ON A $p$\text{-}ADIC HILBERT\text{-}TYPE INTEGRAL OPERATOR AND ITS APPLICATIONS

Huabing Li$^1$ and Jianjun Jin$^{1,\dagger}$

Abstract In this note, we deal with a $p$\text{-}adic Hilbert\text{-}type integral operator induced by a symmetric homogeneous kernel of degree $-1$ and obtain the expression of the norm of this operator. As applications, we establish some new $p$\text{-}adic Hilbert\text{-}type inequalities with best constant factors.

Keywords $p$\text{-}adic field, $p$\text{-}adic Hilbert\text{-}type integral operator, $p$\text{-}adic Hilbert\text{-}type inequalities, norm of operator.


1. Introduction and main result

Let $q > 1$, $\mathbb{R}_+ = (0, +\infty)$, $f$ be a real-valued function on $\mathbb{R}_+$, then we have

$$\left[ \int_0^\infty \left[ \int_0^\infty \frac{f(y)}{x+y} dy \right]^q dx \right]^{\frac{1}{q}} \leq \pi \csc \left( \frac{\pi}{q} \right) \left[ \int_0^\infty |f(x)|^q dx \right]^{\frac{1}{q}},$$

(1.1)

for $f \in L^q(\mathbb{R}_+)$. Here $L^q(\mathbb{R}_+)$ is the usual Lebesgue space on $\mathbb{R}_+$. Inequality (1.1) is well known as Hilbert’s inequality and the constant factor $\pi \csc \left( \frac{\pi}{q} \right)$ in (1.1) is the best possible, see [4]. Hilbert’s inequality can be restated in the language of operator theory. For a measurable kernel $K(x, y)$ on $\mathbb{R}_+^2 = \mathbb{R}_+ \times \mathbb{R}_+$, we define an operator $T$ as: for $f \in L^q(\mathbb{R}_+)$,

$$(Tf)(x) := \int_0^\infty K(x, y)f(y)dy, \quad x \in \mathbb{R}_+. \quad (1.2)$$

Taking the Hilbert kernel $K(x, y) = \frac{1}{x+y}$ in (1.2), we get that $T$ is bounded from $L^q(\mathbb{R}_+)$ to $L^q(\mathbb{R}_+)$ and $\|T\| = \pi \csc \left( \frac{\pi}{q} \right)$.

If we take $K(x, y) = \frac{1}{\max\{x, y\}}$ in (1.2), then we can show that $T$ is bounded from $L^q(\mathbb{R}_+)$ to $L^q(\mathbb{R}_+)$ and $\|T\| = \frac{q^2}{q-1}$. It follows that the following Hardy-Littlewood-Pólya inequality holds for all $f \in L^q(\mathbb{R}_+)$,

$$\left[ \int_0^\infty \left[ \int_0^\infty \frac{f(y)}{\max\{x, y\}} dy \right]^q dx \right]^{\frac{1}{q}} \leq \frac{q^2}{q-1} \left[ \int_0^\infty |f(x)|^q dx \right]^{\frac{1}{q}}.$$

We can obtain some other inequalities with best constant factors similar to Hilbert’s inequality if we take other appropriate kernels, see [4]. We call these

$^1$the corresponding author. Email address: jinjjhb@163.com (J. Jin)

$^\dagger$School of Mathematics Sciences, Hefei University of Technology, Xuancheng Campus, Xuancheng 242000, China
Inequalities Hilbert-type inequalities. Hilbert’s inequality and Hilbert-type inequalities are important in analysis and applications, see [4, 8] and [9]. In the past two decades, this type of inequalities had been generalized and studied in different directions by many mathematicians and a lot of interesting results had been obtained, see for example [1, 3, 5–7, 12, 16], and Yang’s books [13–15] and the references cited therein for more details on this topic.

In this note, we introduce and study a $p$-adic Hilbert-type integral operator induced by a symmetric homogeneous kernel of degree $-1$. We obtain the expression of the norm of this operator. As applications, we establish some new $p$-adic Hilbert-type inequalities with best constant factors.

To state our results, we first recall some basic definitions and notations on $p$-adic analysis.

