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MIXED MONOTONE ITERATIVE TECHNIQUE FOR HILFER FRACTIONAL EVOLUTION EQUATIONS WITH NONLOCAL CONDITIONS*

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Abstract The purpose of this paper is concerned with the existence of mild L-quasi-solutions for Hilfer fractional evolution equations with nonlocal conditions in an ordered Banach spaces E. By employing mixed monotone iterative technique, measure of noncompactness and Sadovskii's fixed point theorem, we obtain the existence of mild L-quasi-solutions for Hilfer fractional evolution equations with noncompact semigroups. Finally, an example is provide to illustrate the feasibility of our main results.

Keywords Mixed monotone iterative technique, coupled L-quasi-upper and lower solutions, Hilfer fractional derivative, measure of noncompactness.

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1. Introduction

Fractional differential equations provide an excellent instrument for the description of memory and hereditary properties of various materials and processes and there has been a significant development in fractional differential equations theory. Hilfer [14] proposed a generalized Riemann-Liouville fractional derivative, for short, Hilfer fractional derivative, which includes Riemann-Liouville fractional derivative and Caputo fractional derivative. This operator appeared in the theoretical simulation of dielectric relaxation in glass forming materials.

In recent years, many authors began to consider Hilfer fractional differential equations, we refer the reader to [1,2,9,11,12,14,15,30]. Hilfer fractional evolution equations has also been widely concerned by many scholars. In [11], Gu and Trujillo investigated a class of Hilfer fractional evolution equations, and established the existence results of mild solutions by using fixed point theorem.

Later, the nonlocal problems have better effects in applications than the initial problem, many contributions have been made in applications of fractional evolution equations with nonlocal conditions, see [20, 23, 24] and the reference therein. In [20], Liang and Yang investigated the exact controllability of the nonlocal Cauchy

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problem for the fractional integro differential evolution equations in Banach spaces

$$\begin{cases} D^q x(t) + Ax(t) = f(t, x(t), Gx(t)) + Bu(t), \quad t \in J, \\ x(0) = \sum\limits_{k=1}^{m} c_k x(t_k), \end{cases}$$

where D^q denotes the Caputo fractional derivative of order $q \in (0,1), -A : D(A) \subset E \to E$ is the infinitesimal generator of a C_0-semigroup $T(t)(t \geq 0)$ of uniformly bounded linear operator, B is a linear bounded operator; f is a given function and the operator is given by

$$Gx(t) = \int_0^t K(t,s)x(s)ds.$$

In [1], Hamdy M. Ahmed et al. studied the existence of mild solutions for Hilfer fractional stochastic integro-differential equations of the form

$$\begin{cases} D_{0+}^{\nu;\mu}[u(t) + F(t, v(t))] + Au(t) = \int_0^t G(s, \eta(s))d\omega(s), \quad t \in J := (0, b], \\ I_{0+}^{(1-\nu)(1-\mu)} u(0) - g(u) = u_0, \end{cases}$$

where $(t, v(t)) = (t, u(t), u(b_1(t))), \ldots, u(b_m(t)))$
and $(t, \eta(t)) = (t, u(t), u(a_1(t))), \ldots, u(a_n(t)))$, $D_{0+}^{\nu;\mu}$ denotes the Hilfer fractional derivative $0 \leq \nu \leq 1, 0 < \mu < 1$, $-A$ is the infinitesimal generator of an analytic semigroup of bounded linear operators $S(t), t \geq 0$ on a separable Hilbert space.

On the other hand, by employing the method of lower and upper to study the existence of extremal mild solution for fractional evolution equation is an interesting issue, which has been attention in [6,21,23,24,26]. In [6], Chen and Li used monotone iterative technique in the presence of coupled lower and upper L-quasi-solutions to discuss the existence of mild solutions to the initial value problem of impulsive evolution equations in an ordered Banach space E:

$$\begin{cases} u'(t) + Au(t) = f(t, u(t), u(t)), \quad t \in J = [0, b], \ t \neq t_k, \\ \Delta u|_{t=t_k} = I_k(u(t_k), u(t_k)), \quad k = 1, 2, \ldots, m, \\ u(0) = u_0, \end{cases}$$

where $A : D(A) \subset E \to E$ is a closed linear operator and $-A$ generates a C_0-semigroup $T(t)(t \geq 0)$ on E, $f \in C(J \times E \times E, E), J = [0, b], b > 0$ is a constant, $0 < t_1 < t_2 < \cdots < t_p, p \in \mathbb{N}$, $I_k \in C(E \times E, E)$ is an impulsive function, $k = 1, 2, \ldots, p; u_0 \in E$.

In [27], Vikram Singh et al. investigated the existence and uniqueness of mild solutions for Sobolev type fractional impulsive differential systems with nonlocal conditions

$$\begin{cases} {}^cD^\beta[Bu(t)] = Au(t) + f(t, u(t), \int_0^t K(t, s, u(s))ds), \ t \in J = [0, a], t \neq t_j, \\ \Delta u|_{t=t_j} = I_j(u(t_j)), \quad j = 1, 2, \ldots, m, m \in \mathbb{N}, \\ {}^LD^{1-\beta}[Tu(0)] = u_0 + g(u(t)). \end{cases}$$

By applying monotone iterative technique combined with the method of lower and upper solutions.

However, there are few papers that study Hilfer fractional evolution equations with nonlocal problems by applying the mixed monotone iterative technique and coupled L-quasi-upper and lower solutions. Motivated above discussion, in this paper, we use the fixed point theorem combined with mixed monotone iterative technique to discuss the existence of mild L-quasi-solutions for Hilfer fractional evolution equations with nonlocal conditions

$$\begin{cases} D_{0+}^{\nu,\mu}u(t) + Au(t) = f(t, u(t), u(t)), \quad t \in (0, b], \\ I_{0+}^{1-\gamma}u(0) = u_0 + \sum\limits_{i=1}^{m} \lambda_i u(\tau_i), \quad \tau_i \in (0, b], \end{cases} \tag{1.1}$$

where $D_{0+}^{\nu,\mu}$ denotes the Hilfer fractional derivative of order μ and type ν, which will be given in the next section, $0 \le \nu \le 1$, $\frac{1}{2} < \mu < 1$, $\gamma = \nu + \mu - \nu\mu$, the state $u(\cdot)$ takes value in a Banach space E with norm $\|\cdot\|$ and $-A : D(A) \subset E \to E$ is the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\ge0}$ of uniformly bounded linear operator in E. $J = [0, b](b > 0), J' = (0, b]$, $f : J' \times E \times E \to E$ is given functions satisfying some assumptions, $u_0 \in E$ and $\tau_i (i = 1, 2, \ldots, m)$ are prefixed points satisfying $0 < \tau_1 \le \cdots \le \tau_m < b$ and λ_i are real numbers. Here the nonlocal condition $I_{0+}^{1-\gamma}u(0) = u_0 + \sum\limits_{i=1}^{m} \lambda_i u(\tau_i)$ can be applied in physical problem better effect than the initial conditions $I_{0+}^{1-\gamma}u(0) = u_0$.

The rest of this paper is organized as follows: In Section 2, we review some Lemmas and notations. In Section 3, we prove the existence of mild L-quasi-solutions for Hilfer fractional differential system (1.1). In Section 4, an example is given to illustrate the effectiveness of the our results.

2. Preliminaries

Throughout this paper, by $C(J, E)$ and $C(J', E)$, we denote the spaces of all continuous functions from J to E and J' to E, respectively. Let E be an ordered Banach space with the norm $\|\cdot\|$ and partial order $\le$, whose positive cone $P = \{x \in E : x \ge \theta\}$ is normal with normal constant N.

Define $C_{1-\gamma}(J, E) = \Big\{u \in C(J', E) : t^{1-\gamma}u(t) \in C(J, E)\Big\}$. Clearly, $C_{1-\gamma}(J, E)$ is a Banach space with the norm $\|u\|_\gamma = \sup_{t\in J'} |t^{1-\gamma}u(t)|$. And $C_{1-\gamma}(J, E)$ is also an ordered Banach space with the partial order $\le$ induced by the positive cone $P' = \{u \in C_{1-\gamma}(J, E) | u(t) \ge \theta, t \in J\}$ which is also normal with the same normal constant N.

First, we recall some definitions and basic results on fractional calculus, for more details see [9, 11, 15, 19, 30].

Definition 2.1. The Riemann-Liouville fractional integral of order γ of a function $f : [0, \infty) \to R$ is defined as

$$I_{0+}^{\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t - s)^{\gamma-1} f(s)ds, \ t > 0, \gamma > 0,$$

provided the right side is point-wise defined on $[0, \infty)$.

Definition 2.2. The Riemann-Liouville derivative of order γ with the lower limit zero for a function $f : [0, \infty) \to R$ can be written as

$$D_{0+}^{\gamma} f(t) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\gamma+1-n}} ds, \quad t > 0, n-1 < \gamma < n.$$

Definition 2.3. The Caputo fractional derivative of order γ for a function $f : [0, \infty) \to R$ can be written as

$$^{c}D_{0+}^{\gamma} f(t) = D_{0+}^{\gamma}\Big[f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0)\Big], \ t > 0, \ n-1 < \gamma < n,$$

where $n = [\gamma] + 1$ and $[\gamma]$ denotes the integer part of γ.

Definition 2.4 (Hilfer fractional derivative see [14])**.** The generazlied Riemann-Liouville fractional derivative of order $0 \leq \nu \leq 1$ and $0 < \mu < 1$ with lower limit a is defined as

$$D_{a+}^{\nu,\mu} f(t) = I_{a+}^{\nu(1-\mu)} \frac{d}{dt} I_{a+}^{(1-\nu)(1-\mu)} f(t)$$

for functions such that the expression on the right hand side exists.

Remark 2.1. For $0 < \mu < 1$, the Laplace transformation of Hilfer fractional derivatives is given by

$$\mathcal{L}[D_{0+}^{\mu,\nu} f(x)](\lambda) = \lambda^{\mu} \mathcal{L}[f(x)](\lambda) - \lambda^{\nu(\mu-1)} (I_{0+}^{(1-\nu)(1-\mu)} f)(0+),$$

where $(I_{0+}^{(1-\nu)(1-\mu)} f)(0+)$ is the Riemann-Liouville fractional integral of order $(1-\nu)(1-\mu)$ in the limits as $t \to 0+$, and

$$\mathcal{L}[f(x)](\lambda) = \int_0^{\infty} e^{-\lambda x} f(x) dx. \tag{2.1}$$

The symbol $\alpha(\cdot)$ is the Kuratowski noncompactness measure defined on bounded subset Ω of E. For any $\Omega \subset C(J, E)$ and $t \in J$, set $\Omega(t) = \{u(t) : u \in B\} \subset E$. If B is bounded in $C(J, E)$, then $\Omega(t)$ is bounded in E, and $\alpha(\Omega(t)) \leq \alpha(\Omega)$.

