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QUALITATIVE ANALYSIS OF A FOURTH ORDER DIFFERENCE EQUATION

H. S. Alayachi[1,†], M. S. M. Noorani[1] and E. M. Elsayed[2,3]

Abstract In this paper, we will investigate some qualitative behavior of solutions of the following fourth order difference equation $x_{n+1} = ax_{n-1} + \frac{bx_{n-1}}{cx_{n-1}-dx_{n-3}}$, $n = 0, 1, ...,$ where the initial conditions x_{-3},x_{-2}, x_{-1} and x_0 are arbitrary real numbers and the values a, b, c and d are defined as positive real numbers.

Keywords Stability, periodicity, global attractor, boundedness, difference equations.

MSC(2010) 39A10.

1. Introduction

Our main objective in this paper is to obtain the qualitative behavior of the solutions of the following recursive equation:

$$x_{n+1} = ax_{n-1} + \frac{bx_{n-1}}{cx_{n-1} - dx_{n-3}}, \quad n = 0, 1, ..., \tag{1.1}$$

where the initial conditions $x_{-3},\ x_{-2,}\ x_{-1}$ and x_0 are arbitrary nonzero real numbers and $a, b, c,$and d are positive constants.

In recent years, the theory of difference equations has been studied by a large number of researchers due to the importance of this field in modeling a large number of real- life problems. Difference equations are used in modeling some natural phenomena that appear in biology, physics, economy, engineering, etc. Difference equations become apparent in the study of discretization methods for differential equations. Some results in the theory of difference equations have been obtained in the corresponding results of differential equations as more or less natural discrete analogues. Some recent studies of the dynamics of difference equations are given as follows. Agarwal and Elsayed [3] studied the periodicity character and global stability and provided a solution form for several special cases of the recursive sequence

$$x_{n+1} = ax_n + \frac{bx_n x_3}{cx_{n-2} + dx_{n-3}}.$$

†the corresponding author. Email address: HSSHAREEF@taibahu.edu.sa (H. S. Alayachi), msn@ukm.edu.my (M. S. M. Noorani), emmelsayed@yahoo.com (E. M. Elsayed)
1School of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, Bangi, Selangor, Malaysia
2Mathematics Department, Faculty of Science, King Abdulaziz University, P. O. Box 80203, Jeddah 21589, Saudi Arabia
3Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt

Cinar [7] investigated the solution of the difference equation

$$x_{n+1} = \frac{ax_{n-1}}{1 + bx_n x_{n-1}}.$$

Ibrahim [25] presented some relevant results of the difference equation

$$x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1}(a + bx_n x_{n-2})}.$$

Elsayed [16] analyzed the global stability and examined the periodic solution of the following difference equation:

$$x_{n+1} = ax_{n-l} + \frac{bx_{n-l}}{cx_{n-l} - dx_{n-k}}.$$

Elabbasy et al. [9] investigated the global stability and periodicity character and gave the solution of the special case of the difference equation

$$x_{n+1} = ax_n - \frac{bx_n}{cx_n - dx_{n-1}}.$$

Additionally, Yalçınkaya [39] addressed the difference equation

$$x_{n+1} = \alpha + \frac{x_{n-m}}{x_n^k}.$$

Yang et al. [40] examined the global and local stability of the equilibrium points of the following recursive equation:

$$x_{n+1} = \frac{ax_{n-1} + bx_{n-2}}{c + dx_{n-1}x_{n-2}}.$$

Other results of the qualitative behavior of difference equations can be obtained in refs. [1]- [42].

2. Some Basic Properties and Definitions

Here, we recall some basic definitions and some theorems that we need in the sequel.

Let I be some interval of real numbers, and the function f have continuous partial derivatives on I^{k+1} ,where $I^{k+1} = I \times I \times \cdots \times I$ $(k+1-$ times). Then, for initial conditions $x_{-k}, x_{-k+1}, ..., x_0 \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k}), \ n = 0, 1, ... \tag{2.1}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

A point $\overline{x} \in I$ is called an equilibrium point of Eq.(2.1) if

$$\overline{x} = f(\overline{x}, \overline{x}, ..., \overline{x}).$$

That is, $x_n = \overline{x}$ for $n \geq 0$ is a solution of Eq.(2.1), or equivalently, $\overline{x}$ is a fixed point of f.

