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EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR $P(X)$-LAPLACIAN DIFFERENTIAL INCLUSIONS INVOLVING CRITICAL GROWTH*

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Abstract This paper concernes with the existence and multiplicity of solutions for $p(x)$-Laplacian differential inclusions involving critical growth. The main tools are the nonsmooth analysis and variational methods. Our main results generalize some recent results in the literature into nonsmooth cases.

Keywords Locally Lipschitz, nonsmooth analysis, hemivariational inequality.

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1. Introduction

The purpose of this paper is to deal with the following quasilinear problem involving variable critical growth with a nonsmooth potential

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) \in \lambda |u|^{q(x)-2} + \partial F(x,u) \qquad \text{for a.e. } \ x \in \Omega, \\ u\big|_{\partial\Omega} = 0, \end{cases} \tag{P_λ}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain in $\mathbb{R}^N$, λ is a positive parameter and $p,\ q : \bar{\Omega} \to \mathbb{R}$ are Lipschitz continuous functions and satisfy

$$1 < p^- = \min_{x\in\bar{\Omega}} p(x) \le p^+ = \max_{x\in\bar{\Omega}} p(x) < N \quad \text{and } p^+ < q(x) \le p^*(x), \quad \forall x \in \bar{\Omega}, \tag{H_0}$$

where

$$p^*(x) = \frac{Np(x)}{N - p(x)}, \quad \forall x \in \bar{\Omega},$$

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and

$$\mathfrak{D} = \{x \in \bar{\Omega} : q(x) = p^*(x)\} \text{ is nonempty.} \tag{H_1}$$

The study of variational problems and differential equations involving variable exponent conditions has been a very important and interesting topic. These problems are very useful in applications and lead many interesting mathematical problems. For example, $p(x)$-Laplacian problems can be found in the thermistor problem [40], the problem of electro-rheological fluid [36], or the problem of image recovery [6]. Of course, $p(x)$-Laplacian problems possess more complicated nonlinearities than p-Laplacian (a constant) problems, for instance, it is inhomogeneous and in general, it does not have the first eigenvalue. In other words, the infimum equals 0 (see [32]). Some related results can be found in [11, 24, 25, 27] and references therein. Moreover the compact embedding theorem of the Lebesgue-Sobolev space $W^{1,p(x)}(\Omega)$ has more strict requirements.

If $\lambda = 0$, then problem (P_λ) becomes into the following form:

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) \in \partial F(x,u) & x \in \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{1.1}$$

Problem (1.1) has been studied by several authors and obtained a few interesting results. For example, Dai and Liu [10] obtained the existence of three solutions for problem (1.1) by a nonsmooth version of three critical points theorem with $\partial F(x,u)$ replaced by $\lambda\partial F(x,u)$. Qian and Shen [35], using the theory of nonsmooth critical points theory, derived the existence and multiplicity of solutions for problem (1.1). Ge et al. [15], employing variational methods combined with suitable truncation techniques based on nonsmooth critical points theory for locally Lipschitz functional, proved the existence of at least five solutions for problem (1.1) under suitable conditions. It is well known that when $p(x) = p$ (a constant) p-Laplacian differential inclusion has been widely studied by lots of authors. Some related results can be found in [3, 7, 8, 14, 16–19, 29–31, 37, 39] and references therein.

However, all the above results did not consider the critical growth of problem (1.1). Very little is known about critical growth nonlinearities for variable exponent problems with nonsmooth potentials. Motivated by this fact, we will consider problem (P_λ) involving critical Sobolev exponent and study the existence and multiplicity of solutions for (P_λ). Compared with the previous works, The critical case brings some new difficulties. In particular, there is no compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{p^*(x)}(\Omega)$. Then, it is not clear that the energy functional associated with (P_λ) satisfies the nonsmooth C-condition. To deal with this difficulty, we will employ a version of the concentration compactness lemma due to Lions for variable exponents found in Bonder and Silva [4] to overcome it. Furthermore, because of the non-differentiability of F, it is very important to find an efficient method to deal with problem (P_λ). In this paper our method relies on the theory of hemivariational inequalities [32–34] and differential inclusions (which involves the generalized gradient of a given locally Lipschitz functional).

In order to introduce our main results, we give our hypotheses on the nonsmooth potential function $F(x,u)$.

(H_F) $F : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ is a function such that $F(x,u) = 0$ and

(i) for all $u \in \mathbb{R}$, $\Omega \ni x \mapsto F(x,u)$ is measurable;

(ii) for a.e. $x \in \Omega$, $u \to F(x,u)$ is locally Lipschitz;

(iii) for a.e. $x \in \Omega$, all $\omega(x) \in \partial F(x,u)$, we have

$$\lim_{|u|\to 0} \frac{\omega}{|u|^{p^+-1}} = 0 \quad \text{and} \quad \lim_{|u|\to +\infty} \frac{\omega}{|u|^{q(x)-1}} = 0;$$

(iv) there exists $q^- > \alpha > p^+$ such that

$$\alpha F(x,u) + F^0(x,u;-u) \leq 0$$

for a.e. $x \in \Omega$ and all $u \in \mathbb{R}$.

Here $F^0(x,u;v)$ denotes the partial generalized directional derivative of $F(x,\cdot)$ at the point $u \in \mathbb{R}$ in the direction $v \in \mathbb{R}$(see Section 2).

The main results are the following:

Theorem 1.1. *If hypotheses (H_0), (H_1) and (H_F) hold, then problem (P_λ) has at least one nontrivial solution.*

Theorem 1.2. *If hypotheses (H_0), (H_1), (H_F) hold and $F(x,-u) = F(x,u)$ for a.e. $x \in \Omega$, $u \in \mathbb{R}$, then problem (P_λ) has at least k-pairs of nontrivial solutions.*

Remark 1.1. Theorems 1.1-1.2 are new as far as we know and it generalizes the results in [1,38] for $p(x)$-Laplacian type problem with critical growth into nonsmooth cases. This means that our conditions are more wider than those in [1,38] and suit more practical applications.

Remark 1.2. In this paper, we apply the concentration compactness principle in [4], which is slightly more general than those in [13] as we do not demand $q(x)$ to be critical everywhere.

