objective in this work is to study the approximate controllability of the system (1.1). Observe that the term $L(y_t)$ in (1.1) is not uniformly bounded since L is a bounded linear operator on $\mathscr{B}$, thus it can not be regarded as a special case of System (1.4), and its approximate controllability can not be investigated straightly by cosine family and the above resolvent condition method as in [31, 40] (where the the nonlinear terms were required to be uniformly bounded). To overcome the difficulties brought by the non-uniform boundedness of $L(y_t)$, as in [25, 35], we are going to discuss this problem by constructing the fundamental solution for the associated linear second order evolution equations with infinite delay. In this manner we will represent the mild solutions of (1.1) via the fundamental solution and Laplace transform. As a result, we can still explore the controllability by applying the technique of the resolvent condition (see the condition (H_3) in Section 3).

On the other hand, in many practical cases the nonlinear function f in (1.1) frequently involves a term of spatial derivative, like in Example 5.2 discussed in Section 5. It can be seen that, when take $X=L^2(0,\pi)$, then the function f in (5.9) is defined on $[0,T]\times\mathscr{C}_{g,\frac{1}{2}}$, not on $[0,T]\times\mathscr{C}_g$. This means that one can not discuss the existence of mild solutions for (5.9) directly on space X, but on $X_{\frac{1}{2}}$. For this reason, in order to study the approximate controllability of System (1.1) for this situation, we shall also restrict this equation in Section 4.2 in the Banach space $X_{\frac{1}{2}}(\subset X)$ induced by fractional power operators $A^{\frac{1}{2}}$ to get the existence result. Namely, we will discuss the existence of mild solutions by applying fractional power operators theory and the $\frac{1}{2}$-norm. We note that, unlike analytic semigroups, there are few similar estimates on the fractional power operators with analytic cosine and sine operators. Fortunately we can take advantage of the assumption (F) due to Travis and Webb [44] (see (H_4') in Section 4.2) to carry on our discussion. We stress that, we do not use Schauder fixed point theorem to prove the existence of solutions in Theorem 4.3 as the compactness of the operator P_λ in space $X_{\frac{1}{2}}$ is unknown.

It is easy to see that our obtained results extend and develop directly the work on approximate controllability of second order evolution systems with infinite delay appearing in literature mainly in two aspects: one is that the considered system (1.1) involves infinite state-dependent delay and a non-uniformly bounded term; the second is that the results can be applied to the control systems in which the nonlinear functions admit derivative terms of spatial variables. To the best of our knowledge, there are seemly few similar results on this kind of second order evolution systems. In addition, it is worth pointing out here that, the fundamental solution theory for second order linear retarded evolution systems with infinite delay founded in this article are theoretically meaningful and can also be applied to discuss other important issues such as qualitative properties and optimal controls of solutions for second order semilinear FDEs with infinite delay.

Subsequently, we first present in Section 2 some notations and properties of strongly continuous cosine operator families. In addition, notations about phase spaces for infinite delay are also introduced in this section. Then in Section 3, we construct the fundamental solution $G(t)$ and discuss some regularity properties for it. After that, we represent the mild solutions of System (1.1) explicitly via the fundamental solution $G(t)$ using Laplace transform techniques. Based on this, in Section 4 we exploit the approximate controllability of (1.1) for two cases respectively by employing the resolvent condition and the uniform boundedness of the function $F(\cdot,\cdot)$, and some sufficient conditions of approximate controllability for (1.1) are obtained. In Section 5 two examples are provided to show the applications of the main results. Finally, Section 6 is an appendix in which we prove for completeness the existence and uniqueness of mild solutions of the corresponding second order linear system. This is the start point of constructing the fundamental solution.

2. Preliminaries

In this section, we collect some concepts, notations, and properties on strongly continuous cosine family and the phase spaces for infinite delay to be used in the whole paper. Let X be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, and $\mathscr{L}(X)$ be the Banach space of bounded linear operators from X into X.

2.1. Basic conceptions of cosine family

First, let us recall in this subsection some definitions and properties of cosine families.

Definition 2.1. The one parameter family $\{C(t): t\in\mathbb{R}\}\subset\mathscr{L}(X)$ satisfying

(i) $C(0)=I$,

(ii) $C(t+s)+C(t-s)=2C(t)C(s)$ for all $t,\ s\in\mathbb{R}$,

(iii) $C(t)x$ is continuous in t on $\mathbb{R}$, for all $x\in X$,

is called a strongly continuous cosine family on X.

The corresponding strongly continuous sine family $\{S(t) : t \in \mathbb{R}\} \subset \mathscr{L}(X)$ is defined by

$$S(t)x = \int_0^t C(s)xds,\ t \in \mathbb{R},\ x \in X. \tag{2.1}$$

The generator of a strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ is the linear operator $A : D(A) \subset X \to X$ given by

$$Ax = \frac{d^2}{dt^2}C(t)x\Big|_{t=0}, \quad \text{for all } x \in D(A), \tag{2.2}$$

with

$$D(A) = \left\{x \in X : C(\cdot)x \in C^2(\mathbb{R}; X)\right\}.$$

There is a necessary and sufficient condition guaranteeing that an operator $(A, D(A))$ generates a strongly continuous cosine family, which is analogous to the Hille-Yoside generation theorem of operator semigroup theory.

Theorem 2.1 (see [45, Proposition 2.7])**.** *A closed, densely defined linear operator $(A, D(A))$ on X is the infinitesimal generator of a strongly continuous cosine family $(C(t))_{t\in\mathbb{R}}$ with $\|C(t)\| \le M_\omega e^{\omega t}$, if and only if the resolvent $R(\lambda^2; A)$ exists for any $\lambda > \omega$, and it is strongly infinitely differentiable, satisfying*

$$\left|\left(\lambda R(\lambda^2; A)\right)^{(n)}\right| \le \frac{Mn!}{(\lambda-\omega)^{n+1}}, \quad for \quad \lambda > \omega \quad and \quad n = 0, 1, 2, \cdots.$$

Next we present some basic properties of cosine families. Let

$$E = \left\{x \in X : C(\cdot)x \in C^1(\mathbb{R}; X)\right\}.$$

It was proved in Kisyński [27] that E is a Banach space endowed with the norm

$$\|x\|_E = \|x\|_X + \sup_{0\le t\le 1} \|AS(t)x\|_X,\ x \in E.$$

Lemma 2.1 (see [45, Proposition 2.2 and Proposition 2.3])**.** *Suppose that $(A, D(A))$ is the infinitesimal generator of a family of cosine operators $\{C(t) : t \in \mathbb{R}\}$, $\{S(t) : t \in \mathbb{R}\}$ is the corresponding sine family. Then*

- (i) *There exist $M_\omega \ge 1$ and $\omega \ge 0$ such that $\|C(t)\|_{\mathscr{L}(X)} \le M_\omega e^{\omega t}$ and hence $\|S(t)\|_{\mathscr{L}(X)} \le M_\omega e^{\omega t}$.*
- (ii) *For any $x \in X$ and $r, s \in \mathbb{R}$, $\int_s^r S(u)xdu \in D(A)$ and $A\int_s^r S(u)xdu = [C(r) - C(s)]x$.*
- (iii) *There exists $M \ge 1$, such that $\|S(s) - S(r)\|_{\mathscr{L}(X)} \le M\left|\int_r^s e^{\omega|\theta|}d\theta\right|$ for all $0 \le r \le s < \infty$.*
- (iv) *If $x \in E$, then $S(t)x \in D(A)$ and $\frac{d}{dt}C(t)x = AS(t)x$.*
- (v) *If $x \in E$, then $\lim_{t\to 0} AS(t)x = 0$.*
- (vi) *If $x \in D(A)$, then $C(t)x \in D(A)$ and $\frac{d^2}{dt^2}C(t)x = AC(t)x = C(t)Ax$.*

The uniform boundedness principle together with (i) above implies that both $\{C(t) : t \in [0, T]\}$ and $\{S(t) : t \in [0, T]\}$ are uniformly bounded, i.e., there exist positive constants M_1 and M_2 such that

$$\|C(t)\| \le M_1 \text{ and } \|S(t)\| \le M_2 \quad \text{for every } t \in [0, T]. \tag{2.3}$$

The following lemma which can be found in [12] gives a characterization of resolvent of the infinitesimal generator of a strongly continuous cosine family.

Lemma 2.2. *Let* $(C(t))_{t\in\mathbb{R}}$ *be a strongly continuous cosine family in* X *satisfying* $\|C(t)\|_X \le M_\omega e^{\omega|t|}$, $t \in \mathbb{R}$, *and* $(A, D(A))$ *its the infinitesimal generator. Then for* $\lambda \in \mathbb{C}$ *with* $Re\lambda > \omega$, $\lambda^2 \in \rho(A)$ *and*

$$\lambda R(\lambda^2, A)x = \int_0^\infty e^{-\lambda t} C(t)x dt, \quad for \quad x \in X, \tag{2.4}$$

$$R(\lambda^2, A)x = \int_0^\infty e^{-\lambda t} S(t)x dt, \quad for \quad x \in X. \tag{2.5}$$

2.2. Fractional power operators

Throughout this paper, we will always assume that the linear operator $(-A, D(-A))$ in System (1.1) satisfies that

(H_0) For any $\lambda > 0$, the resolvent $R(\lambda^2; -A)$ exists, and it is strongly infinitely differentiable, satisfying

$$\left|\left(\lambda R(\lambda^2; -A)\right)^{(n)}\right| \le \frac{n!}{\lambda^{n+1}}, \quad \text{for} \quad \lambda > 0 \quad \text{and} \quad n = 0, 1, 2, \cdots.$$

Then from Theorem 2.1, $(-A, D(-A))$ generates a uniformly bounded (strongly continuous) cosine family $(C(t))_{t\in\mathbb{R}}$ on the Hilbert space X. Moreover, since the spectrum $\sigma(-A)$ is merely contained in the negative real axis, $(-A, D(-A))$ also generates an analytic semigroup $(T(t))_{t\ge 0}$ on X, and actually we have the Weierstrass formula (see [12], Remark 5.11)

$$T(t)x = \frac{1}{(\pi t)^{\frac{1}{2}}} \int_0^\infty e^{-\frac{s^2}{4t}} C(s)x ds, \; t > 0, \; x \in X. \tag{2.6}$$

In the next section we shall, as mentioned in Section 1, study the fractional power theory for the fundamental solutions which will be applied to discuss the controllability problem of System (1.1). For this purpose, in the sequel we present some results on the fractional power theory for cosine families. Let $0 \in \rho(A)$, then we can define in the following way the fractional power A^α, for $0 < \alpha < 1$, as a closed linear operator on its domain $D(A^\alpha)$ (see [38] for more details). Define the bounded linear operator $A^{-\alpha}$ as

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} T(t) dt, \; 0 < \alpha < 1, \tag{2.7}$$

where $\Gamma(\cdot)$ is the well known Γ function given by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$. And A^α $(\alpha \in (0,1))$ is then defined as $A^\alpha = (A^{-\alpha})^{-1}$. Moreover, the subspace $D(A^\alpha)$ is dense in X and the expression

$$\|x\|_\alpha = \|A^\alpha x\|, \qquad x \in D(A^\alpha),$$

defines a norm on $D(A^\alpha)$. Denoting the space $(D(A^\alpha), \|\cdot\|_\alpha)$ by X_α, then it is well known that for each $0 < \alpha \le 1$, X_α is a Banach space, $X_\alpha \hookrightarrow X_\beta$, for $0 < \beta < \alpha \le 1$, and the imbedding is compact whenever $R(\lambda, A) = (\lambda I + A)^{-1}$, the resolvent operator of $-A$, is compact for some $\lambda > 0$. Hereafter, we denote by $C([0,T];X_\alpha)$ the Banach space of continuous functions from $[0,T]$ to X_α with the norm

$$\|x\|_{C_\alpha} = \sup_{0\le t\le T} \|A^\alpha x(t)\|, \quad x \in C([0,T];X_\alpha).$$

In the light of the above statements we can now establish the following result.

