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DYNAMICS OF AN IMPULSIVE STOCHASTIC SIR EPIDEMIC MODEL WITH SATURATED INCIDENCE RATE

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Abstract In this paper, the dynamics of an impulsive stochastic SIR epidemic model with saturated incidence rate are analyzed. The existence and uniqueness of the global positive solution is proved by constructing the equivalent system without pulses. The threshold which determines the extinction and persistence of the disease is obtained. The global attraction of disease-free periodic solution is addressed. Sufficient condition for the existence of a positive periodic solution is established. These results are supported by computer simulations.

Keywords SIR epidemic model, stochastic perturbation, impulsive effect, periodic solution, global attraction.

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1. Introduction

From UK, South America to India, pulse vaccination strategy (PVS) has been widely used as a powerful way to eliminate infectious diseases [1, 18]. PVS is a series of periodic vaccinations applied to susceptible group in a very short period of time [18]. Comparing with routine constant vaccination strategy, theoretical studies suggest that PVS can largely reduce the incidence of disease at lower vaccination rates because it keeps the average number of the susceptible during vaccination interval below the epidemic threshold [33]. Mathematical analysis of PVS begins with Agur et al [1], further investigations can be refered in [6, 7, 11, 14, 20, 28, 30, 33].

Epidemic models in aforementioned papers are all described by the ordinary differential equations. However, environmental noises are ubiquitous in real world and can induce different dynamics in real system [2, 3, 5, 8–10, 34, 35]. Therefore researchers have shown great interest in stochastic epidemic models incorporated with white noises, colore noises or Lévy noises [4, 12, 21–23, 39, 40]. The investigations demonstrate that environmental noises can help to suppress the disease and change the basic reproduction number of the disease [12, 39]. Although we can find intensive studies in impulsive stochastic population models [24, 25, 31, 32, 41], there are few papers about impulsive stochastic epidemic models [13, 36, 37]. And none of them gives the threshold which determines the extinction and persistence of the disease

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because the hybrid of stochastic perturbation and impulsive effects adds an extra level of complexity to deal with. To explore the effect of white noises and inspired by above works, we will study a stochastic SIR epidemic model with saturated incidence rate and pulse vaccinations.

Our model is derived from the deterministic impulsive system in Jin's research [17]. The deterministic SIR epidemic model with pulses is as follows:

$$\begin{cases} S'(t) = \mu K - \frac{\beta S(t)I(t)}{1+mI(t)} - \mu S(t), \quad t \neq k, \quad k \in N, \\ I'(t) = \frac{\beta S(t)I(t)}{1+mI(t)} - (\mu+\alpha+\lambda)I(t), \\ R'(t) = \lambda I(t) - \mu R(t), \end{cases}$$
$$\begin{cases} S(k^+) = (1-p)S(k), \quad t = k, \quad k \in N, \\ I(k^+) = I(k), \\ R(k^+) = R(k) + pS(k), \end{cases} \tag{1.1}$$

where $S(t)$, $I(t)$ and $R(t)$ stand for the population number of the susceptible, infectious and recovery at time t respectively. The parameter μ represents the birth rate (and the natural death rate is assumed to be identical), K is total population size, β denotes the transmission rate, α reflects the disease-related death rate and λ is the recovery rate of the infective individuals. In model (1.1) the period of pulse vaccination is 1, k is the time at which we applied the pulse, and k^- is the time just before applying the pulse. p is the fraction of all the susceptible to whom the vaccine is inoculated at discrete time $t = k$, $k \in N$. All the parameters are positive constants. For deterministic system (1.1), there exists a periodic infection-free solution $(S^*(t), 0, R^*(t))$, where

$$S^*(t) = K\left[1 - \frac{pe^{-\mu(t-k-1)}}{e^{\mu} - 1 + p}\right], \quad R^*(t) = K - S^*(t), \quad k < t \leq k+1.$$

Let $\langle S^* \rangle_1 \triangleq \int_0^1 S^*(s)ds = K[1 - \frac{p(e^{\mu}-1)}{\mu(e^{\mu}-1+p)}]$. Then there is the basic reproduction number $R_0 = \frac{\beta \langle S^* \rangle_1}{\mu+\alpha+\lambda}$. If $R_0 < 1$, the periodic infection-free solution $(S^*(t), 0, R^*(t))$ is globally stable; if $R_0 > 1$, the disease will uniformly persist and system (1.1) has a positive periodic solution [17].

One of the approaches to introduce white noises into biological models is proposed by Imhof and Walcher [16]. They give a detailed and rigorous derivation of a stochastic model by considering a discrete time Markov chain in which the random amount is supposed to be linear to the microbe population. In this paper, our approach to include random perturbation is analogous to that of Imhof and Walcher [16, 21]. Here we assume that the white noises are proportional to $S(t)$, $I(t)$, $R(t)$, directly influencing on the $S'(t)$, $I'(t)$, $R'(t)$ in the model (1.1). By this way, our stochastic model takes the form of

$$\begin{cases} dS(t) = \left[\mu K - \frac{\beta S(t)I(t)}{1+mI(t)} - \mu S(t)\right] dt + \sigma_1 S(t) dB_1(t), \\ dI(t) = \left[\frac{\beta S(t)I(t)}{1+mI(t)} - (\mu+\alpha+\lambda)I(t)\right] dt + \sigma_2 I(t) dB_2(t), \\ S(k^+) = (1-p)S(k), \quad t = k, \quad k \in N, \end{cases} \tag{1.2}$$

where $B_1(t)$ and $B_2(t)$ are independent standard Brownian motions with $B_1(0) = 0$, $B_2(0) = 0$ and $\sigma_i > 0$ represents the intensity of $B_i(t)$, $i = 1, 2$. For the dynamic of group R has no effects on the transmission dynamics, so we omit it in system (1.2). Corresponding to the results of deterministic model, the novelties and contributions of our paper are:

- we give the threshold which determines the extinction and persistence of the disease.
- we verify the global attraction of disease-free periodic solution.
- we demonstrate the existence of positive periodic solution.

The rest of this paper is organized as follows. In Section 2, we demonstrate the existence and uniqueness of global positive solution. In Section 3, we establish the threshold which determines disease to die out or prevail. In Section 4, we show that there is a globally attractive boundary periodic solution for system (1.2). In Section 5, we prove the existence of nontrivial positive periodic solution of the system (1.2). Finally, we summarize the main results in this paper and provide a brief discussion.

Throughout this paper, unless otherwise specified, let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathscr{F}_0$ contains all $\mathbb{P}$-null sets). Let $B_1(t)$ and $B_2(t)$be the Brownian motions defined on this probability space.

For convenience, we always use the following notations. $[\cdot]$ denotes the integer-valued function. If $f(t)$ is an integrable function on $[0, +\infty)$, define $\langle f\rangle_t = \frac{1}{t}\int_0^t f(s)ds$, $t > 0$. Therefore $\int_0^1 S^*(t)dt = \langle S^*\rangle_1$.

If $f(t)$ is a bounded function on $[0, +\infty)$,

$$\check{f} = \sup_{t\in[0,\infty)} f(t), \quad \hat{f} = \inf_{t\in[0,\infty)} f(t).$$

And denote

$$R^l_+ := \{x \in R^l : x_i > 0, \text{ for all } 1 \leq i \leq l\}.$$

2. Existence and uniqueness of global positive solution

In this section, we will prove the existence and uniqueness of global positive solution. Before the proof, we first need the definition of the solution of stochastic differential equation with impulses (ISDE), (see [26] for details).