For a prime number $p$, let $\mathbb{Q}_p$ be the field of $p$-adic numbers. It is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the non-Archimedean $p$-adic norm $| \cdot |_p$. The $p$-adic norm is defined as follows: $|0|_p = 0$; if any non-zero rational number $x$ is represented as $x = p^m \frac{a}{b}$, where $\gamma \in \mathbb{Z}$, $m \in \mathbb{Z}$, $n \in \mathbb{Z}$, and $m$ and $n$ are not divisible by $p$, then $|x|_p = p^{-\gamma}$. Any non-zero $p$-adic number $x \in \mathbb{Q}_p$ can be uniquely represented in the following canonical form $x = p^\gamma \sum_{j=0}^{\infty} a_j p^j$, $\gamma = \gamma(x) \in \mathbb{Z}$, where $a_j$ are integers with $0 \leq a_j \leq p - 1$, $a_0 \neq 0$.

Also, it is not hard to show that the norm satisfies the following properties:

$$|xy|_p = |x|_p |y|_p, \quad |x + y|_p \leq \max\{|x|_p, |y|_p\}.$$  

It follows that, if $|x|_p \neq |y|_p$, then $|x + y|_p = \max\{|x|_p, |y|_p\}$. In what follows, we set $\mathbb{Q}_p^* = \mathbb{Q}_p \backslash \{0\}$ and denote by

$$B_\gamma(a) = \{x \in \mathbb{Q}_p : |x - a|_p \leq p^\gamma\},$$

the ball with center at $a \in \mathbb{Q}_p$ and radius $p^\gamma$, and

$$S_\gamma(a) = \{x \in \mathbb{Q}_p : |x - a|_p = p^\gamma\} = B_\gamma(a) \backslash B_{\gamma - 1}(a).$$

For simplicity, we use $B_\gamma$ and $S_\gamma$ to denote $B_\gamma(0)$ and $S_\gamma(0)$, respectively.

Since $\mathbb{Q}_p$ is a locally compact Hausdorff space, there exists a Haar measure $dx$ on $\mathbb{Q}_p$, which is unique up to positive constant multiple and is translation invariant. We normalize the measure $dx$ by the equality

$$\int_{B_0} dx = |B_0|_H = 1,$$

where $|E|_H$ denotes the Haar measure of a measurable subset $E$ of $\mathbb{Q}_p$. A simple calculation yields that

$$\int_{B_\gamma} dx = |B_\gamma|_H = p^\gamma, \quad \int_{S_\gamma} dx = |S_\gamma|_H = p^\gamma (1 - p^{-1}).$$

We refer the reader to [11] or [10] for a more detailed introduction to the $p$-adic analysis.

Let $q > 1$, $w(x)$ be a non-negative measurable function on $\mathbb{Q}_p^*$, $f$ be a real-valued measurable function on $\mathbb{Q}_p^*$, we define the weighted Lebesgue space $L^q_w(\mathbb{Q}_p^*)$ on $\mathbb{Q}_p^*$ as

$$L^q_w(\mathbb{Q}_p^*) := \left\{ f(x) : ||f||_{q,w} = \left[ \int_{\mathbb{Q}_p^*} |f(x)|^q w(x)dx \right]^{\frac{1}{q}} < \infty \right\}.$$
We write $L^q(Q^*_p)$ and $\|f\|_q$ instead of $L^q_w(Q^*_p)$ and $\|f\|_{q,w}$, respectively, if $w(x) \equiv 1$.

For $r > 1$, let $r'$ be the conjugate of $r$, i.e., $\frac{1}{r} + \frac{1}{r'} = 1$. Let $K(x, y)$ be non-negative and continuous on $\mathbb{R}^2_+$, and satisfy $K(tx, ty) = t^{-1}K(x, y)$, $K(x, y) = K(y, x)$, for any $t, x, y > 0$. Here we say $K(x, y)$ is a symmetric homogeneous function of degree $-1$. We assume that

$$0 < k_p(r) := (1 - p^{-1}) \sum_{-\infty < \gamma < \infty} K(1, p^\gamma) \cdot p^{\gamma + 1} < \infty.$$  

**Remark 1.1.** Noting that $K(x, y)$ is a symmetric homogeneous function of degree $-1$, we see that

$$k_p(r) = (1 - p^{-1}) \left[ K(1, 1) + \sum_{\gamma = 1}^{\infty} K(1, p^\gamma)(p^{\gamma + 1} + p^\gamma) \right] = k_p(r').$$

The following theorem is the main result of this paper.

