

NEWLY FIXED DISC RESULTS USING ADVANCED CONTRACTIONS ON F-METRIC SPACE

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Abstract In this article, we present some fixed disc results for improving contractions in the frame work of F-metric space and to support the new results with examples. There are motivations for volterra Integral equation and findings of the existence of a solution to an initial value problem of second order differential equation.

Keywords Fixed disc, F -metric space, $\alpha - \psi$ contraction, Volterra Integral equation.

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1. Introduction

Let (M, d) be a metric space and T be a self mapping on M . A point $\xi \in M$ is a fixed point of T if $\xi = T\xi$. The mapping T is called a contraction if

$$d(T\xi, T\eta) \leq kd(\xi, \eta)$$

holds for all $\xi, \eta \in M$ where $0 < k < 1$. Jleli and Samet [3] introduced the concept of F -metric space and extended the Banach's contraction principle (see [2]). Banach contraction principle was generalized and opened for the further research in various fields [1, 4]. Recently, the fixed-circle problem has been considered for metric and some generalized metric spaces(see [6, 7, 9, 10] for more details). In [7], the fixed circle results he had taken by using Carsti-type contraction in metric space, in [9], he had given fixed circle results in self-mapping that includes mapping in a given circle onto itself. In this article [12] it is defined new notation F_c - contraction, it was extended known fixed circle theorems in metric space. In this article [6, 10] fixed circle problem studied in the S-metric space. In [13], a completely new fixed circle results had been given by using a modified khan-type contraction condition in S-metric space. Some results had generalized of fixed circles with a geometric view of metric space S_b and metric space N_b parametric were obtained [8, 14]. Further, it has been proposed to be investigated some fixed circle theorems in an extended

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M_b -metric space [5]. On the other hand, an application of the obtained fixed-circle results was given to discontinuous activation functions on metric spaces (see [13]). Hence it is important to study new fixed-circle results using different techniques. In this paper, we study a new class of general Ciric-type contraction and obtain few interesting results in the setting of F-metric space. Let M be an F-metric space and T be a self mapping on M . The circle and disc on F-metric space are respectively defined by:

$$C_{\xi_0, r} = \{\xi \in M : D(\xi, \xi_0) = r\},$$

and

$$D_{\xi_0, r} = \{\xi \in M : D(\xi, \xi_0) \leq r\}.$$

Definition 1.1 ([7]). If $\xi = T\xi$. for every $\xi \in C_{\xi_0, r}$, then $C_{\xi_0, r}$ is said to be a fixed circle of T .

Theorem 1.1 ([12]). Let T be an F_c - contraction on M , If $\exists, \xi_0 \in M$, and r be defined as:

$$r = \min\{d(\xi, T\xi) : \xi \neq T\xi\}.$$

then $C_{\xi_0, r}$ is a fixed circle of T .

More recently, Jleli and Samet [3] explained the concept of F-metric space and presented a generalization of the Banach contraction principle.

Definition 1.2. Let M be a nonempty set and $D : M \times M \rightarrow [0, +\infty]$ which satisfies the required conditions as given below:

$$(D_1) \quad (\xi, \eta) \in M \times M, D(\xi, \eta) = 0 \iff \xi = \eta.$$

$$(D_2) \quad D(\xi, \eta) = D(\eta, \xi) \text{ for all } (\xi, \eta) \in M \times M.$$

$$(D_3) \quad \text{For every } (\xi, \eta) \in M \times M, \aleph \in \mathbb{N}, \aleph \geq 2, \text{ and for every } (u_i)_{i=1}^{\aleph} \subset M \text{ with } (u_1, u_{\aleph}) = (\xi, \eta), \text{ we have}$$

$$D(\xi, \eta) > 0 \Rightarrow \mu(D(\xi, \eta)) \leq \mu \left(\sum_{i=1}^{\aleph-1} D(u_i, u_{i+1}) \right) + \beta,$$

where $\mu \in \mathbf{F}$, is defined as given below and $\beta \in [0, \infty)$, D is F-metric on M , and (M, D) is the F-metric space.