Definition 2.1 ([26]). Consider the following ISDE:

$$\begin{cases} dX(t) = F(t, X(t))dt + G(t, X(t))dB(t), t \neq t_k, k \in N, \\ X(t_k^+) = X(t_k) + B_k X(t_k), k \in N, \end{cases} \tag{2.1}$$

with initial condition $X(0)$. A stochastic process $X(t) = (X_1(t), \ldots, X_n(t))^T$, $t \in R$, is said to be a solution of ISDE (2.1) if

(i) $X(t)$ is $\{\mathscr{F}_t\}$-adapted and is continuous on $[0, t_1)$ and each interval $[t_k, t_{k+1}) \subset R_+$, $k \in N$; $F(t, X(t)) \in L^2(R^+, R_n)$, $G(t, X(t)) \in L^2(R^+, R_n)$, where $L^k(R^+, R_n)$

is all R^n valued measurable $\{\mathscr{F}_t\}$-adapted processes $f(t)$ satisfying $\int_0^T |f(t)|^k dt < \infty$ a.s. for every $T > 0$;
(ii) for each $t_k,\ k \in N,\ X(t_k^+) = \lim_{t\to t_k^+} X(t)$ and $X(t_k^-) = \lim_{t\to t_k^-} X(t)$ exist $X(t_k) = X(t_k^-)$ with probability one;
(iii) for almost all $t \in [0, t_1)$, $X(t)$ obeys the integral equation

$$X(t) = X(0) + \int_0^t F(s, X(s))ds + \int_0^t G(s, X(s))dB(s).$$

And for almost all $[t_k, t_{k+1}), k \in N$, $X(t)$ obeys the integral equation

$$X(t) = X(t_k^+) + \int_{t_k^+}^t F(s, X(s))ds + \int_{t_k^+}^t G(s, X(s))dB(s).$$

Moreover, $X(t)$ satisfies the impulsive conditions at each $t = t_k,\ k \in N$ with probability one.

Theorem 2.1. *For any given initial value $(S(0), I(0)) \in R_+^2$, there is a unique global positive solution $(S(t), I(t)) \in R_+^2$ of system (1.2) on time $t \geq 0$ almost surely, which means the solution will remain in R_+^2 with probability* 1.

Proof. First, let's consider the following SDE without impulse:

$$\begin{cases} dx(t) = \left\{\mu K W^{-1}(t) - [\mu - \ln(1-p)]x(t) - \frac{\beta x(t)y(t)}{1+my(t)}\right\} dt + \sigma_1 x(t) dB_1(t), \\ dy(t) = \left[\frac{\beta W(t)x(t)y(t)}{1+my(t)} - (\mu + \alpha + \lambda)y(t)\right] dt + \sigma_2 y(t) dB_2(t), \end{cases} \tag{2.2}$$

with initial value $(x(0), y(0)) = (S(0), I(0))$, where

$$W(t) = \begin{cases} (1-p)^{[t]-t}, & t \neq k,\ k \in N, \\ (1-p)^{-1}, & t = k. \end{cases}$$

Obviously, $W(t)$ is 1-periodic and left-continuous. By the theory of SDE (see e.g. [29]), system (2.2) has a unique continuous maximal local solution $(x(t), y(t))$ on $[0, \tau_e)$, where τ_e is the explosion time. Then we only need to prove $\tau_e = +\infty$. The proof is similar to that in [39, 42] and therefore we omit it here.

We will show that system (2.2) is equivalent to system (1.2). Let $(S(t), I(t)) = (W(t)x(t), y(t))$. In fact, it is easy to check that $(x(t), y(t))$ are continuous on $(k, k+1) \subset [0, +\infty)$, $k \in N$. And for every interval, $t \neq k$,

$$\begin{aligned} dS(t) =& W'(t)x(t)dt + W(t)dx(t) \\ =& W(t)\left(\mu K W^{-1}(t) - \frac{\beta x(t)y(t)}{1+my(t)} - \mu x(t)\right) dt + \sigma_1 W(t)x(t)dB_1(t) \\ =& \left(\mu K - \frac{\beta S(t)I(t)}{1+mI(t)} - \mu S(t)\right) dt + \sigma_1 S(t) dB_1(t). \end{aligned}$$

For $t = k,\ k \in N$,

$$S(k^-) = \lim_{t\to k^-} W(t)x(t) = (1-p)^{(k-1)-k}x(k) = (1-p)^{-1}x(k) = S(k),$$

$$S(k^+) = \lim_{t\to k^+} W(t)x(t) = (1-p)^{k-k}x(k) = x(k).$$

Thus we have $S(k^+) = (1-p)S(k)$ for $t = k$.

Similarly, we can obtain that

$$dI(t) = \left[\frac{\beta S(t)I(t)}{1+mI(t)} - (\mu+\alpha+\lambda)I(t)\right]dt + \sigma_2 I(t)dB_2(t).$$

Therefore system (2.2) is equivalent to system (1.2).

With the existence and uniqueness of solution $(x(t), y(t))$ to system (2.2) and $(S(t), I(t)) = (W(t)x(t), y(t))$, hence we can hold the existence and uniqueness of solution $(S(t), I(t))$ to system (1.2) for $t \geq 0$ with any given initial value $(S(0), I(0)) \in R^2_+$. The proof is completed. □

3. Extinction and persistence of the disease

In this section, based on the global existence of the solution, we shall explore the threshold which determines the disease to die out or persist.

Let

$$R_0^s = \frac{\beta\langle S^*\rangle_1}{\mu+\alpha+\lambda+\frac{1}{2}\sigma_2^2}.$$

Theorem 3.1. *If $\mu \geq \frac{1}{2}(\sigma_1^2 \vee \sigma_2^2)$ and $R_0^s < 1$, then for any given initial value $(S(0), I(0)) \in R^2_+$, the solution $(S(t), I(t))$ of system (1.2) has the following property:*

$$\limsup_{t\to+\infty} \frac{1}{t}\ln I(t) \leq (\mu+\alpha+\lambda+\frac{1}{2}\sigma_2^2)(R_0^s - 1) < 0 \quad a.s.$$

In other words, the disease will go extinct exponentially with probability one.

Proof. First, we define a 1-periodic auxiliary function $h(t)$ which will be used later. we shall give its explicit form and calculate $\lim_{t\to+\infty}\frac{1}{t}\int_0^t W^{-1}(s)h(s)ds$. In the latter part of the proof, we will establish the threshold which determines the disease to go extinct or prevail.

Define a 1-periodic function $h(t)$ which satisfies $h'(t) - h(t)[\mu - \ln(1-p)] = -W(t)$ in each interval $(k, k+1)$, $k \in N$. By calculation we can get

$$h(t) = \frac{e^{\mu t}(1-p)^{-t}\int_t^{t+1} e^{-\mu s}(1-p)^{[s]}ds}{1-e^{-\mu}(1-p)}.$$

When $t \in [0,1]$,

$$\begin{aligned}
e^{\mu t}\int_t^{t+1} e^{-\mu s}(1-p)^{[s]}ds &= \int_t^{t+1} e^{-\mu(s-t)}(1-p)^{[s]}ds \\
&= \int_0^1 e^{-\mu s}(1-p)^{[s+t]}ds \\
&= \int_0^{1-t} e^{-\mu s}(1-p)^{[s+t]}ds + \int_{1-t}^1 e^{-\mu s}(1-p)^{[s+t]}ds \\
&= \int_0^{1-t} e^{-\mu s}ds + \int_{1-t}^1 e^{-\mu s}(1-p)ds \\
&= \frac{1}{\mu}e^{-\mu}(e^{\mu} - 1 + p + pe^{\mu t}).
\end{aligned}$$

Thus we obtain the explicit expression of $h(t)$,

$$h(t) = \frac{1}{\mu}(1-p)^{-t}\left(1 - \frac{pe^{\mu t}}{e^{\mu} - 1 + p}\right), \quad t \in [0,1]. \tag{3.1}$$