Theorem 2.2. *Let $(C(t))_{t\in\mathbb{R}}$ be the cosine family generated by $(-A, D(-A))$, $(S(t))_{t\in\mathbb{R}}$ be the associated sine family. Then $C(t)$ and $S(t)$ commute with the operator A^α, that is, for each $\alpha \in (0,1)$, $x \in D(A^\alpha)$ and for all $t \ge 0$, there hold*

$$A^\alpha C(t)x = C(t)A^\alpha x \quad \text{and} \quad A^\alpha S(t)x = S(t)A^\alpha x.$$

Proof. If $x \in D(A^\alpha)$, let $y = A^\alpha x$, then $x = A^{-\alpha} y$. Combining (2.6) and (2.7) we have that

$$\begin{aligned} C(t)x &= C(t)A^{-\alpha}y = C(t)\frac{1}{\Gamma(\alpha)}\int_0^\infty s^{\alpha-1}T(s)yds \\ &= C(t)\frac{1}{\Gamma(\alpha)}\int_0^\infty s^{\alpha-1}\frac{1}{(\pi s)^{\frac12}}\int_0^\infty e^{-\frac{r^2}{4s}}C(r)ydrds \\ &= \frac{1}{\Gamma(\alpha)}\int_0^\infty s^{\alpha-1}\frac{1}{(\pi s)^{\frac12}}\int_0^\infty e^{-\frac{r^2}{4s}}C(t)C(r)ydrds \\ &= \frac{1}{\Gamma(\alpha)}\int_0^\infty s^{\alpha-1}T(s)ydsC(t) = A^{-\alpha}C(t)y. \end{aligned}$$

Hence, $C(t)x \in D(A^\alpha)$, and $A^\alpha C(t)x = C(t)A^\alpha x$. By the expression of $S(t)$, it is easy to verify

$$A^\alpha S(t)x = S(t)A^\alpha x, \text{ for each } \alpha \in (0,1),\ x \in D(A^\alpha).$$

The proof is completed. □

2.3. Phase space for infinite delay

In this subsection we turn to introduce some notations on the phase spaces for infinite delay. Throughout this paper, we will employ an axiomatic definition of the phase space $\mathscr{B}$ introduced by Hale and Kato [15]. Adopting the terminologies used in [23], $\mathscr{B}$ will be a linear space of functions mapping $(-\infty,0]$ into X endowed with a seminorm $\|\cdot\|_{\mathscr{B}}$ and satisfies the following axioms:

- (A) If $x : (-\infty, \sigma + a] \to X$, $a > 0$, is continuous on $[\sigma, \sigma + a]$ and $x_\sigma \in \mathscr{B}$, then for every $t \in [\sigma, \sigma + a]$ the followings hold:
  (i) x_t is in $\mathscr{B}$;
  (ii) $\|x(t)\| \le H\|x_t\|_{\mathscr{B}}$;
  (iii) $\|x_t\|_{\mathscr{B}} \le K(t-\sigma)\sup\{\|x(s)\| : \sigma \le s \le t\} + M(t-\sigma)\|x_\sigma\|_{\mathscr{B}}$.
  Here $H \ge 0$ is a constant, $K, M : [0,+\infty) \to [0,+\infty)$, $K(\cdot)$ is continuous and $M(\cdot)$ is locally bounded, and H, $K(\cdot)$, $M(\cdot)$ are independent of $x(t)$.
- (A_1) For the function $x(\cdot)$ in (A), x_t is a $\mathscr{B}$-valued continuous function on $[\sigma, \sigma+a]$.
- (B) The space $\mathscr{B}$ is complete.

We denote by $\mathscr{B}_\alpha$ the set of all the elements in $\mathscr{B}$ which take values in space X_α, that is,

$$\mathscr{B}_\alpha := \{\varphi \in \mathscr{B} :\ \varphi(\theta) \in X_\alpha \text{ for all } \theta \le 0\}.$$

Then $\mathscr{B}_\alpha$ becomes a subspace of $\mathscr{B}$ endowed with the seminorm $\|\cdot\|_{\mathscr{B}_\alpha}$ which is induced by $\|\cdot\|_{\mathscr{B}}$ through $\|\cdot\|_\alpha$. More precisely, for any $\varphi \in \mathscr{B}_\alpha$, the seminorm $\|\cdot\|_{\mathscr{B}_\alpha}$ is defined by $\|A^\alpha\varphi\|$, instead of $\|\varphi\|$. For example, let the phase space $\mathscr{B} = C_r \times L^p(g : X)$, $r \ge 0$, $1 \le p < \infty$ (see [23]), which consists of all classes of functions $\varphi \in (-\infty, 0] \to X$ such that φ is continuous on $[-r, 0]$ and $g\|\varphi(\cdot)\|^p$ is Lebesgue integrable on $(-\infty, -r)$, where $g : (-\infty, -r) \to \mathbb{R}$ is a positive Lebesgue integrable function. The seminorm in $\mathscr{B}$ is defined by

$$\|\varphi\|_{\mathscr{B}} = \sup\{\|\varphi(\theta)\| : -r \le \theta \le 0\} + \left(\int_{-\infty}^{-r} g(\theta)\|\varphi(\theta)\|^p d\theta\right)^{\frac{1}{p}}.$$

Then the seminorm in $\mathscr{B}_\alpha$ is defined by

$$\|\varphi\|_{\mathscr{B}_\alpha} = \sup\{\|A^\alpha\varphi(\theta)\| : -r \le \theta \le 0\} + \left(\int_{-\infty}^{-r} g(\theta)\|A^\alpha\varphi(\theta)\|^p d\theta\right)^{\frac{1}{p}}.$$

See also the space $\mathscr{C}_{g,\frac{1}{2}}$ presented in Section 5. Hence, since X_α is still a Banach space, we see that the subspace $\mathscr{B}_\alpha$ will satisfy the following conditions:

(A') If $x : (-\infty, \sigma + a] \to X_\alpha$, $a > 0$, is continuous on $[\sigma, \sigma + a]$ (in $\alpha-$norm) and $x_\sigma \in \mathscr{B}_\alpha$, then for every $t \in [\sigma, \sigma + a]$ the followings hold:
(i) x_t is in $\mathscr{B}_\alpha$;
(ii) $\|x(t)\|_\alpha \le H\|x_t\|_{\mathscr{B}_\alpha}$;
(iii) $\|x_t\|_{\mathscr{B}_\alpha} \le K(t-\sigma)\sup\{\|x(s)\|_\alpha : \sigma \le s \le t\} + M(t-\sigma)\|x_\sigma\|_{\mathscr{B}_\alpha}$.
Here H, $K(\cdot)$ and $M(\cdot)$ are as in $(A)(iii)$ above.

(A'_1) For the function $x(\cdot)$ in (A'), x_t is a $\mathscr{B}_\alpha$-valued continuous function on $[\sigma, \sigma+a]$.

(B') The space $\mathscr{B}_\alpha$ is complete.

For any $\varphi \in \mathscr{B}_\alpha$, the notation φ_t, $t \le 0$, represents the function $\varphi_t(\theta) = \varphi(t+\theta)$, $\theta \in (-\infty, 0]$. Then, if the function $x(\cdot)$ in axiom (A') with $x_0 = \varphi$, we may extend the mapping $t \to x_t$ to the whole interval $(-\infty, T]$ by setting $x_t = \varphi_t$ for $t \le 0$. On the other hand, for the function $\rho : [0, T] \times \mathscr{B}_\alpha \to (-\infty, T]$, we introduce the set

$$Z(\rho^-) = \{\rho(s, \psi) : \rho(s, \psi) \le 0,\ (s, \psi) \in [0, T] \times \mathscr{B}_\alpha\}$$

and give the following hypothesis on φ_t : the function $t \to \varphi_t$ is continuous from $Z(\rho^-)$ into $\mathscr{B}_\alpha$ and there exists a continuous and bounded function $H^\varphi : Z(\rho^-) \to (0, +\infty)$ such that, for each $t \in Z(\rho^-)$,

$$\|\varphi_t\|_{\mathscr{B}_\alpha} \le H^\varphi(t)\|\varphi\|_{\mathscr{B}_\alpha}.$$

Then we have the following lemma, which plays an important role in our proofs in the next section.

Lemma 2.3 (see [23])**.** *Let $x : (-\infty, T] \to X_\alpha$ be a function such that $x_0 = \varphi$ and the restriction of $x(\cdot)$ to the interval $[0, T]$ is continuous. Then*

$$\|x_s\|_{\mathscr{B}_\alpha} \le (H_1+H_3)\|\varphi\|_{\mathscr{B}_\alpha} + H_2 \sup\{\|x(\theta)\|_\alpha : \theta \in [0, \max\{0, s\}]\},\ s \in Z(\rho^-) \cup [0, T],$$

where

$$H_1 = \sup_{t \in Z(\rho^-)} H^\varphi(t),\ H_2 = \sup_{t \in [0,T]} K(t),\ H_3 = \sup_{t \in [0,T]} M(t).$$

3. Fundamental solution

In this section we will first construct the theory of fundamental solution for the linear system corresponding to Eq. (1.1), then we express the mild solutions of Eq. (1.1) via the established fundamental solution based on Laplace transform arguments. Let us make the following hypotheses on the operators appearing in System (1.1).

(H_1) $B \in \mathscr{L}(U; X)$, i.e., B is a bounded linear operator from U to X.

(H_2) The operator $L : \mathscr{B} \to X$ is a bounded linear operator with $\|L\| = l$ for some $l > 0$.

Consider now the following linear second order functional differential equation on space X associated to System (1.1)

$$\begin{cases} \frac{d^2}{dt^2} y(t) = -Ay(t) + L(y_t), & t \geq 0, \\ y_0 = \varphi, \quad y'(0) = x^0, & t \leq 0, \end{cases} \tag{3.1}$$

where $(-A, D(-A))$ and $L : \mathscr{B} \to X$ are operators given above and $(\varphi, x^0) \in \mathscr{B} \times X$. The mild solutions of System (3.1) can be defined by the cosine family $C(t)$ and sine family $S(t)$ generated by $(-A, D(-A))$ as (see [44])

Definition 3.1. A function $y(\cdot) : (-\infty, T] \to X$, denoted by $y(t; \varphi, x^0)$ to show its dependence on the initial date (φ, x^0), is called a mild solution of Eq. (3.1), if it satisfies that

$$y(t; \varphi, x^0) = \begin{cases} C(t)\varphi(0) + S(t)x^0 + \int_0^t S(t-s)L\left(y_s(\cdot; \varphi, x^0)\right) ds, & t \in [0, T], \\ \varphi(t), \quad t \leq 0. \end{cases} \tag{3.2}$$

Remark 3.1. Note that, if $\varphi(0) \in E$, then the mild solution $y(\cdot; \varphi, x^0)$ is continuously differentiable and verifies

$$y'(t; \varphi, x^0) = AS(t)\varphi(0) + C(t)x^0 + \int_0^t C(t-s)L(y_s(\cdot; \varphi, x^0))ds. \tag{3.3}$$

Obviously, from Lemma 2.1 (iv), one has $y'(0) = x^0$ in this case.

Then, we may establish the following existence and uniqueness result for System (3.1). Its proof is standard but not trivial, and for the sake of completeness we will present the whole proof in details in Appendix, see Section 6.

Theorem 3.1. *For any $(\varphi, x^0) \in \mathscr{B} \times X$ and $T > 0$, there exists a unique mild solution $y(t) = y(t; \varphi, x^0)$ of System (3.1) on $(-\infty, T]$. Moreover, it satisfies that*

$$\|y(t; \varphi, x^0)\| \leq M_* e^{\gamma t} \left(\|\varphi\|_{\mathscr{B}} + \|x^0\|\right), \tag{3.4}$$

for all $t \geq 0$, where $M_* > 1$ and $\gamma \in \mathbb{R}$ are constants.

Making use of the preceding theorem we now set about to construct the theory of fundamental solution. Let $y(\cdot; 0, x^0)$ denote the solution of (3.1) through $(0; 0, x^0)$. For any $x \in X$, we define the fundamental solution $G(t) \in \mathscr{L}(X)$ as

$$G(t)x = \begin{cases} y(t; 0, x), & t \geq 0, \\ 0, & t < 0. \end{cases}$$

That is to say, from (3.2), $G(t)$ is the unique solution of the operator equation

$$G(t)=\begin{cases} S(t)+\int_0^t S(t-s)L\left(G_s\right)ds, & t\geq 0,\\ 0, & t<0, \end{cases} \tag{3.5}$$

where $G_t(\theta):=G(t+\theta)$, $\theta\in(-\infty,0]$. This definition is well guaranteed by Theorem 3.1. For the fundamental solution $G(t)$, one has the following properties.

Theorem 3.2. *For $G(t)$, $t\in\mathbb{R}$, there hold*

(*i*) *$\big(G(t)\big)_{t\in\mathbb{R}}$ is a strongly continuous one-parameter family of bounded linear operators on X and satisfies that*

$$\|G(t)\|\leq M_*e^{\gamma t}, \qquad t\geq 0,$$

where $M_*>1$ and $\gamma\in\mathbb{R}$ are from Theorem 3.1. Clearly, there is some $\overline{M}\geq 1$ such that

$$\|G(t)\|\leq\overline{M}, \qquad t\in[0,T]. \tag{3.6}$$

(*ii*) *If the strongly continuous sine family $\big(S(t)\big)_{t\geq 0}$ is compact, then $G(t)$ is also compact for all $t\geq 0$.*

(*iii*) *For all $x\in X$, the function $G(\cdot)x$ is continuously differentiable for any $t\geq 0$, and*

$$\frac{dG(t)x}{dt}=C(t)x+\int_0^t C(t-s)L(G_s)xds, \qquad \frac{dG(t)x}{dt}\Big|_{t=0}=x. \tag{3.7}$$

Moreover,

$$\|G'(\cdot)\|\leq N, \quad t\in[0,T], \tag{3.8}$$

for $N=M_1+M_1l\overline{M}T$.

