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ROBUST FIXED-TIME CONSENSUS PROTOCOLS FOR MULTI-AGENT SYSTEMS WITH NONLINEAR STATE MEASUREMENTS

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Abstract This paper solves the robust fixed-time consensus problem for multi-agent systems with nonlinear state measurements. Sufficient conditions are established for the proposed protocol to reach fixed-time consensus under time-varying undirected and fixed directed topology with the aid of Lyapunov functions. It is proved that the finite settling time of the presented protocol for robust consensus is uniformly bounded for any initial condition, which makes it possible for people to design and estimate the convergence time off-line. Numerical simulations are preformed to show the effectiveness of our proposed protocol.

Keywords Fixed-time consensus, multi-agent systems, robustness, state measurements.

MSC(2010) 93C10, 93C15, 93D09, 93D40, 93D50.

1. Introduction

Consensus, which means that a group of autonomous agents reaches agreement upon a common assessment or certain quantity of interest, has been intensively studied in physics [11, 23, 24], biology [1, 20, 26], computer science [22], sociology [2, 4, 17] and control engineering [10, 25]. To achieve consensus, every individual evolves by a distributed neighbor-based feedback law to compare its current state with the information coming from its neighbors. Hence, the main challenge in solving the consensus problem lies in how to design the interaction rule, also called the consensus protocol or algorithm.

In the study of consensus, an important indicator for a proposed protocol is the convergence rate. Olfati-Saber and Murray [19] proposed a linear protocol and showed that the second smallest eigenvalue of the Laplacian matrix of interaction graph, also called the algebraic connectivity, qualifies the consensus speed. This finding motivated some researchers to seek proper interaction topology with larger algebraic connectivity [15, 18]. Although these results improved convergence rate, consensus can only be achieved asymptotically, which means that all agents reach agreement as time tends to infinity. In practice, finite-time convergence is more preferable because it can better meet complex practical cases and has stronger robustness against uncertainties [3]. Based on non-smooth stability analysis, people

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have introduced various protocols to make multi-agent systems reach consensus within finite time. Cortés [7] proposed the normalized and signed gradient dynamical systems associated with a differentiable function to solve finite-time consensus. Li etc [16] constructed the finite-time consensus algorithm for leaderless and leader-follower multi-agent systems with external disturbances with the help of a power integrator method. Hua etc [12] investigated the finite-time consensus of second-order multi-agent systems with unknown velocities and disturbances. In our previous work [5], we provided a new class of protocols with the aid of the signum function and sufficient conditions are established for systems to admit finite-time consensus under time-varying undirected and fixed directed topology.

Unfortunately, in the above results, the settling time depends on the initial states of agents and grows unboundedly along with the deviation of initial conditions from the equilibrium, which prohibits their practical applications when the knowledge of initial conditions is unavailable in advance. Fixed-time stability as an extension of finite-time stability was first proposed by Polyakov etc [21] and it guarantees that the settling time is upper bounded by a constant independent of initial values. Fixed-time consensus has been solved by constructing a new class of global continuous protocols for multi-agent systems with undirected topology [27] and directed topology [28]. In addition, Defoort etc [9] tackled the robust fixed-time consensus problem for multi-agent systems with unknown inherent nonlinear dynamics and external disturbances. Zuo etc [29] addressed the fixed-time leader-follower consensus problem for high-order integrator multi-agent systems subject to matched external disturbances.

On the other hand, in some cases, information variables of agents in a multi-agent team may be unobservable and nonlinear sensors are then used for measurements [14]. However, there have been few results on fixed-time consensus problem with nonlinear state feedback.

In this paper, we attempt to solve the robust fixed-time consensus problem for N agents with nonlinear state measurements. Consider the following dynamics

$$\dot{x}_i(t)=\alpha\sum_{j=1}^{N}a_{ij}(h(x_j)-h(x_i))^{\frac{m}{n}}+\beta\sum_{j=1}^{N}a_{ij}(h(x_j)-h(x_i))^{\frac{p}{q}}+\gamma\sum_{j=1}^{N}a_{ij}\mathrm{sign}(h(x_j)-h(x_i)) \tag{1.1}$$

for $i,j\in\mathcal{I}_N=\{1,2,...,N\}$, where $x_i(t)\in\mathbb{R}$ denotes the state (opinion, voltage, or incremental cost) of agent i at time t, $a_{ij}\geq 0$ measures the mutual influence of agent j on agent i. The coefficients α, β, $\gamma>0$ are overall strength, and m, n, p, q are all positive odd integers such that $m>n$, $p<q$. The nonlinear function $h\in C^1$ is strictly increasing. The main contribution of this paper is to investigate the robust fixed-time consensus of multiple agents governed by the dynamics (1.1) under time-varying undirected and fixed directed topology. For this to happen, it is necessary to give the definition of fixed-time consensus mathematically at first.

Definition 1.1. Fixed-time consensus in (1.1) is said to be reached if for any initial condition and $i,j\in\mathcal{I}_N$, there exists a unified settling time $T\in[0,+\infty)$ such that

$$\begin{cases}\lim\limits_{t\to T}|x_i(t)-x_j(t)|=0,\\ x_i(t)=x_j(t),\quad t\geq T.\end{cases}$$

The rest of this paper is organized as follows. Some useful definitions and lemmas are recalled in Section 2. Section 3 presents the main results of this paper. Numerical simulations are carried out in Section 4. Conclusions and future research directions end the paper in Section 5.