Lemma 2.1 ([18])**.** *Let $B \subset C(J, E)$ be bounded and equicontinuous, then $\overline{co}B \subset C(J, E)$ is also bounded and equicontinuous.*

Lemma 2.2 ([17])**.** *Let E be a Banach space, and let $D \subset E$ be bounded. Then there exists a countable set $D_0 \subset D$, such that $\alpha(D) \leq 2\alpha(D_0)$.*

Lemma 2.3 ([10])**.** *Let E be a Banach space, and let $\Omega \subset C(J, E)$ is equicontinuous and bounded, then $\alpha(\Omega(t))$ is continuous on J, and $\alpha(\Omega) = \max_{t \in J} \alpha(\Omega(t))$.*

Lemma 2.4 ([13])**.** *Let $\Omega = \{u_n\}_{n=1}^{\infty} \subset C(J, E)$ be a bounded and countable set and there exists a function $m \in L^1(J, R^+)$ such that for every $n \in N$,*

$$\|u_n(t)\| \leq m(t), \quad a.e. t \in J.$$

Then $\alpha(\Omega(t))$ is Lebesgue integral on J, and

$$\alpha\Big(\Big\{ \int_J u_n(t) dt : n \in \mathbb{N} \Big\}\Big) \leq 2 \int_J \alpha(\Omega(t)) dt.$$

Lemma 2.5 ([11]). *Assume that $-A$ is the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ of uniformly bounded linear operator in E. If $f \in C_{1-\gamma}(J,E)$, for any $u \in C_{1-\gamma}(J,E)$, a function u is a solution of the equation*

$$\begin{cases} D_{0+}^{\nu,\mu}u(t) + Au(t) = f(t,u(t),u(t)), \ t \in J', \\ I_{0+}^{1-\gamma}u(0) = u_0, \end{cases} \tag{2.2}$$

if and only if u satisfies the following integral equation:

$$u(t) = S_{\nu,\mu}(t)u_0 + \int_0^t K_\mu(t-s)f(s,u(s),u(s))ds,$$

where

$$S_{\nu,\mu}(t) = I_{0+}^{\nu(1-\mu)}K_\mu(t), \quad K_\mu(t) = \mu\int_0^\infty \sigma t^{\mu-1}\xi_\mu(\sigma)T(t^\mu\sigma)u_0 d\sigma, \tag{2.3}$$

the function ξ_μ is the function of Wright type:

$$\xi_\mu(\sigma) = \frac{1}{\pi\mu}\sum_{n=1}^{\infty}(-\sigma)^{n-1}\frac{\Gamma(n\mu+1)}{n!}\sin(n\pi\mu), \ \sigma \in (0,\infty).$$

Lemma 2.6 ([11]). *Assume that A generate a C_0-semigroup $\{T(t)\}_{t\geq 0}$ of uniformly bounded linear operator in E and $T(t)$ is continuous in the uniform operator topology for $t > 0$. That is, there exists $M \geq 1$ such that $\sup_{t\in[0,+\infty)}|T(t)| \leq M$. Then the operators $S_{\nu,\mu}(t)$ and $K_\mu(t)$ have the following properties.*

(i) For any fixed $t \geq 0$, $\{S_{\nu,\mu}(t)\}_{t>0}$ and $\{K_\mu(t)\}_{t>0}$ are linear operators, and for any $u \in E$,

$$\|S_{\nu,\mu}(t)u\| \leq \frac{Mt^{\gamma-1}}{\Gamma(\gamma)}\|u\|, \quad \|K_\mu(t)u\| \leq \frac{Mt^{\mu-1}}{\Gamma(\mu)}\|u\|.$$

(ii) The operators $S_{\nu,\mu}(t)$ and $K_\mu(t)$ are strongly continuous for all $t \geq 0$.

(iii) If $T(t)(t \geq 0)$ is an equicontinuous semigroup, then $S_{\nu,\mu}(t)$ and $K_\mu(t)$ are equicontinuous in E for $t > 0$.

Definition 2.5. A function $u \in C_{1-\gamma}(J,E)$ is said to be a mild solution of (2.2) if $u_0 \in E$ the integral equation

$$u(t) = S_{\nu,\mu}(t)u_0 + \int_0^t K_\mu(t-s)f(s,u(s),u(s))ds,$$

is satisfied, for all $t \in J'$.

Next, we present useful lemma which plays an important role in our main results.

Lemma 2.7. *Suppose A is the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ of uniformly bounded linear operator in E, for $0 \leq \nu \leq 1, 0 < \mu < 1$, then*

$$D_{0+}^{\nu,\mu}\Big(S_{\nu,\mu}(t)u_0\Big) = -A\Big(S_{\nu,\mu}(t)u_0\Big),$$

and

$$\begin{aligned} &D_{0+}^{\nu,\mu}\Big(\int_0^t K_\mu(t-s)f(s,u(s),u(s))ds\Big) \\ =& -A\int_0^t K_\mu(t-s)f(s,u(s),u(s))ds + f(t,u(t),u(t)). \end{aligned} \tag{2.4}$$

Proof. Let $\lambda > 0$, we consider the one sided stable probability density as follows

$$\varpi_\mu(\sigma) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\sigma^{-\mu n-1}\frac{\Gamma(n\mu+1)}{n!}\sin(n\pi\mu),\ \sigma\in(0,\infty),$$

whose Laplace transform is given by

$$\int_0^\infty e^{-\lambda\sigma}\varpi_\mu(\sigma)d\sigma = e^{-\lambda^\mu},\quad \mu\in(0,1). \tag{2.5}$$

Then, using (2.5), we have

$$\begin{aligned}
(\lambda^\mu I + A)^{-1}u &= \int_0^\infty e^{-\lambda^\mu s}T(s)u_0 ds = \int_0^\infty \mu t^{\mu-1}e^{-(\lambda t)^\mu}T(t^\mu)udt\\
&= \int_0^\infty\int_0^\infty e^{-(\lambda t\sigma)}\mu t^{\mu-1}\varpi_\mu(\sigma)W(t^\mu)ud\sigma dt\\
&= \mu\int_0^\infty\int_0^\infty e^{-\lambda\theta}\frac{\theta^{\mu-1}}{\sigma^\mu}\varpi_\mu(\sigma)T\Big(\frac{\theta^\mu}{\sigma^\mu}\Big)ud\theta d\sigma\\
&= \int_0^\infty e^{-\lambda\tau}\Big[\mu\int_0^\infty\frac{\tau^{\mu-1}}{\sigma^\mu}\varpi_\mu(\sigma)T\Big(\frac{\tau^\mu}{\sigma^\mu}\Big)ud\sigma\Big]d\tau\\
&= \int_0^\infty e^{-\lambda t}\Big[\mu\int_0^\infty\frac{t^{\mu-1}}{\sigma^\mu}\varpi_\mu(\sigma)T\Big(\frac{t^\mu}{\sigma^\mu}\Big)ud\sigma\Big]dt\\
&= \int_0^\infty e^{-\lambda t}\Big[\mu\int_0^\infty \sigma t^{\mu-1}\xi_\mu(\sigma)T(t^\mu\sigma)ud\sigma\Big]dt\\
&= \int_0^\infty e^{-\lambda t}K_\mu(t)udt,
\end{aligned}\tag{2.6}$$

where ξ_μ is a probability density function defined on $(0,\infty)$ such that

$$\xi_\mu(\sigma) = \frac{1}{\mu}\sigma^{-1-\frac{1}{\mu}}\varpi_\mu(\sigma^{-\frac{1}{\mu}}) \geq 0.$$

Since the Laplace inverse transform of $\lambda^{\nu(\mu-1)}$ is

$$\mathcal{L}^{-1}(\lambda^{\nu(\mu-1)}) = \begin{cases}\frac{t^{\nu(1-\mu)-1}}{\Gamma(\nu(1-\mu))},\ 0<\nu\leq 1,\\ \delta(t),\quad \nu=0,\end{cases}\tag{2.7}$$

where $\delta(t)$ is the Delta function.

It follows from (2.6), (2.7) and Laplace transform, it is obvious to see that

$$\begin{aligned}
\mathcal{L}(S_{\nu,\mu}(t)u_0) &= \mathcal{L}(I_{0+}^{\nu(1-\mu)}K_\mu(t)u_0)\\
&= \mathcal{L}\Big(\frac{t^{\nu(1-\mu)-1}}{\Gamma(\nu(1-\mu))}*K_\mu(t)u_0\Big)\\
&= \mathcal{L}\Big(\mathcal{L}^{-1}(\lambda^{\nu(\mu-1)})*K_\mu(t)u_0\Big)\\
&= \lambda^{\nu(\mu-1)}(\lambda^\mu I+A)^{-1}u_0,
\end{aligned}\tag{2.8}$$

where $*$ denotes the convolution of functions. By Remark 2.2, we obtain

$$\mathcal{L}(D_{0+}^{\nu,\mu}[S_{\nu,\mu}(t)u_0]) = \lambda^\mu\mathcal{L}(S_{\nu,\mu}(t)u_0) - \lambda^{\nu(\mu-1)}u_0$$

$$
\begin{aligned}
&= \lambda^{\mu}\Big[\lambda^{\nu(\mu-1)}(\lambda^{\mu}I + A)^{-1}\Big]u_0 - \lambda^{\nu(\mu-1)}u_0 \\
&= \lambda^{\nu(\mu-1)}(\lambda^{\mu}I + A)^{-1}\Big[\lambda^{\mu} - (\lambda^{\mu} + A)\Big]u_0 \\
&= \lambda^{\nu(\mu-1)}(\lambda^{\mu}I + A)^{-1}\Big[\lambda^{\mu} - \lambda^{\mu} - A\Big]u_0 \\
&= -\lambda^{\nu(\mu-1)}(\lambda^{\mu}I + A)^{-1}Au_0 \\
&= -A\lambda^{\nu(\mu-1)}(\lambda^{\mu}I + A)^{-1}u_0. \qquad (2.9)
\end{aligned}
$$

Combing (2.8) and (2.9) yields

$$
D_{0+}^{\nu,\mu}[S_{\nu,\mu}(t)u_0] = -A[S_{\nu,\mu}(t)u_0].
$$

Similarly, we have

$$
\mathcal{L}\Big(\int_0^t K_{\mu}(t-s)f(s,u(s),u(s))ds\Big) = \mathcal{L}(K_{\mu}(t)) \cdot \mathcal{L}\Big(f(t,u(t),u(t))\Big), \qquad (2.10)
$$

and

$$
\begin{aligned}
&\mathcal{L}\Big(D_{0+}^{\nu,\mu}\Big[\int_0^t K_{\mu}(t-s)f(s,u(s),u(s))ds\Big]\Big) \\
=&\lambda^{\mu}\mathcal{L}\Big(\int_0^t K_{\mu}(t-s)f(s,u(s),u(s))ds\Big) - \lambda^{\nu(\mu-1)} \cdot 0 \\
=&\lambda^{\mu}\mathcal{L}(K_{\mu}(t)) \cdot \mathcal{L}\Big(f(t,u(t),u(t))\Big) \\
=&\lambda^{\mu}(\lambda^{\mu}I + A)^{-1} \cdot \mathcal{L}\Big(f(t,u(t),u(t))\Big) \\
=&(\lambda^{\mu}I + A - A)(\lambda^{\mu}I + A)^{-1} \cdot \mathcal{L}\Big(f(t,u(t),u(t))\Big) \\
=&-A(\lambda^{\mu}I + A)^{-1} \cdot \mathcal{L}\Big(f(t,u(t),u(t))\Big) + \mathcal{L}\Big(f(t,u(t),u(t))\Big). \qquad (2.11)
\end{aligned}
$$

Thus, it follows from (2.10) and (2.11) that

$$
\begin{aligned}
&D_{0+}^{\nu,\mu}\Big[\int_0^t K_{\mu}(t-s)f(s,u(s),u(s))ds\Big] \\
=&-A\int_0^t K_{\mu}(t-s)f(s,u(s),u(s))ds + f(t,u(t),u(t)). \qquad (2.12)
\end{aligned}
$$

□

For the convenience of discussion, we assume that

(H0) Assume A generate a C_0-semigroup $\{T(t)\}_{t\geq 0}$ of uniformly bounded linear operator in E and $T(t)$ is continuous in the uniform operator topology for $t > 0$. That is, there exists $M \geq 1$ such that $\sup_{t\in[0,+\infty)} \|T(t)\| \leq M$.