Let \mathbf{F} be the set of functions $f : (0, \infty) \rightarrow \mathbb{R}$ and satisfying the required conditions as given below:

$$(i) \quad f \text{ is non-decreasing i.e } 0 < s < t \Rightarrow f(s) \leq f(t).$$

$$(ii) \quad \text{For each sequences } t_n \in (0, +\infty), \text{ we have}$$

$$\lim_{n \rightarrow \infty} t_n = 0 \Leftrightarrow \lim_{n \rightarrow +\infty} f(t_n) = -\infty.$$

Definition 1.3 ([3]). If $\{\xi_n\} \rightarrow \xi$ w.r.t the F-metric D , then $\{\xi_n\}$ is F-convergent to ξ .

Definition 1.4 ([3]). A sequence $\{\xi_n\}$ is F-cauchy, if $\forall \epsilon > 0, \exists N$ s.t

$$\lim_{m, n \rightarrow \infty} D(\xi_m, \xi_n) = 0, \quad \text{where } m, n > N.$$

Definition 1.5 ([3]). If every F-cauchy sequence in M is F-convergent to a certain element in M , then M is said to be F-complete.

Samet et al. Introduced the notation of $\alpha - \psi$ contraction [11] an extension of Banach results.

Definition 1.6. It is denoted by ψ and defined as: the set of all nondecreasing functions $\psi : [0, \infty) \rightarrow [0, \infty)$ is defined as given below:

$$\sum_{n=1}^{\infty}(\psi^n(t)) < \infty \text{ for all } t > 0,$$

where $\psi(t) < t$.

Definition 1.7 ([12]). Let $\alpha : M \times M \rightarrow [0, \infty)$ be a mapping where $M \neq \emptyset$. A self mapping T is called α - admissible if $\forall \xi, \eta \in M$, we have $\alpha(\xi, \eta) \geq 1 \Rightarrow \alpha(T\xi, T\eta) \geq 1$.

2. Fixed Disc Results for General Contraction

In this section, we study a new class of general Ciric-type contraction and obtain few interesting results in the setting of F-metric space and let $T : M \rightarrow M$, and r be defined as given below:

$$r = \inf\{D(T\xi, \xi) : T\xi \neq \xi\}.$$

Definition 2.1. Let T be a self mapping on M , if $\exists, \psi \in \Psi$ and $\alpha : M \times M \rightarrow [0, \infty)$ s.t

$$\alpha(\xi, \eta)D^t(T\xi, T\eta) \leq \psi(M^t(\xi, \eta)), \tag{2.1}$$

where $M^t(\xi, \eta) = \max\{D^t(\xi, \eta), D^t(T\xi, T\eta), D^t(\xi, T\xi), D^t(\xi, T\eta), D^t(\eta, T\xi), D^t(\eta, T\eta)\}$, $\forall \xi, \eta \in M$ and $t \in [1, \infty)$. Then T is said to be a generalize Ciric-type $\alpha - \psi$ contractive mapping.

Theorem 2.1. Let T be an generalize Ciric-type $\alpha - \psi$ contractive self mapping on M , if $\exists, \xi_0 \in M$, and T is α - admissible s.t $\alpha(\xi_0, T\xi_0) \geq 1$ and $\xi_0 = T\xi_0$, then $D_{\xi_0, r}$ is a fixed disc of T .

Proof. Define a sequence $\{\xi_n\}$ in M by $\xi_{n+1} = T\xi_n = T^{n+1}\xi_0$ and assume that $\xi_n \neq \xi_{n+1} \forall n \geq 0$. T is α - admissible, we have $\alpha(\xi_0, \xi_1) = \alpha(\xi_0, T\xi_0) \geq 1, \Rightarrow \alpha(T\xi_0, T\xi_1) = \alpha(\xi_1, \xi_2) \geq 1$. By induction, we get $\alpha(\xi_n, \xi_{n+1}) \geq 1 \forall n \in N$, by using the condition (2.1) and take $\xi = \xi_0$ and $\eta = \xi_1$, we get