2. Preliminaries

2.1. Graph theory

A (weighted) directed graph $G(\boldsymbol{A}) = (V, \varepsilon, \boldsymbol{A})$ consists of a node set $V(G) = \{v_1, v_2, ..., v_N\}$, an edge set $\varepsilon \subseteq V \times V$ and an adjacency matrix $\boldsymbol{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$. An edge $(v_i, v_j) \in \varepsilon$ means that agent j can receive information from agent i or v_i and v_j are adjacent. The adjacency matrix $\boldsymbol{A}$ is defined by $a_{ii} = 0$, $a_{ij} > 0$ if $(v_j, v_i) \in \varepsilon$ and $a_{ij} = 0$ $(i \neq j)$ otherwise. A digraph is undirected if and only if $\boldsymbol{A}$ is symmetric. A path on G from v_i to v_j is a sequence of distinct vertices v_i, ..., v_j if consecutive vertices are adjacent. A digraph is called strongly connected if there is a directed path for any two nodes. An undirected graph is called connected if there exists a path for any two distinct nodes. Furthermore, a directed graph $G(\boldsymbol{A})$ is said to satisfy the detail-balanced condition if there exist some scalars $\omega_i > 0$ such that $\omega_i a_{ij} = \omega_j a_{ji}$ for all $i, j \in \mathcal{I}_N$ [6].

Lemma 2.1 (Remark 4, [19]). *Let $\boldsymbol{L_A} = [l_{ij}] \in \mathbb{R}^{N \times N}$ denote the Laplacian matrix of graph $G(\boldsymbol{A})$ with elements*

$$l_{ij} = \begin{cases} \sum\limits_{\substack{k=1,\\ k \neq i}}^{N} a_{ik}, & i = j, \\ -a_{ij}, & i \neq j. \end{cases}$$

Then $\boldsymbol{L_A}$ has the following properties:

(i) 0 is an eigenvalue of $\boldsymbol{L_A}$ and $\mathbf{1}_N = [1, 1, ..., 1]^{\mathrm{T}}$ is the associated eigenvector;

(ii) If $G(\boldsymbol{A})$ is undirected, then $\boldsymbol{x}^{\mathrm{T}} \boldsymbol{L_A} \boldsymbol{x} = \frac{1}{2} \sum\limits_{i=1}^{N} \sum\limits_{j=1}^{N} a_{ij}(x_j - x_i)^2$ for any $\boldsymbol{x} = [x_1, x_2, ..., x_N]^{\mathrm{T}}$, and $\boldsymbol{L_A}$ is positive semi-definite, which implies that all eigenvalues of $\boldsymbol{L_A}$ are nonnegative real numbers;

(iii) For an undirected $G(\boldsymbol{A})$, the algebraic connectivity of $G(\boldsymbol{A})$ is given by

$$\lambda_2(\boldsymbol{L_A}) = \min_{\substack{\boldsymbol{x} \neq \mathbf{0}, \\ \mathbf{1}_N^{\mathrm{T}} \boldsymbol{x} = 0}} \frac{\boldsymbol{x}^{\mathrm{T}} \boldsymbol{L_A} \boldsymbol{x}}{\boldsymbol{x}^{\mathrm{T}} \boldsymbol{x}},$$

where $\lambda_2(\boldsymbol{L_A})$ also equals the second smallest eigenvalue of $\boldsymbol{L_A}$. In addition, $G(\boldsymbol{A})$ is connected if and only if $\lambda_2(\boldsymbol{L_A}) > 0$.

2.2. Mathematical lemmas

Lemma 2.2. *Suppose that $\boldsymbol{\xi} = [\xi_1, \xi_2, ..., \xi_N]^{\mathrm{T}} \in \mathbb{R}^N$ and $\boldsymbol{Q} = [q_{ij}] \in \mathbb{R}^{N \times N}$ is symmetric. If $f : \mathbb{R} \to \mathbb{R}$ is an odd function, then we have*

$$\sum_{i=1}^{N} \sum_{j=1}^{N} q_{ij} \xi_i f(x_j - x_i) = -\frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} q_{ij} (\xi_j - \xi_i) f(x_j - x_i).$$

Proof. It follows from the symmetry that

$$\sum_{i=1}^{N}\sum_{j=1}^{N} q_{ij}\xi_i f(x_j - x_i) = \sum_{i=1}^{N}\sum_{j=1}^{N} q_{ji}\xi_j f(x_i - x_j) = -\sum_{i=1}^{N}\sum_{j=1}^{N} q_{ij}\xi_j f(x_j - x_i),$$

which implies that

$$\begin{aligned}\sum_{i=1}^{N}\sum_{j=1}^{N} q_{ij}\xi_i f(x_j - x_i) =& \frac{1}{2}\left(\sum_{i=1}^{N}\sum_{j=1}^{N} q_{ij}\xi_i f(x_j - x_i) - \sum_{i=1}^{N}\sum_{j=1}^{N} q_{ij}\xi_j f(x_j - x_i)\right)\\ =& -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} q_{ij}(\xi_j - \xi_i) f(x_j - x_i).\end{aligned}$$

□

Lemma 2.3 (Lemma 1, [28]). *If $\xi_1, \xi_2, \ldots, \xi_N \geq 0$, then*

(i) $\left(\sum_{i=1}^{N} \xi_i\right)^p \leq \sum_{i=1}^{N} \xi_i^p, \quad \text{if } 0 < p \leq 1;$

(ii) $N^{1-p}\left(\sum_{i=1}^{N} \xi_i\right)^p \leq \sum_{i=1}^{N} \xi_i^p, \quad \text{if } p > 1.$