Definition 2.1 (Stability).
(i) The equilibrium point $\overline{x}$ of Eq.(2.1) is locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \delta,$$

we have

$$|x_n - \overline{x}| < \epsilon \quad \text{for all} \quad n \geq -k.$$

(ii) The equilibrium point $\overline{x}$ of Eq.(2.1) is locally asymptotically stable if $\overline{x}$ is a locally stable solution of Eq.(2.1) and there exists $\gamma > 0$ such that for all $x_{-k}, x_{-k+1}, ...,$ $x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \gamma,$$

we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iii) The equilibrium point $\overline{x}$ of Eq.(2.1) is a global attractor if for all $x_{-k}, x_{-k+1},...,$ $x_{-1}, x_0 \in I$, we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iv) The equilibrium point $\overline{x}$ of Eq.(2.1) is globally asymptotically stable if $\overline{x}$ is locally stable, and $\overline{x}$ is also a global attractor of Eq.(2.1).
(v) The equilibrium point $\overline{x}$ of Eq.(2.1) is unstable if $\overline{x}$ is not locally stable.

The linearized equation of Eq.(2.1) about the equilibrium $\overline{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial f(\overline{x}, \overline{x}..., \overline{x})}{\partial x_{n-i}} y_{n-i}. \tag{2.2}$$

Now, assume that the characteristic equation associated with Eq.(2.2) is

$$p(\lambda) = p_0\lambda^k + p_1\lambda^{k-1} + ... + p_{k-1}\lambda + p_k = 0, \tag{2.3}$$

where $p_i = \dfrac{\partial f(\overline{x}, \overline{x}..., \overline{x})}{\partial x_{n-i}}$.

Theorem A ([30]). *Assume that* $p_i \in R$ $, i = 1, 2, ...$ *and* $k \in \{0, 1, 2, ...\}$. *Then,*

$$\sum_{i=1}^{k} |p_i| < 1$$

is a sufficient condition for the asymptotic stability of the difference equation

$$y_{n+k} + p_1 y_{n+k-1} + ... + p_k y_n = 0, \quad n = 0, 1 \ldots.$$

Next, we introduce a fundamental theorem to prove the global attractor of the fixed points.

Theorem B ([30]). *Let* $g : [a, b]^{k+1} \to [a, b]$ *be a continuous function, where* k *is a positive integer and* $[a, b]$ *is an interval of real numbers. Consider the difference equation*

$$x_{n+1} = g(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, \ldots. \tag{2.4}$$

Suppose that g satisfies the following conditions.

(1) For each integer i with $1 \leq i \leq k+1$, the function $g(z_1, z_2, ..., z_{k+1})$ is weakly monotonic in z_i for fixed $z_1, z_2, ..., z_{i-1}, z_{i+1}, ..., z_{k+1}$.

(2) If m, M is a solution of the system

$$m = g(m_1, m_2, ..., m_{k+1}), \; M = g(M_1, M_2, ..., M_{k+1}),$$

then $m = M$, where for each $i = 1, 2, ..., k+1$, we set

$$m_i = \left\{ \begin{matrix} m, & \text{if } g \text{ is non-decreasing in } z_i, \\ M, & \text{if } g \text{ is non-increasing in } z_i, \end{matrix} \right\}, \; Mi = \left\{ \begin{matrix} M, & \text{if } g \text{ is non-decreasing in } z_i, \\ m, & \text{if } g \text{ is non-increasing in } z_i. \end{matrix} \right\}.$$

Then, there exists exactly one equilibrium point $\bar{x}$ of Eq. (2.4), and every solution of Eq. (2.4) converges to $\bar{x}$.

3. Local Stability of the Equilibrium Point of Eq.(1.1)

This section studies the local stability character of the equilibrium point of Eq.(1.1). Eq.(1.1) has an equilibrium point given by

$$\overline{x} = a\overline{x} + \frac{b\overline{x}}{c\overline{x} - d\overline{x}}.$$

If $(c-d)(1-a) > 0$, then the only positive equilibrium point of Eq.(1.1) is given by

$$\overline{x} = \frac{b}{(c-d)(1-a)}.$$

Let $f : (0, \infty)^2 \longrightarrow (0, \infty)$ be a continuous function defined by

$$f(u, v) = au + \frac{du}{cu - dv}. \tag{3.1}$$

Therefore, it follows that

$$\frac{\partial f(u,v)}{\partial u} = a - \frac{bdv}{(cu-dv)^2},$$
$$\frac{\partial f(u,v)}{\partial v} = \frac{bdu}{(cu-dv)^2}.$$

Then, we see that

$$\frac{\partial f(\overline{x},\overline{x})}{\partial u} = a - \frac{d(1-a)}{(c-d)} = p_0,$$
$$\frac{\partial f(\overline{x},\overline{x})}{\partial v} = \frac{d(1-a)}{(c-d)} = p_1.$$

Then, the linearized equation of Eq.(1.1) about $\overline{x}$ is

$$y_{n+1} - p_0 y_{n-1} - p_1 y_{n-3} = 0.$$

Theorem 3.1. *Assume that*

$$|ac - d| + d\,|1 - a| < |c - d|\,.$$

Then, the positive fixed point of Eq.(1.1) is locally asymptotically stable.