$$\alpha(\xi_0, \xi_1)D^t(T\xi_0, T\xi_1) \leq \psi(M^t(\xi_0, \xi_1)).$$

Since

$$\begin{aligned} M^t(\xi_0, \xi_1) &= \max\{D^t(\xi_0, \xi_1), D^t(T\xi_0, T\xi_1), D^t(\xi_0, T\xi_0), D^t(\xi_0, T\xi_1), \\ &\quad D^t(\xi_1, T\xi_0), D^t(\xi_1, T\xi_1)\}, \\ M^t(\xi_0, \xi_1) &= \max\{D^t(\xi_0, \xi_1), D^t(\xi_1, \xi_2), D^t(\xi_0, \xi_1), D^t(\xi_0, \xi_2), D^t(\xi_1, \xi_1), D^t(\xi_1, \xi_2)\}, \\ M^t(\xi_0, \xi_1) &= \max\{D^t(\xi_0, \xi_1), D^t(\xi_1, \xi_2), D^t(\xi_0, \xi_2)\}. \end{aligned}$$

If $\max\{D^t(\xi_0, \xi_1), D^t(\xi_1, \xi_2), D^t(\xi_0, \xi_2)\} = D^t(\xi_1, \xi_2)$,

$$\begin{aligned} \alpha(\xi_0, \xi_1)D^t(T\xi_0, T\xi_1) &\leq \psi(D^t(\xi_1, \xi_2)), \\ \alpha(\xi_0, \xi_1)D^t(T\xi_0, T\xi_1) &\leq \psi(D^t(T\xi_0, T\xi_1)) < D^t(T\xi_0, T\xi_1), \end{aligned}$$

which is a contradiction.

$$\text{If } \max\{D^t(\xi_0, \xi_1), D^t(\xi_1, \xi_2), D^t(\xi_0, \xi_2)\} = D^t(\xi_0, \xi_2),$$

$$\begin{aligned}\alpha(\xi_0, \xi_1)D^t(T\xi_0, T\xi_1) &\leq \psi(D^t(\xi_0, \xi_2)), \\ \alpha(\xi_0, \xi_1)D^t(T\xi_0, T\xi_1) &\leq \psi(D^t(\xi_0, T\xi_1)) < D^t(\xi_0, T\xi_1),\end{aligned}$$

which is a contradiction.

$$\text{If } \max\{D^t(\xi_0, \xi_1), D^t(\xi_1, \xi_2), D^t(\xi_0, \xi_2)\} = D^t(\xi_0, \xi_1),$$

$$\alpha(\xi_0, \xi_1)D^t(T\xi_0, T\xi_1) \leq \psi(D^t(\xi_0, T\xi_0)) < \psi(0) < 0,$$

which is a contradiction, hence $D^t(\xi, T\xi) = 0$ and accordingly $T\xi = \xi$, i.e., $D_{\xi_0, r}$ is a fixed disc of T . \square

Corollary 2.1. *Let T be an self-mapping on M , if T is a generalize Ciric-type $\alpha - \psi$ contractive mapping which satisfies the required conditions as given below:*

- (i) T is an $\alpha -$ admissible;
- (ii) $\exists, \xi_0 \in M$ s.t $\alpha(\xi_0, T\xi_0) \geq 1$;
- (iii) T is continuous on M .

Then $D_{\xi_0, r}$ is a fixed disc of T .

Example 2.1. Let $M = \{ 1, 2, e - 1, e, e + 1 \}$ be an F-metric space with

$$D^1(\xi, \eta) = (\xi - \eta)^2,$$

and let $T : M \rightarrow M$ and defined as:

$$T\xi = \begin{cases} e & \text{if } \xi = 1 \\ \xi + 1 & \text{if } \xi \in \{2, e\} \\ \xi & \text{otherwise} \end{cases}$$

$\forall \xi \in M$.