Lemma 2.4 (Lemma 2, [28]). *Consider a scalar system*

$$\dot{y} = -\alpha y^{\frac{m}{n}} - \beta y^{\frac{p}{q}}, \quad y(0) = y_0, \tag{2.1}$$

where m, n, p, q are all positive odd integers satisfying $m > n$, $p < q$ and α, $\beta > 0$. Then, the equilibrium of (2.1) is globally finite-time stable and the settling time is bounded by

$$T = \frac{1}{\alpha}\frac{n}{m-n} + \frac{1}{\beta}\frac{q}{q-p}.$$

Lemma 2.5. *Let $x_i(t)$ be a solution to system (1.1) and $x_{\max}(t) = \max_{i\in\mathcal{I}_N} x_i(t)$, $x_{\min}(t) = \min_{i\in\mathcal{I}_N} x_i(t)$ for $t \geq 0$. Then $x_i(t)$ is bounded and satisfies*

$$x_{\min}(0) \leq x_i(t) \leq x_{\max}(0), \quad i \in \mathcal{I}_N, \ t \geq 0.$$

Proof. For a given time t, we have

$$\begin{aligned}\dot{x}_{\max}(t) =& \alpha\sum_{j=1}^{N} a_{ij}(h(x_j) - h(x_{\max}))^{\frac{m}{n}} + \beta\sum_{j=1}^{N} a_{ij}(h(x_j) - h(x_{\max}))^{\frac{p}{q}}\\ &+ \gamma\sum_{j=1}^{N} a_{ij}\mathrm{sign}(h(x_j) - h(x_{\max})) \leq 0.\end{aligned}$$

Similarly, we can show that $\dot{x}_{\min}(t) \geq 0$. Therefore,

$$x_{\min}(0) \leq x_{\min}(t) \leq x_i(t) \leq x_{\max}(t) \leq x_{\max}(0), \quad i \in \mathcal{I}_N, \ t \geq 0.$$

□

Definition 2.1. Assume that $\boldsymbol{A} \in \mathbb{C}^{N\times N}$ is an Hermitian matrix, then for $\boldsymbol{x} \neq \boldsymbol{0}$,

$$R(\boldsymbol{A};\boldsymbol{x}) = \frac{\boldsymbol{x}^{\mathrm{H}}\boldsymbol{A}\boldsymbol{x}}{\boldsymbol{x}^{\mathrm{H}}\boldsymbol{x}}$$

is called the Rayleigh quotient of $\boldsymbol{A}$.

All eigenvalues of $\boldsymbol{A}$ are real numbers and one can order them as

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N.$$

Then we have

Lemma 2.6 (Theorem 4.2.2, [13])**.**

$$\min_{\boldsymbol{x}\neq\boldsymbol{0}} R(\boldsymbol{A};\boldsymbol{x}) = \lambda_1, \quad \max_{\boldsymbol{x}\neq\boldsymbol{0}} R(\boldsymbol{A};\boldsymbol{x}) = \lambda_N.$$

Definition 2.2. Suppose that $\boldsymbol{A},\boldsymbol{B} \in \mathbb{C}^{N\times N}$ are Hermitian matrices, then for $\boldsymbol{x} \neq \boldsymbol{0}$,

$$R(\boldsymbol{A},\boldsymbol{B};\boldsymbol{x}) = \frac{\boldsymbol{x}^{\mathrm{H}}\boldsymbol{A}\boldsymbol{x}}{\boldsymbol{x}^{\mathrm{H}}\boldsymbol{B}\boldsymbol{x}}$$

is called the generalized Rayleigh quotient of $\boldsymbol{A}$ and $\boldsymbol{B}$.

It is easy to check that all roots of $\det(\mu\boldsymbol{B} - \boldsymbol{A})$ are real numbers and one can order them as

$$\mu_1 \leq \mu_2 \leq \cdots \leq \mu_N.$$

With this, we have

Lemma 2.7. *If the Hermitian matrix $\boldsymbol{B}$ is positive-definite, then*

$$\min_{\boldsymbol{x}\neq\boldsymbol{0}} R(\boldsymbol{A},\boldsymbol{B};\boldsymbol{x}) = \mu_1, \quad \max_{\boldsymbol{x}\neq\boldsymbol{0}} R(\boldsymbol{A},\boldsymbol{B};\boldsymbol{x}) = \mu_N.$$

Proof. Denote $\boldsymbol{y} = \boldsymbol{B}^{\frac{1}{2}}\boldsymbol{x}$. Noting that $\boldsymbol{B}^{\frac{1}{2}}(\boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{A}\boldsymbol{B}^{-\frac{1}{2}})\boldsymbol{B}^{-\frac{1}{2}} = \boldsymbol{A}\boldsymbol{B}^{-1}$, according to Lemma 2.6, we have

$$\max_{\boldsymbol{x}\neq\boldsymbol{0}} \frac{\boldsymbol{x}^{\mathrm{H}}\boldsymbol{A}\boldsymbol{x}}{\boldsymbol{x}^{\mathrm{H}}\boldsymbol{B}\boldsymbol{x}} = \max_{\boldsymbol{x}\neq\boldsymbol{0}} \frac{\boldsymbol{y}^{\mathrm{H}}\boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{A}\boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{y}}{\boldsymbol{y}^{\mathrm{H}}\boldsymbol{y}} = \lambda_N(\boldsymbol{A}\boldsymbol{B}^{-1}) = \mu_N.$$

The same reasoning applies to the other case. □

3. Consensus analysis

In this section, we present the robust fixed-time consensus proofs for the multi-agent system (1.1) under time-varying undirected and fixed directed topology.

3.1. Consensus under time-varying undirected topology

In many practical situations, the information exchange may not be available all the time due to special physical devices, limited sensing range or existence of obstacles. Therefore, it is reasonable to assume that the interaction topology is dynamically changing. For this case, we have the following result.

Theorem 3.1. *Suppose that the time-varying topology $G(\boldsymbol{A}(t))$ is always undirected and connected, and there exists $\lambda_2^* > 0$ such that*

$$\lambda_2^* = \min\left\{\inf_{t\geq 0}\lambda_2(\boldsymbol{L_B}(t)),\ \inf_{t\geq 0}\lambda_2(\boldsymbol{L_C}(t))\right\}, \tag{3.1}$$

where we define $\boldsymbol{B}(t) = \left[a_{ij}^{\frac{2n}{m+n}}(t)\right]$, $\boldsymbol{C}(t) = \left[a_{ij}^{\frac{2q}{p+q}}(t)\right] \in \mathbb{R}^{N\times N}$. *Then fixed-time consensus can be reached in system* (1.1).

