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WONG-ZAKAI APPROXIMATIONS AND ATTRACTORS FOR FRACTIONAL STOCHASTIC REACTION-DIFFUSION EQUATIONS ON UNBOUNDED DOMAINS*

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Abstract In this paper, we investigate the Wong-Zakai approximations induced by a stationary process and the long term behavior of the fractional stochastic reaction-diffusion equation driven by a white noise. Precisely, one of the main ingredients in this paper is to establish the existence and uniqueness of tempered pullback attractors for the Wong-Zakai approximations of fractional stochastic reaction-diffusion equations. Thereafter the upper semi-continuity of attractors for the Wong-Zakai approximation of the equation as $\delta \to 0$ is proved.

Keywords Fractional reaction-diffusion equation, Wong-Zakai approximation, random attractor, upper semi-continuity, white noise.

MSC(2010) 60H15, 37H10, 37L55, 37D10.

1. Introduction

This paper considers the Wong-Zakai approximations and the long term behavior of the non-local, fractional stochastic reaction-diffusion equations on $\mathbb{R}^n$ as following:

$$\frac{\partial u}{\partial t} + (-\Delta)^s u + \lambda u = f(t,x,u) + g(t,x) + h(t,x,u) \circ \frac{dW}{dt}, \quad t > \tau,\ x \in \mathbb{R}^n, \tag{1.1}$$

with initial condition

$$u(\tau, x) = u_\tau(x), x \in \mathbb{R}^n. \tag{1.2}$$

Here, $s \in (0,1)$, $\lambda > 0$ is a fixed constant, $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a smooth functon, $g \in L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^n))$, W is a one-dimensional two-sided Brownian motion, the symbol $\circ$ means that the equation is understood in the sense of Stratonovich's integration.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the classical Wiener probability space, where

$$\Omega = \mathbf{C}_0(\mathbb{R}, \mathbb{R}) := \{\omega \in \mathbf{C}(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}$$

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with the open compact topology, $\mathcal{F}$ is its Borel σ-algebra, and $\mathbb{P}$ is the Wiener measure. The shift operator θ_t is defined on $(\Omega, \mathcal{F}, \mathbb{P})$ by

$$\theta_t \omega(\cdot) = \omega(t + \cdot) - \omega(t).$$

As we all know, the probability measure $\mathbb{P}$ is an ergodic invariant measure for θ_t. $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$ forms a metric dynamical system, see Arnold [1].

For given $\delta \in \mathbb{R}$, set $\mathcal{G}_\delta : \Omega \to \mathbb{R}$ as the random variable:

$$\mathcal{G}_\delta(\omega) = \frac{1}{\delta}\omega(\delta),$$

then we have

$$\mathcal{G}_\delta(\theta_t\omega) = \frac{1}{\delta}(\omega(t+\delta) - \omega(t)). \tag{1.3}$$

By checking, we know that $\mathcal{G}_\delta(\theta_t\omega)$ is a stationary stochastic process. $\mathcal{G}_\delta(\theta_t\omega)$ is an approximation of white noise in the sense

$$\lim_{\delta\to 0} \sup_{t\in[0,T]} \left| \int_0^t \mathcal{G}_\delta(\theta_s\omega) ds - W(t,\omega) \right| = 0, \quad a.s. \tag{1.4}$$

for each $T > 0$, which was first introduced by K. Lu and Q. Wang in [13]. From then on, the same approximation is used in [14] for bounded domains and also in [15] and [17] for unbounded domains. More recently, this approximation was used by Shen, Zhao, Lu and Wang to investigate the invariant manifolds and stable foliations of the Wong-Zakai approximaitons, which turn out to converge to the invariant manifolds and stable foliations of the stochastic evolution equation, respectively in [18]. In the present paper, we also use the same approximation as in (1.4). (1.4) implies that equation (1.1) could be approximated by the following Wong-Zakai equation driven by a multiplicative noise of $\mathcal{G}_\delta(\theta_t\omega)$ as $\delta \to 0$:

$$\frac{\partial u}{\partial t} + (-\Delta)^s u + \lambda u = f(t,x,u) + g(t,x) + h(t,x,u)\mathcal{G}_\delta(\theta_t\omega), \quad t > \tau, x \in \mathbb{R}^n. \tag{1.5}$$

To place our result in context, we review a few highlights from the random attractors of fractional stochastic equations. The authors of [9] and [10] obtained the existence of random attractors. Recently, the authors of [22] established the existence of the random attractors on bounded domains, and the authors of [12] solved the case of unbounded domains by applying diagonal processes for two times and tail-estimate. Moreover, by using the idea of spectral decomposition on bounded domains $\mathcal{O}$ in $\mathbb{R}^n$ along with the uniform tail-estimates of solutions, Wang etc obtained the regularity of random attractors [6]. Note that the noise in the above five papers is either additive or linear multiplicative. On the other hand, for the nonlinear case, there are few results: the existence of random attractors for stochastic PDEs driven by a fractional Brownian motion was proved by the authors of [4], [5]. Also, very recently, Wang etc in [24] established the existence and uniqueness of pullback random attractors for the fractional nonclassical diffusion equations driven by colored noise via using the similar method in [6].

In this paper, strongly motivated by the work of Wang etc [15], we study the long term behavior of equations (1.1) and (1.2). In general, the stochastic equation (1.1) could generate a continuous cocycle only when $h(t,x,u)$ either only depends on t and x or is linear in u. For general nonlinear function $h(t,x,u)$, (1.1) may not generate continuous cocycle, hence the existence of attractor is unclear. Fortunately, we are able to show that (1.5) could generate a continuous cocycle and additionally it has a tempered attractor for a class of nonlinear functions h. Thus we could indirectly investigate the long term behavior of (1.1) via considering (1.5). This result is presented in Theorem 2.1.

To illustrate the advantage of random equation (1.5) over stochastic equation (1.1), in section 3, for linear multiplicative noise, we will prove the solutions of equation (1.5) converge to that of equation (1.1) as $\delta \to 0$ and furthermore we will obtain the upper semi-continuity of the attractors for equation (1.5) in $L^2(\mathbb{R}^n)$. This property is contained in Theorem 3.1.

To obtain the uniform estimates of solutions of equation (1.5) in $H^s(\mathbb{R}^n)$, a strong condition (2.9) for the noise term $h(t,x,u)$ is necessary. Indeed, one needs (2.9) to ensure the regularity of $h(t,x,u)$, which is a key step in the estimate of u. The uniform estimates and the uniform tail-estimates of solutions in $L^2(\mathbb{R}^n)$ will yield the pullback asymptotic compactness of solutions in $H^s(\mathbb{R}^n)$.

The pioneer work of approximating stochastic equations by pathwise deterministic equations could date back to Wong and Zakai [25,26]. So far, there has been a series of nice results about Wong-Zakai approxiamtions, for instance, the readers can consult [2,3,7,8,19].

The remaining part of this paper is organized as follows: In section 2, the existence and uniqueness of pullback random attractors for Wong-Zakai approximations are proved. In the last section, we obtain the upper semi-continuity of attractors of Wong-Zakai approximations (3.11) for multiplicative noise as $\delta \to 0$.

Notations. Before ending this introduction, let us recall some related notions about the integral fractional operator $(-\Delta)^s$. Given $s \in (0,1)$, the fractional Laplace operator $(-\Delta)^s$ is defined by

$$(-\Delta)^s u(x) = C(n,s)\int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,\mathrm{d}y,$$

where $C(n,s)$ is a positive number depending on n and s with

$$C(n,s) = \left(\int_{\mathbb{R}^n} \frac{1-cos(\xi_1)}{|\xi|^{n+2s}}\,\mathrm{d}\xi\right)^{-1}, \quad \xi = (\xi_1,\xi_2,\cdots,\xi_n) \in \mathbb{R}^n. \tag{1.6}$$

For $s \in (0,1)$, the fractional Sobolev space $H^s(\mathbb{R}^n)$ is defined by

$$H^s(\mathbb{R}^n) = \left\{u \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y < \infty\right\},$$

with norm

$$\|u\|_{H^s(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |u(x)|^2 dx + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}}.$$

In this paper, the norm and the inner product of $L^2(\mathbb{R}^n)$ are denoted by $\|\cdot\|$ and $(\cdot,\cdot)$, respectively. The Gagliardo semi-norm of $H^s(\mathbb{R}^n)$ is denoted by

$$\|u\|^2_{\dot{H}^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y, \quad u \in H^s(\mathbb{R}^n).$$

Note that $H^s(\mathbb{R}^n)$ is a Hilbert space, with inner product

$$(u,v)_{H^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} u(x)v(x)dx + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y, \quad \text{for} \quad u,v \in H^s(\mathbb{R}^n).$$

Moreover, by [16] we have

$$\|u\|^2_{H^s(\mathbb{R}^n)} = \|u\|^2 + \frac{2}{C(n,s)}\|(-\Delta)^{\frac{s}{2}}u\|^2, \quad u \in H^s(\mathbb{R}^n), \tag{1.7}$$

thus, $\big(\|u\|^2 + \|(-\Delta)^{\frac{s}{2}}u\|^2)\big)^{\frac{1}{2}}$ is an equivalent norm of $H^s(\mathbb{R}^n)$.

Hereafter, the letters c, c_i and C_i may be different in different lines.

2. Wong-Zakai approximations

Let $\tau, \delta \in \mathbb{R}, \delta \neq 0, s \in (0,1)$. Consider the following random fractional reaction-diffusion equation:

$$\frac{\partial u}{\partial t} + (-\Delta)^s u + \lambda u = f(t,x,u) + g(t,x) + h(t,x,u)\mathcal{G}_\delta(\theta_t\omega), \quad t > \tau, x \in \mathbb{R}^n, \tag{2.1}$$

with initial condition

$$u(\tau,x) = u_\tau(x), \quad x \in \mathbb{R}^n, \tag{2.2}$$

where λ is a positive constant, $g \in L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^n))$.

The nonlinearity function $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is continuous and for all $t, u \in \mathbb{R}, x \in \mathbb{R}^n$,

$$f(t,x,u)u \le -\beta|u|^p + \psi_1(t,x), \tag{2.3}$$

$$|f(t,x,u)| \le \psi_2(t,x)|u|^{p-1} + \psi_3(t,x), \tag{2.4}$$

$$\left|\frac{\partial f}{\partial u}(t,x,u)\right| \le \psi_4(t,x), \tag{2.5}$$

$$|f(t,x,u) - f(t,y,u)| \le |\psi_5(x) - \psi_5(y)|, \tag{2.6}$$

where $\beta > 0, p \ge 2$ are constants, $\psi_1 \in L^1_{loc}(\mathbb{R}, L^1(\mathbb{R}^n)), \psi_2, \psi_4 \in L^\infty_{loc}\big(\mathbb{R}, L^\infty(\mathbb{R}^n)\big)$, $\psi_3 \in L^q_{loc}\big(\mathbb{R}, L^q(\mathbb{R}^n)\big), \frac{1}{p} + \frac{1}{q} = 1, \psi_5 \in H^s(\mathbb{R}^n)$.