**Theorem 1.1.** Let $p$ a prime number, $r > 1$, $q > 1$, $K(x, y)$ satisfy above conditions. Let $w(x) = |x|_p^{\frac{1}{r} - 1}$, we define $p$-adic Hilbert-type integral operator $T_p$ as: for $f \in L^q_w(Q^*_p)$,

$$(T_p f)(y) := \int_{Q^*_p} K(|x|_p, |y|_p)f(x)dx, \quad y \in Q^*_p.$$  

Then we have $T_p$ is bounded from $L^q_w(Q^*_p)$ to $L^q_w(Q^*_p)$ and $\|T_p\| = k_p(r)$, where

$$\|T_p\| := \sup_{f \in L^q_w(Q^*_p)} \frac{\|T_p f\|_{q,w}}{\|f\|_{q,w}}.$$  

It follows that

**Corollary 1.1.** Under the assumptions of Theorem 1.1. Let $f \geq 0$, $f \in L^q_w(Q^*_p)$. Then we have

$$\left[ \int_{Q^*_p} \left( \int_{Q^*_p} K(|x|_p, |y|_p)f(x)dx \right)^q dy \right]^{\frac{1}{q}} \leq k_p(r) \|f\|_{q,w},$$

where the constant factor $k_p(r)$ is the best possible.

## 2. Proof of main result

In this section, we prove Theorem 1.1. The following lemma is needed in our proof.

**Lemma 2.1.** Under the assumption of Theorem 1.1 and let $r'$ and $q'$ be the conjugates of $r$ and $q$, respectively. Denote $W_1(r, q; x), W_2(r', q'; y)$ by

$$W_1(r, q; x) := \int_{Q^*_p} K(|x|_p, |y|_p) \cdot \frac{|x|_p^{\frac{1}{r} - 1}}{|y|_p^{\frac{1}{q}'}} dy, \quad x \in Q^*_p; \quad (2.1)$$  

$$W_2(r', q'; y) := \int_{Q^*_p} K(|y|_p, |x|_p) \cdot \frac{|y|_p^{\frac{1}{q}' - 1}}{|x|_p^{\frac{1}{r}'}} dx, \quad y \in Q^*_p. \quad (2.2)$$
Then we have

\[ W_1(r, q; x) = k_p(r)|x|_{p}^{\frac{q}{p}-1}, \quad x \in \mathbb{Q}_p^*; \]

\[ W_2(r', q'; y) = k_p(r)|y|_{p}^{\frac{q'}{p}-1}, \quad y \in \mathbb{Q}_p^*. \]

**Proof.** Let \( y = xt \) in (2.1), then, by \( dy = |x|_{p}dt \), we have

\[ W_1(r, q; x) = \int_{\mathbb{Q}_p^*} K(|x|_{p}, |xt|_{p}) \cdot \frac{|x|_{p}^{\frac{q}{p}-1}}{|xt|_{p}^{\frac{q}{p}}} |x|_{p}dt \]

\[ = |x|_{p}^{\frac{q}{p}-1} \int_{\mathbb{Q}_p^*} K(1, |t|_{p}) \cdot \frac{1}{|t|_{p}^{\frac{q}{p}}} dt \]

\[ = |x|_{p}^{\frac{q}{p}-1} \sum_{-\infty < \gamma < \infty} \int_{S_{\gamma}} K(1, |t|_{p}) |t|_{p}^{-\frac{q}{p}} dt \]

\[ = |x|_{p}^{\frac{q}{p}-1} (1 - p^{-1}) \sum_{-\infty < \gamma < \infty} K(1, p^\gamma) p^{-\frac{q}{p}} \cdot p^\gamma \]

\[ = |x|_{p}^{\frac{q}{p}-1} (1 - p^{-1}) \sum_{-\infty < \gamma < \infty} K(1, p^\gamma) p^{\gamma} \]

\[ = k_p(r)|x|_{p}^{\frac{q}{p}-1}. \]