Remark 1.3. There exist many functions $F(x,u)$ satisfying hypothesis (H_F). For example, define

$$F(x,u) = \begin{cases} \frac{1}{m}|u|^m, & |u| \leq 2, \\ \frac{1}{n}|u|^n - \frac{2^n}{n} + \frac{2^m}{m}, & |u| > 2, \end{cases}$$

where $m,n \in (p^+,q^-)$ and $p^+ < \alpha < \min\{m,n\}$. Then this function is locally Lipschitz and non-differentiable, and it satisfies hypothesis (H_F).

This paper is organized as follows: in Section 2, some necessary preliminary knowledge is presented. In Section 3, we prove our main results.

2. Preliminaries

We firstly give some basic notations and some definitions.

- $\rightharpoonup$ means weak convergence while $\to$ means strong convergence.
- c, C, c_i and C_i $(i = 1,2,\cdots)$ denote estimated constants (the exact value may be different from line to line). o_n denotes a sequence whose limit is 0 as $n \to \infty$.
- $(X, \|\cdot\|)$ denotes a (real) Banach space and $(X^*, \|\cdot\|_*)$ denotes its topological dual, $|\cdot|_r$ denotes the norm of $L^r(\mathbb{R}^N)$.

Definition 2.1 ([21]). A function $I\colon X \to \mathbb{R}$ is locally Lipschitz if for every $u \in X$ there exist a neighborhood U of u and $L > 0$ such that for every ν, $\eta \in U$,

$$|I(\nu) - I(\eta)| \leq L\|\nu - \eta\|.$$

Definition 2.2 ([21]). Let $I : X \to \mathbb{R}$ be a locally Lipschitz functional. The generalized derivative of I in u along the direction ν is defined by

$$I^0(u;\nu) = \limsup_{\eta \to u, \tau \to 0^+} \frac{I(\eta + \tau\nu) - I(\eta)}{\tau},$$

where u, $\nu \in X$.

It is easy to see that the function $\nu \mapsto I^0(u;\nu)$ is sublinear, continuous and so is the support function of a nonempty, convex and w^*-compact set $\partial I(u) \subset X^*$, defined by

$$\partial I(u) = \{u^* \in X^* : \langle u^*, \nu\rangle_X \leq I^0(u;\nu) \text{ for all } v \in X\}.$$

If $I \in C^1(X)$, then

$$\partial I(u) = \{I'(u)\}.$$

Clearly, these definitions extend those of the Gâteaux directional derivative and gradient.

Definition 2.3 ([21]). We say that I satisfies the nonsmooth C-condition if every sequence $\{u_n\} \subset X$ satisfying

$$I(u_n) \to c \text{ and } (1 + \|u_n\|)m(u_n) \to 0 \text{ as } n \to \infty,$$

has a strongly convergent subsequence, where $m(u_n) = \inf_{u_n^* \in \partial I(u_n)} \|u_n^*\|_{X^*}$.

Definition 2.4. We say that $u \in W_0^{1,p}(\Omega)$ is a weak solution of problem (P_λ), if for all $v \in W_0^{1,p}(\Omega)$ the following hemivariational inequality is satisfied

$$0 \leq \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v \mathrm{d}x - \lambda \int_\Omega |u|^{q(x)-2} uv \mathrm{d}x + \int_\Omega F^0(x, u; -v)\mathrm{d}x.$$

Lemma 2.1 ([5]). *If h is a locally Lipschitz functional, then*

(i) $(-h)^0(u;z) = h^0(u;-z)$ *for all* $u,\ z \in X$*;*

(ii) $h^0(u;z) = \max\{\langle u^*, z\rangle_X : u^* \in \partial h(u)\}$ *for all* $u,\ z \in X$*;*

(iii) *Let* $j : X \to \mathbb{R}$ *be a continuously differentiable function. Then* $\partial j(u) = \{j'(u)\}$*,* $j^0(u;z)$ *coincides with* $\langle j'(u), z\rangle_X$ *and* $(h + j)^0(u;z) = h^0(u;z) + \langle j'(u), z\rangle_X$ *for all* $u,\ z \in X$*;*

(iv) *(Lebourg's mean value theorem) Let u and v be two points in X. Then there exists a point ξ in the open segment between u and v, and $u_\xi^* \in \partial h(\xi)$ such that*

$$h(u) - h(v) = \langle u_\xi^*, u - v\rangle_X;$$

(v) *(Second chain rule) Let Y be a Banach space and $j : Y \to X$ a continuously differentiable function. Then $h \circ j$ is locally Lipschitz and*

$$\partial(h \circ j)(y) \subseteq \partial h(j(y)) \circ j'(y) \quad \text{for all } y \in Y;$$

(vi) $m(u) = \inf_{u^* \in \partial h(u)} \|u^*\|_{X^*}$ *is lower semicontinuous.*

Set

$$C_+(\bar{\Omega}) = \{h \in C(\bar{\Omega}) : \min_{x \in \bar{\Omega}} h(x) > 1\}.$$

Denote by $S(\Omega)$ the set of all measurable real functions defined on Ω. For any $p \in C_+(\bar{\Omega})$ we define the variable exponent Lebesgue space by

$$L^{p(x)}(\Omega) = \left\{ u \in S(\Omega) : \int_\Omega |u(x)|^{p(x)} \mathrm{d}x < \infty \right\}$$

with the norm

$$|u|_{L^{p(x)}(\Omega)} = \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} \mathrm{d}x \leq 1 \right\},$$

then $L^{p(x)}(\Omega)$ is a Banach space. The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\}$$

with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\Omega)},$$

or equivalently

$$\|u\|_{W^{1,p(x)}(\Omega)} = \inf \left\{ \lambda > 0 : \int_\Omega \left(\left| \frac{u(x)}{\lambda} \right|^{p(x)} + \left| \frac{\nabla u(x)}{\lambda} \right|^{p(x)} \right) \mathrm{d}x \leq 1 \right\}$$

for all $u \in W^{1,p(x)}(\Omega)$. Define $W_0^{1,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. We point out that when Ω is bounded, $|\nabla u|_{p(x)}$ is an equivalent norm on $W_0^{1,p(x)}(\Omega)$. The following Hölder type inequality is very useful in the next section.