The noise term $h : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is continuous and for all $t, u \in \mathbb{R}, x \in \mathbb{R}^n$,

$$|h(t,x,u)| \le \varphi_1(t,x)|u|^{p_1-1} + \varphi_2(t,x), \tag{2.7}$$

$$\left|\frac{\partial h}{\partial u}(t,x,u)\right| \le \varphi_3(t,x), \tag{2.8}$$

$$|h(t,x,u) - h(t,y,u)| \le |\varphi_4(x) - \varphi_4(y)|, \tag{2.9}$$

where $2 \le p_1 < p,\ \varphi_1 \in L^{\frac{p}{p-p_1}}_{loc}\big(\mathbb{R}, L^{\frac{p}{p-p_1}}(\mathbb{R}^n)\big),\ \varphi_2 \in L^q_{loc}\big(\mathbb{R}, L^q(\mathbb{R}^n)\big)$, $\varphi_3 \in L^\infty\big(\mathbb{R}, L^\infty(\mathbb{R}^n)\big),\ \varphi_4 \in H^s(\mathbb{R}^n)$. We make the following necessary assumptions:

$$\int_{-\infty}^0 e^{\lambda s}\big(\|g(s+\tau)\|^2 + \|\psi_1(s+\tau)\|_{L^1}\big)\,\mathrm{d}s < \infty, \quad \forall \tau \in \mathbb{R}, \tag{2.10}$$

$$\lim_{r\to-\infty} e^{cr}\int_{-\infty}^{0} e^{\lambda s}\big(\|g(s+\tau)\|^2+\|\psi_1(s+\tau)\|_{L^1}\big)\ \mathrm{d}s=0,\quad \forall c>0. \tag{2.11}$$

For the simplicity of notations, we define

$$\alpha(t)=\|g(t)\|^2+\|\psi_1(t)\|_{L^1}. \tag{2.12}$$

We are ready to state the first main theorem of this paper.

Theorem 2.1. *Suppose* (2.3)-(2.9) *hold, and* (2.10)-(2.11) *are assumed. Then the cocycle* Φ *of equations* (2.1)-(2.2) *addimits a unique* $\mathcal{D}$*-pullback attractor* $\mathcal{A}=\{\mathcal{A}(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$ *in* $L^2(\mathbb{R}^n)$.

Now we comment on the proof of Theorem 2.1. We firstly prove the existence of a continuous cocycle for random fractional reaction-diffusion equations.

2.1. Continuous cocycles

As in the introduction, $(\Omega,\mathcal{F},\mathbb{P},\{\theta_t\}_{t\in\mathbb{R}})$ is a metric dynamical system, then there exists a $\{\theta_t\}_{t\in\mathbb{R}}$-invariant subset of full measure $\Omega_0\subseteq\Omega$ such that for all $\omega\in\Omega_0$,

$$\frac{\omega(t)}{t}\to 0\quad \text{as}\quad t\to\pm\infty. \tag{2.13}$$

For the concision of notation, in this paper, we don't distinguish Ω and Ω_0. By (1.3), we have

$$\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds=\int_\delta^0\frac{\omega(s)}{\delta}ds+\int_t^{t+\delta}\frac{\omega(s)}{\delta}ds. \tag{2.14}$$

Here is a list of properties of $\mathcal{G}_\delta(\theta_t\omega)$, see [14].

Lemma 2.1. *Let* $\tau\in\mathbb{R},\omega\in\Omega,T>0$*. Then for* $\forall\epsilon>0$*, there exists* $\delta_0=\delta_0(\epsilon,\tau,\omega,T)>0$ *such that for all* $0<|\delta|<\delta_0$ *and* $t\in[\tau,\tau+T]$*, we have*

$$\left|\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds-\omega(t)\right|<\epsilon. \tag{2.15}$$

Since $\omega(t)$ is continuous on $[\tau,\tau+T]$, there exists $c_1=c_1(\tau,\omega,T)>0$ such that

$$|\omega(t)|\le c_1,\quad \text{for all}\quad t\in[\tau,\tau+T], \tag{2.16}$$

which together with (2.15) gives that there exists $\delta_1=\delta_1(\tau,\omega,T)>0$, $c_2=c_2(\tau,\omega,T)>0$ such that for all $0<|\delta|<\delta_1$ and $t\in[\tau,\tau+T]$,

$$\begin{aligned}&\left|\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds\right|\\ \le&\left|\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds-\omega(t)\right|+|\omega(t)|\\ \le& c_2.\end{aligned} \tag{2.17}$$

Our present purpose is proving the existence of the continuous cocycle for e-quations (2.1)-(2.2), so we need to prove the well-posedness. To this end, we set $\mathcal{O}_k = \{x \in \mathbb{R}^n : |x| < k\}$ for all $k \in \mathbb{N}$. It is plausible that one utilizes $\mu = \mu(s) \in C^\infty(\mathbb{R}^+)$ as a cut-off function such that

$$\mu(s) = \begin{cases} 1, & 0 \le s \le \frac{1}{2}, \\ 0, & s \ge 1. \end{cases} \tag{2.18}$$

Correspondingly, we shall study the following problem:

$$\frac{\partial u_k}{\partial t} + (-\Delta)^s u_k + \lambda u_k = f(t,x,u_k) + g(t,x) + h(t,x,u_k)\mathcal{G}_\delta(\theta_t\omega), \quad t > \tau, x \in \mathcal{O}_k, \tag{2.19}$$

with boundary condition

$$u_k(t,x) = 0, \quad \forall t > \tau, |x| = k, \tag{2.20}$$

and initial condition

$$u_k(\tau,x) = \mu\left(\frac{|x|}{k}\right) u_\tau(x), \quad x \in \mathcal{O}_k, \tag{2.21}$$

where $u_\tau \in L^2(\mathbb{R}^n)$. To prove the well-posedness of (2.19)-(2.21), for every $k \in \mathbb{N}$, we set $\Theta_k = \{u \in L^2(\mathbb{R}^n) : u = 0 \text{ a.e. on } \mathbb{R}^n \backslash \mathcal{O}_k\}$, $\Lambda_k = \{u \in H^s(\mathbb{R}^n) : u = 0 \text{ a.e. on } \mathbb{R}^n \backslash \mathcal{O}_k\}$. Let $a : H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n) \to \mathbb{R}$,

$$\begin{aligned} & a(u_1,u_2) \\ =& \lambda(u_1,u_2) + ((-\Delta)^{\frac{s}{2}}u_1, (-\Delta)^{\frac{s}{2}}u_2) \\ =& \lambda(u_1,u_2) + \frac{1}{2}C(n,s)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u_1(x)-u_1(y))(u_2(x)-u_2(y))}{|x-y|^{n+2s}} \,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{2.22}$$

Moreover, we define $\mathrm{A} : H^s(\mathbb{R}^n) \to H^{-s}(\mathbb{R}^n)$ as following

$$(\mathrm{A}(u_1), u_2)_{(H^{-s},H^s)} = a(u_1,u_2) \quad \text{for all} \quad u_1,u_2 \in H^s(\mathbb{R}^n). \tag{2.23}$$

By checking that $(\mathrm{A}(u_1), u_2)$ is linear and continuous both in u_1 and u_2, hence $\mathrm{A} : H^s \to H^{-s}$ is well-defined. Under conditions (2.3)-(2.5), and (2.7)-(2.8), by [22], we see that for all $\tau \in \mathbb{R}, \omega \in \Omega$, and $u_\tau \in L^2(\mathbb{R}^n)$, (2.19)-(2.21) is well-posed in $L^2(\mathcal{O}_k)$. Moreover, the solution is $(\mathcal{F}, \mathcal{B}(L^2(\mathcal{O}_k)))$-measurable in $\omega \in \Omega$. Next, we derive the uniform estimates of solutions u_k with respective to $k \in \mathbb{N}$, then letting $k \to \infty$, the existence and uniqueness of solutions of equations (2.1)-(2.2) are obtained.

Lemma 2.2. *Assume* (2.3)-(2.5) *and* (2.7)-(2.8)*, then for all* $\tau \in \mathbb{R}, \omega \in \Omega$, *and* $u_\tau \in L^2(\mathbb{R}^n)$, *equations* (2.1)-(2.2) *admits a unique solution* $u(\cdot,\tau,\omega,u_\tau) \in C([\tau,\infty); L^2(\mathbb{R}^n)) \bigcap L^2_{loc}([\tau,\infty); H^s(\mathbb{R}^n))$.

Proof. Although some essential steps in the proof are similar to those in [12], we still give the details for reader's convenience. We will prove the result in the following three steps:
(1) Uniform estimates
By (2.19), we get that for $t > \tau$,

$$\begin{aligned} & \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{O}_k} |u_k(x)|^2 \,\mathrm{d}x + \int_{\mathcal{O}_k} u_k(x)(-\Delta)^s u_k(x) \,\mathrm{d}x + \lambda\int_{\mathcal{O}_k} |u_k(x)|^2 \,\mathrm{d}x \\ =& \int_{\mathcal{O}_k} f(t,x,u_k)u_k(x) \,\mathrm{d}x + \int_{\mathcal{O}_k} g(t,x)u_k(x) \,\mathrm{d}x + \mathcal{G}_\delta(\theta_t\omega)\int_{\mathcal{O}_k} h(t,x,u_k)u_k(x) \,\mathrm{d}x. \end{aligned} \tag{2.24}$$

For the simplicity of presentation, we define that

$$I(t,x,u)=f(t,x,u)+g(t,x)+h(t,x,u)\mathcal{G}_\delta(\theta_t\omega). \tag{2.25}$$

Because of the boundary condition (2.20), the above (2.24) can be rewritten as

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_k\|^2+\frac{1}{2}C(n,s)\|u_k\|^2_{\dot{H}^s(\mathbb{R}^n)}+\lambda\|u_k\|^2=\int_{\mathbb{R}^n} I(t,x,u_k)u_k(x)\ \mathrm{d}x. \tag{2.26}$$

By (2.3), (2.7) and Young's inequality, we obtain that

$$\begin{aligned}\int_{\mathbb{R}^n} f(t,x,u_k)u_k(x)\ \mathrm{d}x\le&-\beta\int_{\mathbb{R}^n}|u_k(x)|^p\ \mathrm{d}x+\int_{\mathbb{R}^n}\psi_1(t,x)\ \mathrm{d}x\\ \le&-\beta\|u_k\|^p_{L^p}+\|\psi_1(t)\|_{L^1},\end{aligned} \tag{2.27}$$

$$\begin{aligned}\mathcal{G}_\delta(\theta_t\omega)\int_{\mathbb{R}^n} h(t,x,u_k)u_k(x)\ \mathrm{d}x\le&\frac{\beta}{2}\int_{\mathbb{R}^n}|u_k(x)|^p\ \mathrm{d}x\\ &+c_1|\mathcal{G}_\delta(\theta_t\omega)|^{\frac{p}{p-p_1}}\int_{\mathbb{R}^n}|\varphi_1(t,x)|^{\frac{p}{p-p_1}}\ \mathrm{d}x\\ &+c_2|\mathcal{G}_\delta(\theta_t\omega)|^q\int_{\mathbb{R}^n}|\varphi_2(t,x)|^q\ \mathrm{d}x,\end{aligned} \tag{2.28}$$

$$\int_{\mathbb{R}^n} g(t,x)u_k(x)\ \mathrm{d}x\le\frac{\lambda}{4}\|u_k\|^2+\frac{1}{\lambda}\|g(t)\|^2. \tag{2.29}$$

(2.26)-(2.29) implies that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|u_k\|^2+C(n,s,\lambda)\|u_k\|^2_{H^s(\mathbb{R}^n)}+\lambda\|u_k\|^2+\beta\|u_k\|^p_{L^p}\\ \le&C_\lambda\alpha(t)+c_1|\mathcal{G}_\delta(\theta_t\omega)|^{\frac{p}{p-p_1}}\|\varphi_1(t)\|^{\frac{p}{p-p_1}}_{L^{\frac{p}{p-p_1}}}+c_2|\mathcal{G}_\delta(\theta_t\omega)|^q\|\varphi_2(t)\|^q_{L^q}.\end{aligned} \tag{2.30}$$