(*iv*) *If $x\in E$, then the function $G'(\cdot)x$ is still continuously differentiable for any $t\geq 0$ and there hold*

$$\frac{d^2}{dt^2}G(t)x=AS(t)x+\int_0^t AS(t-s)L(G_s)xds \quad \text{and} \quad \frac{d^2}{dt^2}G(t)x\Big|_{t=0}=0. \tag{3.9}$$

(*v*) *$G(t)$ is uniformly continuous on $[0,T]$.*

(*vi*) *If the operator L maps $\mathscr{B}_\alpha$ into $D(A^\alpha)$ and there holds $A^\alpha L=LA^\alpha$, then for all $\alpha\in(0,1)$, $G(t)$, $G'(t)$ commute with the operator A^α, that is, $A^\alpha G(t)=G(t)A^\alpha$ and $A^\alpha G'(t)=G'(t)A^\alpha$ for each $\alpha\in(0,1)$.*
(Here, $A^\alpha L=LA^\alpha$ is understood as, for any $\varphi\in\mathscr{B}_\alpha$, $A^\alpha L(\varphi)=L(A^\alpha\varphi)$. It is readily seen that this commuting property is verified for any systems, also see Example 5.2 in Section 5.)

(*vii*) *If $\|A^{\frac{1}{2}}S(t)\|\leq C$ for some constant $C>0$, then there is $\widehat{M}>0$ such that, for $t\in(0,T]$,*

$$\left\|A^{\frac{1}{2}}G(t)\right\|\leq\widehat{M}.$$

Proof. Assertions (i), (ii) and (iii) follow readily from Theorem 3.1 and the formula (3.5) of $G(\cdot)$.

In order to prove Assertion (iv), firstly, for $x \in E$, by the definition of $S(t)$ and Lemma 2.1, we see $S(t)x \in D(A)$, then $G(\cdot)x \in D(A)$ and the function $G(\cdot)x$ is continuously differentiable. In particular, (3.9) follows from (3.7), and by Assertion (i), we have

$$\begin{aligned}\|G'(t)\| \leq &\|C(t)\| + \|\int_0^t C(t-s)L(G_s)ds\| \\ \leq &M_1 + M_1 l\overline{M}T := N.\end{aligned}$$

Due to Lemma 2.1, $S(t)$ is uniformly continuous on $[0, T]$, then from (3.5) again we deduce that $G(t)$ is also uniformly continuous for $t \in [0, T]$ which proves Assertion (v).

As for (vi), using (3.5) and (3.7) and applying Theorem 2.2 we obtain immediately the properties $A^\alpha G(t) = G(t)A^\alpha$ and $A^\alpha G'(t) = G'(t)A^\alpha$, for each $\alpha \in (0, 1)$.

Finally, applying the assumption to (3.5) we get Assertion (vii) readily. □

Next we further consider the following linear inhomogeneous second-order function differential equation with infinite delay on X:

$$\begin{cases} \dfrac{d^2}{dt^2}y(t) = -Ay(t) + L(y_t) + f(t), & t \geq 0, \\ y_0 = \varphi, \quad y'(0) = x^0, & t \leq 0, \end{cases} \tag{3.10}$$

where the function $f(t) \in L^1(\mathbb{R}^+, X)$. The mild solutions of Eq. (3.10) are represented by cosine and sine families as

$$y(t; \varphi, x^0) = \begin{cases} C(t)\varphi(0) + S(t)x^0 \\ \quad + \displaystyle\int_0^t S(t-s)\left[L\left(y_s(\cdot; \varphi, x^0)\right) + f(s)\right] ds, \quad t \geq 0, \\ \varphi(t), \qquad t \leq 0. \end{cases} \tag{3.11}$$

For the subsequent discussion, we need to represent the mild solutions (3.11) via the fundamental solution $G(t)$ established previously, that is

Theorem 3.3. *For $(\varphi, x^0) \in \mathscr{B} \times X$, the mild solutions (3.11) of System (3.10) can be expressed equivalently by*

$$y(t; \varphi, x^0) = \begin{cases} G'(t)\varphi(0) + G(t)x^0 + \displaystyle\int_0^t G(t-s)\left[L(\tilde{\varphi}_s) + f(s)\right] ds, \ t \geq 0, \\ \varphi(t), \qquad\qquad t \leq 0, \end{cases} \tag{3.12}$$

where the function $\tilde{\varphi}(\cdot)$ is defined as

$$\tilde{\varphi}(t) = \begin{cases} \varphi(t), & t \leq 0, \\ 0, & t > 0. \end{cases}$$

Proof. We prove this theorem by Laplace transform arguments. Thanks to Theorem 3.1 we may calculate the Laplace transform $\hat{y}(\lambda)$ of $y(t)$ for all $\lambda \in \mathbb{C}$ with $\text{Re}\lambda > \max\{0, \gamma\}$.

In fact, due to Lemma 2.2, one has that $\hat{C}(\lambda) = \int_0^{+\infty} e^{-\lambda t}C(t)dt = \lambda R(\lambda^2, -A)$ for $Re\lambda > 0$, and the Laplace transform of $\int_0^t S(t-s)\left[L\left(y_s(\cdot;\varphi,x^0)\right) + f(s)\right]ds$ is computed as

$$\begin{aligned}
&\int_0^{+\infty} e^{-\lambda t}\int_0^t S(t-s)\left[L\left(y_s(\cdot;\varphi,x^0)\right) + f(s)\right]dsdt\\
=&\int_0^{+\infty}\int_0^{+\infty} e^{-\lambda(t+s)}S(t)dt\left[L\left(y_s(\cdot;\varphi,x^0)\right) + f(s)\right]ds\\
=&R(\lambda^2,-A)\int_0^{+\infty} e^{-\lambda s}\left[L\left(y_s(\cdot;\varphi,x^0)\right) + f(s)\right]ds\\
=&R(\lambda^2,-A)\left[\int_0^{+\infty} e^{-\lambda s}L\left(y(s+\theta,\varphi,x^0)\right)ds + \hat{f}(\lambda)\right]\\
=&R(\lambda^2,-A)\left[L\left(\int_\theta^0 \varphi(t)e^{\lambda(\theta-t)}dt\right) + L\Big(e^{\lambda\theta}I\Big)\hat{y}(\lambda) + \hat{f}(\lambda)\right].
\end{aligned}$$

Hence, from (3.11) we get

$$\begin{aligned}
\hat{y}(\lambda) =&\lambda R(\lambda^2,-A)\varphi(0) + R(\lambda^2,-A)x^0\\
&+ R(\lambda^2,-A)\left[L\left(\int_\theta^0 \varphi(t)e^{\lambda(\theta-t)}dt\right) + L\left(e^{\lambda\theta}I\right)\hat{y}(\lambda) + \hat{f}(\lambda)\right]\\
=&R(\lambda^2,-A)\left[\lambda\varphi(0) + x^0 + L\left(\int_\theta^0 \varphi(t)e^{\lambda(\theta-t)}dt\right) + L\left(e^{\lambda\theta}I\right)\hat{y}(\lambda) + \hat{f}(\lambda)\right],
\end{aligned}$$

or

$$\Delta(\lambda)\hat{y}(\lambda) = R(\lambda^2,-A)\left[\lambda\varphi(0) + x^0 + L\left(\int_\theta^0 \varphi(t)e^{\lambda(\theta-t)}dt\right) + \hat{f}(\lambda)\right],$$

where $\Delta(\lambda) := I - R(\lambda^2,-A)L\left(e^{\lambda\theta}I\right)$. It is easy to see that $\Delta(\lambda) \to I$ as $\mathrm{Re}\lambda \to +\infty$, which indicates that there exists a $\lambda_0 > 0$ such that for any $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda > \lambda_0$, the inverse operator $\Delta^{-1}(\lambda)$ exists. Thus, for all λ with $Re(\lambda) > \max\{\gamma, \lambda_0\}$, we get

$$\hat{y}(\lambda) = \Delta^{-1}(\lambda)R(\lambda^2,-A)\left[\lambda\varphi(0) + x^0 + L\left(\int_\theta^0 \varphi(t)e^{\lambda(\theta-t)}dt\right) + \hat{f}(\lambda)\right]. \quad (3.13)$$

On the other hand, by (3.5) and Theorem 3.2 (i), we may similarly calculate the Laplace transform of the operator $G(t)$ as (note that $G(t) = 0$ for $t \leq 0$)

$$\begin{aligned}
\hat{G}(\lambda) &= R(\lambda^2,-A) + R(\lambda^2,-A)L\left(\int_0^{+\infty} G(t+\theta)e^{-\lambda t}dt\right)\\
&= R(\lambda^2,-A)\left[I + L\left(e^{\lambda\theta}I\right)\hat{G}(\lambda)\right].
\end{aligned}$$

So, for all λ with $Re\lambda \geq \lambda_0$, it gives

$$\hat{G}(\lambda) = \Delta^{-1}(\lambda)R(\lambda^2,-A). \quad (3.14)$$

Substituting (3.14) into (3.13), we obtain

$$\hat{y}(\lambda) = \hat{G}(\lambda)\left[\lambda\varphi(0) + x^0 + L\left(\int_\theta^0 \varphi(t)e^{\lambda(\theta-t)}dt\right) + \hat{f}(\lambda)\right]. \quad (3.15)$$

Observe by the definition of function $\tilde{\varphi}(\cdot)$ that

$$L\left(\int_\theta^0 \varphi(t)e^{\lambda(\theta-t)}dt\right) = L\left(\int_0^{-\theta} \varphi(t+\theta)e^{-\lambda t}dt\right) = L\left(\int_0^{+\infty} \tilde{\varphi}_t(\cdot)e^{-\lambda t}dt\right) = (L(\tilde{\varphi}_t(\cdot)))^\wedge$$

and $\hat{G}'(t) = \lambda\hat{G}(\lambda)$. Then employing the inverse transform to (3.15) we arrive at

$$y(t) = G'(t)\varphi(0) + G(t)x^0 + \int_0^t G(t-s)\Big(L(\tilde{\varphi}_s) + f(s)\Big)ds,$$

from which and the uniqueness of Laplace transforms, we get the desired result. □

Remark 3.2. (i) If $(\varphi(0), x^0) \in E \times X$, then by Lemma 2.1, (3.3) and Theorem 3.2, one has readily that

$$\frac{d}{dt}y(t;\varphi,x^0)\Big|_{t=0} = x^0,$$

which manifests that the initial condition $y'(0) = x^0$ is also involved in the expression (3.12).

(ii) The value of this new expression of mild solutions is that $y(\cdot)$ does not appear directly in the right hand side of (3.12).

Accordingly, the mild solutions of Eq. (1.1) expressed by the fundamental solution are defined as

Definition 3.2. A function $y(\cdot) : (-\infty, T] \to X$ is said to be a mild solution of Eq. (1.1), if it is continuous on $[0,T]$ and satisfies that

$$y(t) = \begin{cases} G'(t)\varphi(0) + G(t)x^0 \\ \quad + \displaystyle\int_0^t G(t-s)\Big(L(\tilde{\varphi}_s) + F(s, y_{\rho(s,y_s)}) + Bu(s)\Big)ds, \ t \in [0,T], \\ \varphi(t), \quad t \in (-\infty, 0]. \end{cases}$$

Next we turn to present the concept of approximate controllability.

Definition 3.3. For an initial date $(\varphi, x^0) \in \mathscr{B} \times X$, System (1.1) is said to be approximately controllable on $[0,T]$, if $\mathscr{R}(T;\varphi,x^0)$ is dense in X, i.e.,

$$\overline{\mathscr{R}(T;\varphi,x^0)} = X,$$

where $\mathscr{R}(T;\varphi,x^0) = \left\{y(T;\varphi,x^0,u) : \ u(\cdot) \in L^2([0,T];U)\right\}$.

As mentioned in Section 1, in the sequel, we shall study the approximate controllability for System (1.1) by using a so-called resolvent operator condition (the condition (H_3) below). For this purpose, we introduce the following resolvent operator. Let

$$\Gamma_0^T = \int_0^T G(T-s)BB^*G^*(T-s)ds,$$

where B^* and G^* denote respectively the adjoint operators of B and G, then the resolvent operator $R(\lambda, -\Gamma_0^T) \in \mathscr{L}(X)$ for $\lambda > 0$ is defined as

$$R(\lambda, -\Gamma_0^T) := (\lambda I + \Gamma_0^T)^{-1}.$$

Since the operator Γ_0^T is clearly positive, $R(\lambda, -\Gamma_0^T)$ is well defined. We will always assume that

(H_3) $\lambda R(\lambda, -\Gamma_0^T) \to 0$ as $\lambda \to 0^+$ in the strong operator topology.