Proposition 2.1 ([12,28]). *The space $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$, and $W_0^{1,p(x)}(\Omega)$ are all separable, and reflexive Banach space. The conjugate space of $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$, where $1/p(x) + 1/p'(x) = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ we have*

$$\left| \int_\Omega uv \mathrm{d}x \right| \leq \left(\frac{1}{p^-} + \frac{1}{(p')^-} \right) |u|_{L^{p(x)}(\Omega)} |v|_{L^{p'(x)}(\Omega)} \leq 2|u|_{L^{p(x)}(\Omega)} |v|_{L^{p'(x)}(\Omega)}.$$

Proposition 2.2 ([12,28]). *Set $\rho(u) = \int_\Omega |u(x)|^{p(x)} \mathrm{d}x$. For any $u,\ u_k \in L^{p(x)}(\Omega)$, we have*

(i) *For $u \neq 0$, $|u|_{L^{p(x)}(\Omega)} = \lambda \Leftrightarrow \rho(\frac{u}{\lambda}) = 1$;*

(ii) $|u|_{L^{p(x)}(\Omega)} < 1 (= 1, > 1) \Leftrightarrow \rho(u) < 1 (= 1, > 1)$;

(iii) *If $|u|_{L^{p(x)}(\Omega)} > 1$, then $|u|^{p^-}_{L^{p(x)}(\Omega)} \leq \rho(u) \leq |u|^{p^+}_{L^{p(x)}(\Omega)}$;*

(iv) *If $|u|_{L^{p(x)}(\Omega)} < 1$, then $|u|^{p^+}_{L^{p(x)}(\Omega)} \leq \rho(u) \leq |u|^{p^-}_{L^{p(x)}(\Omega)}$;*

(v) $\lim_{k\to\infty} |u_k|_{L^{p(x)}(\Omega)} = 0 \Leftrightarrow \lim_{k\to\infty} \rho(u_k) = 0$;

(vi) $|u_k|_{L^{p(x)}(\Omega)} \to \infty \Leftrightarrow \rho(u_k) \to \infty$.

From Hölder inequality, we can easily obtain the following proposition:

Proposition 2.3. *Let $h,\ r \in L^{\infty}_{+}(\Omega)$ with $h(x) \le r(x)$ a.e. in Ω and $u \in L^{r(x)}(\Omega)$. Then, $|u|^{h(x)} \in L^{\frac{r(x)}{h(x)}}(\Omega)$ and*

$$||u|^{h(x)}|_{L^{\frac{r(x)}{h(x)}}(\Omega)} \le |u|^{h^+}_{L^{r(x)}(\Omega)} + |u|^{h^-}_{L^{r(x)}(\Omega)}.$$

The following lemma is a variable exponent case of Brézis-Lieb Lemma.

Proposition 2.4. *Suppose that $\{u_n\}$ is bounded in $L^{h(x)}(\Omega)$ and $u_n(x) \to u(x)$ a.e. in Ω. Then, $u \in L^{h(x)}(\Omega)$ and*

$$\lim_{n\to\infty}\left(\int_\Omega |u_n|^{h(x)}\mathrm{d}x - \int_\Omega |u-u_n|^{h(x)}\mathrm{d}x\right) = \int_\Omega |u|^{h(x)}\mathrm{d}x,$$

or equivalently

$$\int_\Omega |u_n|^{h(x)}\mathrm{d}x - \int_\Omega |u-u_n|^{h(x)}\mathrm{d}x = \int_\Omega |u|^{h(x)}\mathrm{d}x + o_n(1).$$

Now, we give our main tools used in this paper.

Theorem 2.1 ([22]). *Assume that X is a reflexive Banach space and $\varphi : X \to \mathbb{R}$ is locally Lipschitz and satisfies the nonsmooth C-condition. Assume further that there exist $u_1 \in X$ and $r > 0$ such that $\|u_1\| > r$ and*

$$\max\{\varphi(0), \varphi(u_1)\} < \inf\{\varphi(v) : \|v\| = r\}.$$

Then φ has a nontrivial critical point $u \in X$ such that $\varphi(u) \ge \inf\{\varphi(v) : \|v\| = r\}$.

Theorem 2.2 ([20]). *Assume that X is a reflexive Banach space and $\varphi : X \to \mathbb{R}$ is even locally Lipschitz and satisfies the nonsmooth C-condition and also*

(i) $\varphi(0) = 0$;

(ii) *There exists a subspace $Y \subseteq X$ of finite codimension and numbers $\beta, \gamma > 0$, such that $\inf\{\varphi(u) : u \in Y \cap \partial B_\gamma(0)\} \ge \beta$, where $B_\gamma = \{u \in X : \|u\| < \gamma\}$ and $\partial B_\gamma = \{u \in X : \|u\| = \gamma\}$;*

(iii) *There is a finite dimensional subspace V of X with dim $V >$codim Y, such that $\varphi(v) \to -\infty$ as $\|v\| \to +\infty$ for any $v \in V$.*

Then φ has at least dim V-codim Y pairs of nontrivial critical points.

3. Main results

Set $X = W_0^{1,p(x)}(\Omega)$. Since X is a reflexive and separable Banach space, there exist $e_j \subset X$ and $e_j^* \subset X^*$ such that

$$X = \overline{\mathrm{span}\{e_j | j = 1, 2, \cdots\}}, \quad X^* = \overline{\mathrm{span}\{e_j^* | j = 1, 2, \cdots\}},$$

and

$$\langle e_i^*, e_j\rangle = \begin{cases} 1, & i = j, \\ 0, & i \ne j. \end{cases}$$

For convenience, we write $X_j = \text{span}\{e_j\}$, $Y_k = \bigoplus_{j=1}^{k} X_j$, $Z_k = \bigoplus_{j=k}^{\infty} X_j$. We define the function I on X by

$$I_\lambda = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\mathrm{d}x - \lambda \int_\Omega \frac{1}{q(x)}|u|^{q(x)}\mathrm{d}x - \int_\Omega F(x,u)\mathrm{d}x.$$

Let $\mathscr{F}(u) = \int_\Omega F(x,u)\mathrm{d}x$. In order to prove our results, we need the following lemmas.