Multiplying (2.30) by $e^{\lambda t}$, then integrating over (τ,t) for $t\ge\tau$, we get

$$\begin{aligned}&\|u_k(t,\tau,\omega,u_\tau)\|^2+\beta\int_\tau^t e^{\lambda(s-t)}\|u_k(s,\tau,\omega,u_\tau)\|^p_{L^p}\ \mathrm{d}s\\ &+C(n,s,\lambda)\int_\tau^t e^{\lambda(s-t)}\|u_k(s,\tau,\omega,u_\tau)\|^2_{H^s(\mathbb{R}^n)}\ \mathrm{d}s\\ \le&e^{\lambda(\tau-t)}\|u_\tau\|^2+C_\lambda\int_\tau^t e^{\lambda(s-t)}\alpha(s)\ \mathrm{d}s\\ &+c_1\int_\tau^t e^{\lambda(s-t)}|\mathcal{G}_\delta(\theta_s\omega)|^{\frac{p}{p-p_1}}\|\varphi_1(s)\|^{\frac{p}{p-p_1}}_{L^{\frac{p}{p-p_1}}}\ \mathrm{d}s\\ &+c_2\int_\tau^t e^{\lambda(s-t)}|\mathcal{G}_\delta(\theta_s\omega)|^q\|\varphi_2(s)\|^q_{L^q}\ \mathrm{d}s.\end{aligned} \tag{2.31}$$

Since $\mathcal{G}_\delta(\theta_t\omega)$ is continuous in t for fixed $\omega\in\Omega$, and $\psi_1\in L^1_{loc}\big(\mathbb{R},L^1(\mathbb{R}^n)\big), g\in L^2_{loc}\big(\mathbb{R},L^2(\mathbb{R}^n)\big),\varphi_1\in L^{\frac{p}{p-p_1}}_{loc}\big(\mathbb{R},L^{\frac{p}{p-p_1}}(\mathbb{R}^n)\big)$, and $\varphi_2\in L^q_{loc}\big(\mathbb{R},L^q(\mathbb{R}^n)\big)$, hence for $T>\tau$, we have

$$\{u_k\}_{k=1}^\infty\quad\text{is bounded in}\quad L^\infty(\tau,T;L^2(\mathbb{R}^n))\bigcap L^2(\tau,T;H^s(\mathbb{R}^n))\bigcap L^p(\tau,T;L^p(\mathbb{R}^n)). \tag{2.32}$$

By (2.23),

$$\{\mathrm{A}(u_k)\}_{k=1}^{\infty} \ \text{ is bounded in } \ L^2(\tau,T;H^{-s}(\mathbb{R}^n)). \tag{2.33}$$

(2.4) and (2.7) imply that

$$\int_{\tau}^{T}\int_{\mathbb{R}^n}|f(t,x,u_k)|^q \,\mathrm{d}x\,\mathrm{d}t \le c_1\int_{\tau}^{T}\int_{\mathbb{R}^n}|u_k|^p \,\mathrm{d}x\,\mathrm{d}t + c_2\int_{\tau}^{T}\int_{\mathbb{R}^n}|\psi_3(t,x)|^q \,\mathrm{d}x\,\mathrm{d}t, \tag{2.34}$$

$$\begin{aligned}\int_{\tau}^{T}\int_{\mathbb{R}^n}|\mathcal{G}_\delta(\theta_t\omega)h(t,x,u_k)|^q \,\mathrm{d}x\,\mathrm{d}t \le & c_1\int_{\tau}^{T}\int_{\mathbb{R}^n}|u_k|^p \,\mathrm{d}x\,\mathrm{d}t \\ & + c_2\int_{\tau}^{T}\int_{\mathbb{R}^n}|\varphi_1(t,x)|^{\frac{p}{p-p_1}} \,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{2.35}$$

Hence, (2.32), (2.34) and (2.35) imply

$$\{f(t,x,u_k)\}_{k=1}^{\infty} \ \text{ and } \ \{\mathcal{G}_\delta(\theta_t\omega)h(t,x,u_k)\}_{k=1}^{\infty} \ \text{ are bounded in } \ L^q(\tau,T;L^q(\mathbb{R}^n)). \tag{2.36}$$

By (2.19), (2.32), (2.33) and (2.36), for fixed $K\in\mathbb{N}$, we get that

$$\left\{\frac{\mathrm{d}u_k}{\mathrm{d}t}\right\}_{k=1}^{\infty} \ \text{ is bounded in } \ L^2(\tau,T;\Lambda_K^*) + L^q\big(\tau,T;L^q(\mathbb{R}^n)\big). \tag{2.37}$$

Note that $1<q\le 2$ and $\frac{1}{p}+\frac{1}{q}=1$, thus

$$\left\{\frac{du_k}{dt}\right\}_{k=1}^{\infty} \ \text{ is bounded in } \ L^q\Big(\tau,T;\big(\Lambda_K\bigcap L^p(\mathbb{R}^n)\big)^*\Big). \tag{2.38}$$

(2) Limiting process
Similar to the method in [12], by (2.32)-(2.38), we can get that there exists $\bar{u}\in L^2(\mathbb{R}^n), u\in L^\infty(\tau,T;L^2(\mathbb{R}^n))\bigcap L^2(\tau,T;H^s(\mathbb{R}^n))\bigcap L^p(\tau,T;L^p(\mathbb{R}^n))$ and $w\in L^q(\tau,T;L^q(\mathbb{R}^n))$ such that up to a subsequence,

$$u_k\to u \quad weak^* \quad \text{in} \quad L^\infty(\tau,T;L^2(\mathbb{R}^n)), \tag{2.39}$$

$$u_k\rightharpoonup u \quad \text{in} \quad L^2(\tau,T;H^s(\mathbb{R}^n))\bigcap L^p(\tau,T;L^p(\mathbb{R}^n)), \tag{2.40}$$

$$f(t,\cdot,u_k)+\mathcal{G}_\delta(\theta_t\omega)h(t,\cdot,u_k)\rightharpoonup w \quad \text{in} \quad L^q(\tau,T;L^q(\mathbb{R}^n)), \tag{2.41}$$

$$\frac{du_k}{dt}\rightharpoonup\frac{du}{dt} \quad \text{in} \quad L^q(\tau,T;(\Lambda_K\bigcap L^p(\mathbb{R}^n))^*),\forall K\in\mathbb{N}, \tag{2.42}$$

and

$$u_k(T,\tau,\omega)\rightharpoonup\bar{u} \quad \text{in} \quad L^2(\mathbb{R}^n). \tag{2.43}$$

Note that the embedding $H^s(\mathcal{O}_k)\hookrightarrow L^2(\mathcal{O}_k)$ is compact and $L^2\big(\mathcal{O}_k\big)\hookrightarrow \big(\Lambda_K\bigcap L^p(\mathbb{R}^n)\big)^*$ is continuous, and by (2.32), (2.38), after an diagonal process about k, we infer from [11]

$$u_k\to u \quad \text{strongly in} \quad L^2(\tau,T;L^2(\mathcal{O}_K)), \quad \forall K\in\mathbb{N}. \tag{2.44}$$

Again, by (2.44) and a diagonal process about K, there exists a subsequence of $\{u_k\}_{k=1}^{\infty}$, which we still denote by$\{u_k\}_{k=1}^{\infty}$ such that

$$u_k\to u \quad \text{a.e.} \quad (t,x)\in(\tau,T)\times\mathbb{R}^n. \tag{2.45}$$

Since f and h are continuous, by (2.45), we have

$$f(t,x,u_k)+\mathcal{G}_\delta(\theta_t\omega)h(t,x,u_k)\to f(t,x,u)+\mathcal{G}_\delta(\theta_t\omega)h(t,x,u) \quad \text{a.e. } (t,x)\in(\tau,T)\times\mathbb{R}^n. \tag{2.46}$$

From (2.36) and (2.46), we have

$$f(t,\cdot,u_k)+\mathcal{G}_\delta(\theta_t\omega)h(t,\cdot,u_k)\rightharpoonup f(t,\cdot,u)+\mathcal{G}_\delta(\theta_t\omega)h(t,\cdot,u) \quad \text{in} \quad L^q(\tau,T;L^q(\mathbb{R}^n)). \tag{2.47}$$

By (2.41) and (2.47), we have

$$w=f(t,\cdot,u)+\mathcal{G}_\delta(\theta_t\omega)h(t,\cdot,u). \tag{2.48}$$

Given $\xi\in H^s(\mathbb{R}^n)\bigcap L^p(\mathbb{R}^n)$, let

$$\xi_K(x)=\mu\left(\frac{|x|}{K}\right)\xi(x), \quad \text{for} \quad x\in\mathbb{R}^n. \tag{2.49}$$

It is obvious that for all $K\in\mathbb{N},\xi_K\in H^s(\mathbb{R}^n)\bigcap L^p(\mathbb{R}^n)$,

$$\xi_K\to\xi \quad \text{as} \quad K\to\infty \quad \text{in} \quad H^s(\mathbb{R}^n)\bigcap L^p(\mathbb{R}^n). \tag{2.50}$$

For every $k>K$, letting $\varphi\in C_0^\infty(\tau,T)$, by (2.19)-(2.21), we get

$$-\int_\tau^T(u_k,\xi_K)\varphi'\,\mathrm{d}t+\int_\tau^T a(u_k,\xi_K)\varphi\,\mathrm{d}t=\int_\tau^T(I(t,\cdot,u_k),\xi_K)\varphi\,\mathrm{d}t. \tag{2.51}$$

In (2.51), letting $k\to\infty$, by (2.39)-(2.41) and (2.48), we get

$$-\int_\tau^T(u,\xi_K)\varphi'\,\mathrm{d}t+\int_\tau^T a(u,\xi_K)\varphi\,\mathrm{d}t=\int_\tau^T(I(t,\cdot,u),\xi_K)\varphi\,\mathrm{d}t. \tag{2.52}$$

In (2.52), also letting $K\to\infty$, by (2.50), we have

$$-\int_\tau^T(u,\xi)\varphi'\,\mathrm{d}t+\int_\tau^T a(u,\xi)\varphi\,\mathrm{d}t=\int_\tau^T(I(t,\cdot,u),\xi)\varphi\,\mathrm{d}t. \tag{2.53}$$

Thus, for all $\xi\in H^s(\mathbb{R}^n)\bigcap L^p(\mathbb{R}^n)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}(u,\xi)+a(u,\xi)=(I(t,\cdot,u),\xi). \tag{2.54}$$

To prove the continuity of $u:[\tau,\infty)\to L^2(\mathbb{R}^n)$, we notice that $u\in L^2\big(\tau,T;H^s(\mathbb{R}^n)\big)\bigcap L^p\big(\tau,T;L^p(\mathbb{R}^n)\big)$ and $\frac{du}{dt}\in L^2\big(\tau,T;H^{-s}(\mathbb{R}^n)\big)+L^q\big(\tau,T;L^q(\mathbb{R}^n)\big)$, thus by [11] we get that $u\in C([\tau,T],L^2(\mathbb{R}^n))$.