The above condition (H_3) is equivalent to the approximate controllability of the linear system

$$\begin{cases} \dfrac{d^2}{dt^2} y(t) = -Ay(t) + L(y_t) + Bu(t), \quad t \in [0, T], \\ y_0 = 0, \quad y'(0) = x^0. \end{cases} \tag{3.16}$$

More precisely, we have that

Theorem 3.4. *The following statements are equivalent:*

(i) *The control system (3.16) is approximately controllable on* $[0, T]$.

(ii) *If* $B^*G^*(t)y = 0$ *for all* $t \in [0, T]$, *then* $y = 0$.

(iii) *The condition* (H_3) *holds.*

This theorem can be prove in the similar way as that of ([4, Theorem 2]) and ([10, Theorem 4.4.17]), so we omit the proof here.

4. Approximate controllability

With the preceding preparation, in this section we devote to investigating the approximate controllability for System (1.1) to obtain the main results of this article .

4.1. The general case

We first discuss the general case, i.e., the approximate controllability problem of System (1.1) under the assumption that the function $F(\cdot, \cdot)$ is defined on $[0, T] \times \mathscr{B}$. To do this, besides the previous assumptions on operators A, L and B, we will also impose the following restrictions on the sine family $(S(t))_{t \geq 0}$ and the nonlinear function $F(\cdot, \cdot)$.

(H_4) The sine family $S(t)$ is compact for any $t \geq 0$.

(H_5) The function $F : [0, T] \times \mathscr{B} \to X$ verifies the following conditions:

(i) For any $y : (-\infty, T] \to X$ satisfying that $y_0 = \varphi$ and the restriction of $y(\cdot)$ to the interval $[0, T]$ is continuous, the function $t \to F(s, y_{\rho(s, y_s)})$ is strongly measurable on $[0, T]$ and $t \to F(s, y_{\rho(s, y_s)})$ is continuous on $Z(\rho^-) \cup [0, T]$ for every $t \in [0, T]$.

(ii) For each $r > 0$, there exists a function $f_r \in C([0, T], \mathbb{R}^+)$ such that

$$\sup_{\|\varphi\|_{\mathscr{B}} \leq r} \|F(t, \varphi)\| \leq f_r(t), \ t \in [0, T], \ \varphi \in \mathscr{B}.$$

And there exist $l_1 > 0$ and $\gamma_1 \in (0, 1)$ such that

$$\|f_r(\cdot)\| \leq l_1(r^{\gamma_1} + 1).$$

We shall prove that, for given $(\varphi, x_0) \in \mathscr{B} \times X$, for any $y^T \in X$, by selected proper control u^λ (for any given $\lambda \in (0,1)$), there exists a mild solution $y^\lambda(\cdot;\varphi,x^0,u^\lambda) : (-\infty,T] \to X$ for System (1.1), such that $y^\lambda(T;\varphi,x^0,u^\lambda) \to y^T$ in X as $\lambda \to 0^+$, which reaches the result.

In what follows, we identify a function $y(\cdot) : [0,T] \to X$ satisfying $y(0) = \varphi(0)$ with its continuous extension to $(-\infty, 0]$ by φ (see (6.5)) so that y_t is always well defined for each $t \in (-\infty, T]$.

Now, let $y^T \in X$, $(\varphi, x^0) \in \mathscr{B} \times X$. For any function $y(\cdot) \in C([0,T];X)$ with $y(0) = \varphi(0)$, we take the control function u^λ, simply denoted by $u(t)$, as

$$
\begin{aligned}
u(t) := {} & B^*G^*(T-t)R(\lambda, -\Gamma_0^T)\cdot \\
& \Big(y^T - G'(T)\varphi(0) - G(T)x^0 - \int_0^T G(T-s)\Big(L(\tilde{\varphi}_s) + F(s, y_{\rho(s,y_s)})\Big)ds\Big).
\end{aligned}
\tag{4.1}
$$

Using this control function, we define the operator P_λ on $C([0,T];X)$ as

$$
(P_\lambda y)(t) = G'(t)\varphi(0) + G(t)x^0 + \int_0^t G(t-s)\Big(L(\tilde{\varphi}_s) + F(s, y_{\rho(s,y_s)}) + Bu(s)\Big)ds,
\tag{4.2}
$$

for $t \in [0,T]$. We begin with proving the following existence result by applying the Schauder fixed point theorem.

Theorem 4.1. *Let $(\varphi, x^0) \in \mathscr{B} \times X$. Assume that the above hypotheses $(H_0)-(H_5)$ are all fulfilled, then, for any $\lambda \in (0,1)$, System (1.1) admits a mild solution $y^\lambda(\cdot) : (-\infty,T] \to X$ which is continuous on $[0,T]$.*

Proof. Let $y^T \in X$ and $(\varphi, x^0) \in \mathscr{B} \times X$ be given, and put

$$
E(b) := \left\{ y(\cdot) \in C([0,T];X) \mid \ y(0) = \varphi(0), \ \|y(t)\| := \sup_{t\in[0,T]} \|y(t)\| \le b \right\}.
$$

Then $E(b)$ is obviously a bounded, closed and convex subset of $C([0,T];X)$.

We will prove the assertion by applying Schauder fixed point theorem that for each $\lambda \in (0,1)$, there is a $b_0 > 0$ such that the operator P_λ given by (4.2) has a fixed point on $E(b_0)$.

Initially, we show that $P_\lambda(E(b_0)) \subset E(b_0)$ for some $b_0 > 0$. If this is not true, then, for every $b > 0$, there exist $y \in E(b)$ and some $t \in [0,T]$ such that $\|(P_\lambda y)(t)\| > b$. Then, noting by (H_3) that (from (H_3) we may assume w.l.o.g. $\left\|R(\lambda, -\Gamma_0^T)\right\| < \frac{1}{\lambda}$ for $\lambda \in (0,1)$)

$$
\begin{aligned}
\|u(t)\| \le {} & \left\|B^*G^*(T-t)R(\lambda,-\Gamma_0^T)\right\| \cdot \Big\| y^T - G'(T)\varphi(0) - G(T)x^0 \\
& - \int_0^T G(T-s)\Big(L(\tilde{\varphi}_s) + F(s, y_{\rho(s,y_s)})\Big)ds \Big\| \\
\le {} & \frac{1}{\lambda}\overline{M}\|B\| \cdot \Big[\left\|y^T\right\| + N\|\varphi(0)\| + \overline{M}\|x^0\| + \overline{M}Tl\|\varphi\|_{\mathscr{B}} \\
& + \overline{M}\int_0^T \|F(s, y_{\rho(s,y_s)})\|ds \Big] \\
:= {} & M_\lambda.
\end{aligned}
$$

Then, by virtue of Theorem 3.2, we can estimate

$$
\begin{aligned}
b &< \|(P_\lambda y)(t)\| \\
&\leq \|G'(t)\varphi(0)\|_{\mathscr{B}} + \left\|G(t)x^0\right\| + \left\|\int_0^t G(t-s)\Big(L(\tilde{\varphi}_s) + F(s, y_{\rho(s,y_s)}) + Bu(s)\Big)ds\right\| \\
&\leq N\|\varphi(0)\| + \overline{M}\left\|x^0\right\| + \overline{M}Tl\|\varphi\|_{\mathscr{B}} + \int_0^t \|G(t-s)\|\left(\|F(s, y_{\rho(s,y_s)})\| + \|B\|M_\lambda\right)ds.
\end{aligned}
$$

For any $y \in E(b)$, it follows from Lemma 2.3 that

$$
\|y_{\rho(s,y_s)}\|_{\mathscr{B}} \leq (H_1 + H_3)\|\varphi\|_{\mathscr{B}} + H_2 b := r.
$$

Hence, using $(H_5)(ii)$ we further obtain that

$$
\begin{aligned}
b < \|(P_\lambda y)(t)\| \leq& N\|\varphi(0)\| + \overline{M}\left\|x^0\right\| + \overline{M}Tl\|\varphi\|_{\mathscr{B}} + \overline{M}T\|f_r(\cdot)\| + \frac{1}{\lambda}\|B\|^2\overline{M}^2 \\
&\cdot\left[\left\|y^T\right\| + N\|\varphi(0)\| + \overline{M}\|x^0\| + \overline{M}Tl\|\varphi\|_{\mathscr{B}} + \overline{M}T\|f_r(\cdot)\|\right] \\
:=& K_1 + K_2 r^{\gamma_1},
\end{aligned}
$$

where K_1, K_2 are constants independent of b. Thus,

$$
b - K_2 r^{\gamma_1} < K_1. \tag{4.3}
$$

However, the left side of (4.3) may go to $+\infty$ as long as $b \to +\infty$ since $\gamma_1 < 1$ by our assumption. This is a contradiction. Therefore, there is an $b_0(\lambda) > 0$ such that P_λ maps $E(b_0)$ into itself.

To prove that P_λ is a compact operator, we first prove that P_λ is continuous on $E(b_0)$. Let $\{y^n\} \subseteq E(b_0)$ with $y^n \to y \ \ (n \to +\infty)$ for some $y \in E(b_0)$, and then, we have that $y^n_{\rho(s,y^n_s)} \to y_{\rho(s,y_s)}$ as $n \to \infty$ for every $s \in Z(\rho^-) \cup [0,T]$. Then for all $s \in Z(\rho^-) \cup [0,T]$ by (A), we have

$$
\begin{aligned}
\|y^n_s - y_s\|_{\mathscr{B}} \leq H_2 \sup_{s+\theta\in[0,T]} \Big\|\left(y^n(s+\theta) - y(s+\theta)\right)\Big\| \leq H_2\|y^n - y\| \\
\to 0 \quad \text{as} \quad n \to +\infty.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\left\|F(s, y^n_{\rho(s,y^n_s)}) - F(s, y_{\rho(s,y_s)})\right\| \\
\leq&\left\|F(s, y^n_{\rho(s,y^n_s)}) - F(s, y_{\rho(s,y^n_s)})\right\| + \left\|F(s, y_{\rho(s,y^n_s)}) - F(s, y_{\rho(s,y_s)})\right\| \\
\to& 0 \quad \text{as} \quad n \to +\infty.
\end{aligned}
$$

Since

$$
\begin{aligned}
\|((P_\lambda y^n)(t) - (P_\lambda y)(t))\| \leq& \left\|\int_0^t G(t-s)\left(F(s, y^n_{\rho(s,y^n_s)}) - F(s, y_{\rho(s,y_s)})\right)ds\right\| \\
&+ \left\|\int_0^t G(t-s)B\left(u^n(s) - u(s)\right)ds\right\|,
\end{aligned}
$$

where u^n and u are the controls corresponding to y^n and y respectively. By carrying on the semilinear estimations as above we may apply the Lebesgue dominated convergence theorem to get that

$$\sup_{0\le t\le T} \|((P_\lambda y^n)(t) - (P_\lambda y)(t))\| \to 0,$$

as $n \to +\infty$, i.e., P_λ is continuous.

Next, we prove that the family $V(\cdot) = \{(P_\lambda y)(\cdot) : y \in E(b_0)\}$ is an equicontinuous family of functions. To this end, let $0 \le t_1 < t_2 \le T$, then

$$\begin{aligned}
&\|(P_\lambda y)(t_2) - (P_\lambda y)(t_1)\| \\
\le &\|G'(t_2) - G'(t_1)\|\, H\|\varphi(0)\|_{\mathscr{B}} + \|G(t_2) - G(t_1)\|\, \|x^0\| \\
&+ \int_0^{t_1} \|G(t_2 - s) - G(t_1 - s)\|\, \|L(\tilde{\varphi}_s)\| ds \\
&+ \int_0^{t_1} \|(G(t_2 - s) - G(t_1 - s))\|\, \left\|F(s, y_{\rho(s,y_s)}) + Bu(s)\right\| ds \\
&+ \int_{t_1}^{t_2} \|G(t_2 - s)\|\, \|L(\tilde{\varphi}_s)\| ds \\
&+ \int_{t_1}^{t_2} \|G(t_2 - s)\|\, \left\|F(s, y_{\rho(s,y_s)}) + Bu(s)\right\| ds,
\end{aligned}$$

from which and the previous estimates we see that $\|(P_\lambda y^n)(t_2) - (P_\lambda y)(t_1)\|$ tends to zero independently of $y \in E(b_0)$ as $t_2 \to t_1$, since we have shown in Theorem 3.2 (v) that $G(t)$ is uniformly continuous on $[0, \infty)$. Therefore, $V(\cdot)$ is equicontinuous on $t \in [0, T]$.