Lemma 3.1. *Assume that F satisfies conditions $(H_F)(i) - (iii)$, then $\mathscr{F}(u)$ is locally Lipschitz, and*

$$\mathscr{F}^0(x,u;u) \le \int_\Omega F^0(x,u;u)\mathrm{d}x.$$

Proof. By virtue of hypothesis $(H_F)(iii)$, for given $\epsilon > 0$, we can find $M_1 = M_1(\epsilon) > 0$, such that

$$|\omega| \le c|u|^{p^+-1} + \epsilon|u|^{q(x)-1} \tag{3.1}$$

for a.e. $x \in \Omega$, all $|u| \ge M_1$ and $\omega(x) \in \partial F(x,u)$. By Lebegue's mean value theorem, we obtain that

$$|F(u) - F(v)| = |\omega^*(u-v)|$$

for some $\omega^* \in \partial F(x, u+\theta(u-v))$ and $\theta \in [0,1]$. Then

$$|F(u) - F(v)| \le (c|z|^{p^+-1} + 2\epsilon|z|^{q(x)-1})|u-v|,$$

where $|z| = \max\{|u|,|v|\}$. For a neighborhood $\delta_u \subset X$ of $u, v \in \delta_u$,

$$|\mathscr{F}(u) - \mathscr{F}(v)| \le c\int_\Omega |z|^{p^+-1}|u-v|\mathrm{d}x + 2\epsilon\int_\Omega |z|^{q(x)-1}|u-v|\mathrm{d}x.$$

From Hölder's inequality and the embedding theorem, we derive

$$\begin{aligned}|\mathscr{F}(u) - \mathscr{F}(v)| &\le c|z|_{p^+}^{p^+-1}|u-v|_{p^+} + 2\epsilon\max\{|z|_{q(x)}^{q^--1}, |z|_{q(x)}^{q^-+1}\}|u-v|_{q(x)} \\ &\le (c\|z\|^{p^+-1} + 2c\epsilon\max\{\|z\|^{q^--1}, \|z\|^{z^+-1}\})\|u-v\|.\end{aligned}$$

Hence $\mathscr{F}(u)$ is locally Lipschitz. Then $I(u)$ is locally Lipschitz. Similar as that in [26, Lemma 2.1], we can obtain that $\mathscr{F}^0(x,u;u) \le \int_\Omega F^0(x,u;u)\mathrm{d}x$. Thus the proof is completed. □

Lemma 3.2. *Assume that F satisfies hypotheses $(H_F)(i)-(iii)$, then every critical point of $u_0 \in X$ of I_λ is a weak solution of problem (P_λ).*

Proof. Since $u \in X$ is a critical point of I_λ, for every $v \in X$, Lemma 3.1 gives

$$\begin{aligned}0 \le I_\lambda^0(u;v) &= \int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla v\mathrm{d}x - \lambda\int_\Omega |u|^{q(x)-2}uv\mathrm{d}x + (-\mathscr{F})^0(x,u;v) \\ &\le \int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla v\mathrm{d}x - \lambda\int_\Omega |u|^{q(x)-2}uv\mathrm{d}x + \int_\Omega F^0(x,u;-v)\mathrm{d}x,\end{aligned}$$

i.e., u is a weak solution of problem (P_λ). □

Lemma 3.3. *If hypothesis (H_F) hold, then any nonsmooth C-condition sequence of I_λ is bounded in X.*

Proof. Let $\{u_n\}_{n\geq 1} \subset X$ be a sequence such that $I_\lambda(u_n) \to c$, and $(1+\|u_n\|)m(u_n) \to 0$ as $n \to +\infty$. Let $A : X \to X^*$ be the nonlinear operator defined by

$$\langle A(u), v\rangle = \int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla v \mathrm{d}x \quad \text{for all} \quad u, v \in X.$$

From [23] we know that A is maximal monotone and

$$u_n^* = A(u_n) - \lambda |u_n|^{q(x)-2}u_n - \omega_n, \tag{3.2}$$

where $\omega_n \in \partial F(x, u_n)$ and $u_n^* \in \partial I_\lambda(u_n)$ for $n \geq 1$.

$(1 + \|u_n\|)m(u_n) \to 0$ deduces

$$|\langle u_n^*, u_n\rangle| \to 0 \tag{3.3}$$

as $n \to +\infty$. We claim that the sequence $\{u_n\}$ is bounded. Indeed, by virtue of (3.2) and (3.3), we derive that

$$I_\lambda^0(u_n; u_n) \geq \langle u_n^*, u_n\rangle_X \geq -\|u_n^*\|_{X^*}\|u_n\| \geq -\alpha\|u_n\| \tag{3.4}$$

for n sufficiently large. Using Lemma 3.1, the above estimation and (H_F)(iv), we obtain that

$$\begin{aligned}
c + 1 + \|u_n\| &\geq I_\lambda(u_n) - \frac{1}{\alpha} I_\lambda^0(u_n; u_n)\\
&= \int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\mathrm{d}x - \lambda \int_\Omega \frac{1}{q(x)}|u_n|^{q(x)}\mathrm{d}x - \int_\Omega F(x, u_n)dx\\
&\quad - \frac{1}{\alpha}\left[\int_\Omega |\nabla u_n|^{p(x)}\mathrm{d}x - \lambda \int_\Omega |u_n|^{q(x)}\mathrm{d}x + (-\mathscr{F})^0(x, u_n; u_n)\right]\\
&\geq \left(\frac{1}{p^+} - \frac{1}{\alpha}\right)\int_\Omega |\nabla u_n|^{p(x)}\mathrm{d}x + \lambda\left(\frac{1}{\alpha} - \frac{1}{q^-}\right)\int_\Omega |u_n|^{q(x)}\mathrm{d}x\\
&\quad - \int_\Omega \left[F(x, u_n) + \frac{1}{\alpha}F^0(x, u_n; -u_n)\right]\mathrm{d}x\\
&\geq \left(\frac{1}{p^+} - \frac{1}{\alpha}\right)\int_\Omega |\nabla u_n|^{p(x)}\mathrm{d}x.
\end{aligned}$$

Once that $\|u_n\| > 1$, it follows from Proposition 2.1 that

$$c\|u_n\|^{p^-} \leq c + 1 + \|u_n\|.$$

Since $p^- > 1$, the above inequality means that $\{u_n\}$ is bounded in X. □

As a consequence of the last result, if $\{u_n\}$ is a nonsmooth C-condition of I_λ, we can extract a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, and $u \in X$ such that

$$\begin{aligned}
&u_n \rightharpoonup u \ \text{ in } \ X, \quad u_n \rightharpoonup u \ \text{ in } \ L^{q(x)}(\Omega),\\
&u_n \to u \ \text{ in } \ L^{r(x)}(\Omega), \quad 1 < r^- \leq r(x) \ll p^*(x).
\end{aligned}$$

By virtue of the concentration compactness lemma of Lions for variable exponents in [4], there exist two nonnegative measures $\mu, \nu \in \Lambda(\Omega)$, a countable set idex set E, points $\{x_j\}_{i\in E}$ in $\mathfrak{D}$ and sequences $\{\mu_j\}_{j\in E}, \{\nu_j\}_{j\in E} \subset [0,+\infty)$, such that

$$|\nabla u_n|^{p(x)} \to \mu \geq |\nabla u|^{p(x)} + \sum_{j\in E} \mu_j \delta_{x_j} \quad \text{in} \quad \Lambda(\Omega),$$
$$|u_n|^{p(x)} \to \nu = |\ u|^{p(x)} + \sum_{j\in E} \nu_j \delta_{x_j} \quad \text{in} \quad \Lambda(\Omega),$$

and

$$S\nu_j^{\frac{1}{p^*(x_j)}} \leq \mu_j^{\frac{1}{p(x_j)}} \quad \forall j \in E,$$

where $S = \inf_{\phi\in C_0^\infty(\Omega)} \frac{\|\nabla\phi\|_{L^{p(x)}(\Omega)}}{\|\phi\|_{L^{q(x)}(\Omega)}}$.