Next, we prove $u(\tau)=u_\tau$ and $u(T)=\bar{u}$. For this aim, we let $\varphi\in C^1([\tau,T])$ and $\xi\in H^s(\mathbb{R}^n)\bigcap L^p(\mathbb{R}^n)$. Similar to (2.51), by (2.19)-(2.21), we get for every $k>K$,

$$\begin{aligned}&(u_k(T),\xi_K)\varphi(T)-(u_k(\tau),\xi_K)\varphi(\tau)\\&=\int_\tau^T(u_k,\xi_K)\varphi'\,\mathrm{d}t-\int_\tau^T a(u_k,\xi_K)\varphi\,\mathrm{d}t+\int_\tau^T(I(t,\cdot,u_k),\xi_K)\varphi\,\mathrm{d}t.\end{aligned} \tag{2.55}$$

By (2.21), (2.39)-(2.41), (2.43), (2.48), in (2.55), letting $k \to \infty$,

$$\begin{aligned}&(\bar{u}, \xi_K)\varphi(T) - (u_\tau, \xi_K)\varphi(\tau) \\ &= \int_\tau^T (u, \xi_K)\varphi' \, \mathrm{d}t - \int_\tau^T a(u, \xi_K)\varphi \, \mathrm{d}t + \int_\tau^T (I(t, \cdot, u), \xi_K)\varphi \, \mathrm{d}t.\end{aligned} \tag{2.56}$$

In (2.56), letting $K \to \infty$, by (2.50), we get for all $\xi \in H^s(\mathbb{R}^n) \bigcap L^p(\mathbb{R}^n)$,

$$(\bar{u}, \xi)\varphi(T) - (u_\tau, \xi)\varphi(\tau) = \int_\tau^T (u, \xi)\varphi' \, \mathrm{d}t - \int_\tau^T a(u, \xi)\varphi \, \mathrm{d}t + \int_\tau^T (I(t, \cdot, u), \xi)\varphi \, \mathrm{d}t. \tag{2.57}$$

By (2.54), we get

$$\begin{aligned}&(u(T), \xi)\varphi(T) - (u(\tau), \xi)\varphi(\tau) \\ &= \int_\tau^T (u, \xi)\varphi' \, \mathrm{d}t - \int_\tau^T a(u, \xi)\varphi \, \mathrm{d}t + \int_\tau^T (I(t, \cdot, u), \xi)\varphi \, \mathrm{d}t.\end{aligned} \tag{2.58}$$

Thus, we get

$$(u(T), \xi)\varphi(T) - (u(\tau), \xi)\varphi(\tau) = (\bar{u}, \xi)\varphi(T) - (u_\tau, \xi)\varphi(\tau). \tag{2.59}$$

Since $\varphi \in C^1([\tau, T])$, firstly, letting $\varphi(\tau) = 1$ and $\varphi(T) = 0$, from (2.59), we get

$$(u(\tau), \xi) = (u_\tau, \xi). \tag{2.60}$$

Then, letting $\varphi(\tau) = 0$ and $\varphi(T) = 1$, from (2.59), we get

$$(u(T), \xi) = (\bar{u}, \xi). \tag{2.61}$$

By (2.60) and (2.61), we get

$$u(\tau) = u_\tau, \quad u(T) = \bar{u} \quad \text{in} \quad L^2(\mathbb{R}^n), \tag{2.62}$$

combined with (2.43), we have

$$u_k(T, \tau, \omega) \rightharpoonup u(T) \quad \text{in} \quad L^2(\mathbb{R}^n). \tag{2.63}$$

Similar to (2.63), we can get that for all $t \geq \tau$, as $k \to \infty$,

$$u_k(t, \tau, \omega) \rightharpoonup u(t) \quad \text{in} \quad L^2(\mathbb{R}^n). \tag{2.64}$$

(3) Uniqueness and measurability of solutions

Suppose u_1 and u_2 are solutions of (2.1)-(2.2), then $u_1 - u_2$ satisfies

$$\frac{\mathrm{d}(u_1 - u_2)}{\mathrm{d}t} + \mathrm{A}(u_1 - u_2) = f(t, x, u_1) - f(t, x, u_2) + \mathcal{G}_\delta(\theta_t\omega)(h(t, x, u_1) - h(t, x, u_2)),$$

which along with (2.5) and (2.8), we get that for every $T > \tau$, there exists $c > 0$, such that for all $t \in [\tau, T]$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_1 - u_2\|^2 \leq c\|u_1 - u_2\|^2.$$

Therefore, we get

$$\|u_1(t,\tau,\omega,u_{1,\tau})-u_2(t,\tau,\omega,u_{2,\tau})\|^2 \le e^{c(t-\tau)}\|u_{1,\tau}-u_{2,\tau}\|^2. \tag{2.65}$$

Thus, the uniqueness and continuity of solutions in initial data in $L^2(\mathbb{R}^n)$ are proved. Since $u_k(t,\tau,\omega)$ is measurable in $\omega\in\Omega$, by (2.64), $u(t,\tau,\omega)$ is also measurable in ω.

By the three steps above, the proof of Lemma 2.2 is completed. □

Now we can define a mapping $\Phi:\mathbb{R}^+\times\mathbb{R}\times\Omega\times L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)$ such that for all $t\in\mathbb{R}^+,\tau\in\mathbb{R},\omega\in\Omega,u_\tau\in L^2(\mathbb{R}^n)$,

$$\Phi(t,\tau,\omega,u_\tau)=u(t+\tau,\tau,\theta_{-\tau}\omega,u_\tau), \tag{2.66}$$

where u is a soluiton of equations (2.1)-(2.2). The uniqueness of the solution shows that Φ is a continuous cocycle in $L^2(\mathbb{R}^n)$ for equations (2.1)-(2.2).

2.2. Pullback random attractors

In this subsection, we prove the existence and uniqueness of attractors of Φ in $L^2(\mathbb{R}^n)$. By [20], our subsequent tasks are to prove the existence of a tempered pullback absorbing set for equations (2.1)-(2.2) in $L^2(\mathbb{R}^n)$ as well as the asymptotic compactness of the solutions. Also, we assume $\varphi_1\in L^\infty(\mathbb{R},L^{\frac{p}{p-p_1}}(\mathbb{R}^n)),\varphi_2\in L^\infty(\mathbb{R},L^q(\mathbb{R}^n))$. In this process, a series of inequalities are derived via delicate computation and analysis.

We recall that a family of bounded nonempty subsets of $L^2(\mathbb{R}^n),\mathrm{D}=\{\mathrm{D}(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}$ is tempered if for all $c>0,\tau\in\mathbb{R}$ and $\omega\in\Omega$,

$$\lim_{t\to-\infty}e^{ct}\|\mathrm{D}(\tau+t,\theta_t\omega)\|=0, \tag{2.67}$$

where $\|\mathrm{D}\|=\sup\limits_{u\in\mathrm{D}}\|u\|$.

Let $\mathcal{D}$ denote the collection of all tempered families of bounded nonempty subsets of $L^2(\mathbb{R}^n)$, i.e.

$$\mathcal{D}=\{\mathrm{D}=\mathrm{D}(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega:\mathrm{D}\ \text{ is tempered}\}. \tag{2.68}$$

Lemma 2.3. *Suppose* (2.3)−(2.5), (2.7)−(2.8) *hold, in addition,* (2.10) *is assumed. Then for all* $\sigma\in\mathbb{R},\tau\in\mathbb{R},\omega\in\Omega,\mathrm{D}=\{\mathrm{D}(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$*, there exists* $T=T(\tau,\omega,\mathrm{D},\sigma)>0$ *such that for all* $t\ge T$*, the solution* u *of equations* (2.1)-(2.2) *satisfies*

$$\begin{aligned}&\|u(\sigma,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2+\int_{-t}^{\sigma-\tau}e^{\lambda(s+\tau-\sigma)}\|u(s+\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2_{H^s}\,\mathrm{d}s\\&+\beta\int_{-t}^{\sigma-\tau}e^{\lambda(s+\tau-\sigma)}\|u(s+\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^p_{L^p}\,\mathrm{d}s\\&\le 1+C_1\int_{-\infty}^{\sigma-\tau}e^{\lambda(s+\tau-\sigma)}(\alpha(s+\tau)+\gamma(s))\,\mathrm{d}s,\end{aligned} \tag{2.69}$$

where

$$\gamma(t)=|\mathcal{G}_\delta(\theta_t\omega)|^{\frac{p}{p-p_1}}+|\mathcal{G}_\delta(\theta_t\omega)|^q, \tag{2.70}$$

$u_{\tau-t}\in\mathrm{D}(\tau-t,\theta_{-t}\omega)$ *and* C_1 *is a positive constant independent of* τ,ω,σ *and* D.

Proof. By (2.3), (2.7) and (2.54), we get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2+C(n,s)\|u(t)\|^2_{\dot{H}^s}+2\lambda\|u(t)\|^2+\beta\|u(t)\|^p_{L^p}\\ \le&2\|\psi_1(t)\|_{L^1}+c_1|\mathcal{G}_\delta(\theta_t\omega)|^{\frac{p}{p-p_1}}\|\varphi_1(t)\|^{\frac{p}{p-p_1}}_{L^{\frac{p}{p-p_1}}}+c_2|\mathcal{G}_\delta(\theta_t\omega)|^q\|\varphi_2(t)\|^q_{L^q}\\ &+2\int_{\mathbb{R}^n}g(t,x)u\ \mathrm{d}x.\end{aligned} \tag{2.71}$$

By Young's inequality,

$$2\int_{\mathbb{R}^n}g(t,x)u(x)dx\le\frac{\lambda}{2}\|u\|^2+\frac{2}{\lambda}\|g(t)\|^2. \tag{2.72}$$

Since $\varphi_1\in L^\infty(\mathbb{R},L^{\frac{p}{p-p_1}}(\mathbb{R}^n)),\varphi_2\in L^\infty(\mathbb{R},L^q(\mathbb{R}^n))$, along with (2.71) and (2.72), we get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2+\lambda\|u(t)\|^2+C(n,s,\lambda)\|u(t)\|^2_{H^s(\mathbb{R}^n)}+\beta\|u(t)\|^p_{L^p}\\ \le&2\|\psi_1(t)\|_{L^1}+\frac{2}{\lambda}\|g(t)\|^2+c\gamma(t).\end{aligned} \tag{2.73}$$

Multiplying (2.73) by $e^{\lambda t}$, then integrating over $(\tau-t,\sigma)$, we get

$$\begin{aligned}&\|u(\sigma,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2+C(n,s,\lambda)\int_{\tau-t}^{\sigma}e^{\lambda(s-\sigma)}\|u(s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2_{H^s(\mathbb{R}^n)}\ \mathrm{d}s\\ &+\beta\int_{\tau-t}^{\sigma}e^{\lambda(s-\sigma)}\|u(s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^p_{L^p}\ \mathrm{d}s\\ \le&e^{\lambda(\tau-t-\sigma)}\|u_{\tau-t}\|^2+C_\lambda\int_{\tau-t}^{\sigma}e^{\lambda(s-\sigma)}\alpha(s)\ \mathrm{d}s+c\int_{\tau-t}^{\sigma}e^{\lambda(s-\sigma)}\gamma(s-\tau)\ \mathrm{d}s.\end{aligned} \tag{2.74}$$

After changing variable, we get

$$\begin{aligned}&\|u(\sigma,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2\\ &+C(n,s,\lambda)\int_{-t}^{\sigma-\tau}e^{\lambda(s+\tau-\sigma)}\|u(s+\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2_{H^s(\mathbb{R}^n)}\ \mathrm{d}s\\ &+\beta\int_{-t}^{\sigma-\tau}e^{\lambda(s+\tau-\sigma)}\|u(s+\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^p_{L^p}\ \mathrm{d}s\\ \le&e^{\lambda(\tau-t-\sigma)}\|u_{\tau-t}\|^2+C_\lambda\int_{-\infty}^{\sigma-\tau}e^{\lambda(s+\tau-\sigma)}\alpha(s+\tau)\ \mathrm{d}s+c\int_{-\infty}^{\sigma-\tau}e^{\lambda(s+\tau-\sigma)}\gamma(s)\ \mathrm{d}s.\end{aligned} \tag{2.75}$$

Since $u_{\tau-t}\in\mathrm{D}(\tau-t,\theta_{-t}\omega)$ and D is tempered, it follows that

$$\begin{aligned}&\lim_{t\to+\infty}e^{\lambda(\tau-t-\sigma)}\|u_{\tau-t}\|^2\\ \le&\lim_{t\to+\infty}e^{\lambda(\tau-t-\sigma)}\|\mathrm{D}(\tau-t,\theta_{-t}\omega)\|^2\\ =&0.\end{aligned}$$

Thus, there exists $\mathrm{T} = \mathrm{T}(\tau, \omega, \mathrm{D}, \sigma) > 0$ such that for all $t \geq \mathrm{T}$,

$$e^{\lambda(\tau - t - \sigma)} \|u_{\tau - t}\|^2 \leq 1. \tag{2.76}$$

By (2.10), (2.75) and (2.76), we can get the desired result. □

Now the existence of $\mathcal{D}$-pullback absorbing sets for equations (2.1)-(2.2) is an immediate consequence of Lemma 2.3.