Now, it remains us to show that, for each $t \in [0, T]$, the set $V(\cdot) = \big\{(P_\lambda y)(\cdot) : y \in E(b_0)\big\}$ is relatively compact in X. In fact, since $S(t)$ is a compact operator for $t \ge 0$, from Theorem 3.2 (ii) $G(t)$ is also compact. As a result, the set $V(\cdot)$ is relatively compact in X for all $t \in [0, T]$.

The above arguments enable us to infer from Arzela-Ascoli theorem that $P_\lambda : E(b_0) \to E(b_0)$ is compact and, consequently, by Schauder fixed point theorem we conclude that there exists a fixed point $y(\cdot)$ for P_λ on $E(b_0)$ whose extension by φ on $(-\infty, 0]$ is (by Definition 3.1) a mild solution for System (1.1). □

Theorem 4.2. *Let $(\varphi, x^0) \in \mathscr{B} \times X$. If the assumptions $(H_0) - (H_5)$ are fulfilled. Additionally, we assume that $F(\cdot,\cdot)$ is uniformly bounded, that is, there is a $K > 0$, such that $\|F(t, \varphi)\| \le K$ for any $(t, \varphi) \in [0, T] \times \mathscr{B}$. Then the System (1.1) is approximately controllable on $[0, T]$.*

Proof. Let $y^T \in X$, $\lambda \in (0, 1)$ and $y^\lambda(\cdot; \varphi, x^0) : (-\infty, T] \to X$ be the mild solution of (1.1) obtained in Theorem 4.1 under the control $u(\cdot)$ given by (4.1). So $y^\lambda(\cdot; \varphi, x^0)$ satisfies that

$$\begin{aligned}
y^\lambda(T; \varphi, x^0) &= G'(T)\varphi(0) + G(T)x^0 + \int_0^T G(T - s)\left(L(\tilde{\varphi}_s) + F(s, y^\lambda_{\rho(s,y_s)})\right) ds \\
&\quad + \int_0^T G(T - s)Bu(s)ds \\
&= G'(T)\varphi(0) + G(T)x^0 + \int_0^T G(T - s)\left(L(\tilde{\varphi}_s) + F(s, y^\lambda_{\rho(s,y_s)})\right) ds
\end{aligned}$$

$$+\int_0^T G(T-s)BB^*G^*(T-s)R(\lambda,-\Gamma_0^T)\cdot\Big(y^T-G'(T)\varphi(0)$$
$$-G(T)x^0-\int_0^T G(T-r)\left(L\left(\tilde{\varphi}_r\right)+F(r,y_{\rho(r,y_r)})\right)dr\Big)\,ds.$$

By the definition of the operator Γ_0^T we have

$$y^\lambda(T;\varphi,x^0)=y^T+\left(\Gamma_0^T R(\lambda,-\Gamma_0^T)-I\right)$$
$$\cdot\left(y^T-G'(T)\varphi(0)-G(T)x^0-\int_0^T G(T-s)\left(L\left(\tilde{\varphi}_s\right)+F(s,y_{\rho(s,y_s)})\right)ds\right).$$

Since $\Gamma_0^T R(\lambda,-\Gamma_0^T)-I=-\lambda R(\lambda,-\Gamma_0^T)$, we get

$$y^\lambda(T;\varphi,x^0)-y^T=-\lambda R(\lambda,-\Gamma_0^T)$$
$$\cdot\left(y^T-G'(T)\varphi(0)-G(T)x^0-\int_0^T G(T-s)\left(L\left(\tilde{\varphi}_s\right)+F(s,y_{\rho(s,y_s)})\right)ds\right). \tag{4.4}$$

By assumption we know that $\left\{F(s,y^\lambda_{\rho(s,y_s)}):\lambda\in(0,1)\right\}$ is bounded uniformly in $\lambda\in(0,1)$ in X, from which it follows that there is a subsequence, still denoted by $F(s,y^\lambda_{\rho(s,y_s)})$, that converges weakly to, say, $f(s)$ in X as $\lambda\to 0^+$ for each $s\in[0,T]$. Meanwhile, the compactness of $G(t),\ t>0$, implies that

$$G(T-s)F(s,y^\lambda_{\rho(s,y_s)})\to G(T-s)f(s)$$

in X, for $s\in[0,T]$. Then from this we infer that

$$\left\|\int_0^T G(T-s)\left(F(s,y^\lambda_{\rho(s,y_s)})-f(s)\right)ds\right\|\to 0, \tag{4.5}$$

as $\lambda\to 0^+$. Thus by (4.4), (4.5) and (H_3) we deduce easily that $y^\lambda(T;\varphi,x^0)\to y^T$ as $\lambda\to 0^+$, and consequently we conclude that, for (φ,x^0) in $\mathscr{B}\times X$, System (1.1) is approximately controllable on $[0,T]$. □

4.2. The case involving spatial derivatives

Note that if the function $F(\cdot,\cdot)$ is defined on $[0,T]\times\mathscr{B}_{\frac{1}{2}}$, then Theorem 4.2 becomes invalid since the mild solutions are obtained in space X by Theorem 4.1 which do not verify $F(\cdot,\cdot)$. Hence, in this part we will discuss the approximate controllability problem of System (1.1) for this situation. More precisely, when F satisfies the assumption (H_5') below, we shall show that, for $(\varphi,x^0)\in\mathscr{B}_{\frac{1}{2}}\times X$ and for any $y^T\in X$, by selecting some $u^\lambda(t)$ for given $\lambda\in(0,1)$, there is a solution $y(\cdot,\varphi,x^0,u^\lambda):(-\infty,T]\to X_{\frac{1}{2}}$ for (1.1) such that $y^\lambda(\cdot,\varphi,x^0,u^\lambda)\to y^T$ in space X.

To this end, we require here that the operator L, the sine family $S(t)$ and the function $F(\cdot,\cdot)$ satisfy respectively the following conditions (H_2'), (H_4') and (H_5') instead of (H_2), (H_4) and (H_5).

(H_2') The operator $L:\mathscr{B}\to X$ is a bounded linear operator with $\|L\|=l$ for some $l>0$. In addition, it maps $\mathscr{B}_{\frac{1}{2}}$ into $D(A^{\frac{1}{2}})$ and $A^{\frac{1}{2}}L=LA^{\frac{1}{2}}$ holds.

(H_4') There is a constant $C > 0$ such that

$$\|A^{\frac{1}{2}}S(t)\| \le C, \text{ for } t \in [0,T].$$

(H_5') The function $F : [0,T] \times \mathscr{B}_{\frac{1}{2}} \to X$ satisfies the Lipschitz condition with respect to the second variable uniformly in $t \in [0,T]$, that is, there exists a constant $k > 0$ such that for any $t \in [0,T]$ and $\varphi_1,\ \varphi_2 \in \mathscr{B}_{\frac{1}{2}}$, there holds

$$\|F(t,\varphi_1) - F(t,\varphi_2)\| \le k\|\varphi_1 - \varphi_2\|_{\mathscr{B}_{\frac{1}{2}}}.$$

Moreover, there is a positive constant K such that

$$\|F(t,\varphi)\| \le K, \quad \text{for all} \quad (t,\varphi) \in [0,T] \times \mathscr{B}_{\frac{1}{2}}.$$

We remark here that, (H_4') is actually a standard assumption for seconder order evolution equations since it coincides with the hypothesis (F) in [44].

As above, we first need to establish the existence result in the subspace $X_{\frac{1}{2}}$.

Theorem 4.3. *Let $(\varphi, x^0) \in \mathscr{B}_{\frac{1}{2}} \times X$. Assume that the above hypotheses (H_0), (H_1), (H_2'), (H_3), (H_4') and (H_5') are satisfied, then, for any $\lambda \in (0,1)$, System (1.1) admits a mild solution $y^\lambda(\cdot) : (-\infty, T] \to X_{\frac{1}{2}}$ with $y^\lambda \Big|_{[0,T]} \in C([0,T]; X_{\frac{1}{2}})$.*

Proof. Let the operator P_λ and the control $u(\cdot)$ be respectively defined in (4.2) and (4.1). As there is no assumption on compactness of $S(t)$, we will prove P_λ has a fixed point on $C([0,T]; X_{\frac{1}{2}})$ by Banach fixed point principle.

At first we show that, for any $\lambda \in (0,1)$, there is a $b_1(\lambda) > 0$ such that $P_\lambda(E(b_1)) \subset E(b_1)$ where

$$E(b_1) := \left\{ y(\cdot) \in C([0,T]; X_{\frac{1}{2}}) \mid\ y(0) = \varphi(0),\ \|y(t)\|_{C_{\frac{1}{2}}} := \sup_{t\in[0,T]} \|A^{\frac{1}{2}} y(t)\| \le b_1 \right\}.$$

In fact, for any $y(\cdot) \in E(b_1)$ with $b_1 > 0$ determined below, from (4.1) we have

$$\begin{aligned}
\|u(t)\| \le & \left\|B^* G^*(T-t) R(\lambda, -\Gamma_0^T)\right\| \Big\| y^T - G'(T)\varphi(0) - G(T)x^0 \\
& - \int_0^T G(T-s)\Big(L(\tilde{\varphi}_s) + F(s, y_{\rho(s,y_s)})\Big) ds \Big\| \\
\le & \frac{1}{\lambda}\overline{M}\|B\| \cdot \Big[\left\|y^T\right\| + N \left\|A^{-\frac{1}{2}}\right\| \|\varphi(0)\|_{\frac{1}{2}} + \overline{M}\|x^0\| \\
& + \overline{M}Tl \left\|A^{-\frac{1}{2}}\right\| \|\varphi\|_{\mathscr{B}_{\frac{1}{2}}} + \overline{M}KT \Big] \\
:= & M_\lambda'.
\end{aligned}$$

Therefore, from (4.2) and Theorem 3.2 $(i)(iii)(vii)$, it follows that

$$\begin{aligned}
\|(P_\lambda y)(t)\|_{\frac{1}{2}} \le & \|G'(t)\varphi(0)\|_{\frac{1}{2}} + \left\|G(t)x^0\right\|_{\frac{1}{2}} \\
& + \left\| \int_0^t G(t-s)\Big(L(\tilde{\varphi}_s) + F(s, y_{\rho(s,y_s)}) + Bu(s)\Big) ds \right\|_{\frac{1}{2}} \\
\le & N \|\varphi(0)\|_{\frac{1}{2}} + \widehat{M} \left\|x^0\right\| + \overline{M}Tl\|\varphi\|_{\mathscr{B}_{\frac{1}{2}}}
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t \|A^{\frac{1}{2}}G(t-s)\|\left(\|F(s,y_{\rho(s,y_s)})\|+\|B\|M'_\lambda\right)ds\\
\leq & N\left\|\varphi(0)\right\|_{\frac{1}{2}}+\widehat{M}\left\|x^0\right\|+\overline{M}Tl\|\varphi\|_{\mathscr{B}_{\frac{1}{2}}}+\widehat{M}KT+\widehat{M}\|B\|M'_\lambda\\
:= & b_1,
\end{aligned}
$$

where we have used the community between $A^{\frac{1}{2}}$ and $G'(t)$ (cf. Theorem 3.2 (vi)). Hence, we have $P_\lambda(E(b_1))\subset E(b_1)$ for such $b_1>0$.