Proof. Since $a_{ij}(t) = a_{ji}(t)$ for all $t \geq 0$, $i, j \in \mathcal{I}_N$, and $(\cdot)^{\frac{m}{n}}$, $(\cdot)^{\frac{p}{q}}$, $\mathrm{sign}(\cdot)$ are all odd functions with respect to $(\cdot)$, it follows from Lemma 2.2 that $\sum_{i=1}^{N}\dot{x}_i(t) = 0$, which implies that the total momentum in the model

$$\bar{x}(t) = \frac{1}{N}\sum_{i=1}^{N}x_i(t)$$

is conserved, that is, $\bar{x}(t) \equiv \bar{x}(0)$. Let $\boldsymbol{\delta}(t) = [\delta_1(t), \delta_2(t), ..., \delta_N(t)]^{\mathrm{T}}$ be the group disagreement vector with $\delta_i(t) = x_i(t) - \bar{x}$, then $\dot{\delta}_i(t) = \dot{x}_i(t)$ and $\sum_{i=1}^{N}\delta_i(t) = 0$.

Consider the following Lyapunov function candidate

$$V(\delta) = \frac{1}{2}\sum_{i=1}^{N}\delta_i^2(t).$$

Differentiating V along the protocol versus time yields

$$\begin{aligned}
\dot{V} =& \sum_{i=1}^{N}\delta_i\dot{\delta}_i \\
=&\alpha\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\delta_i(h(x_j)-h(x_i))^{\frac{m}{n}} + \beta\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\delta_i(h(x_j)-h(x_i))^{\frac{p}{q}} \\
&+ \gamma\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\delta_i\mathrm{sign}(h(x_j)-h(x_i)) \\
=&-\frac{\alpha}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}(\delta_j-\delta_i)(h(x_j)-h(x_i))^{\frac{m}{n}} - \frac{\beta}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}(\delta_j-\delta_i)(h(x_j)-h(x_i))^{\frac{p}{q}} \\
&- \frac{\gamma}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}(\delta_j-\delta_i)\mathrm{sign}(h(x_j)-h(x_i)) \\
=&-\frac{\alpha}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}|\delta_j-\delta_i||h(x_j)-h(x_i)|^{\frac{m}{n}} - \frac{\beta}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}|\delta_j-\delta_i||h(x_j)-h(x_i)|^{\frac{p}{q}} \\
&- \frac{\gamma}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}|\delta_j-\delta_i|
\end{aligned}$$

where the last equality is derived from $h(x)$ being strictly increasing. Since $h \in C^1$, according to the Lagrange mean value theorem and Lemma 2.5, there must exist $\bar{c} = \max\{\dot{h}(x), x \in [x_{\min}(0), x_{\max}(0)]\}$ such that $|\delta_j - \delta_i| = |x_j - x_i| \geq \frac{1}{\bar{c}}|h(x_j) - h(x_i)|$. Invoking Lemma 2.3, we have

$$\begin{aligned}
\dot{V} \leq & -\frac{\alpha}{2\bar{c}}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}|h(x_j) - h(x_i)|^{\frac{m+n}{n}} - \frac{\beta}{2\bar{c}}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}|h(x_j) - h(x_i)|^{\frac{p+q}{q}} \\
= & -\frac{\alpha}{2\bar{c}}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(a_{ij}^{\frac{2n}{m+n}}(h(x_j)-h(x_i))^2\right)^{\frac{m+n}{2n}} - \frac{\beta}{2\bar{c}}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(a_{ij}^{\frac{2q}{p+q}}(h(x_j)-h(x_i))^2\right)^{\frac{p+q}{2q}} \\
\leq & -\frac{\alpha}{2\bar{c}}N^{\frac{n-m}{n}}\left(\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}^{\frac{2n}{m+n}}(h(x_j) - h(x_i))^2\right)^{\frac{m+n}{2n}} \\
& -\frac{\beta}{2\bar{c}}\left(\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}^{\frac{2q}{p+q}}(h(x_j) - h(x_i))^2\right)^{\frac{p+q}{2q}} \\
= & -\frac{\alpha}{2\bar{c}}N^{\frac{n-m}{n}}\left(\frac{\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}^{\frac{2n}{m+n}}(h(x_j)-h(x_i))^2}{\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}^{\frac{2n}{m+n}}(\delta_j-\delta_i)^2}\cdot\frac{\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}^{\frac{2n}{m+n}}(\delta_j - \delta_i)^2}{V}\cdot V\right)^{\frac{m+n}{2n}} \\
& -\frac{\beta}{2\bar{c}}\left(\frac{\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}^{\frac{2q}{p+q}}(h(x_j) - h(x_i))^2}{\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}^{\frac{2q}{p+q}}(\delta_j - \delta_i)^2}\cdot\frac{\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}^{\frac{2q}{p+q}}(\delta_j - \delta_i)^2}{V}\cdot V\right)^{\frac{p+q}{2q}}. \qquad (3.2)
\end{aligned}$$