In the following, we will prove an important estimate for $\{\nu_j\}$. With this aim in mind, we have to prove a technical lemma. Let $\phi \in C_0^\infty(\Omega)$ such that

$$\phi(x) = 1 \quad \text{in} \quad B_{\frac{1}{2}}(0), \quad \text{supp}\ \phi \subset B_1(0) \ \text{ and } \ 0 \leq \phi(x) \leq 1 \ \ \forall x \in \Omega.$$

For each $\epsilon > 0$, define

$$\phi_\epsilon(x) = \phi\left(\frac{x}{\epsilon}\right) \quad \forall x \in \Omega.$$

Lemma 3.4. *For each $y \in \bar{\Omega}$ and $u \in L^{p(x)}(\Omega)$,*

$$\int_\Omega |u(x)\nabla\phi_\epsilon(x-y)|^{p(x)}\mathrm{d}x \leq C \max\left\{\|u\|^{p^-}_{L^{p^*(x)}(B_\epsilon(y))}, \|u\|^{p^+}_{L^{p^*(x)}(B_\epsilon(y))}\right\}, \tag{3.5}$$

where C is a constant independent of ϵ and y.

Proof. Observe that

$$\int_\Omega |u(x)\nabla\phi_\epsilon(x-y)|^{p(x)}\mathrm{d}x = \int_\Omega |u(x)|^{p(x)}\left|\frac{1}{\epsilon}\nabla\phi\left(\frac{x-y}{\epsilon}\right)\right|^{p(x)}\mathrm{d}x$$
$$= \int_{B_\epsilon(x_j)} |u(x)|^{p(x)}\left|\frac{1}{\epsilon}\nabla\phi\left(\frac{x-y}{\epsilon}\right)\right|^{p(x)}\mathrm{d}x$$
$$\leq c\left\| |u|^{p(x)} \right\|_{L^{\frac{p^*(x)}{p(x)}}(B_\epsilon(y))} \left\| \left|\frac{1}{\epsilon}\nabla\phi\left(\frac{\cdot - y}{\epsilon}\right)\right|^{p(x)} \right\|_{L^{\frac{p^*(x)}{p^*(x)-p(x)}}(B_\epsilon(y))}.$$

Making a change of variable, we derive

$$\int_{B_\epsilon(x_j)} \left|\frac{1}{\epsilon}\nabla\phi\left(\frac{x-y}{\epsilon}\right)\right|^{\frac{p(x)p^*(x)}{p^*-p(x)}}\mathrm{d}x = \int_{B_1(0)} \left|\frac{1}{\epsilon}\nabla\phi(z)\right|^{\frac{p(y-\epsilon z)p^*(y-\epsilon z)}{p^*(y-\epsilon z)-p(y-\epsilon z)}} \cdot \epsilon^N \mathrm{d}z$$
$$= \int_{B_1(0)} \left|\frac{1}{\epsilon}\nabla\phi(z)\right|^N \epsilon^N \mathrm{d}z = \int_{B_1(0)} |\nabla\phi(z)|^N \mathrm{d}z.$$

Then (3.5) is satisfied. □

Lemma 3.5. *Under conditions of Lemma 3.3, if $\{u_n\}$ is a nonsmooth C-condition for I_λ and $\{\nu_j\}$ as above, then for each $j \in E$, $\nu_j > \frac{S^N}{\lambda^{\frac{N}{p(x_j)}}}$ or $\nu_j = 0$.*

Proof. For $\forall\epsilon > 0$, we set $\phi_\epsilon \in C_0^\infty(\Omega)$ as in Lemma 3.4. Then $\{\phi_\epsilon(\cdot - x_j)u_n\} \subset X$ for any $j \in E$. After a direct computation, we derive that $\{\phi_\epsilon(\cdot - x_j)u_n\}$ is bounded in X. Hence

$$\langle u_n^*, \phi_\epsilon(\cdot - x_j)u_n\rangle = o_n,$$

where $u_n^* \in \partial I_\lambda(u_n)$. Or equivalently,

$$\begin{aligned}&\int_\Omega |\nabla u_n|^{p(x)}\phi_\epsilon(x-x_j)\mathrm{d}x + \int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n u_n \nabla\phi_\epsilon(x-x_j)\mathrm{d}x + o_n(1)\\ =\ &\lambda\int_\Omega |u_n|^{q(x)}\phi_\epsilon(x-x_j)\mathrm{d}x + \int_\Omega \omega(x,u_n)u_n\phi_\epsilon(x-x_j)\mathrm{d}x,\end{aligned} \tag{3.6}$$

where $\omega(x,u_n) \in \partial F(x,u_n)$. For $\forall\sigma > 0$, it follows from Young's inequality that

$$\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n u_n\nabla\phi_\epsilon(x-x_j)\mathrm{d}x \le \sigma\int_\Omega |\nabla u_n|^{p(x)}\mathrm{d}x + C_\sigma\int_\Omega |u_n\nabla\phi_\epsilon(x-x_j)|^{p(x)}\mathrm{d}x. \tag{3.7}$$

Passing to the limit of $n \to +\infty$ in (3.7), we have

$$\limsup_{n\to\infty}\int_\Omega ||\nabla u_n|^{p(x)-2}\nabla u_n u_n\nabla\phi_\epsilon(x-x_j)|\mathrm{d}x \le \sigma C + C_\sigma\int_\Omega |u\nabla\phi_\epsilon(x-x_j)|^{p(x)}\mathrm{d}x. \tag{3.8}$$