Because $W(t)$ and $h(t)$ are 1-periodic functions, it's easy to prove that

$$\begin{aligned}
\lim_{t\to+\infty} \frac{1}{t}\int_0^t W^{-1}(s)h(s)ds = & \int_0^1 W^{-1}(s)h(s)ds \\
= & \frac{1}{\mu}\int_0^1 \left(1 - \frac{pe^{\mu s}}{e^{\mu} - 1 + p}\right) ds \\
= & \frac{1}{\mu}\left[1 - \frac{p(e^{\mu} - 1)}{\mu(e^{\mu} - 1 + p)}\right] \\
= & \frac{1}{\mu K}\langle S^*\rangle_1.
\end{aligned}$$

By using the similar arguments as in Zhao et al. [42], we can get that if $\mu \geq \frac{1}{2}(\sigma_1^2 \vee \sigma_2^2)$, then

$$\begin{aligned}
&\lim_{t\to+\infty} \frac{1}{t}\int_0^t h(s)x(s)dB_1(s) = 0, && && a.s. \\
&\lim_{t\to+\infty} \frac{1}{t}\int_0^t h(s)y(s)dB_2(s) = 0, && \lim_{t\to+\infty} \frac{1}{t}\int_0^t y(s)dB_2(s) = 0, && a.s. \\
&\lim_{t\to+\infty} \frac{x(t)}{t} = 0, && \lim_{t\to+\infty} \frac{y(t)}{t} = 0, && a.s..
\end{aligned} \tag{3.2}$$

Applying Itô formula, we have

$$\begin{aligned}
d[h(t)x] = & h'(t)xdt + h(t)dx \\
= & \left\{\mu KW^{-1}(t)h(t) + h'(t)x(t) - h(t)[\mu - \ln(1-p)]x(t) - \frac{\beta h(t)x(t)y(t)}{1+my(t)}\right\} dt \\
& + \sigma_1 h(t)x(t)dB_1(t) \\
= & \left(\mu KW^{-1}h(t) - W(t)x(t) - \frac{\beta h(t)x(t)y(t)}{1+my(t)}\right)dt + \sigma_1 h(t)x(t)dB_1(t).
\end{aligned}$$

This combined with (3.2) implies

$$\begin{aligned}
0 = & \lim_{t\to+\infty} \frac{h(t)x(t)}{t} \\
= & \lim_{t\to+\infty} \frac{1}{t}\int_0^t \mu KW^{-1}(s)h(s)ds - \lim_{t\to+\infty} \frac{1}{t}\int_0^t W(s)x(s)ds \\
& - \lim_{t\to+\infty} \frac{1}{t}\int_0^t \frac{\beta h(s)x(s)y(s)}{1+my(s)}ds \\
= & \langle S^*\rangle_1 - \lim_{t\to+\infty} \langle Wx\rangle_t - \lim_{t\to+\infty} \langle \frac{\beta hxy}{1+my}\rangle_t.
\end{aligned} \tag{3.3}$$

Similarly we can get that

$$\begin{aligned} 0 &= \lim_{t\to+\infty} \frac{y(t)}{t} \\ &= \lim_{t\to+\infty} \frac{1}{t}\int_0^t \frac{\beta W(s)x(s)y(s)}{1+my(s)}ds - \lim_{t\to+\infty}(\mu+\alpha+\lambda)\frac{1}{t}\int_0^t y(s)ds \\ &= \lim_{t\to+\infty}\langle\frac{\beta Wxy}{1+my}\rangle_t - (\mu+\alpha+\lambda)\lim_{t\to+\infty}\langle y\rangle_t. \end{aligned} \tag{3.4}$$

By Itô's formula

$$d\ln y(t) = \left[\frac{\beta W(t)x(t)}{1+my(t)} - (\mu+\alpha+\lambda) - \frac{1}{2}\sigma_2^2\right]dt + \sigma_2 dB_2(t).$$

Integrating this from 0 to t and dividing t on the both sides, and combining with (3.4) we have

$$\begin{aligned} \frac{1}{t}\ln y(t) =& \langle\frac{\beta Wx}{1+my}\rangle_t - (\mu+\alpha+\lambda+\frac{1}{2}\sigma_2^2) + \frac{1}{t}(\sigma_2 B_2(t) + \ln y(0)), \\ =& \beta\langle Wx\rangle_t - m\langle\frac{\beta Wxy}{1+my}\rangle_t - (\mu+\alpha+\lambda+\frac{1}{2}\sigma_2^2) + \frac{1}{t}(\sigma_2 B_2(t)+\ln y(0)), \\ =& \beta\langle S^*\rangle_1 - \beta\langle\frac{\beta hxy}{1+my}\rangle_t - m(\mu+\alpha+\lambda)\langle y\rangle_t - (\mu+\alpha+\lambda+\frac{1}{2}\sigma_2^2) + \frac{1}{t}\Phi(t), \\ \le& \beta\langle S^*\rangle_1 - m(\mu+\alpha+\lambda)\langle y\rangle_t - (\mu+\alpha+\lambda+\frac{1}{2}\sigma_2^2) + \frac{1}{t}\Phi(t), \\ \le& \beta\langle S^*\rangle_1 - (\mu+\alpha+\lambda+\frac{1}{2}\sigma_2^2) + \frac{1}{t}\Phi(t). \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} \Phi(t) =& m\left[(\mu+\alpha+\lambda)\langle y\rangle_t - \langle\frac{\beta Wxy}{1+my}\rangle_t\right] + \beta\left(\langle Wx\rangle_t + \langle\frac{\beta hxy}{1+my}\rangle_t - \langle S^*\rangle_1\right)t \\ &+ \sigma_2 B_2(t) + \ln y(0). \end{aligned}$$

In view of (3.3) and (3.4), it follows that $\lim_{t\to+\infty}\frac{1}{t}\Phi(t) = 0$, a.s.

Taking the limit superior of both sides of (3.5), and if $R_0^s < 1$, then it follows that

$$\limsup_{t\to+\infty}\frac{1}{t}\ln y(t) \le (\mu+\alpha+\lambda+\frac{1}{2}\sigma_2^2)(R_0^s - 1) < 0 \quad a.s.. \tag{3.6}$$

For $y(t) = I(t)$, it implies that

$$\lim_{t\to+\infty} I(t) = 0, \quad a.s. \tag{3.7}$$

Therefore (3.6) and (3.7) means the disease $I(t)$ will go extinct exponentially with probability one. The proof is completed. □

Theorem 3.2. *If $\mu \ge \frac{1}{2}(\sigma_1^2 \vee \sigma_2^2)$ and $R_0^s > 1$, then for any given initial value $(S(0), I(0)) \in R_+^2$, the disease I will persist in the sense that:*

$$\frac{\mu(\mu+\alpha+\lambda+\frac{1}{2}\sigma_2^2)(R_0^s-1)}{(\beta+\mu m)(\mu+\alpha+\lambda)} \le \liminf_{t\to+\infty}\langle I\rangle_t \le \limsup_{t\to+\infty}\langle I\rangle_t \le \frac{(\mu+\alpha+\lambda+\frac{1}{2}\sigma_2^2)(R_0^s-1)}{m(\mu+\alpha+\lambda)}.$$

Proof. From the first inequality of (3.5) and Lemma A.2 in [42], it is easy to get

$$\limsup_{t\to+\infty}\langle y\rangle_t \leq \frac{(\mu+\alpha+\lambda+\frac{1}{2}\sigma_2^2)(R_0^s-1)}{m(\mu+\alpha+\lambda)}.$$