Corollary 2.1. *Suppose* $(2.3)-(2.5), (2.7)-(2.8)$ *hold, in addition,* (2.11) *is assumed. Then the continuous cocycle* Φ *of equations* (2.1)-(2.2) *has a closed measurable* $\mathcal{D}$*-pullback absorbing set* $\mathrm{B} = \{\mathrm{B}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$*, where*

$$\mathrm{B}(\tau, \omega) = \{u \in L^2(\mathbb{R}^n) \|u\|^2 \leq \mathrm{R}(\tau, \omega)\}, \quad \textit{for all} \quad \tau \in \mathbb{R}, \omega \in \Omega, \tag{2.77}$$

$$\mathrm{R}(\tau, \omega) = 1 + C_1 \int_{-\infty}^{0} e^{\lambda s} \big(\alpha(s + \tau) + \gamma(s)\big) \, \mathrm{d}s. \tag{2.78}$$

Proof. In (2.69), letting $\sigma = \tau$, we can get that B pullback attracts all elements in $\mathcal{D}$. Next, we prove that B given in (2.77) is tempered. For $\forall c > 0$, we have

$$\begin{aligned} & e^{ct} \|\mathrm{B}(\tau + t, \theta_t \omega)\|^2 \\ \leq & e^{ct} \mathrm{R}(\tau + t, \theta_t \omega) \\ = & e^{ct} + C_1 e^{ct} \int_{-\infty}^{0} e^{\lambda s} \alpha(s + \tau + t) \, \mathrm{d}s + C_1 e^{ct} \int_{-\infty}^{0} e^{\lambda s} \gamma(s + t) \, \mathrm{d}s. \end{aligned} \tag{2.79}$$

It is obvious that

$$\lim_{t \to -\infty} e^{ct} = 0. \tag{2.80}$$

By (2.11), we have

$$\lim_{t \to -\infty} e^{ct} \int_{-\infty}^{0} e^{\lambda s} \alpha(s + \tau + t) ds = 0. \tag{2.81}$$

Set $r = \min\{\lambda, c\}$, then for $t \leq 0$,

$$\begin{aligned} e^{ct} \int_{-\infty}^{0} e^{\lambda s} \gamma(s + t) \, \mathrm{d}s & \leq \int_{-\infty}^{0} e^{r(s+t)} \gamma(s + t) \, \mathrm{d}s \\ & = \int_{-\infty}^{t} e^{rs} \gamma(s) \, \mathrm{d}s. \end{aligned} \tag{2.82}$$

By (2.13)-(2.14), we have

$$\int_{-\infty}^{0} e^{rs} \gamma(s) \, \mathrm{d}s < \infty,$$

along with (2.82), we get

$$\lim_{t \to -\infty} e^{ct} \int_{-\infty}^{0} e^{\lambda s} \gamma(s) \, \mathrm{d}s = 0. \tag{2.83}$$

By (2.79)-(2.81) and (2.83), we get that B is tempered. Furthermore, for each $\tau \in \mathbb{R}, \mathrm{R}(\tau, \cdot) : \Omega \to \mathbb{R}$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable, hence B is also measurable. By above all, the desired result is obtained. □

Next, we will prove the asymptotic compactness of the solutions. To handle this, we first derive the uniform tail-estimation of solutions in $L^2(\mathbb{R}^n)$.

Lemma 2.4. *Assume* (2.3)-(2.5), (2.7)-(2.8) *and* (2.10). *Then for* $\forall\epsilon > 0, \tau \in \mathbb{R}, \omega \in \Omega, \mathrm{D} \in \mathcal{D}$, *there exists* $T = T(\tau, \omega, \mathrm{D}, \epsilon) > 0$ *and* $\mathrm{B} = \mathrm{B}(\tau, \omega, \mathrm{D}, \epsilon) \geq 1$ *such that for all* $t \geq T$ *and* $k \geq \mathrm{B}$,

$$\int_{|x|\geq k} |u(\tau, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})|^2 \,\mathrm{d}x \leq \epsilon, \tag{2.84}$$

where $u_{\tau-t} \in \mathrm{D}(\tau - t, \theta_{-t}\omega)$.

Proof. Let $\nu(s) = 1 - \mu(s)$ for all $s \in \mathbb{R}^+$, where $\mu(s)$ is the function in (2.18), that is,

$$\nu(s) = 1 - \mu(s) = \begin{cases} 0, & 0 \leq s \leq \frac{1}{2}, \\ 1, & s \geq 1. \end{cases}$$

Multiplying (2.1) by $\nu(\frac{|x|}{k})u$ and integrating over $\mathbb{R}^n$, we get

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n} \nu\Big(\frac{|x|}{k}\Big)|u|^2 \,\mathrm{d}x + \int_{\mathbb{R}^n} \nu\Big(\frac{|x|}{k}\Big)u(-\Delta)^s u \,\mathrm{d}x + \lambda\int_{\mathbb{R}^n} \nu\Big(\frac{|x|}{k}\Big)|u|^2 \,\mathrm{d}x \\ &= \int_{\mathbb{R}^n} I(t,x,u)\nu\Big(\frac{|x|}{k}\Big)u \,\mathrm{d}x. \end{aligned} \tag{2.85}$$

Note that

$$\begin{aligned} &-\int_{\mathbb{R}^n} \nu\Big(\frac{|x|}{k}\Big)u(-\Delta)^s u dx = -\Bigg((-\Delta)^{\frac{s}{2}}u, (-\Delta)^{\frac{s}{2}}\Bigg(\nu\Big(\frac{|x|}{k}\Big)u\Bigg)\Bigg) \\ &= -\frac{1}{2}C(n,s)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\nu(\frac{|x|}{k})|u(x)-u(y)|^2}{|x-y|^{n+2s}} \,\mathrm{d}x\,\mathrm{d}y \\ &\quad -\frac{1}{2}C(n,s)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\Big(\nu(\frac{|x|}{k}) - \nu(\frac{|y|}{k})\Big)\big(u(x)-u(y)\big)u(y)}{|x-y|^{n+2s}} \,\mathrm{d}x\,\mathrm{d}y \\ &\leq -\frac{1}{2}C(n,s)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\Big(\nu(\frac{|x|}{k}) - \nu(\frac{|y|}{k})\Big)\big(u(x)-u(y)\big)u(y)}{|x-y|^{n+2s}} \,\mathrm{d}x\,\mathrm{d}y \\ &\leq \frac{1}{2}C(n,s)\int_{\mathbb{R}^n}\Bigg[\int_{\mathbb{R}^n} \frac{\Big|\Big(\nu(\frac{|x|}{k}) - \nu(\frac{|y|}{k})\Big)\big(u(x)-u(y)\big)u(y)\Big|}{|x-y|^{n+2s}} \,\mathrm{d}x\Bigg]\,\mathrm{d}y \\ &\leq \frac{1}{2}C(n,s)\|u\|\sqrt{\int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n} \frac{\big|\nu(\frac{|x|}{k}) - \nu(\frac{|y|}{k})\big|^2}{|x-y|^{n+2s}} \,\mathrm{d}x\Big)\Big(\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} \,\mathrm{d}x\Big)\,\mathrm{d}y} \\ &\leq c_1\frac{\sqrt{L_1}}{k^s}\|u\| \cdot \|(-\Delta)^{\frac{s}{2}}u\|, \end{aligned}$$

where we have borrowed the result in [12]:

$$\int_{\mathbb{R}^n} \frac{\big|\nu(\frac{|x|}{k}) - \nu(\frac{|y|}{k})\big|^2}{|x-y|^{n+2s}} \,\mathrm{d}x \leq \frac{L_1}{k^{2s}}, \tag{2.86}$$

where L_1 is a positive constant independent of k and $y \in \mathbb{R}^n$. Then we get

$$-\int_{\mathbb{R}^n} \nu\Big(\frac{|x|}{k}\Big)u(-\Delta)^s u \,\mathrm{d}x \leq \frac{c_2}{k^s}\|u\|^2_{H^s(\mathbb{R}^n)}. \tag{2.87}$$

Now we deal with the nonlinear term and the noise term, by (2.3) and (2.7), we have

$$\int_{\mathbb{R}^n} f(t,x,u)\nu\Big(\frac{|x|}{k}\Big)u \ \mathrm{d}x \le -\beta \int_{\mathbb{R}^n} \nu\Big(\frac{|x|}{k}\Big)|u|^p \ \mathrm{d}x + \int_{\mathbb{R}^n} \nu\Big(\frac{|x|}{k}\Big)|\psi_1(t,x)| \ \mathrm{d}x, \quad (2.88)$$

$$\begin{aligned}\mathcal{G}_\delta(\theta_t\omega)\int_{\mathbb{R}^n} h(t,x,u)\nu\Big(\frac{|x|}{k}\Big)u \ \mathrm{d}x \le& \frac{\beta}{2}\int_{\mathbb{R}^n} \nu\Big(\frac{|x|}{k}\Big)|u|^p dx \\ &+ c_1|\mathcal{G}_\delta(\theta_t\omega)|^{\frac{p}{p-p_1}} \int_{\mathbb{R}^n} \nu\Big(\frac{|x|}{k}\Big)|\varphi_1(t,x)|^{\frac{p}{p-p_1}} \ \mathrm{d}x \\ &+ c_2|\mathcal{G}_\delta(\theta_t\omega)|^q \int_{\mathbb{R}^n} \nu\Big(\frac{|x|}{k}\Big)|\varphi_2(t,x)|^q \ \mathrm{d}x.\end{aligned} \quad (2.89)$$

In addition,

$$2\int_{\mathbb{R}^n} \nu\Big(\frac{|x|}{k}\Big)u(x)g(t,x) \ \mathrm{d}x \le \frac{\lambda}{2}\int_{\mathbb{R}^n} \nu\Big(\frac{|x|}{k}\Big)|u|^2 \ \mathrm{d}x + \frac{1}{2\lambda}\int_{\mathbb{R}^n} \nu\Big(\frac{|x|}{k}\Big)|g(t,x)|^2 \ \mathrm{d}x. \quad (2.90)$$

By (2.85), and (2.87)-(2.90), one has

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^n} \nu\Big(\frac{|x|}{k}\Big)|u|^2 \ \mathrm{d}x + \lambda\int_{\mathbb{R}^n} \nu\Big(\frac{|x|}{k}\Big)|u|^2 \ \mathrm{d}x + \beta\int_{\mathbb{R}^n} \nu\Big(\frac{|x|}{k}\Big)|u|^p \ \mathrm{d}x \\ \le& \frac{c_1}{k^s}\|u\|^2_{H^s(\mathbb{R}^n)} + 2\int_{\mathbb{R}^n} \nu\Big(\frac{|x|}{k}\Big)|\psi_1(t,x)| \ \mathrm{d}x + \frac{1}{\lambda}\int_{\mathbb{R}^n} \nu\Big(\frac{|x|}{k}\Big)|g(t,x)|^2 \ \mathrm{d}x \\ &+ c_2|\mathcal{G}_\delta(\theta_t\omega)|^{\frac{p}{p-p_1}} \int_{\mathbb{R}^n} \nu\Big(\frac{|x|}{k}\Big)|\varphi_1(t,x)|^{\frac{p}{p-p_1}} \ \mathrm{d}x \\ &+ c_3|\mathcal{G}_\delta(\theta_t\omega)|^q \int_{\mathbb{R}^n} \nu\Big(\frac{|x|}{k}\Big)|\varphi_2(t,x)|^q \ \mathrm{d}x.\end{aligned} \quad (2.91)$$

Since $s \in (0,1), \forall \epsilon > 0$, there exists $\mathrm{K}_1 = \mathrm{K}_1(\epsilon) \ge 1$, such that for all $k \ge \mathrm{K}_1$,

$$\frac{c_1}{k^s}\|u\|^2_{H^s(\mathbb{R}^n)} \le \epsilon\|u\|^2_{H^s(\mathbb{R}^n)}. \quad (2.92)$$