From Lemma 3.4, we obtain

$$\begin{aligned}&\limsup_{n\to\infty}\int_\Omega ||\nabla u_n|^{p(x)-2}\nabla u_n u_n\nabla\phi_\epsilon(x-x_j)|\mathrm{d}x\\ \le\ &\sigma C + CC_\sigma\max\{\|u\|^{p^+}_{L^{p^*(x)}(B_\epsilon(x_j))}, \|u\|^{p^-}_{L^{p^*(x)}(B_\epsilon(x_j))}\}.\end{aligned} \tag{3.9}$$

Furthermore, it follows from hypothesis (H_F)(iii), for $\forall\epsilon > 0$,

$$|\omega(x,u_n)| \le \epsilon|u_n|^{q(x)-1} + C_\epsilon|u_n|^{p^+-1},$$

where $C_\epsilon > 0$ and $\omega(x,u_n) \in \partial F(x,u_n)$. Then

$$\int_\Omega \omega(x,u_n)u_n\phi_\epsilon(x-x_j)\mathrm{d}x \le \epsilon\int_\Omega |u_n|^{q(x)}\phi_\epsilon(x-x_j)\mathrm{d}x + C_\epsilon\int_\Omega |u_n|^{p^+}\phi_\epsilon(x-x_j)\mathrm{d}x. \tag{3.10}$$

On the other hand, from the compactness lemma of Strauss [9]

$$\lim_{n\to\infty}\int_\Omega |u_n|^{p^+}\phi_\epsilon(x-x_j)\mathrm{d}x \ge \int_\Omega |u|^{p^+}\phi_\epsilon(x-x_j)\mathrm{d}x. \tag{3.11}$$

Noting that $\{u_n\}$ is bounded in X, from the Sobolev embedding theorem, setting $\epsilon \to 0$, we have

$$\epsilon\int_\Omega |u_n|^{q(x)}\phi_\epsilon(x-x_j)\mathrm{d}x \to 0. \tag{3.12}$$

By (3.6) and (3.8)-(3.12), one has

$$\begin{aligned}&\lim_{n\to\infty}\int_\Omega |\nabla u_n|^{p(x)}\phi_\epsilon(x-x_j)\mathrm{d}x \le \lambda\lim_{n\to\infty}\int_\Omega |u_n|^{q(x)}\phi_\epsilon(x-x_j)\mathrm{d}x\\ &+C_\epsilon\int_\Omega |u_n|^{p^+}\phi_\epsilon(x-x_j)\mathrm{d}x + \sigma C + CC_\sigma\max\left\{\|u\|^{p^+}_{L^{p^*(x)}(B_\epsilon(x_j))}, \|u\|^{p^-}_{L^{p^*(x)}(B_\epsilon(x_j))}\right\},\end{aligned} \tag{3.13}$$

where C is a constant independent of ϵ and j. Recall that

$$
\begin{aligned}
&|\nabla u_n|^{p(x)} \to \mu \quad \text{and} \quad |u_n|^{q(x)} \to \nu \quad \text{in} \quad \Lambda(\Omega),\\
&\lim_{n\to\infty}\int_\Omega |\nabla u_n|^{p(x)}\phi_\epsilon(x-x_j)\mathrm{d}x = \int_\Omega \phi_\epsilon(x-x_j)\mathrm{d}\mu \geq \int_{B_{\frac{\epsilon}{2}(x_j)}} \phi_\epsilon(x-x_j)\mathrm{d}\mu\\
&\qquad\qquad = \int_{B_{\frac{\epsilon}{2}(x_j)}} \mathrm{d}\mu = \mu_j,
\end{aligned}
$$

and

$$
\begin{aligned}
\lim_{n\to\infty}\int_\Omega |u_n|^{q(x)}\phi_\epsilon(x-x_j)\mathrm{d}x &= \int_{B_{\epsilon(x_j)}} \phi_\epsilon(x-x_j)\mathrm{d}\nu \leq \int_{B_{\epsilon(x_j)}} \mathrm{d}\nu\\
&= \nu(B_\epsilon(x_j)).
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\mu_j \leq &\lim_{n\to+\infty}\int_\Omega |\nabla u_n|^{p(x)}\phi_\epsilon(x-x_j)\mathrm{d}x \leq \lambda\nu(B_\epsilon(x_j))\\
&+C_\epsilon\int_\Omega |u|^{p^+}\phi_\epsilon(x-x_j)\mathrm{d}x+\sigma C+CC_\sigma\max\{\|u\|^{p^+}_{L^{p^*(x)}(B_\epsilon(x_j))}, \|u\|^{p^-}_{L^{p^*(x)}(B_\epsilon(x_j))}\}.
\end{aligned}
$$

Setting $\epsilon \to 0$ after $\sigma \to 0$, we infer that

$$\mu_j \leq \lambda\nu_j.$$

So

$$S\nu^{\frac{1}{p^*(x_j)}} \leq \mu_j^{\frac{1}{p(x_j)}} \leq (\lambda\nu_j)^{\frac{1}{p(x_j)}}.$$

Hence

$$\nu_j \geq \frac{S^N}{\lambda^{\frac{N}{p(x_j)}}} \quad \text{or} \quad \nu_j = 0. \qquad \square$$

Lemma 3.6. *If hypothesis (H_F) holds and $\lambda < 1$, then I_λ satisfies the nonsmooth C-condition for $c < \lambda^{1-\frac{N}{p^+}}\left(\frac{1}{\alpha}-\frac{1}{q^-}\right)S^N$.*

Proof. Since

$$I_\lambda(u_n) = c + o_n(u) \quad \text{and} \quad (1+\|u_n\|)m(u_n) \to 0,$$

it follows from (3.2) that

$$
\begin{aligned}
c &= \lim_{n\to\infty} I_\lambda(u_n) = \lim_{n\to\infty}\left(I_\lambda(u_n) - \frac{1}{\alpha}\langle u_n^*, u_n\rangle\right)\\
&\geq \lim_{n\to\infty}\left(I_\lambda(u_n) - \frac{1}{\alpha}I_\lambda^0(u_n; u_n)\right)\\
&= \lim_{n\to\infty}\Bigg[\int_\Omega\left(\frac{1}{p(x)}-\frac{1}{\alpha}\right)|\nabla u_n|^{p(x)}\mathrm{d}x + \lambda\int_\Omega\left(\frac{1}{\alpha}-\frac{1}{q(x)}\right)|u_n|^{q(x)}\mathrm{d}x\\
&\quad - \mathscr{F}(x,u_n) - \frac{1}{\alpha}\mathscr{F}^0(x,u_n;-u_n)\Bigg]
\end{aligned}
$$

$$\geq \lim_{n\to\infty}\left[\int_\Omega\left(\frac{1}{p(x)}-\frac{1}{\alpha}\right)|\nabla u_n|^{p(x)}\mathrm{d}x+\lambda\int_\Omega\left(\frac{1}{\alpha}-\frac{1}{q(x)}\right)|u_n|^{q(x)}\mathrm{d}x\right.$$
$$\left.-\int_\Omega\left(F(x,u_n)+\frac{1}{\alpha}F^0(x,u_n;-u_n)\right)\mathrm{d}x\right].$$