It is easy to verify that $h(t) \leq \frac{1}{\mu}W(t)$ from Eq. (3.1), then from the third equality of (3.5) and (3.4) we have

$$\begin{aligned}\frac{1}{t}\ln y(t) =&\beta\langle S^*\rangle_1-\beta\langle\frac{\beta hxy}{1+my}\rangle_t-m(\mu+\alpha+\lambda)\langle y\rangle_t-(\mu+\alpha+\lambda+\frac{1}{2}\sigma_2^2)+\frac{1}{t}\Phi(t),\\ \geq&\beta\langle S^*\rangle_1-\frac{\beta}{\mu}\langle\frac{\beta Wxy}{1+my}\rangle_t-m(\mu+\alpha+\lambda)\langle y\rangle_t-(\mu+\alpha+\lambda+\frac{1}{2}\sigma_2^2)+\frac{1}{t}\Phi(t),\\ =&\beta\langle S^*\rangle_1-(\frac{\beta}{\mu}+m)(\mu+\alpha+\lambda)\langle y\rangle_t-(\mu+\alpha+\lambda+\frac{1}{2}\sigma_2^2)+\frac{1}{t}\Phi(t).\end{aligned} \tag{3.8}$$

By Lemma 17 in [38] it can obtain

$$\liminf_{t\to+\infty}\langle y\rangle_t \geq \frac{\mu(\mu+\alpha+\lambda+\frac{1}{2}\sigma_2^2)(R_0^s-1)}{(\beta+\mu m)(\mu+\alpha+\lambda)}.$$

For $I(t)=y(t)$, the claim is proved. □

4. Globally attractive boundary periodic solution

In this section and next section, we will prove the existence and global attraction of the disease-free periodic solution and the existence of the positive periodic solution respectively. However,the periodic solution of SDE is in the sense of distribution. For the convenience of readers, we first present the definition of the periodic solution of SDE and cite a result of the periodic solution of stochastic differential equations without impulses.

Definition 4.1 ([19]). A stochastic process $X(t)$ $(-\infty < t < +\infty)$ is said to be periodic with period T if for every finite sequence of numbers t_1, t_2, $\cdots$,t_n, the joint distribution of random variables $X(t_1+h)$,$\cdots$,$X(t_n+h)$ is independent of h, where $h = kT$ $(k = \pm1; \pm2; \cdots)$.

Consider the following periodic stochastic differential equation without impulse:

$$X(t)=X(t_0)+\int_{t_0}^{t}b(s,X(s))ds+\sum_{r=1}^{k}\int_{t_0}^{t}\sigma_r(s,X(s))dB_r(s), \quad X\in R^l, \tag{4.1}$$

where the vectors $b(s,X)$, $\sigma_1(s,X)$, ...,$\sigma_k(s,X)$ $(X \in R^l)$ are continuous functions of (s,X) and satisfy the conditions:

$$\begin{cases} |b(s,x)-b(s,y)|+\displaystyle\sum_{r=1}^{k}|\sigma_r(s,x)-\sigma_r(s,y)| \leq B|x-y|,\\ |b(s,x)|+\displaystyle\sum_{r=1}^{k}|\sigma_r(s,x)| \leq B(1+|x|),\end{cases} \tag{4.2}$$

where B is a constant. Let $I = \{t : 0 \leq t < +\infty\}$, U be a given open set in R^l and $E = I \times R^l$. Let C^2 denote the class of functions on E which are twice continuously differentiable with respect to x_1, ..., x_l and continuously differentiable with respect to t.

Lemma 4.1 ([19]). *Suppose that the coefficients of system* (4.1) *are T-periodic in t and satisfy the conditions* (4.2) *in every cylinder $I \times U$, and suppose further that there exists a function $V(t, x) \in C^2$ in E which is T-periodic in t, and satisfies the following conditions:*
1. there exists a constant M such that $LV(t, x) \leq -1$, $|x| \geq M$,
2. $\inf_{|x|>R} V(t, x) \to \infty$, as $R \to \infty$,
where the operator L is given by

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^{l} b_i(t, x)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{l} a_{ij}(t, x)\frac{\partial^2}{\partial x_i \partial x_j}, \quad a_{ij} = \sum_{r=1}^{k} \sigma_r^i(t, x)\sigma_r^j(t, x).$$

Then there exists a solution of Eq. (4.1) *which is a T-periodic process.*

Lemma 4.2. *Consider the following linear stochastic differential equation*

$$d\overline{X}(t) = \{\mu K W^{-1}(t) - [\mu - \ln(1 - q)]\overline{X}(t)\}dt + \sigma_1 \overline{X}(t)dB_1(t) \tag{4.3}$$

with initial value $\overline{X}(t) = x(0)$. Then Eq. (4.3) *has a positive periodic solution $\overline{X}_p(t)$ which is globally attractive, i.e. attracts all other positive solutions of Eq.* (4.3).

Proof. First we construct a Lyapunov function to prove the existence of $\overline{X}_p(t)$. The C^2-function $V : R_+ \to R$ takes the following form:

$$V(t, \overline{X}) = \overline{X}(t) - 1 - \ln \overline{X}(t).$$

By Itô's formula

$$\begin{aligned} LV =& -[\mu - \ln(1 - p)]\overline{X} - \frac{\mu K W^{-1}(t)}{\overline{X}} + \mu K W^{-1}(t) + [\mu - \ln(1 - p)] + \frac{\sigma_1^2}{2} \\ \leq& -[\mu - \ln(1 - p)]\overline{X} - \frac{\mu K(1 - p)}{\overline{X}} + \mu K + \mu - \ln(1 - p) + \frac{\sigma_1^2}{2}, \\ \triangleq& \Psi(\overline{X}). \end{aligned}$$

Obviously, $\Psi(\overline{X}) \to -\infty$, as $\overline{X} \to 0^+$ or $\overline{X} \to +\infty$. Take $\epsilon > 0$ small enough and let $U = [\epsilon, 1/\epsilon]$, and we have $LV(t, \overline{X}) < -1$, $\overline{X} \in R_+ \setminus U$. Then from Lemma (4.1), Eq. (4.3) has a positive 1-periodic solution $\overline{X}_p(t)$.

Next we will prove that $\overline{X}_p(t)$ is globally attractive. Now $\overline{X}_p(t)$ satisfies Eq. (4.3), so

$$d(\overline{X}(t) - \overline{X}_p(t)) = -[\mu - \ln(1 - p)](\overline{X}(t) - \overline{X}_p(t)) + \sigma_1(\overline{X}(t) - \overline{X}_p(t))dB_1(t).$$

Thus we have

$$\overline{X}(t) - \overline{X}_p(t) = (\overline{X}(0) - \overline{X}_p(0)) \exp\left\{-\int_0^t [\mu - \ln(1 - p) + \frac{\sigma_1^2}{2}]dt + M(t)\right\}. \tag{4.4}$$

Then it follows that

$$\frac{\ln|\overline{X}(t) - \overline{X}_p(t)|}{t} = \frac{\ln|\overline{X}(0) - \overline{X}_p(0)|}{t} - \frac{1}{t}\int_0^t [\mu - \ln(1 - p) + \frac{\sigma_1^2}{2}]dt + \frac{M(t)}{t},$$

where $M(t) = \sigma_1 B_1(t)$. By the property of Brownian motion [29], we can get that $\lim_{t\to+\infty} M(t)/t = 0$. Take limits in above equation, one can see that,

$$\lim_{t\to+\infty} \frac{\ln|\overline{X}(t) - \overline{X}_p(t)|}{t} = -[\mu - \ln(1-p) + \frac{\sigma_1^2}{2}] < 0.$$

This implies that $\overline{X}(t) \to \overline{X}_p(t)$, a.s., so the periodic solution $\overline{X}_p(t)$ of Eq. (4.3) is globally attractive. □

Theorem 4.1. *If $\mu \geq \frac{1}{2}(\sigma_1^2 \vee \sigma_2^2)$ and $R_0^s < 1$, then system (1.2) has a boundary periodic solution $(S_p(t), 0)$ which is globally attractive.*

Proof. Because $(S(t), I(t)) = (W(t)x(t), y(t))$, we just need to prove that the equivalent system (2.2) has a boundary periodic solution $(x_p(t), 0)$ which is globally attractive.