Note that

$$\begin{aligned}&\int_{\mathbb{R}^n} \nu\Big(\frac{|x|}{k}\Big)|\psi_1(t,x)| \ \mathrm{d}x \\ =& \int_{|x|\le\frac{1}{2}k} \nu\Big(\frac{|x|}{k}\Big)|\psi_1(t,x)| \ \mathrm{d}x + \int_{|x|\ge\frac{1}{2}k} \nu\Big(\frac{|x|}{k}\Big)|\psi_1(t,x)| \ \mathrm{d}x \\ \le& \int_{|x|\ge\frac{1}{2}k} |\psi_1(t,x)| \ \mathrm{d}x.\end{aligned} \quad (2.93)$$

Similarly, we get

$$\int_{\mathbb{R}^n} \nu\Big(\frac{|x|}{k}\Big)|g(t,x)|^2 \ \mathrm{d}x \le \int_{|x|\ge\frac{1}{2}k} |g(t,x)|^2 \ \mathrm{d}x. \quad (2.94)$$

$$\begin{aligned}&|\mathcal{G}_\delta(\theta_t\omega)|^{\frac{p}{p-p_1}} \int_{\mathbb{R}^n} \nu\Big(\frac{|x|}{k}\Big)|\varphi_1(t,x)|^{\frac{p}{p-p_1}} \ \mathrm{d}x + |\mathcal{G}_\delta(\theta_t\omega)|^q \int_{\mathbb{R}^n} \nu\Big(\frac{|x|}{k}\Big)|\varphi_2(t,x)|^q \ \mathrm{d}x \\ \le& |\mathcal{G}_\delta(\theta_t\omega)|^{\frac{p}{p-p_1}} \int_{|x|\ge\frac{1}{2}k} |\varphi_1(t,x)|^{\frac{p}{p-p_1}} \ \mathrm{d}x + |\mathcal{G}_\delta(\theta_t\omega)|^q \int_{|x|\ge\frac{1}{2}k} |\varphi_2(t,x)|^q \ \mathrm{d}x.\end{aligned} \quad (2.95)$$

By (2.91)-(2.95), we get for all $k \geq \mathrm{K}_1$,

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^n}\nu\Big(\frac{|x|}{k}\Big)|u|^2\ \mathrm{d}x+\lambda\int_{\mathbb{R}^n}\nu\Big(\frac{|x|}{k}\Big)|u|^2\ \mathrm{d}x\\ \leq&\epsilon\|u\|^2_{H^s(\mathbb{R}^n)}+c_1\int_{|x|\geq\frac{1}{2}k}(|\psi_1(t,x)|+|g(t,x)|^2)\ \mathrm{d}x\\ &+c_2|\mathcal{G}_\delta(\theta_t\omega)|^{\frac{p}{p-p_1}}\int_{|x|\geq\frac{1}{2}k}|\varphi_1(t,x)|^{\frac{p}{p-p_1}}\ \mathrm{d}x+c_3|\mathcal{G}_\delta(\theta_t\omega)|^q\int_{|x|\geq\frac{1}{2}k}|\varphi_2(t,x)|^q\ \mathrm{d}x.\end{aligned}\tag{2.96}$$

Given $t \in \mathbb{R}^+, \tau \in \mathbb{R}, \omega \in \Omega$, solving $\int_{\mathbb{R}^n}\nu(\frac{|x|}{k})|u|^2dx$ by Gronwall's Lemma and after changing variable, we get

$$\begin{aligned}&\int_{\mathbb{R}^n}\nu\Big(\frac{|x|}{k}\Big)|u(\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})|^2\ \mathrm{d}x\\ \leq&e^{-\lambda t}\|u_{\tau-t}\|^2+\epsilon\int_{-t}^0e^{\lambda s}\|u(s+\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2_{H^s(\mathbb{R}^n)}\ \mathrm{d}s\\ &+c_1\int_{-t}^0e^{\lambda s}\int_{|x|\geq\frac{1}{2}k}(|\psi_1(s+\tau,x)|+|g(s+\tau,x)|^2)\ \mathrm{d}x\ \mathrm{d}s\\ &+c_2\int_{-t}^0e^{\lambda s}|\mathcal{G}_\delta(\theta_s\omega)|^{\frac{p}{p-p_1}}\int_{|x|\geq\frac{1}{2}k}|\varphi_1(s+\tau,x)|^{\frac{p}{p-p_1}}\ \mathrm{d}x\ \mathrm{d}s\\ &+c_3\int_{-t}^0e^{\lambda s}|\mathcal{G}_\delta(\theta_s\omega)|^q\int_{|x|\geq\frac{1}{2}k}|\varphi_2(s+\tau,x)|^q\ \mathrm{d}x\ \mathrm{d}s\end{aligned}\tag{2.97}$$

Since $u_{\tau-t} \in \mathrm{D}(\tau-t, \theta_{-t}\omega)$ and D is tempered, we have

$$\lim_{t\to+\infty}e^{-\lambda t}\|u_{\tau-t}\|^2\leq\lim_{t\to+\infty}e^{-\lambda t}\|\mathrm{D}_{\tau-t}\|^2=0,$$

thus there exists $\mathrm{T}_1 = \mathrm{T}_1(\tau, \omega, \mathrm{D}, \epsilon) > 1$ such that for all $t \geq \mathrm{T}_1$,

$$e^{-\lambda t}\|u_{\tau-t}\|^2\leq\epsilon.\tag{2.98}$$

In Lemma 2.3, setting $\sigma = \tau$, we have that there exists $\mathrm{T}_2 = \mathrm{T}_2(\tau, \omega, \mathrm{D}, \epsilon) \geq \mathrm{T}_1$ such that for all $t \geq \mathrm{T}_2$,

$$\epsilon\int_{-t}^0e^{\lambda s}\|u(s+\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2_{H^s(\mathbb{R}^n)}\ \mathrm{d}s\leq\epsilon\mathrm{R}(\tau,\omega),\tag{2.99}$$

where $\mathrm{R}(\tau, \omega)$ is the same as in (2.78). By (2.10), there exists $\mathrm{K}_2 = \mathrm{K}_2(\tau, \omega, \epsilon) \geq \mathrm{K}_1$ such that for all $k \geq \mathrm{K}_2$,

$$\int_{-\infty}^0e^{\lambda s}\int_{|x|\geq\frac{1}{2}k}(|\psi_1(s+\tau,x)|+|g(s+\tau,x)|^2)\ \mathrm{d}x\ \mathrm{d}s\leq\epsilon.\tag{2.100}$$

By (2.13), (2.14) and $\varphi_1 \in L^{\frac{p}{p-p_1}}_{loc}(\mathbb{R}, L^{\frac{p}{p-p_1}}(\mathbb{R}^n)), \varphi_2 \in L^q_{loc}(\mathbb{R}, L^q(\mathbb{R}^n))$, we have

$$\begin{aligned}&\int_{-\infty}^0e^{\lambda s}\Big(|\mathcal{G}_\delta(\theta_s\omega)|^{\frac{p}{p-p_1}}\int_{|x|\geq\frac{1}{2}k}|\varphi_1(s+\tau,x)|^{\frac{p}{p-p_1}}\ \mathrm{d}x\\ &+|\mathcal{G}_\delta(\theta_s\omega)|^q\int_{|x|\geq\frac{1}{2}k}|\varphi_2(s+\tau,x)|^q\ \mathrm{d}x\Big)\ \mathrm{d}s<\infty,\end{aligned}$$

which implies that there exists $\mathrm{K}_3 = \mathrm{K}_3(\tau, \omega, \lambda, \epsilon) \geq \mathrm{K}_2$ such that for all $k \geq \mathrm{K}_3$,

$$\begin{aligned}&\int_{-\infty}^{0} e^{\lambda s}\Big(|\mathcal{G}_\delta(\theta_s\omega)|^{\frac{p}{p-p_1}} \int_{|x|\geq\frac{1}{2}k} |\varphi_1(s+\tau,x)|^{\frac{p}{p-p_1}}\ \mathrm{d}x \\ &+ |\mathcal{G}_\delta(\theta_s\omega)|^{q} \int_{|x|\geq\frac{1}{2}k} |\varphi_2(s+\tau,x)|^{q}\ \mathrm{d}x\Big)\ \mathrm{d}s \leq \epsilon.\end{aligned} \tag{2.101}$$

Therefore, from (2.97)-(2.101) , we get that for all $t \geq \mathrm{T}_2, k \geq \mathrm{K}_3$,

$$\int_{|x|\geq\frac{1}{2}k} |u(\tau, \tau-t, \theta_{-\tau}\omega, u_{\tau-t})|^2\ \mathrm{d}x \leq 3\epsilon + \epsilon\mathrm{R}(\tau,\omega),$$

which completes the proof. □

Next, we derive uniform estimates of solutions in $H^s(\mathbb{R}^n)$. We further assume that $\psi_4, \varphi_3 \in L^\infty(\mathbb{R}, L^\infty(\mathbb{R}^n))$.

Lemma 2.5. *Assume* (2.5)-(2.6), (2.8)-(2.9) *and* (2.10). *Then for all* $\tau \in \mathbb{R}, \omega \in \Omega, \mathrm{D} \in \mathcal{D}$, *there exists* $T = T(\tau, \omega, \mathrm{D}) > 0$ *such that for all* $t \geq T$, *the solution* u *of equations* (2.1)-(2.2) *satisfies*

$$\|u(\tau, \tau-t, \theta_{-\tau}\omega, u_{\tau-t})\|^2_{H^s(\mathbb{R}^n)} \leq \mathrm{C}_2 + \mathrm{C}_2 \int_{-\infty}^{0} e^{\lambda s}\big(\alpha(s+\tau) + \gamma(s)\big)\ \mathrm{d}s, \tag{2.102}$$

where $u_{\tau-t} \in \mathrm{D}(\tau-t, \theta_{-t}\omega)$ *and* C_2 *is a positive constant independent of* τ, ω *and* D.

Proof. Multiplying (2.1) by $(-\Delta)^s u$ and then integrating over $\mathbb{R}^n$, we get

$$\frac{d}{dt}\|(-\Delta)^{\frac{s}{2}}u\|^2 + 2\|(-\Delta)^s u\|^2 + 2\lambda\|(-\Delta)^{\frac{s}{2}}u\|^2 = 2\big(I(t,x,u), (-\Delta)^s u\big). \tag{2.103}$$

First, we deal with the nonlinear term, by (2.5) and (2.6), we have

$$\begin{aligned}&2(f(t,x,u), (-\Delta)^s u) \\ =&2((-\Delta)^{\frac{s}{2}} f, (-\Delta)^{\frac{s}{2}} u) \\ =&C(n,s)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(f(t,x,u(x)) - f(t,y,u(x))(u(x)-u(y))}{|x-y|^{n+2s}}\ \mathrm{d}x\ \mathrm{d}y \\ &+ C(n,s)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(f(t,y,u(x)) - f(t,y,u(y))(u(x)-u(y))}{|x-y|^{n+2s}}\ \mathrm{d}x\ \mathrm{d}y \\ \leq&C(n,s)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\psi_5(x)-\psi_5(y)||u(x)-u(y)|}{|x-y|^{n+2s}}\ \mathrm{d}x\ \mathrm{d}y \\ &+ C(n,s)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\psi_4(t,y)(u(x)-u(y))^2}{|x-y|^{n+2s}}\ \mathrm{d}x\ \mathrm{d}y \\ \leq&\|\psi_5\|^2_{H^s(\mathbb{R}^n)} + (1 + 2\|\psi_4\|_{L^\infty(\mathbb{R}, L^\infty(\mathbb{R}^n))})\|(-\Delta)^{\frac{s}{2}}u\|^2.\end{aligned} \tag{2.104}$$