By virtue of hypothesis (H_F)(iv), we have

$$c\geq\lambda\left(\frac{1}{\alpha}-\frac{1}{q^-}\right)\lim_{n\to\infty}\int_\Omega|u_n|^{q(x)}\mathrm{d}x.$$

Noting that

$$\lim_{n\to\infty}\int_\Omega|u_n|^{q(x)}\mathrm{d}x=\left[\int_\Omega|u|^{q(x)}\mathrm{d}x+\sum_{j\in E}\nu_j\right]\geq\nu_j\quad\forall j\in E,$$

if $\nu_s>0$ for some $s\in E$, we infer that

$$c\geq\lambda\left(\frac{1}{\alpha}-\frac{1}{q^-}\right)\frac{S^N}{\lambda^{\frac{N}{p(x_s)}}}.$$

So, for $\lambda<1$

$$c\geq\lambda\left(\frac{1}{\alpha}-\frac{1}{q^-}\right)\left(\frac{S}{\lambda^{\frac{1}{p^+}}}\right)^N=\lambda^{1-\frac{N}{p^+}}\left(\frac{1}{\alpha}-\frac{1}{q^-}\right)S^N,$$

which is a contradiction. Then, we must have $\nu_j=0$ for any $j\in E$, leading to

$$\int_\Omega|u_n|^{q(x)}\mathrm{d}x\to\int_\Omega|u|^{q(x)}\mathrm{d}x.$$

From the above equation we derive

$$\int_\Omega|u_n-u|^{q(x)}\mathrm{d}x\to0\qquad\text{as}\quad n\to\infty.$$

Then

$$u_n\to u\quad\text{in}\quad L^{q(x)}(\Omega).\tag{3.14}$$

Since $\langle u_n^*,u_n\rangle=o_n(1)$, we obtain

$$\int_\Omega|\nabla u_n|^{p(x)}\mathrm{d}x=\lambda\int_\Omega|u_n|^{q(x)}\mathrm{d}x+\int_\Omega\omega(x,u_n)u_n\mathrm{d}x+o_n(1).$$

In the following, let us denote by $\{P_n\}$ the following sequence

$$P_n(x)=\langle|\nabla u_n(x)|^{p(x)-2}\nabla u_n(x)-|\nabla u(x)|^{p(x)-2}\nabla u(x),\nabla u_n(x)-\nabla u(x)\rangle.$$

The definition of $\{P_n\}$ means that

$$\int_\Omega P_n\mathrm{d}x=\int_\Omega|\nabla u_n|^{p(x)}\mathrm{d}x-\int_\Omega|\nabla u_n|^{p(x)-2}\nabla u_n\nabla u\mathrm{d}x-\int_\Omega|\nabla u|^{p(x)-2}\nabla u\nabla(u_n-u)\mathrm{d}x.$$

Since $u_n\rightharpoonup u$ in $W_0^{1,p(x)}(\Omega)$, we obtain

$$\int_\Omega|\nabla u_n|^{p(x)-2}\nabla u_n\nabla(u_n-u)\mathrm{d}x\to0$$

as $n \to \infty$, which means that

$$P_n(x) = \int_\Omega |\nabla u_n|^{p(x)} \mathrm{d}x - \int_\Omega |\nabla u_n|^{p(x)-2} \nabla u_n \nabla u \mathrm{d}x + o_n(1).$$

Furthermore, from $\langle u_n^*, u_n \rangle = o_n(1)$, we have

$$\begin{aligned}\int_\Omega P_n \mathrm{d}x =& \lambda \int_\Omega |u_n|^{q(x)} \mathrm{d}x + \int_\Omega \omega(x, u_n) u_n \mathrm{d}x - \lambda \int_\Omega |u_n|^{q(x)-2} u_n u \mathrm{d}x \\ & - \int_\Omega \omega(x, u_n) u \mathrm{d}x + o_n(1),\end{aligned}$$

where $u_n^* \in \partial I_\lambda(u_n)$ and $\omega(x, u_n) \in \partial F(x, u_n)$. Combining (3.14) with the compactness lemma of Strauss [9], we infer that

$$\int_\Omega P_n \mathrm{d}x \to 0 \quad \text{ as } \; n \to \infty.$$

Next, let us discuss the sets

$$\Omega_+ = \{x \in \Omega : p(x) \geq 2\} \quad \text{and} \quad \Omega_- = \{x \in \Omega : 1 < p(x) < 2\}.$$

Recalling that

$$P_n(x) \geq \begin{cases} \dfrac{2^{3-p^+}}{p^+} |\nabla u_n - \nabla u|^{p(x)} & \text{if } p(x) \geq 2, \\ (p^- - 1) \dfrac{|\nabla u_n - \nabla u|^2}{(|\nabla u_n| + |\nabla u|)^{2-p(x)}} & \text{if } 1 < p(x) < 2, \end{cases}$$

we derive

$$\int_{\Omega_+} |\nabla u_n - \nabla u|^{p(x)} \mathrm{d}x \to 0 \quad \text{as} \;\; n \to \infty. \tag{3.15}$$

Applying Hölder's inequality, one has

$$\int_{\Omega_-} |\nabla u_n - \nabla u|^{p(x)} \mathrm{d}x \leq c \|f_n\|_{L^{\frac{2}{p(x)}}(\Omega_-)} \|h_n\|_{L^{\frac{2}{2-p(x)}}(\Omega_-)},$$

where

$$\begin{aligned} f_n(x) &= \frac{|\nabla u_n(x) - \nabla u(x)|^{p(x)}}{(|\nabla u_n(x)| + |\nabla u(x)|)^{\frac{p(x)(2-p(x))}{2}}}, \\ h_n(x) &= (|\nabla u_n(x)| + |\nabla u(x)|)^{\frac{p(x)(2-p(x))}{2}}. \end{aligned}$$

A direct computation shows that $\{\|h_n\|_{L^{\frac{2}{2-p(x)}}(\Omega_-)}\}$ is a bounded sequence and

$$\int_{\Omega_-} |f_n|^{\frac{2}{p(x)}} \mathrm{d}x \leq C \int_{\Omega_-} P_n(x) \mathrm{d}x. \tag{3.16}$$

Hence

$$\int_{\Omega_-} |\nabla u_n - \nabla u|^{p(x)} \mathrm{d}x \to 0$$

as $n \to \infty$. (3.15) and (3.16) imply that $u_n \to u$ in X as $n \to \infty$. $\square$

Proof of Theorem 1.1. We firstly prove the following claim.