To prove the global attraction of boundary periodic solution $(x_p(t), 0)$, we should prove that $y(t)$ tends to 0 and $x(t)$ tends to $x_p(t)$ respectively for any solution $(x(t), y(t))$ under assumed conditions.

If $\mu \geq \frac{1}{2}(\sigma_1^2 \vee \sigma_2^2)$ and $R_0^s < 1$ are satisfied, then from theorem 3.1 we know that $\lim_{t\to+\infty} y(t) = 0$ a.s. Combining with Eq. (3.2) that $\lim_{t\to+\infty} x(t)/t = 0$ a.s., for any arbitrary small $\tau > 0$, there exists a $t_0 = t_0(\omega)$ and a set $\Omega_\tau \in \Omega$ such that $P(\Omega_\tau) > 1 - \tau$, $y(t) < e^{-ct}$ and $x(t)y(t) < \tau$ for $t > t_0$, $\omega \in \Omega_\tau$, where $c = (\mu + \alpha + \lambda + \frac{1}{2}\sigma_2^2)(1 - R_0^s)$. Now from the first equation in System (2.2), we obtain that for $t > t_0$, $\omega \in \Omega_\tau$

$$\begin{aligned} dx(t) &= \left\{\mu K W^{-1}(t) - \frac{\beta x(t)y(t)}{1 + my(t)} - [\mu - \ln(1-p)]x(t)\right\} dt + \sigma_1 x(t) dB_1(t) \\ &\geq \{\mu K W^{-1}(t) - \beta\tau - [\mu - \ln(1-p)]x(t)\}dt + \sigma_1 x(t) dB_1(t). \end{aligned}$$

Let $\underline{X}(t)$ be the solution of the equation

$$d\underline{X}(t) = \{\mu K W^{-1}(t) - \beta\tau - [\mu - \ln(1-p)]\underline{X}(t)\}dt + \sigma_1 \underline{X}(t) dB_1(t),$$

with initial value $\underline{X}(0) = x(0)$. Then it follows from the stochastic comparison theorem that for almost all $\omega \in \Omega_\tau$,

$$\underline{X}(t) \leq x(t) \leq \overline{X}(t), \quad t \geq t_0,$$

where $\overline{X}(t)$ is the solution of Eq.(4.3) with $\overline{X}(0) = x(0)$. Let τ tend to zero, then $\lim_{t\to+\infty} |\underline{X}(t) - \overline{X}(t)| = 0$ a.s.. Thus we conclude that

$$\lim_{t\to+\infty} |x(t) - \overline{X}(t)| = 0, \ a.s..$$

This together with the global attraction of $\overline{X}_p(t)$ of Lemma 4.2, yeilds

$$\lim_{t\to+\infty} |x(t) - \overline{X}_p(t)| = 0, \ a.s..$$

Obviously, $x_p(t) = \overline{X}_p(t)$. Then the boundary periodic solution $(x_p(t), 0)$ of System (2.2) is globally attractive. For $(S(t), I(t)) = (W(t)x(t), y(t))$, the boundary periodic solution $(S_p(t), 0) = (W(t)x_p(t), 0)$ of System (1.2) is also globally attractive. The proof is completed.

□

5. Existence of the nontrivial positive periodic solution

Theorem 5.1. *If $R_0^s > 1$ holds, then there exists a positive 1-periodic solution of system (1.2).*

Proof. We just need to prove the existence of a periodic solution of the equivalent system (2.2) without impulses.

Since any $(x(0), y(0)) \in R_+^2$ system (2.2) has a unique global positive solution, we take R_+^2 as the whole space. It is clear that the coefficients of system (2.2) satisfy the local Lipschitz condition. Next we will testify the conditions (1), (2) of Lemma 4.1.

Take $\theta \in (0,1)$ and M satisfying:

$$\mu - \frac{\theta}{2}(\sigma_1^2 \vee \sigma_2^2) > 0, \quad \check{H} - M(\mu + \alpha + \lambda + \frac{1}{2}\sigma_2^2)(R_0^s - 1) \leq -2, \tag{5.1}$$

where the function $H(x)$ is given in Eq. (5.2).

Construct a C^2-function $V : R_+^2 \to R$ in the following form:

$$V(t,x,y) = M\left[-\ln y - \beta h(t)x - (m + \frac{\beta}{\mu})y + \omega(t)\right] - \ln x + \frac{1}{1+\theta}[x + (1-p)y]^{(1+\theta)}$$

where $h(t)$ is given in Theorem 3.1. Here $\omega(t)$ is a function defined on $[0, +\infty)$ satisfying $\omega(0) = 0$ and

$$\dot{\omega}(t) = \beta\mu K W^{-1}(t)h(t) - \beta\langle S^* \rangle_1.$$

For $W(t)$ and $h(t)$ are 1-periodic functions, $\omega(t)$ is obviously a 1-periodic function on $[0, +\infty)$. Hence $V(t,x,y)$ is 1-periodic in t.

In order to confirm the condition (2) of Lemma 4.1, it is obvious that we only need to prove that

$$\inf_{(t,x,y)\in[0,+\infty)\times(R_+^2\setminus U_k)} V(t,x) \to \infty, \quad \text{as } k \to \infty,$$

where $U_k = (\frac{1}{k}, k) \times (\frac{1}{k}, k)$, which is clearly established since

$$\begin{aligned} -\ln x \to +\infty \quad &\text{as} \quad x \to 0^+, \\ -\ln y \to +\infty \quad &\text{as} \quad y \to 0^+, \end{aligned}$$

and

$$[x + (1-p)y]^{(1+\theta)} \to +\infty \quad \text{as} \quad x \to +\infty \text{ or } y \to +\infty.$$

Therefore it is easy to see that $V(t,x,y)$ satisfies the condition (2) of Lemma 4.1.

Next we will find a closed set $U \subset R_+^2$ such that $LV(t,x,y) \leq -1$, $(x,y) \in R_+^2 \setminus U$. Denote $V_1 = -\ln y - \beta h(t)x - (m + \frac{\beta}{\mu})y + \omega(t)$, $V_2 = -\ln x$, $V_3 = \frac{1}{1+\theta}[x + (1 - p)y]^{(1+\theta)}$. Then $LV = MLV_1 + LV_2 + LV_3$.