The self-mapping T and generalize Ciric-type $\alpha - \psi$ Contraction: If we take $\psi(t) = \frac{t}{2}$, $\xi_0 = e$, $\xi = 1$ and $\alpha(\xi, \xi_0) = 1$ and put $t = 1$. Then we have to prove that T is a generalized Ciric-type $\alpha - \psi$ contractive mapping on M .

Now

$$\begin{aligned}M^1(\xi, \xi_0) &= \max\{D^1(\xi, \xi_0), D^1(T\xi, T\xi_0), D^1(\xi, T\xi), D^1(\xi, T\xi_0), \\ &\quad D^1(\xi_0, T\xi), D^1(\xi_0, T\xi_0)\} \\ M^1(1, e) &= \max\{D^1(1, e), D^1(T1, Te), D^1(1, T1), D^1(1, Te), \\ &\quad D^1(e, T1), D^1(e, Te)\} \\ M^1(1, e) &= \max\{D^1(1, e), D^1(e, e + 1), D^1(1, e), D^1(1, e + 1), \\ &\quad D^1(e, e), D^1(e, e + 1)\} \\ M^1(1, e) &= \max\{(e - 1)^2, 1, (e - 1)^2, e^2, 0, 1\} \\ M^1(1, e) &= \max\{(e - 1)^2, 1, e^2\} = e^2.\end{aligned}$$

Then

$$\alpha(\xi, \xi_0)D^1(T\xi, T\xi_0) = \alpha(1, e)D^1(T1, Te) = 1 \leq \psi(M^1(1, e)).$$

Hence T is a generalize Ciric-type $\alpha - \psi$ contractive mapping. Also, we get

$$D^1(1, T1) = 2.9241$$

for $\xi = 1$

$$D^1(\xi_0, T\xi_0) = 1.$$

Also, we obtain

$$r = \min\{D(T\xi, \xi) : T\xi \neq \xi\} = \min\{2.9241, 1\} = 1.$$

So, fixes the disc $D(2.71, 1) = \{e-1, e, e+1\}$ and the circle $C(2.71, 1) = \{e-1, e+1\}$.

3. Fixed Disc Results for Interpolative Ciric-type Contraction

In this section, we study a new class of Interpolative Ciric-type Contraction and obtain some results in the area of F-metric space.

Definition 3.1. Let T be a self-mapping on M, if $\exists, \psi \in \Psi$ and $\alpha : M \times M \rightarrow [0, \infty)$ s.t

$$\alpha(\xi, \eta)D^t(T\xi, T\eta) \leq \psi(M^t(\xi, \eta)), \tag{3.1}$$

where $M^t(\xi, \eta) = \max\{D^{\alpha t}(\xi, \eta)D^{(1-\alpha)t}(\xi, T\xi), D^{\beta t}(T\xi, T\eta)D^{(1-\beta)t}(\eta, T\eta)\}, \forall \xi, \eta \in M, t \in [1, \infty)$ and $\alpha, \beta \in [0, 1)$. Then T is said to be a interpolative Ciric-type $\alpha - \psi$ contractive mapping.

Theorem 3.1. Let T be an interpolative Ciric type $\alpha - \psi$ contractive self mapping on M, if $\exists, \xi_0 \in M$, and T is $\alpha -$ admissible s.t $\alpha(\xi_0, T\xi_0) \geq 1$ and $\xi_0 = T\xi_0$, then $D_{\xi_0, r}$ is a fixed disc of T.

Proof. Define a sequence $\{\xi_n\}$ in M by $\xi_{n+1} = T\xi_n = T^{n+1}\xi_0$ and assume that $\xi_n \neq \xi_{n+1} \forall n \geq 0$. T is $\alpha -$ admissible, we have $\alpha(\xi_0, \xi_1) = \alpha(\xi_0, T\xi_0) \geq 1, \Rightarrow \alpha(T\xi_0, T\xi_1) = \alpha(\xi_1, \xi_2) \geq 1$. By induction, we get $\alpha(\xi_n, \xi_{n+1}) \geq 1 \forall n \in N$, by using the condition (3.1) and take $\xi = \xi_0$ and $\eta = \xi_1$, we get