Next we prove that P_λ^n is a contraction mapping on $E(b_1)$. Indeed, for any y^1, $y^2\in E(b_1)$, from (H_5') it follows that

$$
\begin{aligned}
&\|u(t,y^1)-u(t,y^2)\|\\
=&\left\|B^*G^*(T-t)R(\lambda,-\Gamma_0^T)\int_0^T G(T-s)\Big(F(s,y^1_{\rho(s,y^1_s)})-F(s,y^2_{\rho(s,y^2_s)})\Big)ds\right\|\\
\leq&\frac{1}{\lambda}\|B\|\overline{M}^2k\int_0^T\|y^1_{\rho(s,y^1_s)}-y^2_{\rho(s,y^2_s)}\|_{\mathscr{B}_{\frac{1}{2}}}ds.
\end{aligned}
$$

By (A') and Lemma 2.3, for all $s\in Z(\rho^-)\cup[0,T]$, we have

$$
\left\|y^1_s-y^2_s\right\|_{\mathscr{B}_{\frac{1}{2}}}\leq H_2\sup_{s+\theta\in[0,T]}\left\|\left(y^1(s+\theta)-y^2(s+\theta)\right)\right\|_{\frac{1}{2}}\leq H_2\left\|y^1-y^2\right\|_{C_{\frac{1}{2}}},
$$

from which it yields that

$$
\|u(t,y^1)-u(t,y^2)\|\leq\frac{1}{\lambda}\|B\|\overline{M}^2kTH_2\|y^1-y^2\|_{C_{\frac{1}{2}}}.
$$

Therefore,

$$
\begin{aligned}
&\left\|\left(P_\lambda y^1-P_\lambda y^2\right)(t)\right\|_{\frac{1}{2}}\\
\leq&\int_0^t\left\|A^{\frac{1}{2}}G(t-s)\right\|\left\|B(u(s,y^1_s)-B(u(s,y^2_s)\right\|ds\\
&+\int_0^t\left\|A^{\frac{1}{2}}G(t-s)\right\|k\left\|y^1_{\rho(s,y^1_s)}-y^2_{\rho(s,y^2_s)}\right\|_{\mathscr{B}_{\frac{1}{2}}}ds\\
\leq&\int_0^t\widehat{M}\frac{1}{\lambda}\|B\|^2\overline{M}^2kH_2T\left\|y^1-y^2\right\|_{C_{\frac{1}{2}}}ds+\int_0^t\widehat{M}kH_2\left\|y^1-y^2\right\|_{C_{\frac{1}{2}}}ds\\
=&\int_0^t\widehat{M}kH_2\left(\frac{1}{\lambda}\|B\|^2\overline{M}^2T+1\right)\left\|y^1-y^2\right\|_{C_{\frac{1}{2}}}ds.
\end{aligned}
$$

Put $W(\lambda)=\widehat{M}kH_2\left(\frac{1}{\lambda}\|B\|^2\overline{M}^2T+1\right)$, then the above formula becomes

$$
\left\|\left(P_\lambda y^1-P_\lambda y^2\right)(t)\right\|_{\frac{1}{2}}\leq\int_0^t W(\lambda)\left\|y^1-y^2\right\|_{C_{\frac{1}{2}}}ds \tag{4.6}
$$

or

$$
\left\|\left(P_\lambda y^1-P_\lambda y^2\right)(t)\right\|_{\frac{1}{2}}\leq W(\lambda)\left\|y^1-y^2\right\|_{C_{\frac{1}{2}}}t. \tag{4.7}
$$

Hence,

$$
\sup_{\tau\in[0,t]}\left\|\left(P_\lambda y^1-P_\lambda y^2\right)(\tau)\right\|_{\frac{1}{2}}\leq W(\lambda)\left\|y^1-y^2\right\|_{C_{\frac{1}{2}}}t.
$$

It then implies that

$$\begin{aligned}
&\left\|\left(P_\lambda^2 y^1\right)(\cdot)-\left(P_\lambda^2 y^2\right)(\cdot)\right\|_{C([0,t];X_{\frac{1}{2}})}\\
\leq &\sup_{0\leq\tau\leq t}\int_0^\tau W(\lambda)\left\|(P_\lambda y^1)-(P_\lambda y^2)\right\|_{C([0,s];X_{\frac{1}{2}})}ds\\
\leq &\sup_{0\leq\tau\leq t}\int_0^\tau W^2(\lambda)s\left\|y^1-y^2\right\|_{C([0,s];X_{\frac{1}{2}})}ds\\
\leq &\frac{1}{2}t^2W^2(\lambda)\left\|y^1-y^2\right\|_{C([0,t];X_{\frac{1}{2}})}.
\end{aligned}$$

Now for any integer $n\geq 1$, by iteration, it follows from (4.6) and (4.7) that

$$\begin{aligned}
\left\|\left(P_\lambda^n y^1\right)(\cdot)-\left(P_\lambda^n y^2\right)(\cdot)\right\|_{C([0,T];X_{\frac{1}{2}})}\leq&\frac{W^n(\lambda)T^n}{n!}\left\|y^1-y^2\right\|_{C([0,T];X_{\frac{1}{2}})}\\
:=&b_n\left\|y^1-y^2\right\|_{C([0,T];X_{\frac{1}{2}})}.
\end{aligned}$$

It is easy to compute that

$$\frac{b_{n+1}}{b_n}=\frac{W^{n+1}(\lambda)T^{n+1}}{(n+1)!}\cdot\frac{n!}{W^n(\lambda)T^n}=\frac{W(\lambda)T}{n+1}\longrightarrow 0,\text{ as } n\to\infty.$$

Thus, for sufficiently large n, there must have

$$b_n=\frac{W^n(\lambda)T^n}{n!}<1.$$

Therefore, P_λ^n is a contraction map on $C([0,T];X_{\frac{1}{2}})$ and hence P_λ itself has a unique fixed point $y(\cdot)\in C([0,T];X_{\frac{1}{2}})$. Clearly its extension by φ on $(-\infty,0]$ is a mild solution for System (1.1). The proof is completed. □

We now present the following the approximate controllability result for System (1.1).

Theorem 4.4. *Let $(\varphi,x^0)\in\mathscr{B}_{\frac{1}{2}}\times X$. If the assumptions (H_0), (H_1), (H_2'), (H_3), (H_4') and (H_5') hold true, then the system (1.1) is approximately controllable on $[0,T]$.*

Proof. Since we have just obtained the mild solution $y(\cdot,\varphi,x^0,u^\lambda):(-\infty,T]\to X_{\frac{1}{2}}$ for (1.1) for each $\lambda\in(0,1)$, as in the proof of Theorem 4.2, we only need to prove that $y^\lambda(\cdot,\varphi,x^0,u^\lambda)\to y^T$ in space X (not in $X_{\frac{1}{2}}$). But obviously the proof of this assertion is very similar to that of Theorem 4.2, so we omit it here. □

5. Examples

In this section, we apply the results established above to study the controllability of semilinear wave equation with state-dependent delay. Specifically, we discuss the approximate controllability problem of the following controlled systems.

Example 5.1. Consider the boundary value problem of semilinear retarded wave system.

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}z(t,x) = \dfrac{\partial^2}{\partial x^2}z(t,x) + \displaystyle\int_{-\infty}^{t-1}\int_0^\pi a(s-t,x)z(s,y)dyds + Bu(t,x) \\ \qquad\qquad + f\left(t, z(t-r(\|z(t,x)\|),x)\right), \quad 0 < t \leq 2, \quad 0 \leq x \leq \pi, \\ z(t,0) = z(t,\pi) = 0, \quad 0 \leq t \leq 2, \\ z(\theta,x) = \varphi(\theta,x), \quad \theta \leq 0, \quad 0 \leq x \leq \pi, \\ \dfrac{\partial z(0,x)}{\partial t} = z_0(x), \end{cases} \tag{5.1}$$

where the functions $a(\cdot,\cdot)$, $f(\cdot,\cdot)$ and $\varphi(\cdot,\cdot)$ are functions to be described below.

As stated in the introduction part, the second term on the right-hand side in the first equation is not uniformly bounded and hence the approximate controllability of (5.1) can not be obtained by directly the same methods (concerning cosine and sine operator families) as in Ref. [40]. However, as what follows, Theorem 4.2 is well applied to this system due to theory of fundamental solution and its approximate controllability can be obtained.

Let us first represent this problem as the form of System (1.1). For this, take $X = L^2(0,\pi)$ and define $Z(t)(\cdot) := z(t,\cdot)$ and $\phi(\theta)(\cdot) := \varphi(\theta,\cdot)$. Let $A : D(A) \to X$ be the operator given by

$$(Az)(x) = -\frac{d^2 z(x)}{dx^2},$$

with the domain

$$D(A) = \Big\{ z(\cdot) \in X : \ \ z \in H^2(0,\pi), \ z(0) = z(\pi) = 0 \Big\}.$$

Then the spectrum of $-A$ consists of eigenvalues $-n^2$, $n \in \mathbb{N}^+$, with the corresponding normalized eigenvectors $e_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx)$, $n = 1, 2, \cdots$. And the following properties hold.

(i) If $x \in D(A)$, then

$$Ax = \sum_{n=1}^{\infty} n^2 \langle x, e_n \rangle e_n,$$

for every $x \in D(A)$.

(ii) $(-A, D(-A))$ generates a compact analytic semigroup $(T(t))_{t\geq 0}$ on X expressed by

$$T(t)x = \sum_{n=1}^{\infty} e^{-n^2 t}\langle x, e_n \rangle e_n.$$

(iii) The operators $C(t)$ defined by

$$C(t)x = \sum_{n=1}^{\infty} \cos(nt)\langle x, e_n \rangle e_n, \tag{5.2}$$

is the cosine family in X generated by $(-A, D(-A))$, and the associated sine family is given by

$$S(t)x = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n}\langle x, e_n \rangle e_n. \tag{5.3}$$

It is clear that $C(\cdot)x$ and $S(\cdot)x$ are periodic functions, and $\|C(t)\| \leq 1$, $\|S(t)\| \leq 1$, $t \in \mathbb{R}$. Moreover, $S(t)$ is a compact and self-adjoint operator for each $t \geq 0$.

(iv) The operator $A^{\frac{1}{2}}$ is given by

$$A^{\frac{1}{2}}x = \sum_{n=1}^{\infty} n\langle x, e_n\rangle e_n, \tag{5.4}$$

on the space $D(A^{\frac{1}{2}}) = \left\{x(\cdot) \in X,\ \sum_{n=1}^{\infty} n\langle x, e_n\rangle e_n \in X\right\}$. Moreover, from (5.3) and (5.4) we see that $\left\|A^{\frac{1}{2}}S(t)\right\| \leq 1$, for $t > 0$, which implies that (H_4') holds. And clearly, $A^{\frac{1}{2}}S(t) = S(t)A^{\frac{1}{2}}$.

We take the phase space $\mathscr{B} = \mathscr{C}_g$, where the space $\mathscr{C}_g$ is defined as: let g be a continuous function on $(-\infty, 0]$ with $g(0) = 1$, $\lim\limits_{\theta\to-\infty} g(\theta) = \infty$, and g is decreasing on $(-\infty, 0]$, then

$$\mathscr{C}_g = \left\{\varphi \in C((-\infty, 0]; X) : \sup_{s\leq 0} \frac{\|\varphi(s)\|}{g(s)} < \infty\right\},$$

and the norm is defined by, for $\varphi \in \mathscr{C}_g$,

$$|\varphi|_g = \sup_{s\leq 0} \frac{\|\varphi(s)\|}{g(s)}.$$

It is known that $\mathscr{C}_g$ satisfies the axioms (A), (A_1) and (B) (see [40]). We may choose a proper g such that $H, K(\cdot), M(\cdot) \leq 1$ (see [40]). Thus we have $H_2 \leq 1$, $H_3 \leq 1$.

We impose the following conditions on System (5.1):

(a_1) The function $a(\theta, \cdot) \in C([0, \pi])$ for any $\theta \leq 0$ and there holds

$$l := \pi \int_0^{\pi} \left(\int_{-\infty}^{-1} g(\theta)|a(\theta, x)|d\theta\right)^2 dx < \infty. \tag{5.5}$$

(a_2) The functions $r : [0, \infty) \to [0, \infty)$ and $f(\cdot, \cdot) : [0, 2] \times \mathbb{R} \to \mathbb{R}$ are continuous, and there exist constants $\gamma_1 \in (0, 1)$, $K > 0$ such that, for any $t \in [0, 2]$ and $\varphi \in \mathscr{C}_g$

$$\left(\int_0^{\pi} |f(t, \varphi(\theta)(x))|^2 dx\right)^{\frac{1}{2}} \leq l_1(\|\varphi\|_{\mathscr{C}_g}^{\gamma_1} + 1),$$

and

$$\left|f(t, x)\right| \leq K,$$

for any $(t, x) \in [0, 2] \times \mathbb{R}$.

(a_3) The function $\varphi(t, x)$ belongs to $\mathscr{C}_g$, and $z_0(\cdot) \in X$.

Now define the operator $L : \mathscr{C}_g \to X$,the map $F(\cdot, \cdot) : [0, 2] \times \mathscr{C}_g \to X$ and the state-dependent function $\rho(\cdot, \cdot) : [0, 2] \times \mathscr{C}_g \to (-\infty, T)$, respectively, as

$$L(\phi)(x) = L(\phi(\theta, x)) = \int_{-\infty}^{-1} \int_0^{\pi} a(\theta, x)\phi(\theta, y)dyd\theta,$$

$$F(t,\phi) = f(t,\phi(\theta)(x)),$$
$$\rho(t,\phi)(x) = t - r(\|\phi(0)\|)(x),$$

for any $t \in [0,2]$ and $\phi \in \mathscr{C}_g$. Then under these notations System (5.1) is rewritten into the form of (1.1). Evidently, the assumption (a_2) ensures that the function F satisfies the hypothesis (H_5) and (a_1) implies that $L : \mathscr{C}_g \to X$ is linear and bounded because we have

$$\begin{aligned}
\|L(\phi)\|^2 \leq & \int_0^\pi \left(\int_{-\infty}^{-1} \int_0^\pi |a(\theta,x)\phi(\theta,y)| dy d\theta \right)^2 dx \\
\leq & \int_0^\pi \left(\int_{-\infty}^{-1} g(\theta)|a(\theta,x)| \frac{\|\phi(\theta)\|}{g(\theta)} d\theta \right)^2 dx \\
\leq & \pi \int_0^\pi \left(\int_{-\infty}^{-1} g(\theta)|a(\theta,x)| d\theta \right)^2 dx |\phi(\theta)|_g^2.
\end{aligned}$$

So (H_2) holds with l given by (5.5).