Let $\underline{c} = \min\{\dot{h}(x), x \in [x_{\min}(0), x_{\max}(0)]\}$, then $|h(x_j) - h(x_i)| \geq \underline{c}|x_j - x_i|$, which leads to

$$\frac{\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}^{\frac{2n}{m+n}}(h(x_j) - h(x_i))^2}{\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}^{\frac{2n}{m+n}}(\delta_j - \delta_i)^2} \geq \frac{\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}^{\frac{2n}{m+n}}\underline{c}^2(\delta_j - \delta_i)^2}{\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}^{\frac{2n}{m+n}}(\delta_j - \delta_i)^2} = \underline{c}^2 \qquad (3.3)$$

and

$$\frac{\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}^{\frac{2q}{p+q}}(h(x_j) - h(x_i))^2}{\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}^{\frac{2q}{p+q}}(\delta_j - \delta_i)^2} \geq \frac{\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}^{\frac{2q}{p+q}}\underline{c}^2(\delta_j - \delta_i)^2}{\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}^{\frac{2q}{p+q}}(\delta_j - \delta_i)^2} = \underline{c}^2. \qquad (3.4)$$

Then by Lemma 2.1 and (3.1), we have

$$\frac{\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}^{\frac{2n}{m+n}}(t)(\delta_j - \delta_i)^2}{V} = \left.\frac{2\boldsymbol{\delta}^{\mathrm{T}}\boldsymbol{L_B}(t)\boldsymbol{\delta}}{\frac{1}{2}\boldsymbol{\delta}^{\mathrm{T}}\boldsymbol{\delta}}\right|_{\substack{\boldsymbol{\delta}\neq\mathbf{0},\\ \mathbf{1}_N^{\mathrm{T}}\boldsymbol{\delta}=0}} \geq 4\lambda_2(\boldsymbol{L_B}(t)) \geq 4\lambda_2^* \qquad (3.5)$$

and

$$\frac{\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}^{\frac{2q}{p+q}}(t)(\delta_j-\delta_i)^2}{V} = \left.\frac{2\boldsymbol{\delta}^{\mathrm{T}}\boldsymbol{L_C}(t)\boldsymbol{\delta}}{\frac{1}{2}\boldsymbol{\delta}^{\mathrm{T}}\boldsymbol{\delta}}\right|_{\substack{\boldsymbol{\delta}\neq\mathbf{0},\\ \mathbf{1}_N^{\mathrm{T}}\boldsymbol{\delta}=0}} \geq 4\lambda_2(\boldsymbol{L_C}(t)) \geq 4\lambda_2^*. \tag{3.6}$$

Substituting (3.3), (3.4), (3.5) and (3.6) into (3.2), one has

$$\dot{V} \leq -\frac{\alpha}{2\bar{c}}N^{\frac{n-m}{n}}(4\underline{c}^2\lambda_2^*)^{\frac{m+n}{2n}}V^{\frac{m+n}{2n}} - \frac{\beta}{2\bar{c}}(4\underline{c}^2\lambda_2^*)^{\frac{p+q}{2q}}V^{\frac{p+q}{2q}}. \tag{3.7}$$

Let $y = \sqrt{4\underline{c}^2\lambda_2^* V}$, then (3.7) becomes

$$\dot{y} \leq -\frac{\underline{c}^2\lambda_2^*}{\bar{c}}\left(\alpha N^{\frac{n-m}{n}}y^{\frac{m}{n}} + \beta y^{\frac{p}{q}}\right).$$

By Lemma 2.4 and the comparison principle of differential equations, we can conclude that system (1.1) can achieve fixed-time consensus with the settling time upper bounded by

$$T = \frac{\bar{c}}{\underline{c}^2\lambda_2^*}\left(\frac{N^{\frac{m-n}{n}}}{\alpha}\frac{n}{m-n} + \frac{1}{\beta}\frac{q}{q-p}\right).$$

This completes the proof. □

We have thus obtained a sufficient condition for system (1.1) to admit fixed-time consensus when the topology is time-varying undirected. The following result focuses on the fixed topology case where we provide a more flexible condition to guarantee consensus.

3.2. Consensus under fixed directed topology

We are now in a position to discuss the robust fixed-time consensus under time-invariant directed topology. The main result of this part is the following theorem.

Theorem 3.2. *Suppose that the fixed topology $G(\boldsymbol{A})$ is strongly connected, detail-balanced and satisfies $a_{ij} > 0$ for $i \neq j$. Then fixed-time consensus can be reached in system* (1.1).

Proof. In this case, there exists a positive vector $\boldsymbol{\omega} = [\omega_1, \omega_2, ..., \omega_N]^{\mathrm{T}}$ satisfying $\omega_i a_{ij} = \omega_j a_{ji}$ for all $i, j \in \mathcal{I}_N$. Let $\boldsymbol{D} = \mathrm{diag}(\omega_1, \omega_2, ..., \omega_N)$, then $G(\boldsymbol{DA})$ is connected and undirected. It follows from Lemma 2.2 that $\sum_{i=1}^{N}\omega_i\dot{x}_i(t) = 0$, which implies the weighted center in the model

$$\bar{x}(t) = \frac{\sum_{i=1}^{N}\omega_i x_i(t)}{\sum_{i=1}^{N}\omega_i}$$

remains time-invariant. Let $\boldsymbol{\delta}(t) = [\delta_1(t), \delta_2(t), ..., \delta_N(t)]^{\mathrm{T}}$ be the group disagreement vector with $\delta_i(t) = x_i(t) - \bar{x}$, then $\dot{\delta}_i(t) = \dot{x}_i(t)$ and $\sum_{i=1}^{N}\omega_i\delta_i = 0$.