Similarly, we can obtain that \( W_2(r', q'; y) = k_p(r)|y|_{p}^{\frac{q'}{p}-1} \). The lemma is proved. \( \square \)

Now, we start to prove Theorem 1.1. For \( f \in L^q_{w}(\mathbb{Q}_p^*), \) by using the Hölder’s inequality and Lemma 2.1, we get that for \( y \in \mathbb{Q}_p^*; \)

\[ \left| \int_{\mathbb{Q}_p^*} K(|x|_{p}, |y|_{p}) f(x) dx \right| \]

\[ \leq \int_{\mathbb{Q}_p^*} \left\{ [K(|x|_{p}, |y|_{p})]^{\frac{1}{q}} \frac{|x|_{p}^{\frac{q-1}{p}}}{|y|_{p}^{\frac{q-1}{p}}} |f(x)| \right\} \left\{ [K(|x|_{p}, |y|_{p})]^{\frac{1}{q}} \frac{|y|_{p}^{\frac{1}{q}}}{|x|_{p}^{\frac{1}{q}}} \right\} dx \]

\[ \leq W_2^{\frac{1}{q}} (r', q'; y) \left\{ \int_{\mathbb{Q}_p^*} K(|x|_{p}, |y|_{p}) \frac{|x|_{p}^{\frac{q-1}{p}}}{|y|_{p}^{\frac{q-1}{p}}} |f(x)|^q dx \right\}^{\frac{1}{q}} \]

\[ = [k_p(r)]^{\frac{1}{q}} |y|_{p}^{\frac{1}{q}-\frac{1}{q}} \left\{ \int_{\mathbb{Q}_p^*} K(|x|_{p}, |y|_{p}) \frac{|x|_{p}^{\frac{q-1}{p}}}{|y|_{p}^{\frac{q-1}{p}}} |f(x)|^q dx \right\}^{\frac{1}{q}}. \]

Then

\[ \| T^p f \|_{q,w} = \left\{ \int_{\mathbb{Q}_p^*} |y|_{p}^{\frac{1}{q}-1} \left| \int_{\mathbb{Q}_p^*} K(|x|_{p}, |y|_{p}) f(x) dx \right|^q dy \right\}^{\frac{1}{q}} \]

\[ \leq [k_p(r)]^{\frac{1}{q}} \left\{ \int_{\mathbb{Q}_p^*} \int_{\mathbb{Q}_p^*} K(|x|_{p}, |y|_{p}) \frac{|x|_{p}^{\frac{q-1}{p}}}{|y|_{p}^{\frac{q-1}{p}}} |f(x)|^q dxdy \right\}^{\frac{1}{q}} \]
It follows that $T^p$ is bounded from $L^q_w(Q^*_p)$ to $L^q_w(Q^*_p)$ and $\|T^p\| \leq k_p(r)$.

We next show that $\|T^p\| = k_p(r)$. Let $\varepsilon = p^{-N}$, $N \in \mathbb{N}$, then $|\varepsilon|_p = p^N$. Set $f_\varepsilon(x) = 0$, then $0 < |x|_p < 1$ and $f_\varepsilon(x) = |x|^{\frac{1}{p} - \frac{\varepsilon}{2}}$, when $|x|_p \geq 1$. Then we get that

$$
|f_\varepsilon|_{q,w}^q = \int_{|x|_p \geq 1} |x|_p^{1-\varepsilon} dx = (1 - p^{-1}) \sum_{\gamma=0}^\infty p^{\gamma p^{\gamma(1-\varepsilon)}} = \frac{1 - p^{-1}}{1 - p^{-\varepsilon}}.
$$

and

$$
T^p f_\varepsilon = \int_{|x|_p \geq 1} K(|x|_p, |y|_p) |x|_p^{\frac{1}{p} - \frac{\varepsilon}{2}} dx.
$$