(H1) $\lambda_i > 0 (i = 1, 2, \ldots, m)$ and $\sum\limits_{i=1}^{m} \lambda_i < \frac{\Gamma(\gamma)}{Mb^{\gamma-1}}$.

In view of [6] and [20], we present the following lemma.

Lemma 2.8. *Assume that (H0) and (H1) holds. For any $u \in C_{1-\gamma}(J,E)$ such that $f(\cdot,u(\cdot),u(\cdot)) \in C_{1-\gamma}(J\times E\times E,E)$, then the problem (1.1) has a unique mild solution $u \in C_{1-\gamma}(J)$ given by*

$$\begin{aligned} u(t) =&S_{\nu,\mu}(t)\overline{\Theta}u_0 + \sum_{i=1}^{m}\lambda_i S_{\nu,\mu}(t)\overline{\Theta}\int_0^{\tau_i} K_\mu(\tau_i - s)f(s,u(s),u(s))ds \\ &+ \int_0^t K_\mu(t-s)f(s,u(s),u(s))ds, \end{aligned} \tag{2.13}$$

where $\overline{\Theta} = \Big[I - \sum_{i=1}^m \lambda_i S_{\nu,\mu}(\tau_i)\Big]^{-1}$.

Proof. By assumption (H1), we have

$$\Big\|\sum_{i=1}^m \lambda_i S_{\nu,\mu}(t)\Big\| \le \sum_{i=1}^m |\lambda_i|\cdot\Big\|S_{\nu,\mu}(t)\Big\| \le \sum_{i=1}^m |\lambda_i|\frac{Mb^{\gamma-1}}{\Gamma(\gamma)} < 1.$$

By operator spectrum theorem, the operator $\overline{\Theta} := \Big(I - \sum_{i=1}^m \lambda_i S_{\nu,\mu}(\tau_i))\Big)^{-1}$ exists and is bounded. Furthermore, by Neumann expression, we obtain

$$\|\overline{\Theta}\| \le \sum_{i=0}^{\infty}\|\sum_{i=1}^m \lambda_i S_{\nu,\mu}(\tau_i)\|^n = \frac{1}{1-\|\sum_{i=1}^m \lambda_i S_{\nu,\mu}(\tau_i)\|} \le \frac{1}{1-\frac{Mb^{\gamma-1}}{\Gamma(\gamma)}\sum_{i=1}^m \lambda_i}.$$

According to Definition 2.5, a solution of system (2.2) can be expressed by

$$u(t) = S_{\nu,\mu}(t)I_{0+}^{1-\gamma}u(0) + \int_0^t K_\mu(t-s)f(s,u(s),u(s))ds. \tag{2.14}$$

Next, we substitute $t = \tau_i$ into (2.13) and by applying λ_i to both side of (2.13), we have

$$\lambda_i u(\tau_i) = \lambda_i S_{\nu,\mu}(\tau_i)I_{0+}^{1-\gamma}u(0) + \lambda_i\int_0^{\tau_i} K_\mu(\tau_i - s)f(s,u(s),u(s))ds. \tag{2.15}$$

Thus, we have

$$\begin{aligned} I_{0+}^{1-\gamma}u(0) &= u_0 + \sum_{i=1}^m \lambda_i u(\tau_i) \\ &= u_0 + \sum_{i=1}^m \lambda_i S_{\nu,\mu}(\tau_i)I_{0+}^{1-\gamma}u(0) + \sum_{i=1}^m \lambda_i\int_0^{\tau_i} K_\mu(\tau_i - s)f(s,u(s),u(s))ds \\ &= u_0 + \sum_{i=1}^m \lambda_i S_{\nu,\mu}(\tau_i)I_{0+}^{1-\gamma}u(0) + \sum_{i=1}^m \lambda_i\int_0^{\tau_i} K_\mu(\tau_i - s)f(s,u(s),u(s))ds. \end{aligned}$$

Since $I - \sum_{i=1}^m \lambda_i S_{\nu,\mu}(\tau_i)$ has a bounded inverse operator $\overline{\Theta}$, which implies

$$\begin{aligned} I_{0+}^{1-\gamma}u(0) &= \Big[I - \sum_{i=1}^m \lambda_i S_{\nu,\mu}(\tau_i)\Big]^{-1}\Big(u_0 + \sum_{i=1}^m \lambda_i\int_0^{\tau_i} K_\mu(\tau_i - s)f(s,u(s),u(s))ds\Big) \\ &= \overline{\Theta}u_0 + \sum_{i=1}^m \lambda_i\int_0^{\tau_i}\overline{\Theta}K_\mu(\tau_i - s)f(s,u(s),u(s))ds. \end{aligned} \tag{2.16}$$

Submitting (2.16) to (2.14), we obtain that (2.13). It is imply that u is also a solution of the integral of Eq.(2.13) when u is a solution of system (2.12).

The necessity has been proved. Next, we will prove its sufficiency. Applying $I_{0+}^{1-\gamma}$ to both side of (2.12), and by Lemma 2.7, we have

$$\begin{aligned} I_{0+}^{1-\gamma}u(t) =& I_{0+}^{1-\gamma}\Big(S_{\nu,\mu}(t)\overline{\Theta}u_0 + \sum_{i=1}^{m}\lambda_i S_{\nu,\mu}(t)\overline{\Theta}\int_0^{\tau_i}K_\mu(\tau_i - s)f(s,u(s),u(s))ds \\ &+ \int_0^t K_\mu(t-s)f(s,u(s),u(s))ds\Big). \end{aligned}$$

Therefore, we have

$$\begin{aligned} \lim_{t\to 0} I_{0+}^{1-\gamma}u(t) =& \lim_{t\to 0} I_{0+}^{1-\gamma}S_{\nu,\mu}(t)\overline{\Theta}u_0 \\ &+ \sum_{i=1}^{m}\lambda_i \lim_{t\to 0} I_{0+}^{1-\gamma}S_{\nu,\mu}(t)\overline{\Theta}\int_0^{\tau_i}K_\mu(\tau_i - s)f(s,u(s),u(s))ds \\ =& I_{0+}^{1-\gamma}(\lim_{t\to 0} S_{\nu,\mu}(t)(\overline{\Theta}u_0) \\ &+ I_{0+}^{1-\gamma}\lim_{t\to 0} S_{\nu,\mu}(t)\sum_{i=1}^{m}\lambda_i\overline{\Theta}\int_0^{\tau_i}K_\mu(\tau_i - s)f(s,u(s),u(s))ds \\ =& I_{0+}^{1-\gamma}\Big(\frac{\overline{\Theta}u_0}{\Gamma(\gamma)}t^{\gamma-1}\Big) \\ &+ I_{0+}^{1-\gamma}\Big(\frac{\sum_{i=1}^{m}\lambda_i\overline{\Theta}\int_0^{\tau_i}K_\mu(\tau_i - s)f(s,u(s),u(s))ds}{\Gamma(\gamma)}t^{\gamma-1}\Big) \\ =& \overline{\Theta}u_0 + \sum_{i=1}^{m}\lambda_i\overline{\Theta}\int_0^{\tau_i}K_\mu(\tau_i - s)f(s,u(s),u(s))ds. \end{aligned} \tag{2.17}$$

Substituting $t = \tau_i$ into (2.12), we have

$$\begin{aligned} u(\tau_i) =& S_{\nu,\mu}(\tau_i)\overline{\Theta}u_0 + \sum_{i=1}^{m}\lambda_i S_{\nu,\mu}(\tau_i)\overline{\Theta}\int_0^{\tau_i}K_\mu(\tau_i - s)f(s,u(s),u(s))ds \\ &+ \int_0^{\tau_i}K_\mu(\tau_i - s)f(s,u(s),u(s))ds. \end{aligned}$$

Then, we obtain

$$\begin{aligned} & u_0 + \sum_{i=1}^{m}\lambda_i u(\tau_i) \\ =& u_0 + \sum_{i=1}^{m}\lambda_i S_{\nu,\mu}(\tau_i)\overline{\Theta}u_0 \\ &+ \sum_{i=1}^{m}\lambda_i\sum_{i=1}^{m}\lambda_i S_{\nu,\mu}(\tau_i)\overline{\Theta}\int_0^{\tau_i}K_\mu(\tau_i - s)f(s,u(s),u(s))ds \\ &+ \sum_{i=1}^{m}\lambda_i\int_0^{\tau_i}K_\mu(\tau_i - s)f(s,u(s),u(s))ds \end{aligned}$$

$$
\begin{aligned}
&=\Big(I+\sum_{i=1}^{m}\lambda_i S_{\nu,\mu}(\tau_i)\overline{\Theta}\Big)\Big(u_0+\sum_{i=1}^{m}\lambda_i\int_0^{\tau_i}K_\mu(\tau_i-s)f(s,u(s),u(s))ds\Big)\\
&=\Big(\overline{\Theta}^{-1}+\sum_{i=1}^{m}\lambda_i S_{\nu,\mu}(\tau_i)\Big)\Big(\overline{\Theta}u_0+\sum_{i=1}^{m}\lambda_i\overline{\Theta}\int_0^{\tau_i}K_\mu(\tau_i-s)f(s,u(s),u(s))ds\Big)\\
&=\overline{\Theta}u_0+\sum_{i=1}^{m}\lambda_i\overline{\Theta}\int_0^{\tau_i}K_\mu(\tau_i-s)f(s,u(s),u(s))ds. \qquad (2.18)
\end{aligned}
$$

It follows (2.16) and (2.17) that $I_{0+}^{1-\gamma}u(0)=u_0+\sum_{i=1}^{m}\lambda_i u(\tau_i)$.