From $h'(t)-h(t)[\mu-\ln(1-p)]=-W(t)$ and $h(t)\leq \frac{1}{\mu}W(t)$, we can have

$$\begin{aligned}
LV_1=&-\frac{\beta W(t)xy}{1+my}+\mu+\alpha+\lambda+\frac{1}{2}\sigma_2^2-\beta h(t)\Big\{\mu K W^{-1}(t)-[\mu-\ln(1-p)]x\\
&-\frac{\beta xy}{1+my}\Big\}-\beta h'(t)x-(m+\frac{\beta}{\mu})\frac{\beta W(t)xy}{1+my}+(m+\frac{\beta}{\mu})(\mu+\alpha+\lambda)y+\beta\frac{\mu K h(t)}{W(t)}\\
&-\beta\langle S^*\rangle_1\\
=&-\beta W(t)x+m\frac{\beta W(t)xy}{1+my}+\mu+\alpha+\lambda+\frac{1}{2}\sigma_2^2-\beta\frac{\mu K h(t)}{W(t)}+\beta h(t)[\mu-\ln(1-p)]x\\
&+\frac{\beta^2 h(t)xy}{1+my}-\beta h'(t)x-m\frac{\beta W(t)xy}{1+my}-\frac{1}{\mu}\frac{\beta^2 W(t)xy}{1+my}+(m+\frac{\beta}{\mu})(\mu+\alpha+\lambda)y\\
&+\beta\frac{\mu K h(t)}{W(t)}-\beta\langle S^*\rangle_1\\
\leq&-\beta\langle S^*\rangle_1+\mu+\alpha+\lambda+\frac{1}{2}\sigma_2^2+(m+\frac{\beta}{\mu})(\mu+\alpha+\lambda)y\\
\leq&-(\mu+\alpha+\lambda+\frac{1}{2}\sigma_2^2)(R_0^s-1)+(m+\frac{\beta}{\mu})(\mu+\alpha+\lambda)y,
\end{aligned}$$

Direct calculation implies that

$$\begin{aligned}
LV_2=&-\frac{\mu K W^{-1}(t)}{x}+\frac{\beta W(t)y}{1+my}+\mu-\ln(1-p)+\frac{1}{2}\sigma_1^2\\
\leq&-\frac{\mu(1-p)K}{x}+\frac{\beta}{m(1-p)}+\mu-\ln(1-p)+\frac{1}{2}\sigma_1^2,
\end{aligned}$$

And by elementary inequality $(a+b)^\theta\leq\max\{1,2^{\theta-1}\}(a^\theta+b^\theta)$, $\theta\geq0$, and inequality $(a+b)^{-\theta}\leq a^{-\theta}$, $\theta\geq0$, we can obtain

$$\begin{aligned}
LV_3=&[x+(1-p)y]^\theta\Big\{\mu K W^{-1}(t)-[1-(1-p)W(t)]\frac{\beta xy}{1+my}-[\mu-\ln(1-p)]x\\
&-(\mu+\alpha+\lambda)(1-p)y\Big\}+\frac{\theta}{2}[x+(1-p)y]^{\theta-1}[\sigma_1^2x^2+\sigma_2^2(1-p)^2y^2]\\
\leq&\mu K[x+(1-p)y]^\theta-[x+(1-p)y]^\theta[\mu x+\mu(1-p)y]\\
&+\frac{\theta}{2}[x+(1-p)y]^{\theta-1}\sigma_1^2x^2+\frac{\theta}{2}[x+(1-p)y]^{\theta-1}\sigma_2^2(1-p)^2y^2\\
\leq&2^\theta\mu K x^\theta+2^\theta\mu K y^\theta-\mu x^{1+\theta}-\mu(1-p)^{1+\theta}y^{1+\theta}+\frac{\theta}{2}\sigma_1^2x^{1+\theta}+\frac{\theta}{2}\sigma_2^2(1-p)^{1+\theta}y^{1+\theta}\\
=&2^\theta\mu K x^\theta+2^\theta\mu K y^\theta-(\mu-\frac{\theta}{2}\sigma_1^2)x^{1+\theta}-(\mu-\frac{\theta}{2}\sigma_2^2)(1-p)^{1+\theta}y^{1+\theta}.
\end{aligned}$$

Hence, $LV\leq f(x)+g(y)$, where

$$\begin{aligned}
H(x)=&-\frac{\mu(1-p)K}{x}+2^\theta\mu K x^\theta-(\mu-\frac{\theta}{2}\sigma_1^2)x^{1+\theta}+\frac{\beta}{m(1-p)} \qquad (5.2)\\
&+\mu-\ln(1-p)+\frac{1}{2}\sigma_1^2,\\
J(y)=&-M(\mu+\alpha+\lambda+\frac{1}{2}\sigma_2^2)(R_0^s-1)+(m+\frac{\beta}{\mu})(\mu+\alpha+\lambda)y \qquad (5.3)
\end{aligned}$$

$$+ 2^\theta \mu K y^\theta - (\mu - \frac{\theta}{2}\sigma_2^2)(1-p)^{1+\theta} y^{1+\theta}.$$

In view of (5.1), we can obtain

$$H(x) + \check{J} \to -\infty, \quad \text{as } x \to +\infty, \text{ or } x \to 0$$
$$\check{H} + J(y) \to -\infty, \quad \text{as } y \to +\infty,$$

and

$$\check{H} + J(y) \to \check{H} - M(\mu + \alpha + \lambda + \frac{1}{2}\sigma_2^2)(R_0^s - 1) \le -2, \quad \text{as } y \to 0.$$

Take $\kappa > 0$ small enough, and let $U := [\kappa, \frac{1}{\kappa}] \times [\kappa, \frac{1}{\kappa}]$. It follows that

$$LV \le -1, \quad (x, y) \in R_+^2 \setminus U.$$

Thus there is a nontrivial positive periodic solution to system (2.2). For $(S(t), I(t)) = (W(t)x(t), y(t))$, then system (1.2) also has a nontrivial positive periodic solution. The proof is complete. □

6. Numerical simulations

In this section we give numerical simulations by Milstein's Higher Order Method [15]. We assume that the unit of time is one year and the population sizes are measured in unit of 1 million. The examples are just numerical experiments to confirm our results.

Example 6.1. To illustrate the threshold of disease and the effects of the environment white noises, we choose the parameters in deterministic system and stochastic system as follows:

$$\mu = 0.08,\ K = 1,\ \beta = 0.87,\ \alpha = 0.05,\ \lambda = 0.22,\ p = 0.1.$$

Let initial value be $(S(0), I(0)) = (0.7, 0.2)$. We have four different cases.

(a) For deterministic system (1.1), $R_0 = 1.0738 > 1$, the disease will persist.

(b) For stochastic system (1.2), $\sigma_1 = 0.02$, $\sigma_2 = 0.02$ and $R_0^s = 1.0732 > 1$. By Theorem 3.2, I(t) will persist.

(c) For stochastic system (1.2), $\sigma_1 = 0.02$, $\sigma_2 = 0.30$ and $R_0^s = 0.9515 < 1$. By Theorem 3.1, I(t) will tend to zero exponentially with probability one.

(d) For stochastic system (1.2), $\sigma_1 = 0.02$, $\sigma_2 = 0.55$ and $R_0^s = 0.7498 < 1$. By Theorem 3.1, I(t) will tend to zero exponentially with probability one.