Proof. It follows by Theorem A that Eq.(1.1) is asymptotically stable if

$$|p_1| + |p_0| < 1.$$

That is,

$$\left|a - \frac{d(1-a)}{(c-d)}\right| + \left|\frac{d(1-a)}{(c-d)}\right| < 1,$$

then,

$$|a(c-d) - d(1-a)| + |d(1-a)| < |c - d|\,.$$

Thus

$$|ac - d| + d\,|1 - a| < |c - d|\,.$$

According to Theorem A, the fixed point of Eq.(1.1) is asymptotically stable. Hence, the proof is complete. □

4. Global Attractivity of the Equilibrium Point of Eq.(1.1)

In this section, we investigate the global attractivity character of the solutions of Eq.(1.1).

Theorem 4.1. *The fixed point $\overline{x}$ of Eq.(1.1) is a global attractor if $ac > d$.*

Proof. Let α and β be real numbers and assume that $g : [\alpha, \beta]^2 \longrightarrow [\alpha, \beta]$ is a function defined by Eq.(3.1).Then,

$$\frac{\partial g(u,v)}{\partial u} = a - \frac{bdv}{(cu - dv)^2},$$
$$\frac{\partial g(u,v)}{\partial v} = \frac{bdu}{(cu - dv)^2}.$$

Case (1) If $a - \frac{bdv}{(cu-dv)^2} > 0$, then we can easily see that the function $g(u, v)$ is increasing in u, v. Suppose that (m, M) is a solution of the system

$$m = g(m, m) \text{ and } M = g(M, M).$$

Then from, Eq.(1.1), we see that

$$m = am + \frac{bm}{cm - dm},$$
$$M = aM + \frac{bM}{cM - dM}.$$

This result gives

$$(M - m) = a(M - m),\ a \neq 1.$$

Thus,

$$M = m.$$

It follows by Theorem B that $\overline{x}$ is a global attractor of Eq.(1.1). Therefore, the proof is complete.

Case (2) If $a - \frac{bdv}{(cu-dv)^2} < 0$, let α and β be a real numbers and assume that $g : [\alpha, \beta]^2 \longrightarrow [\alpha, \beta]$ is a function defined by

$$g(u, v) = au + \frac{bu}{cu - dv}.$$

Then, we can easily see that the function $g(u, v)$ is decreasing in u and increasing in v. Suppose that (m, M) is a solution of the system

$$M = g(m, M) \text{ and } m = g(M, m).$$

Then, from Eq.(1), we see that

$$\begin{aligned}
m &= aM + \frac{bM}{cM - dm}, \\
M &= am + \frac{bm}{cm - dM}, \\
cMm - acM^2 - dm^2 + adMm &= bM, \\
cMm - acm^2 - dM^2 + adMm &= bm,
\end{aligned}$$

then

$$(M^2 - m^2)(d - ac) = b(M - m), \quad ac > d.$$

Thus,

$$M = m.$$

It follows by Theorem B that $\overline{x}$ is a global attractor of Eq.(1.1). Hence, the proof is complete. □

5. Existence of Periodic Solutions

In this section, we study the existence of periodic solutions of Eq.(1.1). The following theorem states the necessary and sufficient condition that this equation does not have periodic solutions of prime period two.

Theorem 5.1. *Eq.(1.1) has no positive solutions of prime period two.*

Proof. First suppose that there exist prime period two solutions

$$..., p, q, p, q...,$$

of Eq.(1.1). Then,

$$\begin{aligned}
p &= ap + \frac{bp}{cp - dp}, \\
q &= aq + \frac{bq}{cq - dq}.
\end{aligned}$$

Then,

$$p(1-a)=\frac{b}{c-d},$$
$$q(1-a)=\frac{b}{c-d}.$$

This result contradicts the fact that $p\neq q$. Hence, this completes the proof. □

6. Special case of Eq.(1.1)

In this section, we study the following special case of Eq.(1.1)

$$x_{n+1}=x_{n-1}+\frac{x_{n-1}}{x_{n-1}-x_{n-3}}, \tag{6.1}$$

where the initial conditions $x_{-3},\ x_{-2},\ x_{-1}$ and x_0 are arbitrary real numbers with $x_{-3}\neq x_{-1}$ and $x_{-2}\neq x_0$.