Next, by using $D_{0+}^{\nu,\mu}$ to both sides of (2.12) and Lemma 2.9, we have

$$
\begin{aligned}
&D_{0+}^{\nu,\mu}u(t)\\
=&D_{0+}^{\nu,\mu}\Big[S_{\nu,\mu}(t)\overline{\Theta}u_0+\sum_{i=1}^{m}\lambda_i S_{\nu,\mu}(t)\overline{\Theta}\int_0^{\tau_i}K_\mu(\tau_i-s)f(s,u(s),u(s))ds\\
&+\int_0^{t}K_\mu(t-s)f(s,u(s),u(s))ds\Big]\\
=&D_{0+}^{\nu,\mu}\Big[S_{\nu,\mu}(t)\overline{\Theta}u_0+\sum_{i=1}^{m}\lambda_i S_{\nu,\mu}(t)\overline{\Theta}\int_0^{\tau_i}K_\mu(\tau_i-s)f(s,u(s),u(s))ds\Big]\\
&+D_{0+}^{\nu,\mu}\Big[\int_0^{t}K_\mu(t-s)f(s,u(s),u(s))ds\Big]\\
=&\Big[\overline{\Theta}u_0+\sum_{i=1}^{m}\lambda_i\overline{\Theta}\int_0^{\tau_i}K_\mu(\tau_i-s)f(s,u(s),u(s))ds\Big]D_{0+}^{\nu,\mu}\Big[S_{\nu,\mu}(t)\Big]\\
&+D_{0+}^{\nu,\mu}\Big[\int_0^{t}K_\mu(t-s)f(s,u(s),u(s))\Big]\\
=&-\Big[\overline{\Theta}u_0+\sum_{i=1}^{m}\lambda_i\overline{\Theta}\int_0^{\tau_i}K_\mu(\tau_i-s)f(s,u(s),u(s))ds\Big]AS_{\nu,\mu}(t)\\
&-A\int_0^{t}K_\mu(t-s)f(s,u(s),u(s))ds+f(t,u(t),u(t))\\
=&-A\Big(S_{\nu,\mu}(t)\overline{\Theta}u_0+\sum_{i=1}^{m}\lambda_i S_{\nu,\mu}(t)\overline{\Theta}\int_0^{\tau_i}K_\mu(\tau_i-s)f(s,u(s),u(s))ds\\
&+\int_0^{t}K_\mu(t-s)f(s,u(s),u(s))ds\Big)+f(t,u(t),u(t))\\
=&-Au(t)+f(t,u(t),u(t)).
\end{aligned}
$$

Hence,

$$D_{0+}^{\nu,\mu}u(t)+Au(t)=f(t,u(t),u(t)).$$

This proof is completed. □

From Lemma 2.8, we adopt the following definition of mild solution of the problem (1.1).

Definition 2.6. A function $u\in C_{1-\gamma}(J,E)$ is said to be a mild solution of the problem (1.1), if it satisfies the operator equation

$$\begin{aligned} u(t) =& S_{\nu,\mu}(t)\overline{\Theta}u_0 + \sum_{i=1}^{m} \lambda_i S_{\nu,\mu}(t)\overline{\Theta}\int_0^{\tau_i} K_\mu(\tau_i - s)f(s,u(s),u(s))ds \\ &+ \int_0^t K_\mu(t-s)f(s,u(s),u(s))ds, \quad t\in J', \end{aligned} \tag{2.19}$$

where the operators $S_{\nu,\mu}(t)$ and $K_\mu(t)$ are given by (2.3).

Definition 2.7. A C_0-semigroup $\{T(t)\}_{t\geq 0}$ in E is said to be positive, if order inequality $T(t)x \geq \theta$ holds for each $x\geq\theta, x\in E$ and $t\geq 0$.

Remark 2.2. For any $C\geq 0$, $-(A+CI)$ also generates a C_0-semigroup $S(t) = e^{-Ct}T(t)(t\geq 0)$ on E. And $S(t)(t\geq 0)$ is a positive C_0-semigroup if $T(t)(t\geq 0)$ is a positive C_0-semigroup. For the detail, see [17, 25].

For $u\in E$, we define two families $\{S^*_{\nu,\mu}(t)\}_{t\geq}$ and $\{K^*_\mu(t)\}_{t\geq 0}$ of operators by

$$S^*_{\nu,\mu}(t)u = I_{0+}^{\nu(1-\mu)}K^*_\mu(t)u, \quad K^*_\mu(t)u = \mu\int_0^\infty \sigma t^{\mu-1}\xi_\mu(\sigma)S(t^\mu\sigma)ud\sigma$$

where $\xi_\mu(\sigma)$ is given by (2.3).

Since $T(t)(t\geq 0)$ is positive, by Remark 2.4, it is easy know that $S(t)(t\geq 0)$ is also positive. And by the definition of $\xi_\mu(\sigma)$, the operators $S^*_{\nu,\mu}(t)$ and $K^*_\mu(t)$ are also positive for all $t\geq 0$.

To prove our main result, for any $C>0$, we consider the following the system

$$\begin{cases} D_{0+}^{\nu,\mu}u(t) + (A+CI)u(t) = f(t,u(t),u(t)) + Cu(t), \quad t\in(0,b], \\ I_{0+}^{(1-\nu)(1-\mu)}u(0) = u_0 + \sum\limits_{i=1}^{m}\lambda_i u(\tau_i), \quad \tau_i\in(0,b]. \end{cases} \tag{2.20}$$

First, we assume that

(F0) For any $C\geq 0$, $-(A+CI)$ also generates a C_0-semigroup $S(t) = e^{-Ct}T(t)(t\geq 0)$ on E and $S(t)$ is continuous in the uniform operator topology for $t>0$. That is, there exists $M^*\geq 1$ such that $\sup_{t\in[0,+\infty)}\|S(t)\|\leq M^*$.

(F1) $\lambda_i > 0 (i=1,2,\ldots,m)$ and $\sum\limits_{i=1}^{m}\lambda_i < \frac{\Gamma(\gamma)}{M^* b^{\gamma-1}}$.

By assumption (F1), we have

$$\Big\|\sum_{i=1}^{m}\lambda_i S^*_{\nu,\mu}(t)\Big\| \leq \frac{M^* b^{\gamma-1}}{\Gamma(\gamma)}\sum_{i=1}^{m}\lambda_i < 1.$$

By operator spectrum theorem, the operator $I - \sum_{i=1}^{m}\lambda_i S^*_{\nu,\mu}(\tau_i))$ has a bounded inverse operator

$$\Theta := \Big(I - \sum_{i=1}^{m}\lambda_i S^*_{\nu,\mu}(\tau_i))\Big)^{-1}.$$

Furthermore, by Neumann expression, $\overline{\Theta}$ can be expressed by

$$\Theta = \sum_{i=0}^{\infty}\Big(\sum_{i=1}^{m}\lambda_i S^*_{\nu,\mu}(\tau_i)\Big)^n.$$

By the positivity of C_0-semigroup $S(t)(t \geq 0)$, it is easy know that $S^*_{\nu,\mu}(t)$ is positive, we have

$$\Theta u = \sum_{i=0}^{\infty} \Big(\sum_{i=1}^{m} \lambda_i S^*_{\nu,\mu}(\tau_i) \Big)^n u \geq u \geq \theta, \ \forall u \geq \theta.$$

So, Θ is a positive operator, and

$$\|\Theta\| \leq \sum_{i=0}^{\infty} \| \sum_{i=1}^{m} \lambda_i S^*_{\nu,\mu}(\tau_i) \|^n = \frac{1}{1 - \| \sum_{i=1}^{m} \lambda_i S^*_{\nu,\mu}(\tau_i) \|} \leq \frac{1}{1 - \frac{M^* b^{\gamma-1}}{\Gamma(\gamma)} \sum_{i=1}^{m} \lambda_i}.$$

In view of Lemma 2.8, we present the following lemma.

Lemma 2.9. *Assume that (F0) and (F1) holds. For any $u \in C_{1-\gamma}(J, E)$ such that $f(\cdot, u(\cdot), u(\cdot)) \in C_{1-\gamma}(J \times E \times E, E)$, then the problem (2.20) has a unique mild solution $u \in C_{1-\gamma}(J)$ given by*

$$\begin{aligned} u(t) =& S^*_{\nu,\mu}(t)\Theta u_0 + \sum_{i=1}^{m} \lambda_i S^*_{\nu,\mu}(t)\Theta \int_0^{\tau_i} K^*_\mu(\tau_i - s)[f(s, u(s), u(s)) + Cu(s)]ds \\ &+ \int_0^t K^*_\mu(t-s)[f(s, u(s), u(s)) + Cu(s)]ds, \end{aligned} \tag{2.21}$$

where $\Theta = \Big[I - \sum_{i=1}^{m} \lambda_i S^*_{\nu,\mu}(\tau_i) \Big]^{-1}$.

From Lemma 2.9 and Definition 2.7, we state the following definition of mild solution of the problem (2.20).

Definition 2.8. A function $u \in C_{1-\gamma}(J, E)$ is said to be a mild solution of the problem (2.20), if for any $u \in C_{1-\gamma}(J, E)$, the integral equation

$$\begin{aligned} u(t) =& S^*_{\nu,\mu}(t)\Theta u_0 + \sum_{i=1}^{m} \lambda_i S^*_{\nu,\mu}(t)\Theta \int_0^{\tau_i} K^*_\mu(\tau_i - s)[f(s, u(s), u(s)) + Cu(s)]ds \\ &+ \int_0^t K^*_\mu(t-s)[f(s, u(s), u(s)) + Cu(s)]ds, \end{aligned}$$

is satisfied.

In the following, we will state some lemmas whose proofs are similar to those of the paper [11]. Here, we omit it.

Lemma 2.10. *Under assumption $(F0)$, the operators $S^*_{\nu,\mu}(t)$ and $K^*_\mu(t)$ have the following properties.*

(i) for any fixed $t > 0$, $\{K^*_\mu(t)\}_{t>0}$, and $\{S^*_{\nu,\mu}(t)\}_{t>0}$ are linear operators, and for any $u \in E$

$$\|K^*_\mu(t)\| \leq \frac{M^* t^{\mu-1}}{\Gamma(\mu)}, \qquad \|S^*_{\nu,\mu}(t)\| \leq \frac{M^* t^{\gamma-1}}{\Gamma(\gamma)}.$$

(ii) The operators $\{K^*_\mu(t)\}_{t>0}$ and $\{S^*_{\nu,\mu}(t)\}_{t>0}$ are strongly continuous for $t > 0$.

(iii) If $S(t)(t \geq 0)$ is an equicontinuous semigroup, then $S^*_{\nu,\mu}(t)$ and $K^*_\mu(t)$ are equicontinuous in E for $t > 0$.