With σ_2 in denominator, the white noise σ_2 decreases the basic reproduction number of disease. From case (a), (b) and (c) in Fig. 1, we can know that in the deterministic impulsive model (1.1), $I(t)$ tends to 0 if and only if $R_0 = \frac{\beta\langle S^*\rangle_1}{\mu+\alpha+\lambda} < 1$, while in the ISDE SIR model (1.2), $I(t)$ tends to 0 if $R_0^s = \frac{\beta\langle S^*\rangle_1}{\mu+\alpha+\lambda+\frac{1}{2}\sigma_2^2} = \frac{\mu+\alpha+\lambda}{\mu+\alpha+\lambda+\frac{1}{2}\sigma_2^2} R_0 < 1$. In other words, the conditions for I(t) to become extinct in the ISDE SIR model are weaker than that in the corresponding deterministic impulsive model. Furthermore, from Theorem 3.1 one can see that $I(t)$ tends to 0 exponentially in a speed $e^{(\mu+\alpha+\lambda+\frac{1}{2}\sigma_2^2)(R_0^s-1)}$ when $R_0^s < 1$. In theory, the bigger σ_2 is, the faster the

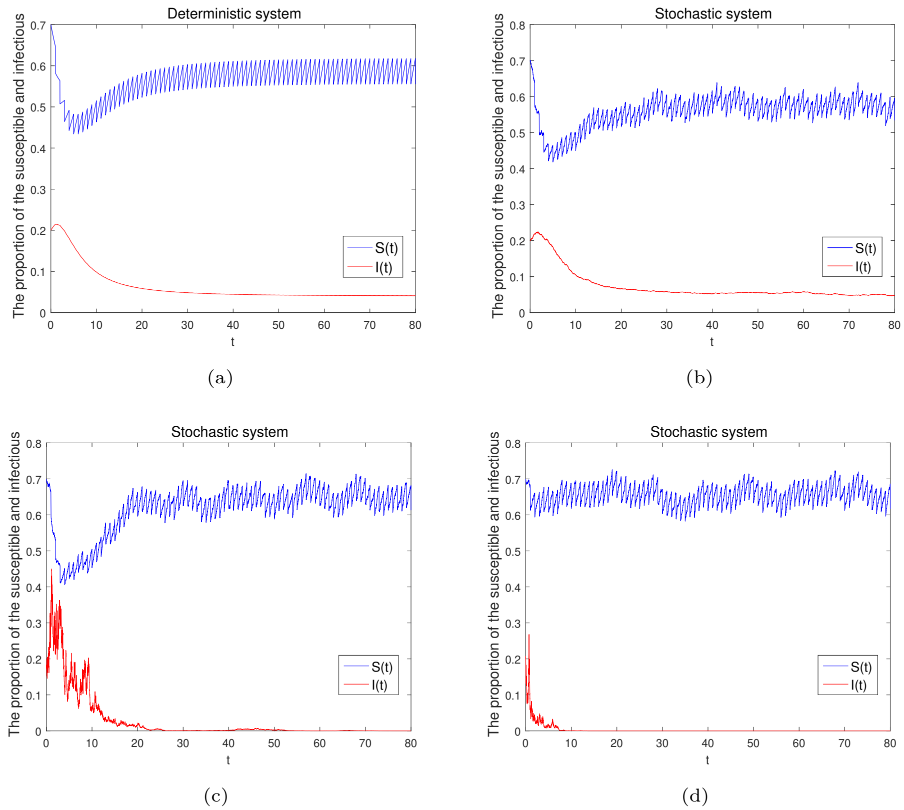


Figure 1. The pathway simulations of $S(t)$ and $I(t)$ for the deterministic system (1.1) and the stochastic system (1.2) with initial value $(S(0), I(0)) = (0.7, 0.2)$. (a) Solutions to deterministic system (1.1) with $R_0 = 1.0738$; (b) Solutions to stochastic system (1.2) with $\sigma_1 = 0.02$, $\sigma_2 = 0.02$, $R_0^s = 1.0732$,; (c) Solutions to stochastic system (1.2) with $\sigma_1 = 0.02$, $\sigma_2 = 0.3$, $R_0^s = 0.9515$; (d) Solutions to stochastic system (1.2) with $\sigma_1 = 0.02$, $\sigma_2 = 0.55$, $R_0^s = 0.7498$.

disease goes extinct, which is illustrated in case (c) and (d) in Fig. 1. In general, this means the environmental noises can help to suppress the spread of disease.

Example 6.2. Here we will give demonstrations of the existence of the boundary periodic solution $(S_p(t), 0)$ of System (1.2) and show it is globally attractive when the disease becomes extinct. To satisfy the conditions of Theorem 4.1, we choose the same parameters

$$\mu = 0.08,\ K = 1,\ \beta = 0.87,\ \alpha = 0.05,\ \lambda = 0.22,\ p = 0.1.$$

And

(a) $\sigma_1 = 0.020$, $\sigma_2 = 0.30$, with initial value $S(0) = 0.7, I(0) = 0.2$;

(b) $\sigma_1 = 0.015$, $\sigma_2 = 0.55$, with initial value $S(0) = 0.7, I(0) = 0.2$;

(c) $\sigma_1 = 0.020$, $\sigma_2 = 0.30$, with initial values $S(0) = 0.6, I(0) = 0.2$; $S(0) = 0.7, I(0) = 0.2$ and $S(0) = 0.8, I(0) = 0.2$;

(d) $\sigma_1 = 0.015$, $\sigma_2 = 0.55$, with initial values $S(0) = 0.6, I(0) = 0.2$; $S(0) = 0.7, I(0) = 0.2$ and $S(0) = 0.8, I(0) = 0.2$.

As illustrated in Fig 1, above parameters can make sure the disease will die out in stochastic system. In addition, to simulate the disease-free periodic solution in the deterministic system (as showed in case (a) and (b) of Fig 2.), we decrease the transmission rate $\beta = 0.5$ for deterministic system (therefore the basic reproduction number $R_0 = 0.6171 < 1$).

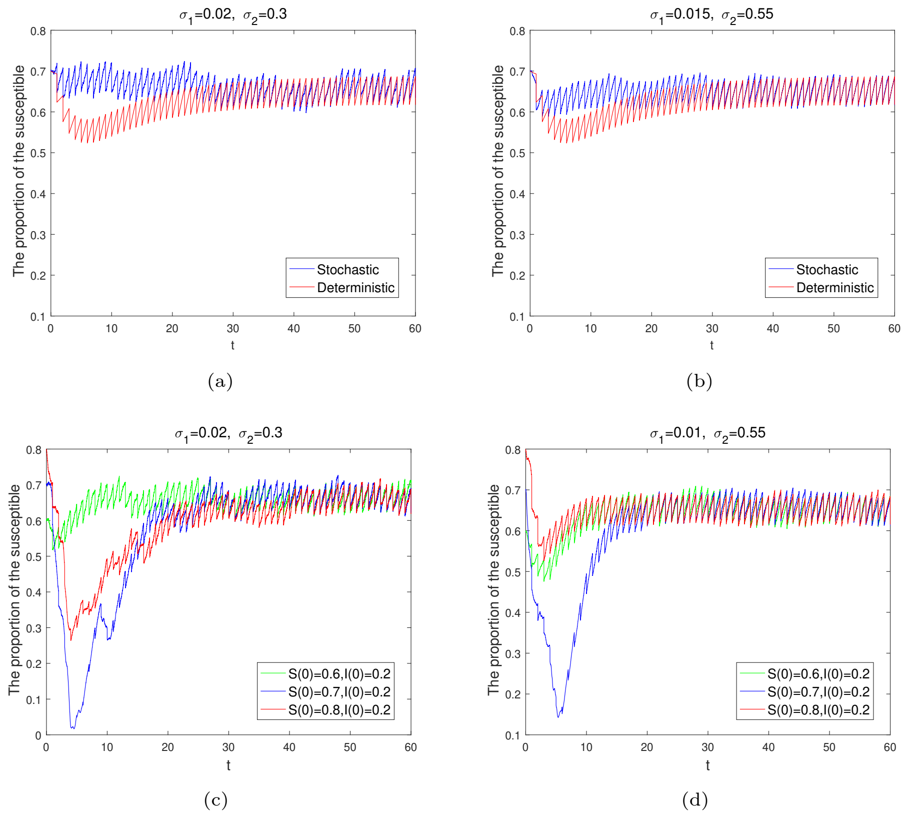


Figure 2. Simulations of the boundary periodic solution and global attraction for the stochastic system (1.2) when disease goes extinct. (a) the positive boundary periodic solution to stochastic system (1.2) with $\sigma_1 = 0.02$, $\sigma_2 = 0.03$ and initial value $(S(0), I(0)) = (0.7, 0.2)$; (b) the positive boundary periodic solution to stochastic system (1.2) with $\sigma_1 = 0.015$, $\sigma_2 = 0.55$ and initial value $(S(0), I(0)) = (0.7, 0.2)$; (c) global attraction for the stochastic system (1.2) with $\sigma_1 = 0.02$, $\sigma_2 = 0.3$ and different initial values $(S(0), I(0)) = (0.6, 0.2)$, $(S(0), I(0)) = (0.7, 0.2)$, $(S(0), I(0)) = (0.8, 0.2)$; (d) global attraction for the stochastic system (1.2) with $\sigma_1 = 0.015$, $\sigma_2 = 0.55$ and different initial values $(S(0), I(0)) = (0.6, 0.2)$, $(S(0), I(0)) = (0.7, 0.2)$, $(S(0), I(0)) = (0.8, 0.2)$.