Here, as in [25, 36], we take

$$U = \left\{ u = \sum_{n=2}^{\infty} u_n e_n : \ \sum_{n=2}^{\infty} u_n^2 < +\infty \right\},$$

with the norm

$$\|u\| = \left(\sum_{n=2}^{\infty} u_n^2 \right)^{\frac{1}{2}}.$$

Then U is a Hilbert space. Now define the linear continuous operator B from U into X as

$$Bu = 2u_2 e_1(x) + \sum_{n=2}^{\infty} u_n e_n(x), \quad \text{for} \quad u = \sum_{n=2}^{\infty} u_n e_n \in U.$$

It is easy to compute that

$$B^* v = (2v_1 + v_2) e_2(x) + \sum_{n=3}^{\infty} v_n e_n(x), \tag{5.6}$$

for $v = \sum_{n=1}^{\infty} v_n e_n(x) \in X$. So (H_1) is verified too.

According to Theorem 4.2, to obtain the approximate controllability for System (5.1), it remains to verify the condition (H_3) since (H_4) is guaranteed by Property (iii) of $S(t)$ above. As one can see that, generally speaking, it is difficult for us to obtain the explicit expression of the fundamental solutions $G(t)$ associated to the linear system. Fortunately, however, we are able to express it through the relationship between $G(t)$ and $S(t)$ on the interval $[0,1]$, and this is enough to ensure the condition (H_3) holds in this situation. Indeed, the mild solutions on the interval $[0,1]$ of the linear equation

$$\begin{cases} \dfrac{d^2}{dt^2} Z(t) = -AZ(t) + L(Z_t) + f(t), & t \in [0,2], \\ Z_0 = 0, \quad Z'(0) = x^0, \end{cases}$$

are given by the sine family $S(t)$ as

$$Z(t) = S(t)x^0 + \int_0^t S(t-s)f(s)ds, \qquad t \in [0,1],$$

since $L(z_s)(x) = \int_{-\infty}^{-1}\int_0^\pi a(\theta,x)z(s+\theta,y)dyd\theta = 0$ for $s \in [0,1]$. On the other hand, however, from Theorem 3.3, the mild solutions can also be represented as

$$Z(t) = G(t)x^0 + \int_0^t G(t-s)f(s)ds.$$

This indicates that $S(t) = G(t)$, for $t \in [0,1]$, thus we have (note that $S(t)$ is self-adjoint)

$$G^*(t) = G(t) = S(t) = S^*(t), \qquad \text{for } t \in [0,1]. \tag{5.7}$$

Let now $\|B^*G^*(t)z\| = 0,$ for all $t \in [0,2]$, then

$$\|B^*G^*(t)z\| = 0, \qquad t \in [0,1].$$

Hence by (5.7), it gives

$$\|B^*S^*(t)z\| = 0, \qquad t \in [0,1]. \tag{5.8}$$

Combining (5.3) and (5.8) we may calculate directly that, for $z = \sum_{n=1}^{+\infty} z_n e_n(x) \in X$ and $t \in [0,1]$,

$$B^*S^*(t)z = \left(\sin t z_1 + \frac{\sin 2t}{2} z_2\right) e_2 + \sum_{n=3}^{\infty} \frac{\sin(nt)}{n} z_n e_n, \qquad t \in [0,1].$$

Hence from (5.8) we get

$$\left(\sin t z_1 + \frac{\sin 2t}{2} z_2\right)^2 + \sum_{n=3}^{\infty} \left(\frac{\sin(nt)}{n}\right)^2 z_n^2 = 0, \qquad t \in [0,1],$$

which implies immediately that $z_n = 0,\ n = 1, 2, \cdots$. So, by virtue of Theorem 3.4, (H_3) holds true. Consequently, applying Theorem 4.2 we deduce that System (5.1) is approximately controllable on the interval [0,2].

Example 5.2. Consider the control system of the following semilinear retarded wave equation.

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} z(t,x) = \dfrac{\partial^2}{\partial x^2} z(t,x) + \displaystyle\int_{-\infty}^{t-1}\int_0^\pi a(s-t,x)z(s,y)dyds + Bu(t,x) \\ \qquad\qquad + f\left(t, z(t-r(\|z(t,x)\|),x), \dfrac{\partial z}{\partial x}(t-r(\|z(t,x)\|),x)\right), \\ \qquad\qquad 0 < t \le 2,\ 0 \le x \le \pi, \\ z(t,0) = z(t,\pi) = 0, \quad 0 \le t \le 2, \\ z(\theta,x) = \varphi(\theta,x), \quad \theta \le 0, \quad 0 \le x \le \pi, \\ \dfrac{\partial z(0,x)}{\partial t} = z_0(x), \end{cases} \tag{5.9}$$

where the functions $a(\cdot,\cdot)$ and $r(\cdot)$ are functions as in Example 5.1.

Since there is a spatial derivative term in the function f as we mentioned previously, the approximate controllability problem should be obtained by applying Theorem 4.4.

Hence we take the phase space $\mathscr{C}_{g,\frac{1}{2}}$ $(\subset \mathscr{C}_g)$, which is defined by

$$\mathscr{C}_{g,\frac{1}{2}} = \left\{ \varphi \in C((-\infty,0];X_{\frac{1}{2}}) : \sup_{s\leq 0} \frac{\|A^{\frac{1}{2}}\varphi(s)\|}{g(s)} < \infty \right\},$$

endowed with the norm $|\varphi|_{g,\frac{1}{2}} = \sup_{s\leq 0} \frac{\|A^{\frac{1}{2}}\varphi(s)\|}{g(s)}$. Clearly, $\mathscr{C}_{g,\frac{1}{2}}$ satisfies correspondingly the axioms (A'), (A_1') and (B').

We proceed the similar discussion as in Example 5.1 with the same notations. Obviously it suffices to verify the conditions (H_2') and (H_5') so that Theorem 4.4 applies. For this we assume in System (5.9) that

(a_1') The function $a(\theta,\cdot) \in C^1([0,\pi])$ for any $\theta \leq 0$ and satisfies (5.5)

(a_2') The function $f(\cdot,\cdot,\cdot) : [0,2]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is Lipschitz continuous with respect to the last two variables, and there exists constant $K > 0$ such that

$$\left|f(t,x,y)\right| \leq K,$$

for any $(t,x,y) \in [0,2]\times\mathbb{R}\times\mathbb{R}$.

Note that, by (a_1') one has, for any $\phi \in \mathscr{C}_{g,\frac{1}{2}}$,

$$\langle L(\phi), e_n\rangle = \frac{1}{n}\left\langle \int_{-\infty}^{-1}\int_0^{\pi} \frac{\partial}{\partial x} a(\theta,x)\phi(\theta,y)dyd\theta, \tilde{e}_n(x)\right\rangle,$$

where $\tilde{e}_n(x) = \sqrt{\frac{2}{\pi}}\cos(nx)$, $n = 1,2,\cdots$. Hence we see that L maps $\mathscr{C}_{g,\frac{1}{2}}$ into $D(A^{\frac{1}{2}})$ and (H_2') holds true. Meanwhile, (a_2') implies F verifies the condition (H_5'). Therefore, in view of Theorem 4.4, we infer that, for any initial functions $\varphi(\theta,\cdot) \in \mathscr{C}_{g,\frac{1}{2}}$ and $z_0(\cdot) \in X$, System (5.9) is approximately controllable on the interval [0,2] as well.

6. Appendix

In this section we prove Theorem 3.1 for the sake of completeness.

Proof of Theorem 3.1. We shall show the existence and uniqueness of solutions of (3.1) by the contractive mapping theorem. For the functions $K(\cdot)$ and $M(\cdot)$ in Axiom $(A)(iii)$ we put, for any $b \in (0,T]$,

$$K_b := \max_{t\in[0,b]} K(t) \qquad \text{and} \qquad M_b := \sup_{t\in[0,b]} M(t). \tag{6.1}$$

Moreover, we may take some $T_1 > 0$ such that

$$k_0 := M_2 l T_1 K_{T_1} < 1, \tag{6.2}$$

and let

$$\rho := \frac{(M_1 H + M_2 l T_1 M_{T_1})\|\varphi\|_{\mathscr{B}} + M_2\left\|x^0\right\|}{1-k_0}. \tag{6.3}$$

Define the set $E(T_1, \rho)$ by

$$E(T_1,\rho) := \left\{ y(\cdot) \in C([0,T_1];X) \;\middle|\; y(0) = \varphi(0) \text{ and } \quad \|y\|_C := \sup_{t\in[0,T_1]} \|y(t)\| \le \rho \right\}.$$

Clearly, $E(T_1, \rho)$ is closed, bounded and convex. On the set $E(T_1, \rho)$ we define an operator Q as

$$(Qy)(t) = C(t)\varphi(0) + S(t)x^0 + \int_0^t S(t-s)L(\tilde{y}_s(\cdot;\varphi,x^0))ds, \quad t \in [0,T_1], \tag{6.4}$$

for any $y \in E(T_1, \rho)$, where $\tilde{y}(\cdot,\cdot,\cdot)$ is given by

$$\tilde{y}(t;\varphi,x^0) = \begin{cases} y(t), & \text{if} \quad t \in [0,T_1], \\ \varphi(t), & \text{if} \quad t \in (-\infty,0]. \end{cases} \tag{6.5}$$

Then by virtue of (2.3) and (H_2) we have, for any $t \in [0, T_1]$,

$$\begin{aligned} \|Q(y)(t)\| &\le \|C(t)\| \;\; \|\varphi(0)\| + \|S(t)\| \;\; \left\|x^0\right\| + \int_0^t \|S(t-s)\| \;\; \left\|L\left(\tilde{y}_s(\cdot;\varphi,x^0)\right)\right\| ds \\ &\le M_1\|\varphi(0)\| + M_2\left\|x^0\right\| + M_2 l T_1 \|\tilde{y}_s(\cdot;\varphi,x^0)\|_{\mathscr{B}}. \end{aligned}$$

Using the Axiom (A) and System (6.1) we get

$$\begin{aligned} \|Q(y)(t)\| &\le (M_1H + M_2 l T_1 M_{T_1})\|\varphi\|_{\mathscr{B}} + M_2\left\|x^0\right\| + M_2 l T_1 K_{T_1}\|y\|_C \\ &\le \rho, \quad \text{(by the definition (6.3) for } \rho). \end{aligned}$$

Thus

$$\|Q(y)\|_C \le \rho,$$

which shows that Q maps $E(T_1, \rho)$ into itself since clearly $(Qy)(0) = \varphi(0)$. On the other hand, for any y^1, $y^2 \in E(T_1, \rho)$, by virtue of (6.2) it yields that, for $t \in [0, T_1]$,

$$\begin{aligned} \left\|\left(Qy^1\right)(t) - \left(Qy^2\right)(t)\right\| &= \left\|\int_0^t S(t-s)L\left(\tilde{y}_s^1(\cdot) - \tilde{y}_s^2(\cdot)\right)ds\right\| \\ &\le M_2 l \int_0^t \left\|\tilde{y}_s^1(\cdot) - \tilde{y}_s^2(\cdot)\right\|_{\mathscr{B}} ds \\ &\le M_2 l T_1 K_{T_1}\left\|y^1 - y^2\right\|_C \\ &= k_0\left\|y^1 - y^2\right\|_C. \end{aligned}$$

Hence

$$\left\|Qy^1 - Qy^2\right\|_C \le k_0\left\|y^1 - y^2\right\|_C,$$

which from (6.2) implies that Q is a contractive mapping on $E(T_1, \rho)$. Therefore, by the well-known Banach fixed point theorem, there exists a unique fixed point $y(\cdot)$ for the operator Q in $E(T_1, \rho)$. Now we may extend $y(\cdot)$ on $(-\infty, T_1]$ by $y|_{(-\infty,0]} = \varphi$ so that $y(\cdot) : (-\infty, T_1] \to X$ is a mild solution of Eq. (3.1) defined on $(-\infty, T_1]$. Then, using the same technique we can obtain a fixed point $y(\cdot)$ on interval $[T_1, 2T_1]$ for operator Q, which may also be extended to a mild solution on $(-\infty, 2T_1]$ for Eq. (3.1).

Repeating these arguments on $[2T_1, 3T_1]$, $[3T_1, 4T_1]$, $\cdots$, in finite steps we can prove the existence of a mild solution of Eq. (3.1) on $(-\infty, T]$.