Consider the following Lyapunov function candidate

$$V(\delta) = \frac{1}{2}\sum_{i=1}^{N}\omega_i\delta_i^2(t).$$

Differentiating V along the protocol versus time yields

$$\begin{aligned}
\dot{V} =& \sum_{i=1}^{N}\omega_i\delta_i\dot{\delta}_i \\
=&\alpha\sum_{i=1}^{N}\sum_{j=1}^{N}\omega_i a_{ij}\delta_i(h(x_j)-h(x_i))^{\frac{m}{n}} + \beta\sum_{i=1}^{N}\sum_{j=1}^{N}\omega_i a_{ij}\delta_i(h(x_j)-h(x_i))^{\frac{p}{q}} \\
&+ \gamma\sum_{i=1}^{N}\sum_{j=1}^{N}\omega_i a_{ij}\delta_i\mathrm{sign}(h(x_j)-h(x_i)) \\
=&-\frac{\alpha}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\omega_i a_{ij}(\delta_j-\delta_i)(h(x_j)-h(x_i))^{\frac{m}{n}} - \frac{\beta}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\omega_i a_{ij}(\delta_j-\delta_i)(h(x_j) \\
&- h(x_i))^{\frac{p}{q}} - \frac{\gamma}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\omega_i a_{ij}(\delta_j-\delta_i)\mathrm{sign}(h(x_j)-h(x_i)) \\
=&-\frac{\alpha}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\omega_i a_{ij}|\delta_j-\delta_i||h(x_j)-h(x_i)|^{\frac{m}{n}} - \frac{\beta}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\omega_i a_{ij}|\delta_j-\delta_i||h(x_j) \\
&- h(x_i)|^{\frac{p}{q}} - \frac{\gamma}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\omega_i a_{ij}|\delta_j-\delta_i|.
\end{aligned}$$

Choose $\bar{c}$ and $\underline{c}$ as the same meanings in the proof of Theorem 3.1, then

$$\begin{aligned}
\dot{V}\leq& -\frac{\alpha}{2\bar{c}}\sum_{i=1}^{N}\sum_{j=1}^{N}\omega_i a_{ij}|h(x_j)-h(x_i)|^{\frac{m+n}{n}} - \frac{\beta}{2\bar{c}}\sum_{i=1}^{N}\sum_{j=1}^{N}\omega_i a_{ij}|h(x_j)-h(x_i)|^{\frac{p+q}{q}} \\
=& -\frac{\alpha}{2\bar{c}}\sum_{i=1}^{N}\sum_{j=1}^{N}\left((\omega_i a_{ij})^{\frac{2n}{m+n}}(h(x_j)-h(x_i))^2\right)^{\frac{m+n}{2n}} \\
&- \frac{\beta}{2\bar{c}}\sum_{i=1}^{N}\sum_{j=1}^{N}\left((\omega_i a_{ij})^{\frac{2q}{p+q}}(h(x_j)-h(x_i))^2\right)^{\frac{p+q}{2q}} \\
\leq& -\frac{\alpha}{2\bar{c}}N^{\frac{n-m}{n}}\left(\sum_{i=1}^{N}\sum_{j=1}^{N}(\omega_i a_{ij})^{\frac{2n}{m+n}}(h(x_j)-h(x_i))^2\right)^{\frac{m+n}{2n}} \\
&- \frac{\beta}{2\bar{c}}\left(\sum_{i=1}^{N}\sum_{j=1}^{N}(\omega_i a_{ij})^{\frac{2q}{p+q}}(h(x_j)-h(x_i))^2\right)^{\frac{p+q}{2q}}
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{\alpha}{2\bar{c}} N^{\frac{n-m}{n}} \left(\frac{\sum_{i=1}^{N}\sum_{j=1}^{N} (\omega_i a_{ij})^{\frac{2n}{m+n}} (h(x_j)-h(x_i))^2}{\sum_{i=1}^{N}\sum_{j=1}^{N} (\omega_i a_{ij})^{\frac{2n}{m+n}} (\delta_j-\delta_i)^2} \cdot \frac{\sum_{i=1}^{N}\sum_{j=1}^{N} (\omega_i a_{ij})^{\frac{2n}{m+n}} (\delta_j-\delta_i)^2}{V} \cdot V \right)^{\frac{m+n}{2n}} \\
&\quad - \frac{\beta}{2\bar{c}} \left(\frac{\sum_{i=1}^{N}\sum_{j=1}^{N} (\omega_i a_{ij})^{\frac{2q}{p+q}} (h(x_j)-h(x_i))^2}{\sum_{i=1}^{N}\sum_{j=1}^{N} (\omega_i a_{ij})^{\frac{2q}{p+q}} (\delta_j-\delta_i)^2} \cdot \frac{\sum_{i=1}^{N}\sum_{j=1}^{N} (\omega_i a_{ij})^{\frac{2q}{p+q}} (\delta_j-\delta_i)^2}{V} \cdot V \right)^{\frac{p+q}{2q}},
\end{aligned}
$$

where

$$
\frac{\sum_{i=1}^{N}\sum_{j=1}^{N} (\omega_i a_{ij})^{\frac{2n}{m+n}} (h(x_j)-h(x_i))^2}{\sum_{i=1}^{N}\sum_{j=1}^{N} (\omega_i a_{ij})^{\frac{2n}{m+n}} (\delta_j-\delta_i)^2} \geq \underline{c}^2
$$

and

$$
\frac{\sum_{i=1}^{N}\sum_{j=1}^{N} (\omega_i a_{ij})^{\frac{2q}{p+q}} (h(x_j)-h(x_i))^2}{\sum_{i=1}^{N}\sum_{j=1}^{N} (\omega_i a_{ij})^{\frac{2q}{p+q}} (\delta_j-\delta_i)^2} \geq \underline{c}^2.
$$