Lemma 2.11 (Sadovskii fixed point theorem)**.** *Let D ba a convex, closed and bounded subset of a Banach space E and $Q : D \to D$ be a condensing map. Then Q has one fixed point in D.*

Lemma 2.12 ([31]). *Let $a \geq 0, \mu > 0, c(t)$ and $u(t)$ be the nonnegative locally integrable functions on $0 \leq t < T < +\infty$, such that*

$$u(t) \leq c(t) + a\int_0^t (t-s)^{\mu-1}u(s)ds,$$

then

$$u(t) \leq c(t) + \int_0^t \Big[\sum_{n=1}^{\infty} \frac{(a\Gamma(\mu))^n}{\Gamma(n\mu)}(t-s)^{n\mu-1}c(s)\Big]ds, \quad 0 \leq t < T.$$

3. Main results

For $v, w \in C_{1-\gamma}(J,E)$ with $v \leq w$, we use $[v,w]$ to denote the order interval $\{u \in C_{1-\gamma}\}(J,E)|v \leq u \leq w\}$ in $C_{1-\gamma}(J,E)$, and $[v(t), w(t)]$ to denote the order interval $u \in E|v(t) \leq u(t) \leq w(t), t \in J$ in E. In this section, we will discuss the existence of extremal mild solutions for problem (1.1).

Definition 3.1. An abstract function $u \in C_{1-\gamma}(J,E)$ is called a solution of the problem (1.1) if $u(t)$ satisfies all the equalities of (1.1).

Definition 3.2. Let $L \geq 0$ be a constant. If functions $v_0, w_0 \in C_{1-\gamma}(J,E)$ satisfies

$$\begin{aligned} &D_{0+}^{\nu,\mu} v_0(t) + Av_0(t) \leq f(t, v_0(t), w_0(t)) + L(v_0(t) - w_0(t)), \quad t \in J, \\ &I_{0+}^{1-\gamma} v_0(0) \leq u_0 + \sum_{i=1}^m \lambda_i v_0(\tau_i), \end{aligned} \tag{3.1}$$

$$\begin{aligned} &D_{0+}^{\nu,\mu} w_0(t) + Aw_0(t) \geq f(t, w_0(t), v_0(t)) + L(w_0(t) - v_0(t)), \quad t \in J, \\ &I_{0+}^{1-\gamma} w_0(0) \geq u_0 + \sum_{i=1}^m \lambda_i w_0(\tau_i), \end{aligned} \tag{3.2}$$

we call v_0, w_0 coupled lower and upper L-quasi-solution of the problem (1.1). Only choosing $=$ in (3.1) and (3.2), we call (v_0, w_0) coupled L-quasi-solution pair of the problem (1.1). Furthermore, if $u_0 := v_0 = w_0$, we call u_0 a solution of the problem (1.1).

Theorem 3.1. *Assume that E be an ordered Banach space and its positive cone P is normal, and $-A$ generates a positive C_0-semigroup $\{T(t)\}_{t\geq 0}$ on E, $f \in C(J \times E \times E, E)$ and $u_0 \in E$. If the problem (1.1) has a lower solution $v_0 \in C_{1-\gamma}(J,E)$ and an upper solution $w_0 \in C_{1-\gamma}(J,E)$ with $v_0 \leq w_0$. Suppose also that the conditions (F0), (F1) and the following conditions*

(F2) There exist a constant $C > 0$ and $L \geq 0$ such that

$$f(t, u_2, v_2) - f(t, u_1, v_1) \geq -C(u_2 - u_1) - L(v_1 - v_2),$$

for any $t \in J$, and $v_0(t) \leq u_1 \leq u_2 \leq w_0(t)$, $v_0(t) \leq v_2 \leq v_1 \leq w_0(t)$.

(F3) There exist a constant $L_1 > 0$ such that for all $t \in J$,

$$\alpha(\{f(t, u_n, v_n)\}) \leq L_1(\alpha(\{u_n\}) + \alpha(\{v_n\})),$$

and increasing or decreasing sequences $\{u_n\} \subset [v_0(t), w_0(t)]$, $\{v_n\} \subset [v_0(t), w_0(t)]$.

(F4) Let $v_n = Q(v_{n-1}, w_{n-1})$, $w_n = Q(w_{n-1}, v_{n-1})$, $n = 1, 2, \dots$, such that the sequence $v_n(0)$ and $w_n(0)$ are convergent.

are satisfied, then the problem (1.1) has minimal and maximal coupled mild L-quasi-solutions between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 and w_0 respectively.

Proof. Since $C > 0$, the problem (1.1) can be written as the system (2.20). By (2.21), we can define operator $Q : [v_0, w_0] \times [v_0, w_0] \to C_{1-\gamma}(J, E)$ as follows

$$\begin{aligned}Q(u,v)(t) =& S^*_{\nu,\mu}(t)\Theta u_0\\ &+\sum_{i=1}^{m} \lambda_i S^*_{\nu,\mu}(t)\Theta \int_0^{\tau_i} K^*_\mu(\tau_i-s)[f(s,u(s),v(s))+(C+L)u(s)-Lv(s)]ds\\ &+\int_0^t K^*_\mu(t-s)[f(s,u(s),v(s))+(C+L)u(s)-Lv(s)]ds, \quad t\in J'.\end{aligned} \tag{3.3}$$

Since f is continuous, it is easily see that the map $Q :\to C_{1-\gamma}(J, E)$ is continuous. And by Lemma 2.9, the mild solutions of the problem (1.1) are equivalent to the fixed points of the operator Q. We will divide the proof in the following steps.

Step 1. We show $Q : [v_0, w_0] \times [v_0, w_0] \to C_{1-\gamma}(J, E)$ is a mixed monotone operator.

In fact, for $\forall t \in J'$, $v_0(t) \le u_1(t) \le u_2(t) \le w_0, v_0(t) \le v_2(t) \le v_1(t) \le w_0(t)$, by the assumptions (F2) and (F3), we have

$$f(t,u_1(t),v_1(t)) + (C+L)u_1(t) - Lv_1(t) \le f(t,u_2(t),v_2(t)) + (C+L)u_2(t) - Lv_2(t).$$

So

$$\begin{aligned}&\int_0^t K^*_\mu(t-s)[f(s,u_1(s),v_1(s)) + (C+L)u_1(s) - Lv_1(t)]ds\\ \le &\int_0^t K^*_\mu(t-s)[f(s,u_2(s),v_2(s)) + (C+L)u_2(s) - Lv_2(s)]ds.\end{aligned}$$

Thus, from (3.3) we have $Q(u_1, v_1) \le Q(u_2, v_2)$.

Step 2. We show that $v_0 \le Q(v_0, w_0), Q(w_0, v_0) \le w_0$. Let $h(t) = D^{\nu,\mu}_{0+}v_0(t) + Av_0(t) + Cv_0(t)$, $h \in C_{1-\gamma}(J, E)$ and $h(t) \le f(t, v_0, w_0) + (C + L)v_0 - Lw_0, t \in J'$. By Definition 2.7 and 3.2, we have

$$\begin{aligned}v_0(t) =& S^*_{\nu,\mu}(t)v_0(0) + \int_0^t K^*_\mu(t-s)h(s)ds\\ \le& S^*_{\nu,\mu}(t)\Theta u_0\\ &+\sum_{i=1}^{m} \lambda_i S^*_{\nu,\mu}(t)\Theta \int_0^{\tau_i} K^*_\mu(\tau_i-s)[f(s,v_0(s),w_0(s))+(C+L)v_0(s)-Lw_0(s)]ds\\ &+\int_0^t K^*_\mu(t-s)[f(s,v_0(s),w_0(s)) + (C+L)v_0(s) - Lw_0(s)]ds\\ =& Q(v_0,w_0)(t), \quad t\in J'.\end{aligned}$$

It implies that $v_0 \le Q(v_0, w_0)$. Similarly, it can prove that $Q(w_0, v_0) \le w_0$. Thus, $Q : [v_0, w_0] \times [v_0, w_0] \to [v_0, w_0]$ is a continuous mixed monotone operator.

Now, we define two sequences $\{v_n\}$ and $\{w_n\}$ in $[v_0, w_0]$ by the iterative scheme

$$v_n = Q(v_{n-1}, w_{n-1}), \quad w_n = Q(w_{n-1}, v_{n-1}), \quad n = 1, 2, \ldots. \tag{3.4}$$

Then from the monotonicity of Q, we have

$$v_0 \le v_1 \le v_2 \le \cdots \le v_n \le \cdots \le w_n \le \cdots \le w_2 \le w_1 \le w_0. \tag{3.5}$$

Step 3. We prove that $\{v_n\}$ and $\{w_n\}$ are convergent in J'.

For convenience, we denote $B = \{v_n : n \in \mathbb{N}\} + \{w_n : n \in \mathbb{N}\}$ and $B_1 = \{v_n : n \in \mathbb{N}\}$, $B_2 = \{w_n : n \in \mathbb{N}\}$, $B_{10} = \{v_{n-1} : n \in \mathbb{N}\}$, $B_{20} = \{w_{n-1} : n \in \mathbb{N}\}$. Then $B_1 = Q(B_{10}, B_{20})$ and $B_2 = Q(B_{20}, B_{10})$. From $B_{10} = B_1 \bigcup\{v_0\}$ and $B_{20} = B_2 \bigcup\{w_0\}$ it follows that $\alpha(B_{10}(t)) = \alpha(B_1(t))$ and $\alpha(B_{20}(t)) = \alpha(B_2(t))$ for $t \in J'$. Let $\varphi(t) := \alpha(B(t)), t \in J'$, we will show that $\varphi(t) \equiv 0$ in J'.

For $t \in J'$, from (3.1), using Lemma 2.2, assumption (F3) and (F4), we have

$$\begin{aligned}
&\varphi(t) = \alpha(B(t)) = \alpha(B_1(t) + B_2(t))\\
=&\alpha(Q(B_{10}, B_{20})(t) + Q(B_{20}, B_{10})(t))\\
=&\alpha\Big(\Big\{S^*_{\nu,\mu}(t)\Theta u_0 + \sum_{i=1}^{m} \lambda_i S^*_{\nu,\mu}(t)\Theta \int_0^{\tau_i} K^*_\mu(\tau_i - s)\\
&\times [f(s, v_{n-1}(s), w_{n-1}(s)) + (C + L)v_{n-1}(s) - Lw_{n-1}(s)]ds\\
&+ \int_0^t K^*_\mu(t - s)[f(s, v_{n-1}(s), w_{n-1}(s)) + (C + L)v_{n-1}(s) - Lw_{n-1}(s)]ds\\
&+ S^*_{\nu,\mu}(t)\Theta u_0\\
&+\sum_{i=1}^{m} \lambda_i S^*_{\nu,\mu}(t)\Theta\int_0^{\tau_i} K^*_\mu(\tau_i-s)[f(s, w_{n-1}(s), v_{n-1}(s))+(C+L)w_{n-1}(s)-Lv_{n-1}(s)]ds\\
&+ \int_0^t K^*_\mu(t - s)[f(s, w_{n-1}(s), v_{n-1}(s)) + (C + L)w_{n-1}(s) - Lv_{n-1}(s)]ds\Big\}\Big)\\
\le&\frac{M^* b^{\gamma-1}}{\Gamma(\gamma)}\alpha\Big(\Big\{\Theta u_0 + \sum_{i=1}^{m} \lambda_i\Theta \int_0^{\tau_i} K^*_\mu(\tau_i - s)[f(s, v_{n-1}(s), w_{n-1}(s))\\
&+ (C + L)v_{n-1}(s) - Lw_{n-1}(s)]ds + \Theta u_0\\
&+ \sum_{i=1}^{m} \lambda_i\Theta \int_0^{\tau_i} K^*_\mu(\tau_i-s)[f(s, w_{n-1}(s), v_{n-1}(s))+(C+L)w_{n-1}(s)-Lv_{n-1}(s)]ds\Big\}\Big)\\
&+ \frac{2M^* b^{\mu-1}}{\Gamma(\mu)} \int_0^t \alpha\Big(\Big\{f(s, v_{n-1}(s), w_{n-1}(s))\\
&+ f(s, w_{n-1}(s), v_{n-1}(s))+C(v_{n-1}(s)+w_{n-1})\Big\}\Big)ds\\
\le&\frac{M^* b^{\gamma-1}}{\Gamma(\gamma)}\Big[\alpha\Big(\Big\{v_n(0)\Big\}\Big) + \alpha\Big(\Big\{w_n(0)\Big\}\Big)\Big]\\
&+ \frac{2M^* b^{\mu-1}(L_1 + C)}{\Gamma(\mu)} \int_0^t (\alpha(B_{10}(s)) + \alpha(B_{20}(s)))ds\\
\le&\frac{4M^* b^{\mu-1}(L_1 + C)}{\Gamma(\mu)} \int_0^t \varphi(s)ds.
\end{aligned}$$