Next we use the Gronwall's inequality to prove the uniqueness of the solutions of (3.1). Let $y^1(t;\varphi,x^0)$ and $y^2(t;\varphi,x^0)$ be two solutions through $(0;\varphi,x^0)$, then, for any $t \in [0,T]$,

$$\begin{aligned}\left\|y^2(t;\varphi,x^0) - y^1(t;\varphi,x^0)\right\| &\le \left\|\int_0^t S(t-s)L\left(y_s^2(\cdot) - y_s^1(\cdot)\right) ds\right\| \\ &\le M_2 l \int_0^t \left\|y_s^2 - y_s^1\right\|_{\mathscr{B}} ds \\ &\le M_2 l K_{T_1} \int_0^t \sup_{0\le\tau\le s} \left\|y^2(\tau) - y^1(\tau)\right\| ds.\end{aligned}$$

Hence

$$\sup_{0\le\tau\le t} \left\|y^2(\tau) - y^1(\tau)\right\| \le M_2 l K_{T_1} \int_0^t \sup_{0\le\tau\le s} \left\|y^2(\tau) - y^1(\tau)\right\| ds.$$

The Gronwall's inequality implies that $\sup_{0\le\tau\le t} \left\|y^2(\tau) - y^1(\tau)\right\| \equiv 0$, thus $y^1(t) = y^2(t)$ for all t in $[0,T]$, and consequently $y^1(t;\varphi,x^0) = y^2(t;\varphi,x^0)$ for $t \in (-\infty, T]$. Actually, it is easy to see that, the mild solution $y(t;\varphi,x^0)$ may exist on the whole $(-\infty, +\infty)$.

Finally we show the estimate (3.4) by applying Gronwall's inequality once again. For any $(\varphi, x^0) \in \mathscr{B} \times X$, from the expression (3.2) it follows immediately that, for $t \in [0,T]$,

$$\begin{aligned}&\left\|y(t;\varphi,x^0)\right\| \\ \le& M_1 \left\|\varphi(0)\right\| + M_2 \left\|x^0\right\| + M_2 l \int_0^t \left\|\tilde{y}_s(\theta;\varphi,x^0)\right\|_{\mathscr{B}} ds \\ \le& (M_1 H + M_2 l T M_T) \left\|\varphi\right\|_{\mathscr{B}} + M_2 \left\|x^0\right\| + M_2 l K_T \int_0^t \sup_{0\le\tau\le s} \left\|y(\tau;\varphi,x^0)\right\| ds,\end{aligned}$$

or

$$\begin{aligned}\sup_{0\le s\le t} \left\|y(s;\varphi,x^0)\right\| \le& (M_1 H + M_2 l T M_T) \left\|\varphi\right\|_{\mathscr{B}} + M_2 \left\|x^0\right\| \\ &+ M_2 l K_T \int_0^t \sup_{0\le\tau\le s} \left\|y(\tau;\varphi,x^0)\right\| ds.\end{aligned}$$

Thus from Gronwall's inequality we infer that

$$\begin{aligned}\sup_{0\le\tau\le t} \left\|y(\tau;\varphi,x^0)\right\| &\le \left((M_1 H + M_2 l T M_T) \left\|\varphi\right\|_{\mathscr{B}} + M_2 \left\|x^0\right\|\right) \ e^{(M_2 l K_T)t} \\ &:= m(T)\left(\left\|\varphi\right\|_{\mathscr{B}} + \left\|x^0\right\|\right).\end{aligned}$$

Now, for any $t > 0$, let $t \in ((n-1)T, nT)$ $(n \in \mathbb{N})$, then proceeding inductively as above, we obtain easily that

$$\left\|y(t;\varphi,x^0)\right\| \le m^n(T)\left(\left\|\varphi\right\|_{\mathscr{B}} + \left\|x^0\right\|\right),$$

which, letting $\gamma = \frac{\ln m(T)}{T}$, immediately yields that, for any $t > 0$,

$$\left\|y(t;\varphi,x^0)\right\| \le M_* e^{\gamma t}\left(\left\|\varphi\right\|_{\mathscr{B}} + \left\|x^0\right\|\right)$$

with $M_* \ge 1$ and $\gamma \in \mathbb{R}$. So (3.4) is proved. □

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References

[1] W. Aiello, H. I. Freedman and J. Wu, *Analysis of a model representing stage-structured population growth with state-dependent time delay*, SIAM J. Appl. Math., 1992, 52, 855–869.

[2] F. Andradea, C. Cuevasa and H. R. Henríquez, *Periodic solutions of abstract functional differential equations with state-dependent delay*, Math. Meth. Appl. Sci., 2016, 39 , 3897–3909.

[3] A. Baliki, M. Benchohra and J. R. Graef, *Global existence and stability for second order functional evolution equations with infinite delay*, Electr. J. Qual. Theory Diff. Equ., 2016, 1-122, 1–10.

[4] A. E. Bashirov and N. I. Mahmudov, *On concepts of controllability for linear deterministic and stochastic systems*, SIAM J. Control Optim., 1999, 37(6), 1808–1821.

[5] M. Benchohra and S. K. Ntouyas, *Controllability of second-order differential inclusion in Banach spaces with nonlocal conditions*, J. Optim. Theory Appl., 2000, 107(3), 559–571.

[6] M. Buger and M. R. W. Martin, *The escaping disaster: a problem related to state-dependent delay*, J. Appl. Math. Phys., 2004, 55 (4), 547–574.

[7] Y. Chang and W. Li, *Controllability of second-order differential and integrodifferential inclusions in Banach spaces*, J. Optim. Theory Appl., 2006, 126, 77–87.

[8] F. Chen, D. Sun and J. Shi, *Periodicity in a food-limited population model with toxicants and state-dependent delays*, J. Math. Anal. Appl., 2003, 288 (1), 136–146.

[9] I. Chueshovand A. Rezounenko, *Dynamics of second order in time evolution equations with state-dependent delay*, Nonl. Anal. (TMA), 2015, 123, 126–149.

[10] R. Curtain and H. J. Zwart, *An Introduction to Infinite Dimensional Linear Systems Theory*, Springer-Verlag, New York, 1995.

[11] S. Das, D. N. Pandey and N. Sukavanam, *Approximate Controllability of a Second Order Neutral Differential Equation with State Dependent Delay*, Diff. Equ. Dyn. Syst., 2016, 24 (2), 201–214.

[12] H. O. Fattorini, *Ordinary differential equations in linear topological space I*, J. Diff. Equ., 1969, 5 (1), 72–105.

[13] H. O. Fattorini, *Second-Order Linear Differential Equations in Banach Space*, North Holland Mathematics Studies 108, North Holland, 1985.

[14] X. Fu and J. Zhang, *Approximate Controllability of Neutral Functional Differential Systems with State-Dependent Delay*, Chin. Ann. Math., 2016, 37B, 291–308.

[15] J. Hale and J. Kato, *Phase space for retarded equations with infinite delay*, Funk. Ekvac., 1978, 21 (1), 11–41.

[16] F. Hartung, T. Krisztin, H. O. Walther and J. Wu, *Functional differential equations with state-dependent delays: theory and applications*, Handbook of differential equations: ODE (Ch 5), 2006, 3, 435–545.

[17] E. M. Hernández, *Existence of solutions for a second-order abstract functional differential equation with state-dependent delay*, Electr. J. Diff. Equ., 2007, 21, 1–10.

[18] E. Hernández, K. Azevedo and V. Rolnik, *Well-posedness of abstract differential equations with state-dependent delay*, Math. Nachr., 2018, 291 (13), 2045–2056.

[19] E. Hernández and M. A. Mckibben, *On state-dependent delay partial neutral functional differential equations*, Appl. Math. Comp., 2007, 186 (1), 294–301.

[20] E. Hernández, D. O'Regan and K. Azevedo, *On second order differential equations with state-dependent delay*, Appl. Anal., 2018, 97(15), 2610–2617.

[21] E. Hernández, M. Pierri and J. Wu, *$C^{1+\alpha}$-strict solutions and wellposedness of abstract differential equations with state dependent delay*, J. Diff. Equ., 2016, 261 (12), 6856–6882.

[22] H. R. Henríquez and E. Hernández M, *Approximate controllability of second-order distributed implicit functional systems*, Nonl. Anal., 2009, 70 (2), 1023–1039.

[23] Y. Hino, S. Murakami and T. Naito, *Functional Differential Equations with Infinite Delay*, Springer, Berlin, 1991.

[24] Q. Hu and J. Wu, *Global Hopf bifurcation for differential equations with state-dependent delay*, J. Diff. Equ., 2010, 248 (12), 2801–2840.

[25] J. Jeong, Y. Kwun and J. Park, *Approximate controllability for semilinear retarded functional differential equations*, J. Dyn. Control Syst., 1999, 5 (3), 329–346.

[26] H. Khatibzadeh and G. Morosanu, *Asymptotically periodic solutions to some second-order evolution and difference equations*, Appl. Anal., 2015, 94 (5), 1042–1050.

[27] J. Kisynski, *On cosine operator functions and one parameter group of operetors*, Studia Math., 1972, 49, 93–105.

[28] N Kosovalic, Y. Chen and J. Wu, *Algebraic-delay differential systems:C_0-extendable submanifolds and linearization*, Trans. Amer. Math. Soc., 2017, 369(5), 3387–3419.

[29] N. Kosovalic, F. M. G. Magpantay, Y. Chen and J. Wu, *Abstract algebraic-delay differential systems and age structured population dynamics*, J. Diff. Equ., 2015, 255(3), 593–609.

[30] T. Krisztin and A. Rezounenkob, *Parabolic partial differential equations with discrete state-dependent delay: Classical solutions and solution manifold*, J. Diff. Equ., 2016, 260(5), 4454–4472.

[31] M. Li and C. Kou, *Existence results for second-order impulsive neutral functional differential equations with nonlocal conditions*, J. Discr. Dyn. Nat. Soc., 2009, 2, 1–11.

[32] F. Liang and Z. Guo, *Asymptotic behavior for second order stochastic evolution equations with memory*, J. Math. Anal. Appl., 2014, 419 (2), 1333–1350.

[33] J. Mahaffy, J. Belair and M. Mackey, *Hematopoietic model with moving boundary condition and state-dependent delay: Applications in Erythropoiesis*, J. Theor. Biol., 1998, 190(2), 135–146.

[34] F. Z. Mokkedem and X. Fu, *Approximate controllability of semi-linear neutral integro-differential systems with finite delay*, Appl. Math. Comp., 2014, 242, 202–215.

[35] F. Z. Mokkedem and X. Fu, *Approximate controllability of a semi-linear neutral evolution system with infinite delay*, Int. J. Rob. Nonl. Control, 2017, 27, 1122–1146.

[36] K. Naito, *Controllability of semilinear control systems dominated by the linear part*, SIAM J. Control Optim., 1987, 25, 715–722.

[37] S. Nicaise and C. Pignotti, *Stability results for second-order evolution equations with switching time-delay*, J. Dyn. Diff. Equ., 2014, 26, 781–803.

[38] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

[39] R. Sakthivel and E. R. Anandhi, *Approximate controllability of impulsive differential equations with state-dependent delay*, Int. J. Control, 2010, 83 (2), 387–393.

[40] R. Sakthivel, E. R. Anandhi and N. I. Mahmudov, *Approximate controllability of second-order systems with state-dependent delay*, Numer. Funct. Anal. Optim., 2008, 29 (11–12), 1347–1362. 1158-1164.

[41] R. Sakthivel and Y. Ren, *Approximate controllability of fractional differential equations with state-dependent delay*, Result. Math., 2013, 63(3–4), 949–963.

[42] R. Sakthivel, Y. Ren and N. I. Mahmudov, *Approximate controllability of second-order stochastic differential equations with impulsive effects*, Modern Phys. Lett. B, 2010, 24(14), 1559–1572.

[43] D. Sanjukta, D. N. Pandey and N. Sukavanam, *Existence of solution and approximate controllability for neutral differential equation with state dependent delay*, Int. J. Part. Diff. Equ., 2014, 3, 1–12.

[44] C. C. Travis and G. F. Webb, *Cosine families and abstract nonlinear second order differential equations*, Acta Math. Acad. Sci. Hungar., 1978, 32(1), 75–96.

[45] C. C. Travis and G. F. Webb, *Second order differential equations in Banach space*, Nonlinear Equations in Abstract Spaces, 1987, 331–361.

[46] L. Wang, *Approximate controllability for integrodifferential equations with multiple delays*, J. Optim. Theory Appl., 2009, 143 (1), 185–206.

[47] Z. Yan, *Approximate controllability of fractional neutral integrodifferential inclusions with state-dependent delay in Hilbert spaces*, IMA J. Math. Control Inf., 2013, 30, 443–462.