Set $\boldsymbol{E} = \left[(\omega_i a_{ij})^{\frac{2n}{m+n}}\right]$, $\boldsymbol{F} = \left[(\omega_i a_{ij})^{\frac{2q}{p+q}}\right] \in \mathbb{R}^{N\times N}$. By Lemma 2.7, we have

$$
\frac{\sum_{i=1}^{N}\sum_{j=1}^{N} (\omega_i a_{ij})^{\frac{2n}{m+n}} (\delta_j-\delta_i)^2}{V} = \left.\frac{2\boldsymbol{\delta}^{\mathrm{T}}\boldsymbol{L_E}\boldsymbol{\delta}}{\frac{1}{2}\boldsymbol{\delta}^{\mathrm{T}}\boldsymbol{D}\boldsymbol{\delta}}\right|_{\substack{\boldsymbol{\delta}\neq\boldsymbol{0},\\ \boldsymbol{\delta}\perp\boldsymbol{\omega}}} \geq 4\min_{\boldsymbol{\delta}\neq\boldsymbol{0}} R(\boldsymbol{L_E},\boldsymbol{D};\boldsymbol{\delta}) = 4\mu_1^* > 0
$$

and

$$
\frac{\sum_{i=1}^{N}\sum_{j=1}^{N} (\omega_i a_{ij})^{\frac{2q}{p+q}} (\delta_j-\delta_i)^2}{V} = \left.\frac{2\boldsymbol{\delta}^{\mathrm{T}}\boldsymbol{L_F}\boldsymbol{\delta}}{\frac{1}{2}\boldsymbol{\delta}^{\mathrm{T}}\boldsymbol{D}\boldsymbol{\delta}}\right|_{\substack{\boldsymbol{\delta}\neq\boldsymbol{0},\\ \boldsymbol{\delta}\perp\boldsymbol{\omega}}} \geq 4\min_{\boldsymbol{\delta}\neq\boldsymbol{0}} R(\boldsymbol{L_F},\boldsymbol{D};\boldsymbol{\delta}) = 4\mu_2^* > 0,
$$

where μ_1^*, μ_2^* are smallest roots of $\det(\mu\boldsymbol{D}-\boldsymbol{L_E}) = 0$ and $\det(\mu\boldsymbol{D}-\boldsymbol{L_F}) = 0$, respectively. Summarizing what we have obtained leads to

$$
\dot{V} \leq -\frac{\alpha}{2\bar{c}} N^{\frac{n-m}{n}} (4\underline{c}^2\mu^*)^{\frac{m+n}{2n}} V^{\frac{m+n}{2n}} - \frac{\beta}{2\bar{c}} (4\underline{c}^2\mu^*)^{\frac{p+q}{2q}} V^{\frac{p+q}{2q}}, \tag{3.8}
$$

where $\mu^* = \min\{\mu_1^*, \mu_2^*\}$. Let $y = \sqrt{4\underline{c}^2\mu^* V}$, then (3.8) becomes

$$
\dot{y} \leq -\frac{\underline{c}^2\mu^*}{\bar{c}} \left(\alpha N^{\frac{n-m}{n}} y^{\frac{m}{n}} + \beta y^{\frac{p}{q}}\right).
$$

By Lemma 2.4 and the comparison principle of differential equations, we can conclude that system (1.1) can achieve fixed-time consensus with the settling time upper bounded by

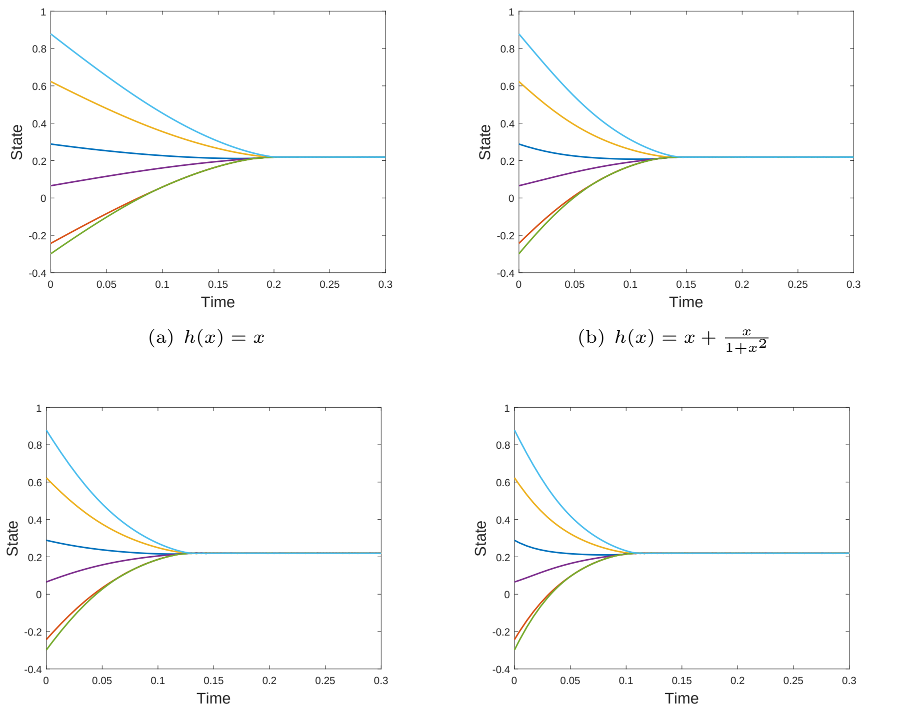
$$
T = \frac{\bar{c}}{\underline{c}^2\mu^*} \left(\frac{N^{\frac{m-n}{n}}}{\alpha}\frac{n}{m-n} + \frac{1}{\beta}\frac{q}{q-p}\right).
$$

This completes the proof. □

Remark 3.1. In fact, an undirected topology is always detail-balanced with the corresponding coefficients $\omega_i = 1$ for all $i \in \mathcal{I}_N$. That is to say, Theorem 3.2 is more general than Theorem 3.1 when the topology is time-invariant.

4. Numerical simulations

In what follows, some illustrative examples are performed to show the effectiveness of our proposed consensus protocol. For demonstration purpose, we take $N = 6$, $\alpha = \beta = 1$, $\gamma = 0.01$ and initial states are randomly chosen from $(-1, 1)$. Four different functions $h(x) = x$, $h(x) = x + \frac{x}{1+x^2}$, $h(x) = 2x + xe^{-x^2}$ and $h(x) = \frac{x}{1-2e^{-x}}$ are selected to use for state measurements.