Hence by Lemma 2.12, $\varphi(t) \equiv 0$ in J. Hence, for any $t \in J, \{v_n(t)\} + \{w_n(t)\}$ is precompact. So $\{v_n(t)\}, \{w_n(t)\}$ are precompact. Combing this with the monotonicity

(3.5), we easily prove that $\{v_n(t)\}$ and $\{w_n(t)\}$ are convergent, i.e., $\lim_{n\to\infty} v_n(t) = \underline{u}(t)$, $t \in J$. Similarly, $\lim_{n\to\infty} w_n(t) = \overline{u}(t)$, $t \in J$.

Evidently $\{v_n(t)\}, \{w_n(t)\} \in C_{1-\gamma}(J,E)$, so $\underline{u}(t), \overline{u}(t)$ are bounded integrable in J. Since for any $t \in J$, we have

$$\begin{aligned} v_n(t) =& Q(v_{n-1}, w_{n-1})(t) \\ =& S^*_{\nu,\mu}(t)\Theta u_0 + \sum_{i=1}^{m} \lambda_i S^*_{\nu,\mu}(t)\Theta \int_0^{\tau_i} K^*_\mu(\tau_i - s)(f(s, v_{n-1}(s), w_{n-1}(s)) \\ &+ (C+L)v_{n-1}(s) - Lw_{n-1}(s))ds \\ &+ \int_0^t K^*_\mu(t-s)(f(s, v_{n-1}(s), w_{n-1}(s)) + (C+L)v_{n-1}(s) - Lw_{n-1}(s))ds \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} w_n(t) =& Q(w_{n-1}, v_{n-1})(t) \\ =& S^*_{\nu,\mu}(t)\Theta u_0 + \sum_{i=1}^{m} \lambda_i S^*_{\nu,\mu}(t)\Theta \int_0^{\tau_i} K^*_\mu(\tau_i - s)(f(s, w_{n-1}(s), v_{n-1}(s)) \\ &+ (C+L)w_{n-1}(s) - Lv_{n-1}(s))ds \\ &+ \int_0^t K^*_\mu(t-s)(f(s, w_{n-1}(s), v_{n-1}(s)) + (C+L)w_{n-1}(s) - Lv_{n-1}(s))ds. \end{aligned} \tag{3.7}$$

If $n \to \infty$ in (3.6) and (3.7), by the Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned} \underline{u}(t) =& Q(\underline{u}(t)) \\ =& S^*_{\nu,\mu}(t)\Theta u_0 \\ &+ \sum_{i=1}^{m} \lambda_i S^*_{\nu,\mu}(t)\Theta \int_0^{\tau_i} K^*_\mu(\tau_i - s)[f(s, \underline{u}(s), \overline{u}(s)) + (C+L)\underline{u}(s) - L\overline{u}(s)]ds \\ &+ \int_0^t K^*_\mu(t-s)[f(s, \underline{u}(s), \overline{u}(s)) + (C+L)\underline{u}(s) - L\overline{u}(s)]ds, \end{aligned}$$

and

$$\begin{aligned} \overline{u}(t) =& Q(\underline{u}(t)) \\ =& S^*_{\nu,\mu}(t)\Theta u_0 + \sum_{i=1}^{m} \lambda_i S^*_{\nu,\mu}(t)\Theta \int_0^{\tau_i} K^*_\mu(\tau_i - s)[f(s, \overline{u}(s), \underline{u}(s)) \\ &+ (C+L)\overline{u}(s) - L\underline{u}(s)]ds \\ &+ \int_0^t K^*_\mu(t-s)[f(s, \overline{u}(s), \underline{u}(s)) + (C+L)\overline{u}(s) - L\underline{u}(s)]dsds. \end{aligned}$$

Thus, we have $\underline{u}(t), \overline{u}(t) \in C_{1-\gamma}(J,E)$, and $\underline{u} = Q\underline{u}, \overline{u} = Q\overline{u}$. Combing this with monotonicity (3.5), we see that $v_0 \le \underline{u} \le \overline{u} \le w_0$. By the monotonicity of Q, it is easy to see that $\underline{u}$ and $\overline{u}$ are the minimal and maximal coupled fixed points of Q in $[v_0, w_0]$. Therefore, $\underline{u}$ and $\overline{u}$ are the minimal and maximal coupled mild L-quasi-solutions of the problem (1.1) in $[v_0, w_0]$, respectively.. $\square$

Remark 3.1. If we replace positive cone P is normal by positive cone P is regular. Then the conclusion in Theorem 3.1 is also valid. For more detail, see [6].

As a supplement to Theorem 3.1, we further discuss the existence of mild solutions for the problem (1.1) in weakly sequentially complete Banach space, we only need to verify the conditions (F1) and (F2) are satisfied.

Corollary 3.1. *Assume that E be an ordered and weakly sequentially complete Banach space and its positive cone P is normal, and $-A$ generates a positive C_0-semigroup $\{T(t)\}_{t\geq 0}$ on E, $f \in C(J \times E \times E, E)$ and $u_0 \in E$. If the problem (1.1) has coupled lower and upper L-quasi-solution v, w_0 with $v_0 \leq w_0$. Suppose also that the conditions (F0)-(F4) are satisfied. Then the problem(1.1) has minimal and maximal coupled mild L-quasi-solutions between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 and w_0 respectively.*

Proof. In view of Theorem 3.1, if E is weakly sequentially complete, the condition (F3) and (F4) holds automatically. And by Theorem 2.2 in [8], any monotonic and order bounded sequence is precompact. By the monotonicity (3.3), it is east to see that $v_n(t)$ and $w_n(t)$ are convergent on J. Thus, $v_n(0)$ and $w_n(0)$ are convergent, i.e. condition (F4) holds. For $t \in J$, let $\{u_n\}$ and $\{v_n\}$ be increasing or decreasing sequences obeying condition (F3), then by condition (F1), $\{f(t, u_n, v_n) + Cu_n - Lv_n\}$ is a monotone and order-bounded sequence. By the property of measure of noncompactness, we have

$$\alpha(\{f(t, u_n, v_n)\}) \leq \alpha(\{f(t, u_n, v_n) + Cu_n - Lv_n\}) + C\alpha(\{u_n\}) + L\alpha(\{v_n\}) = 0,$$

and (F3) holds and by Theorem 3.1, our conclusion is valid. □

Now, we discuss the exists of mild solution to the problem (1.1) between the minimal and maximal coupled mild L-quasi-solutions $\underline{u}$ and $\overline{u}$. If we replace the assumptions (F3) by the following assumptions:

(F3)* The exists a $L_1 > 0$ such that

$$\alpha(f(t, D_1, D_2)) \leq L_1(\alpha(D_1) + \alpha(D_2)),$$

for any $t \in J$, where $D_1 = \{v_n\}$ and $D_2 = \{w_\}$ are countable sets in $[v_0(t), w_0(t)]$.

We have the following results.

Theorem 3.2. *Assume that E be an ordered Banach space and its positive cone P is normal, and $-A$ generates a positive and equicontinuous C_0-semigroup $\{T(t)\}_{t\geq 0}$ on E, $f \in C(J\times E\times E, E)$ and $u_0 \in E$. If the problem (1.1) has coupled lower and upper L-quasi-solutions $v_0 \in C_{1-\gamma}(J, E)$ and $w_0 \in C_{1-\gamma}(J, E)$ with $v_0 \leq w_0$. Suppose also that the conditions (F0)-(F2), (F3)* are satisfied. Then the problem(1.1) has minimal and maximal coupled mild L-quasi-solutions u and u between v_0 and w_0, and at least has one mild solution between $\underline{u}$ and $\overline{u}$ in $[v_0, w_0]$, and*

$$v_n(t) \to \underline{u}(t), \quad w_n(t) \to \overline{u}(t), \ (n \to +\infty), t \in J,$$

where $v_n = Q(v_{n-1}, w_{n-1})$, $w_n = Q(w_{n-1}, v_{n-1})$, $n = 1, 2, \ldots,$ which satisfy

$$v_0(t) \leq v_1(t) \leq \cdots v_n(t) \leq \cdots \underline{u}(t) \leq \overline{u}(t) \leq \cdots \leq w_n(t) \leq \cdots w_1(t) \leq w_0(t), \forall t \in J.$$

Proof. It is easy to see that $(F3)^* \Rightarrow (H3)$. Hence, by Theorem 3.1, the problem (1.1) has minimal and maximal coupled mild L-quasi-solutions $\underline{u}$ and $\overline{u}$ between v_0

and w_0. Next, we prove the existence of the mild solution of the equation between v_0 and w_0. Let $Au = Q(u,u)$, clearly, we know that $A : [v_0, w_0] \to [v_0, w_0]$ is continuous and the mild solution of the problem (1.1) is equivalent to fixed point of operator A. First, we will prove that $A : [v_0, w_0] \to C_{1-\gamma}(J,E)$ is an equicontinuous operator. Since $T(t)(t \geq 0)$ is a equicontinuous C_0-semigroup, and $S(t)(t \geq 0)$ is also a equicontiuous C_0-semigroup. By the normality of the cone P, there exists $\overline{M} > 0$ such that

$$\|f(t,u(t),v(t)) + (C+L)u(t) - Lv(t)\| \leq \overline{M}, \quad u \in [v_0, w_0].$$