$$\alpha(\xi_0, \xi_1)D^t(T\xi_0, T\xi_1) \leq \psi(M^t(\xi_0, \xi_1)).$$

Since $M^t(\xi_0, \xi_1) = \max\{D^{\alpha t}(\xi_0, \xi_1)D^{(1-\alpha)t}(\xi_0, T\xi_0), D^{\beta t}(T\xi_0, T\xi_1)D^{(1-\beta)t}(\xi_1, T\xi_1)\},$

$$M^t(\xi_0, \xi_1) = \max\{D^{\alpha t}(\xi_0, \xi_1)D^{(1-\alpha)t}(\xi_0, \xi_1), D^{\beta t}(\xi_1, \xi_2)D^{(1-\beta)t}(\xi_1, \xi_2)\},$$

$$M^t(\xi_0, \xi_1) = \max\{D^{\alpha t+t-\alpha t}(\xi_0, \xi_1), D^{\beta t+t-\beta t}(\xi_1, \xi_2)\},$$

$$M^t(\xi_0, \xi_1) = \max\{D^t(\xi_0, \xi_1), D^t(\xi_1, \xi_2)\}.$$

If $\max\{D^t(\xi_0, \xi_1), D^t(\xi_1, \xi_2)\} = D^t(\xi_1, \xi_2),$

$$\alpha(\xi_0, \xi_1)D^t(T\xi_0, T\xi_1) \leq \psi(D^t(\xi_1, \xi_2)),$$

$$\alpha(\xi_0, \xi_1)D^t(T\xi_0, T\xi_1) \leq \psi(D^t(T\xi_0, T\xi_1)) < D^t(T\xi_0, T\xi_1),$$

which is a contradiction.

$$\text{If } \max\{D^t(\xi_0, \xi_1), D^t(\xi_1, \xi_2)\} = D^t(\xi_0, \xi_1),$$

$$\alpha(\xi_0, \xi_1)D^t(T\xi_0, T\xi_1) \leq \psi(D^t(\xi_0, T\xi_0)) < \psi(0) < 0,$$

which is a contradiction, hence $D^t(\xi, T\xi) = 0$ and thus $T\xi = \xi$, i.e., $D_{\xi_0, r}$ is a fixed disc of T . \square

Example 3.1. Let $M = \{0, 1, 2, e - 1, e, e + 1\}$ be an F-metric space with

$$D^1(\xi, \eta) = (\xi - \eta)^2.$$

Let $T : M \rightarrow M$ and define as:

$$T\xi = \begin{cases} \frac{1}{2}\xi^2, & \text{if } \xi = 1 \\ 0, & \text{otherwise,} \end{cases}$$

$\forall \xi \in M$. If we take $\psi(t) = \frac{1}{2}$, $\xi_0 = e$, $\xi = 1$ and $\alpha(\xi, \eta) = 1$, $\alpha, \beta \in [0, 1)$ and put $t = 1$. Then we have to prove that T is a interpolative Ciric-type $\alpha - \psi$ contractive mapping on M . Since

$$M^1(\xi, \eta) = \max\{D^{1\alpha}(\xi, \eta)D^{1(1-\alpha)}(\xi, T\xi), D^{1\beta}(T\xi, T\eta)D^{1(1-\beta)}(\eta, T\eta)\},$$

Now

$$(T\xi(t) - T\eta(t))^2 = \frac{1}{4}(\xi^2 - \eta^2)^2 \leq (\xi - \eta)^2,$$

for $\xi = 1$. Consider

$$\begin{aligned} \alpha(\xi, \eta)(T^2\xi - T^2\eta)^2 &= (T(T\xi(t)) - T(T\eta(t)))^2 = \left(T\left(\frac{\xi^2}{2}\right) - T\left(\frac{\eta^2}{2}\right)\right)^2 \\ &= \left(\frac{1}{8}\xi^4 - \frac{1}{8}\eta^4\right)^2 = \frac{1}{64}(\xi^4 - \eta^4)^2 \\ &= \frac{1}{128}(\xi^4 - \eta^4)^2 + \frac{1}{128}(\xi^4 - \eta^4)^2 \\ &\leq \frac{1}{8}(\xi - \eta)^2 + \frac{1}{8}(T\xi(t) - T\eta(t))^2 \\ &\leq \frac{1}{4} \max\{(\xi - \eta)^2, (T\xi(t) - T\eta(t))^2\}, \end{aligned}$$