Similarly, for the noise term, we have

$$\begin{aligned}&2\mathcal{G}_\delta(\theta_t\omega)\big(h(t,x,u), (-\Delta)^s u\big) \\ \leq&|\mathcal{G}_\delta(\theta_t\omega)|\|\varphi_4\|^2_{H^s(\mathbb{R}^n)} + |\mathcal{G}_\delta(\theta_t\omega)|(1 + 2\|\varphi_3\|_{L^\infty(\mathbb{R}, L^\infty(\mathbb{R}^n))})\|(-\Delta)^{\frac{s}{2}}u\|^2.\end{aligned} \tag{2.105}$$

In addition, we have

$$2(g(t,x),(-\Delta)^s u) \leq \|g(t)\|^2 + \|(-\Delta)^s u\|^2. \tag{2.106}$$

By (2.103)-(2.106), we get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|(-\Delta)^{\frac{s}{2}}u\|^2 + 2\|(-\Delta)^s u\|^2 + 2\lambda\|(-\Delta)^{\frac{s}{2}}u\|^2\\ \leq&(1 + 2\|\psi_4\|_{L^\infty(\mathbb{R},L^\infty(\mathbb{R}^n))} + |\mathcal{G}_\delta(\theta_t\omega)| + 2|\mathcal{G}_\delta(\theta_t\omega)|\|\varphi_3\|_{L^\infty(\mathbb{R},L^\infty(\mathbb{R}^n))})\|(-\Delta)^{\frac{s}{2}}u\|^2\\ &+ \|\psi_5\|^2_{H^s(\mathbb{R}^n)} + \|g(t)\|^2 + |\mathcal{G}_\delta(\theta_t\omega)|\|\varphi_4\|^2_{H^s(\mathbb{R}^n)}.\end{aligned} \tag{2.107}$$

Since $\mathcal{G}_\delta(\theta_t\omega)$ is continuous in t for fixed ω, and to apply Lemma 2.3, we choose c_1, c_2, such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|(-\Delta)^{\frac{s}{2}}u\|^2 + \lambda\|(-\Delta)^{\frac{s}{2}}u\|^2 \leq c_1\|(-\Delta)^{\frac{s}{2}}u\|^2 + c_2 + \|g(t)\|^2. \tag{2.108}$$

Given $t \in \mathbb{R}^+, \tau \in \mathbb{R}, \omega \in \Omega$, letting $s \in (\tau-1,\tau)$, multiplying (2.108) by $e^{\lambda t}$ and then integrating over (s,τ), we get

$$\begin{aligned}&\|(-\Delta)^{\frac{s}{2}}u(\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2\\ \leq&e^{\lambda(s-\tau)}\|(-\Delta)^{\frac{s}{2}}u(s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2\\ &+ c_1\int_s^\tau e^{\lambda(\xi-\tau)}\|(-\Delta)^{\frac{s}{2}}u(\xi,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2 \,\mathrm{d}\xi\\ &+ \int_s^\tau e^{\lambda(\xi-\tau)}(c_2 + \|g(\xi)\|^2) \,\mathrm{d}\xi\\ \leq&e^{\lambda(s-\tau)}\|(-\Delta)^{\frac{s}{2}}u(s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2\\ &+ c_1\int_{\tau-1}^\tau e^{\lambda(\xi-\tau)}\|(-\Delta)^{\frac{s}{2}}u(\xi,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2 \,\mathrm{d}\xi\\ &+ \int_{\tau-1}^\tau e^{\lambda(\xi-\tau)}(c_2 + \|g(\xi)\|^2) \,\mathrm{d}\xi.\end{aligned}$$

Then integrating with respect to s over $(\tau-1,\tau)$, we obtain

$$\begin{aligned}&\|(-\Delta)^{\frac{s}{2}}u(\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2\\ \leq&\int_{\tau-1}^\tau e^{\lambda(s-\tau)}\|(-\Delta)^{\frac{s}{2}}u(s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2 \,\mathrm{d}s\\ &+ c_1\int_{\tau-1}^\tau e^{\lambda(\xi-\tau)}\|(-\Delta)^{\frac{s}{2}}u(\xi,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2 \,\mathrm{d}\xi\\ &+ \int_{\tau-1}^\tau e^{\lambda(\xi-\tau)}(c_2 + \|g(\xi)\|^2) \,\mathrm{d}\xi.\end{aligned}$$

After changing the variables, we have

$$\begin{aligned}&\|(-\Delta)^{\frac{s}{2}}u(\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2\\ \leq&\int_{-1}^0 e^{\lambda s}\|(-\Delta)^{\frac{s}{2}}u(s+\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|^2 \,\mathrm{d}s\end{aligned}$$

$$+ c_1 \int_{-1}^{0} e^{\lambda\xi} \|(-\Delta)^{\frac{s}{2}} u(\xi+\tau, \tau-t, \theta_{-\tau}\omega, u_{\tau-t})\|^2 \, \mathrm{d}\xi$$
$$+ \int_{-1}^{0} e^{\lambda\xi}(c_2 + \|g(\xi+\tau)\|^2) \, \mathrm{d}\xi, \tag{2.109}$$

which is combined with Lemma 2.3, we obtain the desired result. □

Finally, we prove the asymptotic compactness of the solutions.

Lemma 2.6. *Assume* (2.3)-(2.9) *and* (2.11). *Then for all* $\tau \in \mathbb{R}, \omega \in \Omega, \mathrm{D} \in \mathcal{D}$, *the sequence* $\{\Phi(t_n, \tau-t_n, \theta_{-t_n}\omega, u_{0,n})\}_{n=1}^{\infty}$ *has a convergent subsequence in* $L^2(\mathbb{R}^n)$ *as* $t_n \to \infty$ *and* $u_{0,n} \in \mathrm{D}(\tau-t_n)$.

Proof. $\forall \epsilon > 0$, we need to show that $\{\Phi(t_n, \tau-t_n, \theta_{-t_n}\omega, u_{0,n})\}_{n=1}^{\infty}$ has a finite cover of balls with radius ϵ in $L^2(\mathbb{R}^n)$. By (2.66) we have

$$\Phi(t_n, \tau-t_n, \theta_{-t_n}\omega, u_{0,n}) = u(\tau, \tau-t_n, \theta_{\tau}\omega, u_{0,n}). \tag{2.110}$$

By the uniform tail-estimation of solutions, there exist $\mathrm{T}_1 = \mathrm{T}_1(\tau, \omega, \mathrm{D}, \epsilon) > 0$ and $\mathrm{K} = \mathrm{K}(\tau, \omega, \mathrm{D}, \epsilon) \geq 1$, such that for all $t \geq \mathrm{T}_1$,

$$\int_{|x|\geq k} |u(\tau, \tau-t, \theta_{-\tau}\omega, u_0)|^2(x) dx < \frac{\epsilon}{8}. \tag{2.111}$$

On the other hand, by the uniform estimates of solutions in $H^s(\mathbb{R}^n)$, there exists $\mathrm{T}_2 = \mathrm{T}_2(\tau, \omega, \mathrm{D}, \epsilon) \geq \mathrm{T}_1$ and $c(\tau, \omega) > 0$ such that for all $t \geq \mathrm{T}_2$,

$$\|u(\tau, \tau-t, \theta_{-\tau}\omega, u_0)\|^2_{H^s(\mathcal{O}_k)} \leq c(\tau, \omega).$$

Since $t_n \to \infty$, there exists $\mathrm{N} = \mathrm{N}(\tau, \omega, \mathrm{D}, \epsilon) \geq 1$, such that for all $n \geq \mathrm{N}, t_n \geq \mathrm{T}_2$,

$$\|u(\tau, \tau-t_n, \theta_{-\tau}\omega, u_0)\|^2_{H^s(\mathcal{O}_k)} \leq c(\tau, \omega), \tag{2.112}$$

where $u_0 \in \mathrm{D}(\tau-t_n)$. Since $u_{0,n} \in \mathrm{D}(\tau-t_n)$, we get for all $n \geq \mathrm{N}$,

$$\|u(\tau, \tau-t_n, \theta_{-\tau}\omega, u_{0,n})\|^2_{H^s(\mathcal{O}_k)} \leq c(\tau, \omega). \tag{2.113}$$

By the compactness of embedding $H^s(\mathcal{O}_k) \hookrightarrow L^2(\mathcal{O}_k)$, we find that $\{u(\tau, \tau-t_n, \theta_{-\tau}\omega, u_{0,n})\}_{n=1}^{\infty}$ is precompact in $L^2(\mathcal{O}_k)$. This shows that $\{u(\tau, \tau-t_n, \theta_{-\tau}\omega, u_{0,n})\}_{n=1}^{\infty}$ has a finite cover of balls with radius $\frac{\epsilon}{4}$ in $L^2(\mathcal{O}_k)$. Combined with (2.111), we get $\{u(\tau, \tau-t_n, \theta_{-\tau}\omega, u_{0,n})\}_{n=1}^{\infty}$ has a finite cover of balls with radius ϵ in $L^2(\mathbb{R}^n)$. This completes the proof. □

3. Upper semi-continuity of attractors for multiplicative noise

In present section, for the case of linear multiplicative noise, we focus on the upper semi-continuity of attractors for the Wong-Zakai approximation as $\delta \to 0$.

3.1. Equation driven by white noise

This subsection concerns about the fractional reaction-diffusion equation (1.1) when $h(t, x, u) = u$:

$$\frac{\partial u}{\partial t} + (-\Delta)^s u + \lambda u = f(t, x, u) + g(t, x) + u \circ \frac{dW}{dt}, \quad t > \tau, \quad x \in \mathbb{R}^n, \tag{3.1}$$

with initial condition

$$u(\tau, x) = u_\tau(x), \quad x \in \mathbb{R}^n. \tag{3.2}$$

By the standard process, we introduce a new variable:

$$v(t, \tau, \omega, v_\tau) = e^{-\omega(t)} u(t, \tau, \omega, u_\tau). \tag{3.3}$$

Then we get

$$\frac{\partial v}{\partial t} + (-\Delta)^s v + \lambda v = e^{-\omega(t)} f(t, x, e^{\omega(t)} v) + e^{-\omega(t)} g(t, x), \tag{3.4}$$

$$v(\tau, x) = v_\tau(x), \quad x \in \mathbb{R}^n, \tag{3.5}$$

where $v_\tau(x) = e^{-\omega(\tau)} u_\tau(x)$.

By the transform (3.3), we could do some computation which is similar to [12] but easier, to get the following result: First, equation (3.4) and (3.5) admit a unique solution v with $v(\cdot, \tau, \omega, v_\tau) \in C\big([\tau, \infty); H^s(\mathbb{R}^n)\big) \cap L^2_{loc}\big([\tau, \infty); L^2(\mathbb{R}^n)\big)$ which is continuous in v_τ and $\Big(\mathcal{F}, \mathcal{B}\big(L^2(\mathbb{R}^n)\big)\Big)$ -measurable in ω. Thus, a continuous cocycle Φ_1 could be defined as follows:

$$\begin{aligned} &\Phi_1(t, \tau, \omega, u_\tau) \\ =& u(t + \tau, \tau, \theta_{-\tau}\omega, u_\tau) \\ =& e^{\omega(t) - \omega(-\tau)} v(t + \tau, \tau, \theta_{-\tau}\omega, v_\tau), \end{aligned} \tag{3.6}$$

for all $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$, $\omega \in \Omega$. Additionally the cocycle Φ_1 admits a unique random attractor $\mathcal{A}_1$.

For our later usage in proving the convergence of solutions, some necessary results are listed as follows:

Lemma 3.1. *Assume* (2.3)-(2.5) *and* (2.10)*. Then for all $\sigma \in \mathbb{R}$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, $D \in \mathcal{D}$, there exists $T = T(\tau, \omega, D, \sigma) > 0$ such that for all $t \geq T$, the solution u of equations* (3.1)-(3.2) *satisfies*

$$\begin{aligned} & e^{-2\omega(\sigma-\tau)} \|u(\sigma, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})\|^2 \\ & + \int_{-t}^{\sigma-\tau} e^{\lambda(s+\tau-\sigma) - 2\omega(s)} \|u(s + \tau, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})\|^2_{H^s(\mathbb{R}^n)} \, \mathrm{d}s \\ \leq & 1 + C_1 \int_{-\infty}^{\sigma-\tau} e^{\lambda(s+\tau-\sigma) - 2\omega(s)} \alpha(s + \tau) \, \mathrm{d}s, \end{aligned} \tag{3.7}$$

where $u_{\tau-t} \in D(\tau - t, \theta_{-t}\omega)$ and C_1 is a positive constant independent of σ, τ, ω, and D.