Theorem 6.1. *Let $\{x_n\}_{n=-3}^{\infty}$ be the solution of Eq.(6.1) satisfying $x_{-3}=t,\ x_{-2}=s,\ x_{-1}=k,\ x_0=h$. Then, for $n=0,1,2...$*

$$x_{4n-3}=\frac{\left[nk^2+k(n^2-t(2n-1))+(n-1)t(t-n)\right]}{k-t},$$
$$x_{4n-2}=\frac{\left[nh^2+h(n^2-s(2n-1))+(n-1)s(s-n)\right]}{h-s},$$
$$x_{4n-1}=\frac{\left[(n+1)k^2+k(n(n+1)-t(2n+1))+nt(t-n)\right]}{k-t},$$
$$x_{4n}=\frac{\left[(n+1)h^2+h(n(n+1)-s(2n+1))+ns(s-n)\right]}{h-s}.$$

Proof. For $n=0$, the result holds. Now suppose that $n>o$ and that our assumption holds for $n-1$. That is ,

$$x_{4n-7}=\frac{\left[(n-1)k^2+((n-1)^2-t(2n-3))+(n-2)t(t-(n-1))\right]}{k-t},$$
$$x_{4n-6}=\frac{\left[(n-1)h^2+((n-1)^2-s(2n-3))+(n-2)s(s-(n-1))\right]}{h-s},$$
$$x_{4n-5}=\frac{\left[nk^2+k(n(n-1)-t(2n-1))+(n-1)t(t-(n-1))\right]}{k-t},$$
$$x_{4n-4}=\frac{\left[nh^2+h(n(n-1)-s(2n-1))+(n-1)s(s-(n-1))\right]}{k-t}.$$

Now, it follows from Eq.(6.1) that

$$\begin{aligned}&x_{4n-3}\\ =&x_{4n-5}+\frac{x_{4n-5}}{x_{4n-5}-x_{4n-7}}\\ =&\frac{\left[nk^2+k(n(n-1)-t(2n-1))+(n-1)t(t-(n-1))\right]}{k-t}\end{aligned}$$

$$+\frac{\frac{\left[nk^2+k(n(n-1)-t(2n-1))+(n-1)t(t-(n-1))\right]}{k-t}}{\frac{[nk^2+k(n(n-1)-t(2n-1))+(n-1)t(t-(n-1))]}{k-t}-\frac{[(n-1)k^2+((n-1)^2-t(2n-3))+(n-2)t(t-(n-1))]}{k-t}}$$

$$=\frac{\left[nk^2+k(n(n-1)-t(2n-1))+(n-1)t(t-(n-1))\right]}{k-t}$$

$$+\frac{\left[nk^2+k(n(n-1)-t(2n-1))+(n-1)t(t-(n-1))\right]}{[nk^2+k(n(n-1)-t(2n-1))+(n-1)t(t-(n-1))]-[(n-1)k^2+((n-1)^2-t(2n-3))+(n-2)t(t-(n-1))]}$$

$$=\frac{\left[nk^2+k(n(n-1)-t(2n-1))+(n-1)t(t-(n-1))\right]}{k-t}$$

$$+\frac{\left[nk^2+k(n(n-1)-t(2n-1))+(n-1)t(t-(n-1))\right]}{[k^2+k(n(n-1)-t(2n-1))+t(t-(n-1))-((n-1)^2-t(2n-3))]}$$

$$=\left[nk^2+k(n(n-1)-t(2n-1))+(n-1)t(t-(n-1))\right]$$

$$\left[\frac{1}{k-t}+\frac{1}{[k^2+k(n(n-1)-t(2n-1))+t(t-(n-1))-((n-1)^2-t(2n-3))]}\right]$$

$$=\left[nk^2+k(n(n-1)-t(2n-1))+(n-1)t(t-(n-1))\right]$$

$$\frac{\left[k^2+t(t-(n-1))+k(n(n-1)-t(2n-1))-((n-1)^2-t(2n-3))+k-t\right]}{(k-t)\left[k^2+k(n(n-1)-t(2n-1))+t(t-(n-1))-((n-1)^2-t(2n-3))\right]}$$