$$\alpha(\xi, \eta)D^1(T\xi, T\eta) \leq \psi(M^1(\xi, \eta)).$$

Hence T is a interpolative Ciric-type $\alpha - \psi$ contractive self mapping on M .

Definition 3.2. Let T be a self mapping on M , if $\exists, \psi \in \Psi$ and $\alpha : M \times M \rightarrow [0, \infty)$ s.t

$$\alpha(\xi, \eta)D(T\xi, T\eta) \leq \psi(M^t(\xi, \eta)), \quad (3.2)$$

where $M^t(\xi, \eta) = \max\left\{D^{\alpha t}(\eta, T\eta)D^{(1-\alpha)t}(T\xi, T\eta), \frac{D^{\beta t}(\xi, T\eta)D^{(1-\beta)t}(\eta, T\xi)}{1+D^t(\xi, \eta)}\right\}$, $\forall \xi, \eta \in M$, $t \in [1, \infty)$ and $\alpha, \beta \in [0, 1)$. Then T is said to be a Rational Ciric-type $\alpha - \psi$ contraction.

Theorem 3.2. Let T be an rational Ciric-type contractive self mapping on M , if $\exists, \xi_0 \in M$, and T is $\alpha -$ admissible s.t $\alpha(\xi_0, T\xi_0) \geq 1$ and $\xi_0 = T\xi_0$, then $D_{\xi_0, r}$ is a fixed disc of T .

Proof. Define a sequence $\{\xi_n\}$ in M by $\xi_{n+1} = T\xi_n = T^{n+1}\xi_0$ and assume that $\xi_n \neq \xi_{n+1} \forall n \geq 0$. T is α -admissible, we have $\alpha(\xi_0, \xi_1) = \alpha(\xi_0, T\xi_0) \geq 1, \Rightarrow \alpha(T\xi_0, T\xi_1) = \alpha(\xi_1, \xi_2) \geq 1$. By induction, we get $\alpha(\xi_n, \xi_{n+1}) \geq 1 \forall n \in N$, by using the condition (3.2) and take $\xi = \xi_0$ and $\eta = \xi_1$, we get

$$\begin{aligned} \alpha(\xi_0, \xi_1)D^t(T\xi_0, T\xi_1) &\leq \psi(M^t(\xi_0, \xi_1)), \\ M^t(\xi_0, \xi_1) &= \max \left\{ D^{\alpha t}(\xi_1, \xi_2)D^{(1-\alpha)t}(\xi_1, \xi_2), \frac{D^{\beta t}(\xi_0, \xi_2)D^{(1-\beta)t}(\xi_1, \xi_1)}{1 + D^t(\xi_0, \xi_1)} \right\}, \\ M^t(\xi_0, \xi_1) &= D^t(\xi_1, \xi_2), \\ \alpha(\xi_0, \xi_1)D^t(T\xi_0, T\xi_1) &\leq \psi(D^t(\xi_1, \xi_2)), \\ \alpha(\xi_0, \xi_1)D^t(T\xi_0, T\xi_1) &\leq \psi(D^t(T\xi_0, T\xi_1)) < D^t(T\xi_0, T\xi_1), \end{aligned}$$

which is a contradiction, therefore $D^t(\xi, T\xi) = 0$ and wherefore $T\xi = \xi$, i.e., $D_{\xi_0, r}$ is a fixed disc of T . □