(a) $h(x) = x$ (b) $h(x) = x + \frac{x}{1+x^2}$

(c) $h(x) = 2x + xe^{-x^2}$ (d) $h(x) = \frac{x}{1-2e^{-x}}$

Figure 1. Consensus of (1.1) with different state measurements under time-varying undirected topology.

To illustrate the result of Theorem 3.1, we choose parameters $m = 5$, $n = 3$, $p = 1$, $q = 3$ and take the Cucker-Smale potential function [8]

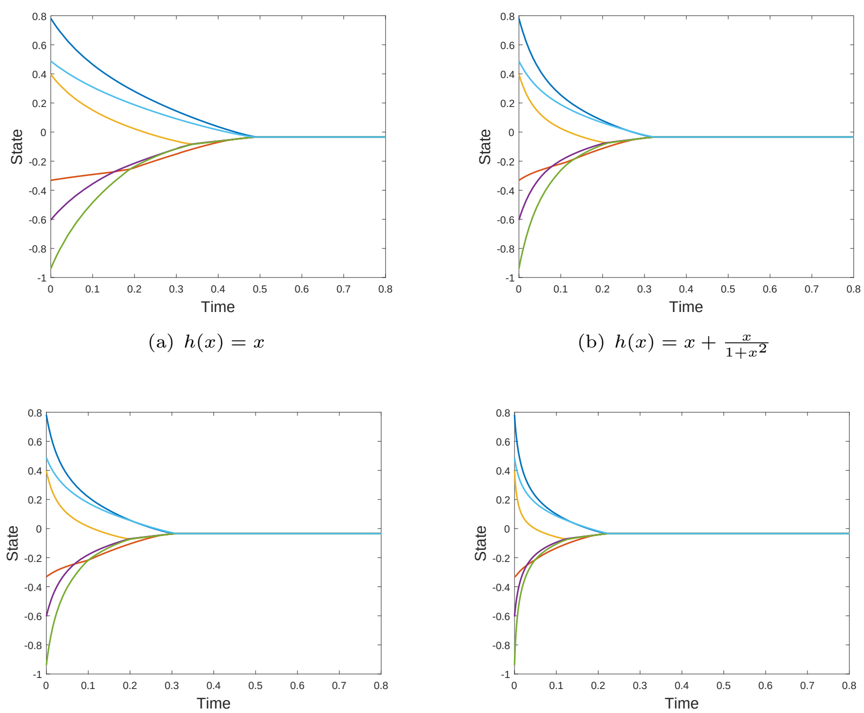
$$a_{ij}(t) = I(|x_j(t) - x_i(t)|)$$

with $I(r) = \frac{1}{1+r^2}$ whose symmetry makes the network topology undirected all the time. Consensus behaviors under four protocol functions are shown in Figure 1. We can see that all the protocols enable the states of six agents to reach agreement within finite time. Meanwhile, we compute the convergence time out numerically and results are 0.1967 for $h(x) = x$, 0.1406 for $h(x) = x + \frac{x}{1+x^2}$, 0.1269 for $h(x) = 2x + xe^{-x^2}$ and 0.1087 for $h(x) = \frac{x}{1-2e^{-x}}$.

Next, we consider the result of Theorem 3.2. Assume that system (1.1) has a fixed directed topology modeled by the detail-balanced adjacency matrix

$$\begin{bmatrix} 0 & 0.3 & 0.2 & 0.4 & 0.5 & 0.1 \\ 0.1 & 0 & 0.1 & 0.2 & 0.5 & 0.25 \\ 0.4 & 0.6 & 0 & 0.5 & 1 & 0.15 \\ 0.8 & 1.2 & 0.5 & 0 & 0.6 & 0.7 \\ 0.1 & 0.3 & 0.1 & 0.06 & 0 & 0.1 \\ 0.2 & 1.5 & 0.15 & 0.7 & 1 & 0 \end{bmatrix}$$

with balanced coefficients $\omega_1 = 1$, $\omega_2 = 3$, $\omega_3 = \omega_4 = \omega_6 = 0.5$, $\omega_5 = 5$ and then choose parameters $m = 7$, $n = 3$, $p = 1$, $q = 5$. Initial values are also chosen from $(-1, 1)$ but different from those in Figure 1. It can be seen from Figure 2 that all the protocols can achieve consensus within finite time. Specifically, the convergence time for $h(x) = x$ is 0.4961, for $h(x) = x + \frac{x}{1+x^2}$ is 0.3277, for $h(x) = 2x + xe^{-x^2}$ is 0.3114 and for $h(x) = \frac{x}{1-2e^{-x}}$ is 0.2265.



(a) $h(x) = x$ (b) $h(x) = x + \frac{x}{1+x^2}$

(c) $h(x) = 2x + xe^{-x^2}$ (d) $h(x) = \frac{x}{1-2e^{-x}}$

Figure 2. Consensus of (1.1) with different state measurements under fixed directed topology.

5. Conclusions

In this paper, a general form of protocols for multi-agent systems with nonlinear state measurements is proposed to solve the robust fixed-time consensus problem. Sufficient conditions are established to guarantee fixed-time consensus under time-varying undirected and fixed directed topology. It is shown that the convergence time is upper bounded by a constant determined by intrinsic parameters and the function used for state measurements but is independent of initial conditions. Thus, the settling time can be assigned in advance via appropriately choosing design parameters for arbitrary initial values. Finally, we have numerically illustrated that our proposed protocol is with high efficiency and good consensus performance for multi-agent systems. However, the detail-balanced condition is too strict to satisfy for a network topology. Consensus of multi-agent systems under less rigorous topology is currently under investigations.

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