For any $u \in C_{1-\gamma}(J,E)$, let $y(t) = t^{1-\gamma}u(t)$, for $t_1 = 0, 0 < t_2 \leq b$, we get

$$\begin{aligned}
&\|y(t_2) - y(0)\| = \|t_2^{1-\gamma}Q(u,v)(t_2)\| \\
\leq &\Big\|t_2^{1-\gamma}S^*_{\nu,\mu}(t_2)\Big\|(\Theta u_0) + \sum_{i=1}^m \lambda_i \Theta \|t_2^{1-\gamma}S^*_{\nu,\mu}(t_2)\| \int_0^{\tau_i} K^*_\mu(\tau_i - s) \\
&\times [f(t,u(t),v(t)) + (C+L)u(t) - Lv(t)]ds \\
&+ t_2^{1-\gamma}\Big\| \int_0^{t_2} K^*_\mu(t_2 - s)[f(t,u(t),v(t)) + (C+L)u(t) - Lv(t)]ds\Big\| \\
\leq &\Big\|t_2^{1-\gamma}S^*_{\nu,\mu}(t_2)\Big\|(\Theta u_0) + \overline{M}\sum_{i=1}^m \lambda_i \Theta \|t_2^{1-\gamma}S^*_{\nu,\mu}(t_2)\| \int_0^{\tau_i} K^*_\mu(\tau_i - s)ds \\
&+ \overline{M}\Big\| \int_0^{t_2} t_2^{1-\gamma}K^*_\mu(t_2 - s)ds\Big\| \\
\to &0, \quad \text{as } t_2 \to t_1 = 0.
\end{aligned}$$

For $0 < t_1 < t_2 \leq b$, by (3.1), we get that

$$\begin{aligned}
&\|y(t_2) - y(t_1)\| \leq \Big\|t_2^{1-\gamma}Q(u,v)(t_2) - t_1^{1-\gamma}Q(u,v)(t_1)\Big\| \\
\leq &\Big\|t_2^{1-\gamma}S^*_{\nu,\mu}(t_2) - t_1^{1-\gamma}S^*_{\nu,\mu}(t_1)\Big\|(\Theta u_0) + \Big\|t_2^{1-\gamma}S^*_{\nu,\mu}(t_2) - t_1^{1-\gamma}S^*_{\nu,\mu}(t_1)\Big\| \\
&\times \sum_{i=1}^m \lambda_i \Theta \int_0^{\tau_i} K^*_\mu(\tau_i - s)[f(s,u(s),v(s)) + (C+L)u(s) - Lv(s)]ds \\
&+ \int_0^{t_2} t_2^{1-\gamma}K^*_\mu(t_2 - s)[f(s,u(s),v(s)) + (C+L)u(s) - Lv(s)]ds \\
&- \int_0^{t_1} t_1^{1-\gamma}K^*_\mu(t_1 - s)[f(s,u(s),v(s)) + (C+L)u(s) - Lv(s)]ds \\
\leq &\Big(\Big\|t_2^{1-\gamma}S^*_{\nu,\mu}(t_2) - t_2^{1-\gamma}S^*_{\nu,\mu}(t_1)\Big\| \\
&+ \Big\|t_2^{1-\gamma}S^*_{\nu,\mu}(t_1) - t_1^{1-\gamma}S_{\nu,\mu}(t_1)\Big\|\Big)(\Theta u_0) + \Big\|t_2^{1-\gamma}S^*_{\nu,\mu}(t_2) - t_1^{1-\gamma}S^*_{\nu,\mu}(t_1)\Big\| \\
&\times \sum_{i=1}^m \lambda_i \Theta \int_0^{\tau_i} K^*_\mu(\tau_i - s)[f(s,u(s),v(s)) + (C+L)u(s) - Lv(s)]ds \\
&+ \Big\| \int_{t_1}^{t_2} t_2^{1-\gamma}K^*_\mu(t_2 - s)[f(s,u(s),v(s)) + (C+L)u(s) - Lv(s)]ds\Big\| \\
&+ \Big\| \int_0^{t_1} t_2^{1-\gamma}K^*_\mu(t_2 - s)[f(s,u(s),v(s)) + (C+L)u(s) - Lv(s)]ds
\end{aligned}$$

$$
\begin{aligned}
&-\int_0^{t_1} t_1^{1-\gamma} K_\mu^*(t_2-s)[f(s,u(s),v(s))+(C+L)u(s)-Lv(s)]ds\Big\|\\
&+\Big\|\int_0^{t_1} t_1^{1-\gamma} K_\mu^*(t_2-s)[f(s,u(s),v(s))+(C+L)u(s)-Lv(s)]ds\\
&-\int_0^{t_1} t_1^{1-\gamma} K_\mu^*(t_1-s)[f(s,u(s),v(s))+(C+L)u(s)-Lv(s)]ds\Big\|\\
=&J_1+J_2+J_3+J_4+J_5+J_6,
\end{aligned}
$$

where

$$
\begin{aligned}
J_1 &= \Big(\Big\|t_2^{1-\gamma} S_{\nu,\mu}^*(t_2) - t_2^{1-\gamma} S_{\nu,\mu}^*(t_1)\Big\|\Big)(\Theta u_0),\\
J_2 &= \Big(\Big\|t_2^{1-\gamma} S_{\nu,\mu}^*(t_1) - t_1^{1-\gamma} S_{\nu,\mu}^*(t_1)\Big\|\Big)(\Theta u_0),\\
J_3 &= \Big\|t_2^{1-\gamma} S_{\nu,\mu}^*(t_2) - t_1^{1-\gamma} S_{\nu,\mu}^*(t_1)\Big\| \sum_{i=1}^m \lambda_i \Theta \int_0^{\tau_i} K_\mu^*(\tau_i - s)[f(s,u(s),v(s))\\
&\quad + (C+L)u(s) - Lv(s)]ds,\\
J_4 &= \Big\|\int_{t_1}^{t_2} t_2^{1-\gamma} K_\mu^*(t_2-s)[f(s,u(s),v(s))+(C+L)u(s)-Lv(s)]ds\Big\|,\\
J_5 &= \Big\|\int_0^{t_1} t_2^{1-\gamma} K_\mu^*(t_2-s)[f(s,u(s),v(s))+(C+L)u(s)-Lv(s)]ds\\
&\quad -\int_0^{t_1} t_1^{1-\gamma} K_\mu^*(t_2-s)[f(s,u(s),v(s))+(C+L)u(s)-Lv(s)]ds\Big\|,\\
J_6 &= \Big\|\int_0^{t_1} t_1^{1-\gamma} K_\mu^*(t_2-s)[f(s,u(s),v(s))+(C+L)u(s)-Lv(s)]ds\\
&\quad -\int_0^{t_1} t_1^{1-\gamma} K_\mu^*(t_1-s)[f(s,u(s),v(s))+(C+L)u(s)-Lv(s)]ds\Big\|.
\end{aligned}
$$

Here we calculate

$$
\Big\|t_2^{1-\gamma} Q(u,v)(t_2) - t_1^{1-\gamma} Q(u,v)(t_1)\Big\| \le \sum_{i=1}^6 \|J_i\|.
$$

Therefore, it is not difficult to see that $\|J_i\|$ tend to 0, when $t_2 - t_1 \to 0, i = 1,2,\dots,6$.

For J_1, by Lemma 2.10, we get

$$
\begin{aligned}
J_1 &= \Big(\Big\|t_2^{1-\gamma} S_{\nu,\mu}^*(t_2) - t_2^{1-\gamma} S_{\nu,\mu}^*(t_1)\Big\|\Big)(\Theta u_0)\\
&\le \Big\|t_2^{1-\gamma}\big(S_{\nu,\mu}^*(t_2) - S_{\nu,\mu}(t_1)\big)\Big\|(\Theta u_0) \to 0, \quad \text{as } t_2 \to t_1.
\end{aligned}
$$

For J_2, by Lemma 2.10, we get

$$
\begin{aligned}
J_2 &= \Big(\Big\|t_2^{1-\gamma} S_{\nu,\mu}^*(t_1) - t_1^{1-\gamma} S_{\nu,\mu}^*(t_1)\Big\|\Big)(\Theta u_0)\\
&\le \frac{M^* b^{\gamma-1}}{\Gamma(\gamma)} \Big\|t_2^{1-\gamma} - t_1^{1-\gamma}\Big\| \|\Theta u_0\|
\end{aligned}
$$

$$\leq \frac{M^* b^{\gamma-1}}{\Gamma(\gamma)} \Big\|(t_2-t_1)^{1-\gamma}\Big\| \|\Theta u_0\| \to 0, \quad \text{as } t_2 \to t_1.$$

For J_3, by Lemma 2.10, we have

$$\begin{aligned} J_3 =& \sum_{i=1}^{m} \lambda_i \Theta \Big\| t_2^{1-\gamma} S^*_{\nu,\mu}(t_1) - t_1^{1-\gamma} S^*_{\nu,\mu}(t_1) \Big\| \int_0^{\tau_i} K^*_\mu(\tau_i - s)[f(s,u(s),v(s)) \\ &+ (C+L)u(s) - Lv(s)]ds \\ \leq & \frac{\overline{M} \sum\limits_{i=1}^{m} |\lambda_i|}{1 - M^* \sum\limits_{i=1}^{m} |\lambda_i|} \Big\| t_2^{1-\gamma} S^*_{\nu,\mu}(t_1) - t_1^{1-\gamma} S^*_{\nu,\mu}(t_1) \Big\| \int_0^{\tau_i} K^*_\mu(\tau_i - s)ds \\ \to & 0, \quad \text{as } t_2 \to t_1. \end{aligned}$$

For J_4, by Lemma 2.10, we have

$$\begin{aligned} J_4 &= \Big\| \int_{t_1}^{t_2} t_2^{1-\gamma} K^*_\mu(t_2 - s)[f(s,u(s),v(s)) + (C+L)u(s) - Lv(s)]ds \Big\| \\ &\leq \overline{M} \int_{t_1}^{t_2} t_2^{1-\gamma} K^*_\mu(t_2 - s)ds \\ &\to 0, \quad \text{as } t_2 \to t_1. \end{aligned}$$

For J_5, by Lemma 2.10, we have

$$\begin{aligned} J_5 =& \Big\| \int_0^{t_1} t_2^{1-\gamma} K^*_\mu(t_2 - s)[f(s,u(s),v(s)) + (C+L)u(s) - Lv(s)]ds \\ &- \int_0^{t_1} t_1^{1-\gamma} K^*_\mu(t_2 - s)[f(s,u(s),v(s)) + (C+L)u(s) - Lv(s)]ds \Big\| \\ \leq & \frac{2M^*}{\Gamma(\mu)} \int_0^{t_1} \Big[t_2^{1-\gamma}(t_2-s)^{\mu-1} - t_1^{1-\gamma}(t_1-s)^{\mu-1} \Big] [f(s,u(s),v(s)) \\ &+ (C+L)u(s) - Lv(s)]ds. \end{aligned}$$

Noting that

$$\begin{aligned} &\int_0^{t_1} \Big[t_2^{1-\gamma}(t_2-s)^{\mu-1} - t_1^{1-\gamma}(t_1-s)^{\mu-1} \Big] [f(s,u(s),v(s)) + (C+L)u(s) - Lv(s)]ds \\ \leq & \int_0^{t_1} t_2^{1-\gamma}(t_2-s)^{\mu-1}[f(s,u(s),v(s)) + (C+L)u(s) - Lv(s)]ds. \end{aligned}$$

and

$$\int_0^{t_1} \Big[t_2^{1-\gamma}(t_2-s)^{\mu-1} - t_1^{1-\gamma}(t_1-s)^{\mu-1} \Big] [f(s,u(s),v(s)) + (C+L)u(s) - Lv(s)]ds$$

exists, and by Lebesgue dominated convergence Theorem, we have

$$\int_0^{t_1} \Big[t_2^{1-\gamma}(t_2-s)^{\mu-1} - t_1^{1-\gamma}(t_1-s)^{\mu-1} \Big] [f(s,u(s),v(s)) + (C+L)u(s) - Lv(s)]ds$$

$\to 0, \quad \text{as } t_2 \to t_1.$

It is easy to see that $\lim_{t_2\to t_1} J_5 = 0$.