Theorem 3.3. Let T be an contractive self mapping on M , if $\exists, \xi_0 \in M$, and T is α -admissible s.t $\alpha(\xi_0, T\xi_0) \geq 1$ and $\xi_0 = T\xi_0$, and satisfied the required condition as given below:

$$\begin{aligned} \alpha(\xi, \eta)D^t(T\xi, T\eta) &\leq \psi(M^t(\xi, \eta)), \tag{3.3} \\ M^t(\xi, \eta) &= \max \left\{ D^{\alpha t}(\xi, \eta)D^{(1-\alpha)t}(\xi, T\xi), D^{\beta t}(\eta, T\eta)D^{(1-\beta)t}(T\xi, T\eta), \right. \\ &\quad \left. \frac{1}{2} \left[D^{\gamma t}(\xi, \eta)D^{(1-\gamma)t}(T^2\xi, T\eta) + D^{\delta t}(\xi, T\eta)D^{(1-\delta)t}(T\xi, \eta) \right] \right\}, \end{aligned}$$

$\forall \xi, \eta \in M, t \in [1, \infty)$ and $\alpha, \beta, \gamma, \delta \in [0, 1)$, then T fixes the disc $D_{\xi_0, r}$.

Proof. Define a sequence $\{\xi_n\}$ in M by $\xi_{n+1} = T\xi_n = T^{n+1}\xi_0$ and assume that $\xi_n \neq \xi_{n+1} \forall n \geq 0$. T is α -admissible, we have $\alpha(\xi_0, \xi_1) = \alpha(\xi_0, T\xi_0) \geq 1, \Rightarrow \alpha(T\xi_0, T\xi_1) = \alpha(\xi_1, \xi_2) \geq 1$. By induction, we get $\alpha(\xi_n, \xi_{n+1}) \geq 1 \forall n \in N$, by using the condition (3.3) and take $\xi = \xi_0$ and $\eta = \xi_1$, we get

$$\alpha(\xi_0, \xi_1)D^t(T\xi_0, T\xi_1) \leq \psi(M^t(\xi_0, \xi_1)).$$

Since

$$\begin{aligned} M^t(\xi_0, \xi_1) &= \max \left\{ D^{\alpha t}(\xi_0, \xi_1)D^{(1-\alpha)t}(\xi_0, T\xi_0), D^{\beta t}(\xi_1, T\xi_1)D^{(1-\beta)t}(T\xi_0, T\xi_1), \right. \\ &\quad \left. \frac{1}{2} \left[D^{\gamma t}(\xi_0, \xi_1)D^{(1-\gamma)t}(T^2\xi_0, T\xi_1) + D^{\delta t}(\xi_0, T\xi_1)D^{(1-\delta)t}(T\xi_0, \xi_1) \right] \right\}, \\ M^t(\xi_0, \xi_1) &= \max \left\{ D^{\alpha t}(\xi_0, \xi_1)D^{(1-\alpha)t}(\xi_0, \xi_1), D^{\beta t}(\xi_1, \xi_2)D^{(1-\beta)t}(\xi_1, \xi_2), \right. \\ &\quad \left. \frac{1}{2} \left[D^{\gamma t}(\xi_0, \xi_1)D^{(1-\gamma)t}(\xi_2, \xi_2) + D^{\delta t}(\xi_0, \xi_2)D^{(1-\delta)t}(\xi_1, \xi_1) \right] \right\}, \\ M^t(\xi_0, \xi_1) &= \max \{ D^{\alpha t+t-\alpha t}(\xi_0, \xi_1), D^{\beta t+t-\beta t}(\xi_1, \xi_2) \}, \\ M^t(\xi_0, \xi_1) &= \max \{ D^t(\xi_0, \xi_1), D^t(\xi_1, \xi_2) \}. \end{aligned}$$

If $\max \{ D^t(\xi_0, \xi_1), D^t(\xi_1, \xi_2) \} = D^t(\xi_1, \xi_2)$,

$$\alpha(\xi_0, \xi_1)D^t(T\xi_0, T\xi_1) \leq \psi(D^t(\xi_1, \xi_2)),$$

$$\alpha(\xi_0, \xi_1)D^t(T\xi_0, T\xi_1) \leq \psi(D^t(T\xi_0, T\xi_1)) < D^t(T\xi_0, T\xi_1),$$

which is a contradiction.