From Lemma 3.1, one immediately has the following two results.

Corollary 3.1. *Suppose* (2.3)-(2.5) *hold. Also,* (2.11) *is assumed. Then the cocycle* Φ_1 *of equations* (3.1)-(3.2) *has a closed measurable* $\mathcal{D}$*-pullback absorbing set* $B_1 = \{B_1(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$*, where*

$$B_1(\tau, w) = \{u \in L^2(\mathbb{R}^n) : \|u\|^2 \le R_1(\tau,\omega)\}, \quad \text{for all} \quad \tau \in \mathbb{R}, \quad \omega \in \Omega, \tag{3.8}$$

$$R_1(\tau,\omega) = 1 + C_2 \int_{-\infty}^{0} e^{\lambda s - 2\omega(s)} \alpha(s+\tau)\ \mathrm{d}s. \tag{3.9}$$

Corollary 3.2. *Suppose* (2.3)-(2.5) *hold. In addition,* (2.10) *is assumed. Then for all* $\tau \in \mathbb{R}$*,* $\omega \in \Omega$*, and* $T > 0$*, there exists* $c = c(\tau,\omega,T) > 0$ *such that for all* $t \in [\tau, \tau+T]$*, the solution* u *of* (3.1)-(3.2) *satisfies*

$$\|u(t,\tau,\omega,u_\tau)\|^2 + \int_\tau^t \|u(s,\tau,\omega,u_\tau)\|_{L^p}^p\ \mathrm{d}s \le c\|u_\tau\|^2 + c\int_\tau^t \alpha(s+\tau)\mathrm{d}s. \tag{3.10}$$

Corollary 3.2 is an essential estimate in proving the convergence of solutions.

3.2. Equations driven by colour noise

Now, we consider the following approximated equation of equation (3.1):

$$\frac{\partial u_\delta}{\partial t} + (-\Delta)^s u_\delta + \lambda u_\delta = f(t,x,u_\delta) + g(t,x) + u_\delta \mathcal{G}_\delta(\theta_t\omega), \quad t > \tau, \quad x \in \mathbb{R}^n, \tag{3.11}$$

with initial condition

$$u_\delta(\tau, x) = u_{\delta,\tau}(x), \quad x \in \mathbb{R}^n. \tag{3.12}$$

By Section 2, we know that for any $\delta \neq 0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $u_{\delta,\tau} \in L^2(\mathbb{R}^n)$, equations (3.11) and (3.12) have a unique solution $u_\delta(t,\tau,\omega,u_{\delta,\tau})$ which is $\Big(\mathcal{F}, \mathcal{B}\big(L^2(\mathbb{R}^n)\big)\Big)$-measurable in ω and continuous in initial data $u_{\delta,\tau}$ in $L^2(\mathbb{R}^n)$, thus, we can define a continuous cocycle Φ_δ^1 for (3.11) and (3.12). Moreover, the cocycle Φ_δ^1 has a unique $\mathcal{D}$-pullback attractor $\mathcal{A}_\delta^1 = \{A_\delta^1(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ in $L^2(\mathbb{R}^n)$.

Next, the main result in this section is placed here:

Theorem 3.1. *Suppose* (2.3)-(2.6), (2.10) *and* (2.11) *hold. Then for all* $\tau \in \mathbb{R}$*,* $\omega \in \Omega$*,*

$$\lim_{\delta \to 0} d_{L^2(\mathbb{R}^n)}\big(\mathcal{A}_\delta^1(\tau,\omega), \mathcal{A}_1(\tau,\omega)\big) = 0. \tag{3.13}$$

To prove this theorem, we firstly prove the convergence of solutions u_δ to solutions u. Thus, similar to (3.3), also by (2.15), we introduce another variable:

$$v_\delta(t,\tau,\omega,v_{\delta,\tau}) = e^{-\int_0^t \mathcal{G}_\delta(\theta_s\omega)\ \mathrm{d}s} u_\delta(t,\tau,\omega,u_{\delta,\tau}). \tag{3.14}$$

Then (3.11) and (3.12) can be rewritten as

$$\frac{\partial v_\delta}{\partial t} + (-\Delta)^s v_\delta + \lambda v_\delta = e^{-\int_0^t \mathcal{G}_\delta(\theta_s\omega)\ \mathrm{d}s} f(t,x,e^{\int_0^t \mathcal{G}_\delta(\theta_s\omega)\ \mathrm{d}s} v_\delta) + e^{-\int_0^t \mathcal{G}_\delta(\theta_s\omega)\ \mathrm{d}s} g(t,x), \tag{3.15}$$

with initial condition

$$v_\delta(\tau,x) = v_{\delta,\tau}(x), \quad x \in \mathbb{R}^n, \tag{3.16}$$

where $v_{\delta,\tau}(x) = e^{-\int_0^\tau \mathcal{G}_\delta(\theta_s\omega)\ \mathrm{d}s} u_{\delta,\tau}(x)$.

The remaining part of the proof is a consequence of the results in Section 2, thus we list necessary lemmas and omit the details of proof.

Lemma 3.2. *Assume* (2.3)-(2.5) *and* (2.10)*. Then for all* $\tau \in \mathbb{R}$, $\omega \in \Omega$, *there exist* $\delta_0 = \delta_0(\tau, \omega, T) > 0$, $c = c(\tau, \omega, T) > 0$, *such that for all* $0 < |\delta| < \delta_0$, $t \in [\tau, \tau + T]$, *the solution* u_δ *of* (3.11) *and* (3.12) *satisfies*

$$\|u_\delta(t, \tau, \omega, u_{\delta,\tau})\|^2 + \int_\tau^t \|u_\delta(s, \tau, \omega, u_{\delta,\tau})\|^2_{H^s(\mathbb{R}^n)} \, \mathrm{d}s + \int_\tau^t \|u_\delta(s, \tau, \omega, u_{\tau-t})\|^p_{L^p} \, \mathrm{d}s$$
$$\leq c\|u_{\delta,\tau}\|^2 + c\int_\tau^t \alpha(s)\mathrm{d}s. \tag{3.17}$$

Lemma 3.3. *Assume* (2.3)-(2.5), (2.10) *. Then for any* $\delta \neq 0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ *and* $D \in \mathcal{D}$, *there exists* $T = T(\tau, \omega, D, \delta) > 0$, *such that for all* $t \geq T$, *the solution* u_δ *of* (3.11) *satisfies*

$$\begin{aligned} &\|u_\delta(\tau, \tau - t, \theta_{-\tau}\omega, u_{\delta,\tau-t})\|^2 \\ &+ \int_{-t}^0 e^{\lambda s + 2\int_s^0 \mathcal{G}_\delta(\theta_r\omega)\ \mathrm{d}r} \|u_\delta(s + \tau, \tau - t, \theta_{-\tau}\omega, u_{\delta,\tau-t})\|^2_{H^s(\mathbb{R}^n)} \, \mathrm{d}s \\ &\leq 1 + C_3 \int_0^{-\infty} e^{\lambda s + 2\int_s^0 \mathcal{G}_\delta(\theta_r\omega)\ \mathrm{d}r} \alpha(s + \tau) \, \mathrm{d}s, \end{aligned} \tag{3.18}$$

where $u_{\delta,\tau-t} \in D(\tau - t, \theta_{-t}\omega)$ *and* C_3 *is a positive constant independent of* τ, ω, D, δ.

From Lemma 3.3 and Lemma 3.7 in [14], we get the following results immediately.

Corollary 3.3. *Suppose* (2.3)-(2.5), (2.10) *and* (2.11) *hold. Then* Φ^1_δ *of equations* (3.11) *and* (3.12) *has a closed measurable* $\mathcal{D}$*-pullback absorbing set* $B^1_\delta = \{B^1_\delta(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, *where*

$$B^1_\delta(\tau, \omega) = \{u_\delta \in L^2(\mathbb{R}^n) : \|u_\delta\|^2 \leq R^1_\delta(\tau, \omega)\}, \quad \text{for all} \ \ \tau \in \mathbb{R}, \ \omega \in \Omega. \tag{3.19}$$

$$R^1_\delta(\tau, \omega) = 1 + C_4 \int_{-\infty}^0 e^{\lambda s + 2\int_s^0 \mathcal{G}_\delta(\theta_r\omega)\ \mathrm{d}r} \alpha(s + \tau) \, \mathrm{d}s. \tag{3.20}$$

Moreover, we have for all $\tau \in \mathbb{R}$, $\omega \in \Omega$,

$$\lim_{\delta \to 0} R^1_\delta(\tau, \omega) = R_1(\tau, \omega), \tag{3.21}$$

where $R_1(\tau, \omega)$ *is defined as in* (3.9).

Lemma 3.4. *Suppose* (2.3)-(2.5) *and* (2.10) *hold. Then for* $\forall \varepsilon > 0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, *there exists* $\delta_0 = \delta_0(\omega) > 0$, $T = T(\tau, \omega, \varepsilon) > 0$ *and* $K = K(\tau, \omega, \varepsilon) \geq 1$, *such that for all* $0 < |\delta| < \delta_0$, $t \geq T$, *and* $k \geq K$,

$$\int_{|x| \geq k} |u_\delta(\tau, \tau - t, \theta_{-\tau}\omega, u_{\delta,\tau-t})|^2 \, \mathrm{d}x \leq \varepsilon, \tag{3.22}$$

where $u_{\delta,\tau-t} \in B^1_\delta(\tau - t, \theta_{-\tau}\omega)$ *with* B^1_δ *given by* (3.19).

As for the uniform compactness of $\mathcal{A}^1_\delta$, we have the following result similar to Lemma 4.8 in [15].

Lemma 3.5. *Suppose* (2.3)-(2.6), (2.10) *and* (2.11) *hold. Then for all* $\tau \in \mathbb{R}$, $\omega \in \Omega$, *if* $\delta_n \to 0$ *and* $u_n \in \mathcal{A}_{\delta_n}(\tau, \omega)$, *then* $\{u_n\}_{n=1}^\infty$ *is precompact in* $L^2(\mathbb{R}^n)$.

At the end of this section, we establish the convergence of solutions, which is crucial in proving the upper semi-continuity of attractors.

Lemma 3.6. *Suppose* (2.3)-(2.5) *hold. Then for* $\forall \varepsilon > 0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, $T > 0$, *there exist* $\delta_0 = \delta_0(\tau, \omega, T, \varepsilon) > 0$ *and* $c = c(\tau, \omega, T) > 0$ *such that for all* $0 < |\delta| < \delta_0$ *and* $t \in [\tau, \tau + T]$,

$$\begin{aligned}
&\|u_\delta(t, \tau, \omega, u_{\delta,\tau}) - u(t, \tau, \omega, u_\tau)\|^2 \\
\le& c\|u_{\delta,\tau} - u_\tau\|^2 \\
&+ c\varepsilon\left(1 + \|u_{\delta,\tau}\|^2 + \|u_\tau\|^2 + \int_\tau^t \left(\|\psi_3(s)\|_{L^q}^q + \|\psi_1(s)\|_{L^1} + \|g(s)\|^2\right) \mathrm{d}s\right).
\end{aligned} \tag{3.23}$$

By checking that Corollary 3.3, Lemma 3.5 and Lemma 3.6 satisfy Proposition 2.2 in [15], the proof of the upper semi-continuity of the $\mathcal{A}_\delta^1$ is completed.

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