$$=\frac{\left[nk^2+k(n^2-t(2n-1))+(n-1)t(t-n)\right]}{k-t}.$$

Additionally, according to Eq.(6.1) ,

$$x_{4n-1}$$

$$=x_{4n-3}+\frac{x_{4n-3}}{x_{4n-3}-x_{4n-5}}$$

$$=\frac{\left[nk^2+k(n^2-t(2n-1))+(n-1)t(t-n)\right]}{k-t}$$

$$+\frac{\frac{\left[nk^2+k(n^2-t(2n-1))+(n-1)t(t-n)\right]}{k-t}}{\frac{[nk^2+k(n^2-t(2n-1))+(n-1)t(t-n)]}{k-t}-\frac{[nk^2+k(n(n-1)-t(2n-1))+(n-1)t(t-(n-1))]}{k-t}}$$

$$=\frac{\left[nk^2+k(n^2-t(2n-1))+(n-1)t(t-n)\right]}{k-t}$$

$$+\frac{\left[nk^2+k(n^2-t(2n-1))+(n-1)t(t-n)\right]}{k(n^2-t(2n-1))-k(n(n-1)-t(2n-1))}$$

$$=\frac{\left[nk^2+k(n^2-t(2n-1))+(n-1)t(t-n)\right]}{k-t}$$

$$+\frac{\left[nk^2+k(n^2-t(2n-1))+(n-1)t(t-n)\right]}{nk}$$

$$=\left[nk^2+k(n^2-t(2n-1))+(n-1)t(t-n)\right](\frac{1}{k-t}+\frac{1}{nk})$$

$$=\left[nk^2+k(n^2-t(2n-1))+(n-1)t(t-n)\right]\left[\frac{k(n-t)}{nk(k-t)}\right]$$

$$=\frac{\left[(n+1)k^2+k(n(n+1)-t(2n+1))+nt(t-n)\right]}{k-t}.$$

Also the other relations can be proofed similarly. $\square$

7. Numerical examples

We now present some numerical examples to confirm the theoretical work.

Example 7.1. We plot the stability of our equation under the values $a = 0.2$, $b = 1$, $c = 5$, $d = 0.5$, $x_{-3} = 5$, $x_{-2} = 0.3$, $x_{-1} = 2$ and $x_0 = -1$. See Figure 1.

Example 7.2. This example shows the global stability of our equation under the values $a = 0.5$, $b = 2$, $c = 6$, $d = 0.1$, $x_{-3} = 6$, $x_{-2} = -5$, $x_{-1} = 4$ and $x_0 = -3$. See Figure 2.

Example 7.3. An unstable solution of Eq.(6.1) is shown in Figure 3 under the values $a = 0.2$, $b = 1$, $c = 5$, $d = 0.5$, $x_{-3} = 5$, $x_{-2} = 0.3$, $x_{-1} = 2$ and $x_0 = -1$.

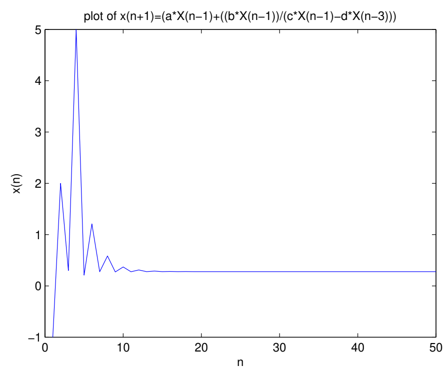

Figure 1.

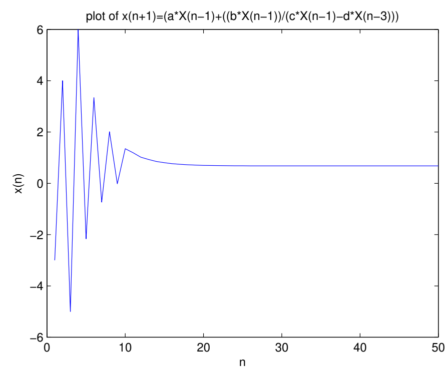

Figure 2.

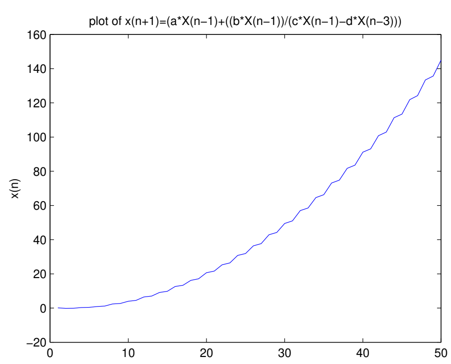

Figure 3.

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