Although Example 6.2 and Example 6.3 are just pathway simulations, they can also confirm our results from another aspect.

It shows that when the disease becomes extinct, the disease-free solution $S^*(t)$ of the deterministic model will display periodic behavior after some time. The stochastic solution $S(t)$ of stochastic model (1.2) will fluctuate in a very small neighborhood around the deterministic periodic solution when the white noise σ_1 is not so big, which indicates the existence of the positive stochastic boundary periodic solution. It also shows that the amplitude of the oscillation around the trajectory of the deterministic periodic solution depends on the intensity of white noise ($\sigma_1 = 0.020$ or $\sigma_1 = 0.015$).

We note that the pathways in case (c) and (d) of Figure 2 overlap each other very well which implies that wherever $S(t)$ start from, the density functions of $S(t)$ converge to the disease-free periodic solution respectively.

In summary, the simulations in Figure 2 confirm our conclusion that the disease-free periodic solution is global attractive under assumed conditions.

Moreover, the global attraction of stochastic periodic solution is analogue to the convergence of density functions to a a stationary distribution. But it is hard to demonstrate because it is a spectrum of density functions. So here we use pathway simulation to substitute it. However, if readers are interested in the convergence of density functions to a stationary distribution, it could be found in Figure 1 and 2 in Lin's paper [27].

Example 6.3. In order to show the existence of the nontrivial periodic solution of System (1.2), we describe the dynamic behaviors of deterministic system and stochastic system in phase portrait respectively by choosing initial value $S(0) = 0.7, I(0) = 0.2$ and parameters as following:

$$\mu = 0.08,\ K = 1,\ \beta = 0.87,\ \alpha = 0.05,\ \lambda = 0.22,\ p = 0.1,$$

and

$$(a) \text{ deterministic system; } (b)\ \sigma_1 = 0.02,\ \sigma_2 = 0.02;\ (c)\ \sigma_1 = 0.01,\ \sigma_2 = 0.01.$$

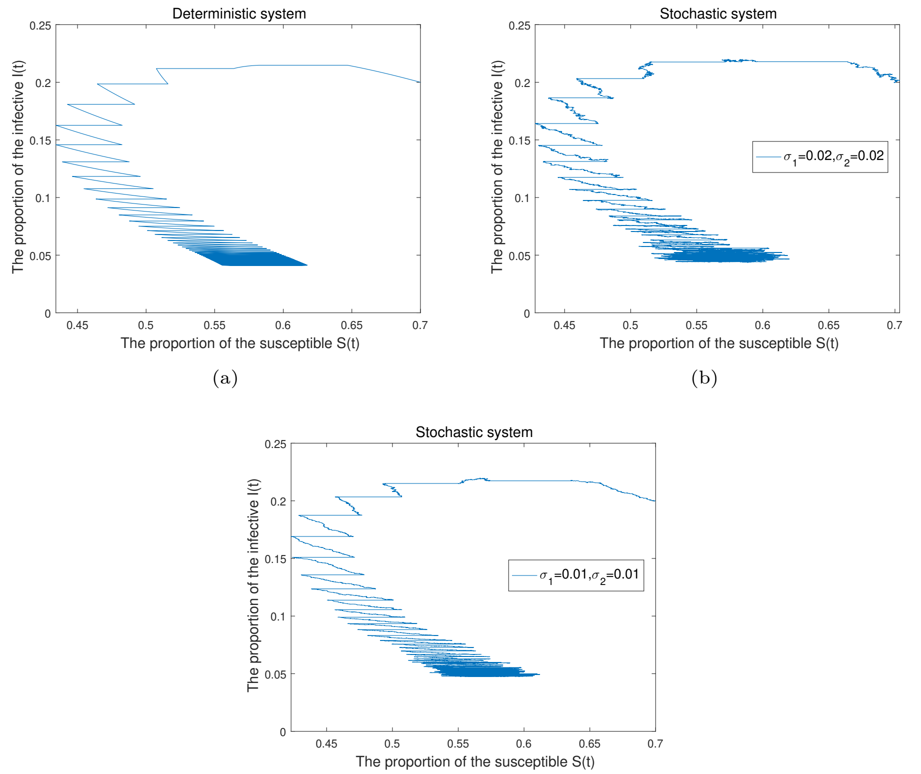
Therefore, the condition of Theorem 5.1 holds. We can see that, after a while, the trajectory of the deterministic solution goes into periodic orbit and the pathways of stochastic solution also show some measure of periodic behavior but with oscillations. Same as Example 2, the fluctuation of the stochastic pathways also depend on the intensities of white noises ($\sigma_1 = 0.02, \sigma_2 = 0.02$ or $\sigma_1 = 0.01, \sigma_1 = 0.02$).

7. Conclusion

In this paper, we study a stochastic SIR model with pulse vaccinations in which we assume random effects directly influence the susceptible, infective and indirectly the recovered group. First, we transform the impulsive stochastic model into equivalent stochastic system without pulses. Then we establish the threshold R_0^s: under extra mild condition $\mu \geq \frac{1}{2}(\sigma_1^2 \vee \sigma_2^2)$, if $R_0^s < 1$ then the disease will go extinct; if $R_0^s > 1$, then the disease will prevail. We also prove that: if $\mu \geq \frac{1}{2}(\sigma_1^2 \vee \sigma_2^2)$ and $R_0^s < 1$ are satisfied, then there exists a disease-free periodic solution which is globally attractive; if $R_0^s > 1$, then there exists at least one positive periodic solution which means the disease will persist.

The environmental noises play an important role in determining the epidemic dynamics. It follows from Theorem 3.1 and Figure 1. that white noises reduce the basic reproduction number R_0 and the disease will die out if $R_0^s < 1$ even as $R_0 > 1$. Therefore, white noises help to suppress the spread of disease. According to Theorem 4.1 and Theorem 5.1, the existences of disease-free periodic solution and positive periodic solution are governed by R_0^s which indicates that the noises can influence the long time behavior of the disease.

Pulse vaccinations also have effects on the dynamic behavior of disease. From Equation (3.3), one can see that $\langle S\rangle_t \leq \langle S^*\rangle_1 + o(t)$, where $o(t)$ is an infinitesimal of t. It means that PVS can also largely reduce the susceptible in the existence of



(a) (b) (c)

Figure 3. Simulations of phase trajectories of positive periodic solutions $(S(t), I(t))$. (a) the deterministic system (1.1); (b) the stochastic system (1.2) with with $\sigma_1 = 0.02$, $\sigma_2 = 0.02$; (c) the stochastic system (1.2) with with $\sigma_1 = 0.01$, $\sigma_2 = 0.01$.

environment noises. Moreover, Theorem 4.1 and Theorem 5.1 verify the existence of periodic solutions even as the impulsive stochastic model (1.2) has no periodic coefficient, which implies the periodicity comes from the periodic pulse vaccinations.

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