Claim 1. There exists $\rho_0 > 0$ such that for all $0 < \rho < \rho_0$, we have $\inf\{I_\lambda : \|u\| = \rho\} > 0$.

From hypothesis (H_F)(iii), for any $\epsilon > 0$, we obtain

$$F(x,u) \leq \epsilon |u|^{p^+} + C_\epsilon |u|^{q(x)}$$

for a.e. $x \in \Omega$, where $C_\epsilon > 0$. Hence, for all $u \in X$ with $\|u\| < 1$

$$\begin{aligned}
I_\lambda(u) &= \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \mathrm{d}x - \lambda \int_\Omega |u|^{q(x)} \mathrm{d}x - \int_\Omega F(x,u) \mathrm{d}x \\
&\geq \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \mathrm{d}x - \lambda \int_\Omega |u|^{q(x)} \mathrm{d}x - \epsilon \int_\Omega |u|^{p^+} \mathrm{d}x - C_\epsilon \int_\Omega |u|^{q(x)} \mathrm{d}x \\
&\geq \left(\frac{1}{p^+} - \epsilon C \right) \|u\|^{p^+} - (\lambda + C_\epsilon) C \|u\|^{q^-}.
\end{aligned}$$

Choosing $\epsilon = \frac{1}{2p^+ C}$, we obtain that

$$I_\lambda(u) \geq \frac{1}{2p^+} \|u\|^{p^+} - (\lambda + C_\epsilon) C \|u\|^{q^-}.$$

Noting that $p^+ < q^-$, there exists $\rho_0 > 0$ such that for all $0 < \rho < \rho_0$, we derive $\inf\{I_\lambda(u) : \|u\| = \rho\} > 0$.

Claim 2. There exists $u_1 \in X$ such that $I_\lambda(u_1) < 0$. By virtue of hypothesis (H_F)(iii), for any $\epsilon > 0$, we have

$$|F(x,u)| \leq C_\epsilon |u|^{p(x)} + \epsilon |u|^{q(x)} \tag{3.17}$$

for a.e. $x \in \Omega$. Through (3.17), for $v \in X \setminus \{0\}$ and $t > 1$, we obtain that

$$\begin{aligned}
I_\lambda(tv) &= \int_\Omega \frac{1}{p(x)} |t\nabla v|^{p(x)} \mathrm{d}x - \lambda \int_\Omega |tv|^{q(x)} \mathrm{d}x - \int_\Omega F(x,tv) \mathrm{d}x \\
&\leq t^{p^+} \int_\Omega \frac{1}{p(x)} |\nabla v|^{p(x)} \mathrm{d}x - \lambda \int_\Omega |tv|^{q(x)} \mathrm{d}x + C_\epsilon t^{p^+} \int_\Omega |v|^{p(x)} \mathrm{d}x + \epsilon \int_\Omega |tv|^{q(x)} \mathrm{d}x \\
&\leq t^{p^+} \left(\int_\Omega \frac{1}{p(x)} |\nabla v|^{p(x)} \mathrm{d}x + C_\epsilon \int_\Omega |v|^{q(x)} \mathrm{d}x \right) - t^{q^-} (\lambda - \epsilon) \int_\Omega |v|^{q(x)} \mathrm{d}x.
\end{aligned}$$

Choosing $0 < \epsilon < \lambda$, and noting that $p^+ < q^-$, we can find t sufficiently large such that $I_\lambda(t_0 v) < 0$ and set $u_1 = t_0 v$, then u_1 is the desired element. Since $I_\lambda(0) = 0$ and $0 < \rho < \rho_0$, by virtue of Claim 1, we obtain

$$\inf\{I_\lambda(u) : \|u\| = \rho\} > \max\{I_\lambda(0), I_\lambda(u_1)\}.$$

By Lemma 3.6, the nonsmooth C-condition is fulfilled. It follows from Theorem 2.1 we obtain that I_λ has at least one nontrivial critical point $\hat{u} \in X$, i.e., a nontrivial solution of problem (P_λ). □

Proof of Theorem 1.2. We claim that $I_\lambda(u) \to -\infty$ as $\|u\| \to +\infty$, for any

$u \in Y_k$. Assume that $\|u\| > 1$. By virtue of (3.17), setting $0 < \epsilon < \lambda$, we derive

$$
\begin{aligned}
I_\lambda(u) &= \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \mathrm{d}x - \lambda \int_\Omega |u|^{q(x)} \mathrm{d}x - \int_\Omega F(x,u) \mathrm{d}x \\
&\leq \frac{1}{p^-} \|u\|^{p^+} - \lambda \int_\Omega |u|^{q(x)} \mathrm{d}x + C_\epsilon \int_\Omega |u|^{p(x)} \mathrm{d}x + \epsilon \int_\Omega |u|^{q(x)} \mathrm{d}x \\
&\leq \frac{1}{p^-} \|u\|^{p^+} - (\lambda - \epsilon) |u|_{q(x)}^{q^-} + C_\epsilon |u|_{p(x)}^{p^+}.
\end{aligned}
$$

Noting that Y_k is a finite dimensional space, then all norms in Y_k are equivalent. Since $p^+ < q^-$, we obtain that $I_\lambda(u) \to -\infty$ as $\|u\| \to +\infty$. Recalling that $I_\lambda(0) = 0$ and I_λ is even with $V = Y_k$ ($\dim Y_k = k$) and $Y = X$ ($\operatorname{codim} Y = 0$), from Lemma 3.6 and Claim 1 in Theorem 1.1, we infer that I_λ has k-pairs of nontrivial solutions for problem (P_λ). □

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