$$\text{If } \max\{D^t(\xi_0, \xi_1), D^t(\xi_1, \xi_2)\} = D^t(\xi_0, \xi_1),$$

$$\alpha(\xi_0, \xi_1)D^t(T\xi_0, T\xi_1) \leq \psi(D^t(\xi_0, T\xi_0)) < \psi(0) < 0,$$

which is a contradiction. Hence $D^t(\xi, T\xi) = 0$ and consequently $T\xi = \xi$, i.e., $D_{\xi_0, r}$ is a fixed disc of T . \square

(2) Existence and Solution of Volterra Integral Equation of Second Order Differential Equation with initial problem:

Take

$$y''(t) - y(t) = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

Let us consider

$$y''(t) = u_1(\xi).$$

Integrating both sides 0 to ξ in above equation

$$y'(t) - y'(0) = \int_0^\xi u_1(t)dt,$$

using the initial condition $y'(0) = 0$, we get

$$y'(t) - 0 = \int_0^\xi u_1(t)dt,$$

$$y'(t) = \int_0^\xi u_1(t)dt.$$

Again integrating on both side 0 to ξ in above, we have

$$y(t) - y(0) = \int_0^\xi \int_0^\xi u_1(t)dt,$$

using the initial condition $y(0) = 1$, we get

$$y(t) - 1 = \int_0^\xi \int_0^\xi u_1(t)dt,$$

$$y(t) = 1 + \int_0^\xi (\xi - t)u_1(t)dt,$$

$$u_1(\xi) - \left(1 + \int_0^\xi (\xi - t)u_1(t)dt\right) = 0,$$

$$u_1(\xi) = \left(1 + \int_0^\xi (\xi - t)u_1(t)dt\right).$$

Suppose that the following condition hold:

(i) $u_1, u_2 : [0, 1] \times R \rightarrow [0, \infty)$ are continuous functions, such that there exist $\forall u_1, u_2 \in M$

$$\int_0^\xi |u_1(t) - u_2(t)| dt < \frac{\psi(M^t(u_1, u_2))}{(\xi - t)\alpha(u_1, u_2)}.$$

Since

$$M^t(u_1, u_2) = \max\{D^t(u_1, u_2), D^t(Tu_1, Tu_2), D^t(u_1, Tu_1), D^t(u_1, Tu_2), \\ D^t(u_2, Tu_1), D^t(u_2, Tu_2)\},$$

where $M = C[0, 1]$ the space of continuous functions defined on $[0, 1]$ to R with

$$D(u_1, u_2) = e^{|u_1(t) - u_2(t)|}, \quad \forall u_1, u_2 \in M.$$

Now

$$|Tu_1 - Tu_2| = \left| \left(1 + \int_0^\xi (\xi - t)u_1(t)dt \right) - \left(1 + \int_0^\xi (\xi - t)u_2(t)dt \right) \right|, \\ |Tu_1 - Tu_2| \leq \int_0^t |u_1(t) - u_2(t)|, \\ |Tu_1 - Tu_2| \leq \frac{\psi(M^t(u_1, u_2))}{\alpha(u_1, u_2)}, \\ \alpha(u_1, u_2)|Tu_1 - Tu_2| \leq \psi(M^t(u_1, u_2)).$$

Hence

$$\alpha(u_1, u_2)D^t(Tu_1, Tu_2) \leq \psi(M^t(u_1, u_2)),$$

where $\psi(t) = \frac{t}{2}$ for given differential equation exists a solution.

Conclusion

In this Paper presented a new method for finding the results of fixed disc using the Ciric techniques in the F-metric space. The results obtained with our method can also be considered as fixed disc results. We have used fixed disc examples to resolve the contraction problems. Our research can be extended to new exciting research areas of fixed disc theory. New ideas lead to further research and applications. We will also implement these concepts in different spaces.

Competing interest

The authors declare that they have no competing interests.

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