For J_6, by Lemma 2.10, we have

$$\begin{aligned}J_6 =&\Big\| \int_0^{t_1} t_1^{1-\gamma} K_\mu^*(t_2-s)[f(s,u(s),v(s)) + (C+L)u(s) - Lv(s)]ds \\ &- \int_0^{t_1} t_1^{1-\gamma} K_\mu^*(t_1-s)[f(s,u(s),v(s)) + (C+L)u(s) - Lv(s)]ds\Big\| \\ \le& \overline{M}\Big\|K_\mu^*(t_2-s) - K_\mu^*(t_1-s)\Big\| \int_0^{t_1} t_1^{1-\gamma} ds \\ &\to 0, \quad \text{as } t_2 \to t_1.\end{aligned}$$

In conclusion,

$$\|y(t_2) - y(t_1)\| \le \Big\|t_2^{1-\gamma} Q(u,v)(t_2) - t_1^{1-\gamma} Q(u,v)(t_1)\Big\| \to 0,$$

as $t_2 \to t_1$, i.e,

$$\Big\|Q(u,v)(t_2) - Q(u,v)(t_1)\Big\|_\gamma \to 0, \qquad \text{as } t_2 \to t_1,$$

which means that $Q : [v_0,w_0] \times [v_0,w_0] \to [v_0,w_0]$ is equicontinuous. Thus, $A : [v_0,w_0] \to [v_0,w_0]$ is also equicontinuous.

So, for any $D \subset [v_0,w_0]$, $A(D) \subset [v_0,w_0]$ is bounded and equicontinuous. Therefore, by Lemma 2.2, there exists a countable set $D_0 = \{u_n\} \subset D$ such that

$$\alpha(A(D)) \le 2\alpha(A(D_0)). \tag{3.8}$$

For $t \in J$, by the definition of the operator Q, we have

$$\begin{aligned}&\alpha(A(D_0(t))) \\ =&\alpha\Big(\Big\{S_{\nu,\mu}^*(t)\Theta u_0 + \sum_{i=1}^m \lambda_i S_{\nu,\mu}^*(t)\Theta \int_0^{\tau_i} K_\mu^*(\tau_i - s)[f(s,u_n(s),u_n(s)+Cu_n(s)))]ds \\ &+ \int_0^t K_\mu^*(t-s[f(s,u_n(s),u_n(s)+Cu_n(s)))]ds\Big\}\Big) \\ \le&\frac{2(M^*)^2\sum_{i=1}^m \lambda_i b^{\mu+\gamma-2}(L_1+C)}{\Gamma(\gamma)\Gamma(\mu)(1-M^*\sum_{i=1}^m)} \int_0^{\tau_i} \alpha(D_0(s))ds \\ &+ \frac{2M^* b^{\mu-1}(L_1+C)}{\Gamma(\mu)} \int_0^t \alpha(D_0(s))ds \\ \le&\frac{2(M^*)^2\sum_{i=1}^m \lambda_i b^{\mu+\gamma-1}(L_1+C)}{\Gamma(\gamma)\Gamma(\mu)(1-\sum_{i=1}^m \lambda_i)}\alpha(D) + \frac{2M^* b^\mu (L_1+C)}{\Gamma(\mu)}\alpha(D) \\ \le&\frac{2M^* b^\mu (L_1+C)}{\Gamma(\mu)}\Big[\frac{b^{\gamma-1}M^*\sum_{i=1}^m \lambda_i}{\Gamma(\gamma)(1-\sum_{i=1}^m \lambda_i)} + 1\Big]\alpha(D) \\ =&\frac{2M^* b^\mu (L_1+C)}{\Gamma(\mu)}\Big[\frac{(b^{\gamma-1}-\Gamma(\gamma))M^*\sum_{i=1}^m \lambda_i + \Gamma(\gamma)}{\Gamma(\gamma)(1-M^*\sum_{i=1}^m \lambda_i)}\Big]\alpha(D).\end{aligned}$$

Since $A(D_0)$ is bounded and equicontinuous, we know from Lemma 2.3 that

$$\alpha(A(D_0)) = \max_{t\in I} \alpha(A(D_0)(t)).$$

And by (3.8), we have

$$\alpha(A(D)) \le \eta\alpha(D),$$

where

$$\eta = \frac{2M^* b^\mu (L_1 + C)}{\Gamma(\mu)}\Big[\frac{(b^{\gamma-1} - \Gamma(\gamma))M^* \sum_{i=1}^m \lambda_i + \Gamma(\gamma)}{\Gamma(\gamma)(1 - M^* \sum_{i=1}^m \lambda_i)}\Big].$$

(i) If $\eta < 1$, then the operator $A : [v_0, w_0] \to [v_0, w_0]$ is condensing, by Lemma 2.11, A has fixed point u in $[v_0, w_0]$, so u is the mild solution of the problem (1.1) in $[v_0, w_0]$.

(ii) If $\eta \ge 1$. Divide $J = [0, b]$ into n equal parts, let $\Delta_n : 0 = t'_0 < t'_1 < \cdots < t'_n = b$ and $t'_i (i = 1, 2, \ldots, n-1)$ such that

$$\frac{2M^* \|\Delta_n\|^\mu (L_1 + C)}{\Gamma(\mu)}\Big[\frac{(\|\Delta_n\|^{\gamma-1} - \Gamma(\gamma))M^* \sum_{i=1}^m \lambda_i + \Gamma(\gamma)}{\Gamma(\gamma)(1 - M^* \sum_{i=1}^m \lambda_i)}\Big] < 1. \tag{3.9}$$

By (i) and (3.9), the problem (1.1) has mild solution $u_1(t)$ in $[0, t'_1]$; Again by (i) and (3.9), if Eq. (1.1) with $u(t'_1) = u_1(t'_1)$ as initial value, then it has mild solution $u_2(t)$ in $[t'_1, t'_2]$ and satisfies $u_2(t'_1) = u_1(t'_1)$. Thus, the mild solution of the equation continuously extend from $[0, t'_1]$ to $[0, t'_2]$; Continuing such a process, the mild solution of the equation can be continuously extend to J. So, we obtain a mild solution $u \in C(J, E)$ of the problem (1.1), which satisfies $u(t) = u_i(t), t'_{i-1} \le t \le t'_i,\ i = 1, 2, \ldots, n$.

Finally, since $u = Au = Q(u, u), v_0 \le u \le w_0$, by the mixed monotonicity of $Qv_1 = Q(v_0, w_0) \le Q(u, u) \le Q(w_0, v_0) = w_1$. Similarly, $v_2 \le u \le w_2$, in general, $v_n \le u \le w_n$, letting $n \to \infty$, we get $\underline{u} \le u \le \overline{u}$. Therefore, the problem (1.1) at least has one mild solution between $\underline{u}$ and $\overline{u}$. □

4. Examples

In this section, we present an example, which illustrate the applicability of our main results.

Example 4.1. We consider the following fractional partial differential equation

$$\begin{cases} D_{0+}^{\nu,\mu} u(t,x) = \Delta u(t,x) + f(t,x,u(t,x),u(t,x)), & (t,x) \in J \times \Omega, \\ I_{0+}^{(1-\nu)(1-\mu)} u(0,x) = u_0 + \sum_{i=1}^m \lambda_i u(\tau_i, x), \end{cases} \tag{4.1}$$

where $D_{0+}^{\nu,\mu}$ is the Hilfer fractional derivative, $0 \le \nu \le 1, 0 < \mu < 1$, $t \in J = [0, b]$, $\lambda_i \ne 0, i = 1, 2, \ldots, m$, integer $\mathbb{N} \ge 1$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with a sufficiently smooth boundary $\partial\Omega$, $f : J \times E \times E \to E$ is continuous.

Let $E = L^p(\Omega)$ with $1 < p < \infty$, $P = \{u \in L^p(\Omega)\} : u(x) \ge 0, q.e.x \in \Omega\}$, and define the operator $A : D(A) \subset E \to E$ as follows:

$$D(A) = W^{2,p} \cap W_0^{1,p}(\Omega), \quad Au = -\Delta u.$$

Then E is a Banach space, P is a normal cone of E, and $-A$ generates a positive C_0-semigroup $T(t)(t \geq 0)$ in E (see [25]). Let $f(t, u(t), u(t)) = f(t, x, u(t, x), u(t, x))$, $u_0 = u_0(\cdot)$, then the problem (4.3) can be written as the abstract (1.1).

Theorem 4.1. *If the following conditions*

(H5) Let $u_0(x) \geq 0, x \in \Omega$, and there exists a function $w = w(t, x) \in C_{1-\gamma}(J \times \Omega)$ such that

$$\begin{cases} D_{0+}^{\nu,\mu} w(t, x) \geq \Delta w(t, x) + f(t, x, w(t, x), w(t, x)), \\ I_{0+}^{(1-\nu)(1-\mu)} w(0, x) \geq u_0 + \sum_{i=1}^{m} \lambda_i w(\tau_i, x). \end{cases} \tag{4.2}$$

(H6) There exist a constant $C > 0$ and $L \geq 0$ such that

$$f(t, x, u_2, v_2) - f(t, x, u_1, v_1) \geq -C(u_2 - u_1) - L(v_1 - v_2),$$

for any $t \in J$, and $v_0(t) \leq u_1 \leq u_2 \leq w_0(t)$, $v_0(t) \leq v_2 \leq v_1 \leq w_0(t)$.

(H7) $\lambda_i > 0 (i = 1, 2, \ldots, m)$ and $\sum_{i=1}^{m} \lambda_i < \frac{\Gamma(\gamma)}{M^* b^{\gamma-1}}$.

(H8) There exists a constant $L_1 > 0$ such that

$$\alpha(\{f(t, u_n, v_n)\}) \leq L_1(\alpha(\{u_n\}) + \alpha(\{v_n\})),$$

for $\forall t \in J$, and increasing or decreasing monotonic sequences $\{u_n\} \subset [v_0(t), w_0(t)]$ and $\{v_n\} \subset [v_0(t), w_0(t)]$.

Then the problem (4.3) has minimal and maximal mild solutions between 0 and $w(x, t)$, which can be obtained by a monotone iterative procedure starting from 0 and $w(t)$, respectively.

Proof. Assumption (H5) implies that $v_0 \equiv 0$ and $w_0 \equiv w(x, t)$ are lower and upper solutions of the problem (4.3), respectively, and from (H6)-(H8), it is easy to verify that all conditions (F1)-(F3) are satisfied. So our conclusion follows from Theorem 3.1. □

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