Then we have

$$
\|T^p f_\varepsilon\|_{q,w}^q = \int_{Q^*_p} |y|_p^{\frac{1}{p} - 1} \left( \int_{|x|_p \geq 1} K(|x|_p, |y|_p) |x|_p^{\frac{1}{p} - \frac{\varepsilon}{2}} dx \right)^q dy
= \int_{Q^*_p} |y|_p^{1-\varepsilon} \left( \int_{|t|_p \geq \frac{1}{|y|_p}} K(1, |t|_p) |t|_p^{\frac{1}{p} - \frac{\varepsilon}{2}} dt \right)^q dy
\geq \int_{|y|_p \geq |\varepsilon|_p} |y|_p^{1-\varepsilon} \left( \int_{|t|_p \geq \frac{1}{|y|_p}} K(1, |t|_p) |t|_p^{\frac{1}{p} - \frac{\varepsilon}{2}} dt \right)^q dy
= \left( \frac{1 - p^{-1}}{1 - p^{-\varepsilon}} \right)^p \int_{|t|_p \geq \frac{1}{|y|_p}} K(1, |t|_p) |t|_p^{\frac{1}{p} - \frac{\varepsilon}{2}} dt.
$$

It follows that

$$
\|T^p\| \geq \frac{\|T^p f_\varepsilon\|_{q,w}}{\|f_\varepsilon\|_{q,w}} \geq \frac{\sqrt{\varepsilon \varepsilon}}{1 - p^{-\varepsilon}} \int_{|t|_p \geq \frac{1}{|y|_p}} K(1, |t|_p) |t|_p^{\frac{1}{p} - \frac{\varepsilon}{2}} dt. \tag{2.3}
$$

Let $A_N = \{ t \in Q^*_p : |t|_p \geq \frac{1}{|y|_p} \} = \{ t \in Q^*_p : |t|_p \geq \frac{1}{p^N} \}$, then

$$
\int_{|t|_p \geq \frac{1}{|y|_p}} K(1, |t|_p) |t|_p^{\frac{1}{p} - \frac{\varepsilon}{2}} dt = \int_{Q^*_p} K(1, |t|_p) \chi_{A_N}(t) |t|_p^{\frac{1}{p} - \frac{\varepsilon}{p^N}} dt.
$$

On the other hand, it is clean that for any $t \in Q^*_p$,

$$
K(1, |t|_p) \chi_{A_N}(t) |t|_p^{\frac{1}{p} - \frac{\varepsilon}{p^N}} \to K(1, |t|_p) |t|_p^{\frac{1}{p}}, \quad N \to \infty.
$$

and $\sqrt{\varepsilon \varepsilon} \to 1$, $N \to \infty$.

Thus, by Fatou’s lemma and (2.3), we obtain that

$$
\|T^p\| \geq \int_{Q^*_p} K(1, |t|_p) |t|_p^{\frac{1}{p}} dt = k_p(r).
$$

Hence we have $\|T^p\| = k_p(r)$. Theorem 1.1 is proved.
3. Some new $p$-adic Hilbert-type inequalities

In this section, we establish some $p$-adic Hilbert-type inequalities with the best constant factors. For $r, q > 1$, let $r'$ and $q'$ be the conjugates of $r$ and $q$, respectively.

(1) Setting

$$K(x, y) = \frac{\ln \frac{y}{x}}{\max\{x, y\}}.$$  

We have

$$k_p(r) = (1 - p^{-1}) \left[ \sum_{\gamma=1}^{\infty} \frac{\gamma \ln p^\gamma}{p^\gamma} + \frac{\gamma \ln p^\gamma}{p^\gamma} \right]$$

$$= [(1 - p^{-1}) \ln p] \left[ \frac{p^\frac{1}{r}}{(p^\frac{1}{r} - 1)^2} + \frac{p^\frac{1}{r'}}{(p^\frac{1}{r'} - 1)^2} \right].$$

By Theorem 1.1, we have the following inequality holds for all $f \in L^q(\mathbb{Q}_p^\ast)$,

$$\left[ \int_{\mathbb{Q}_p^\ast} |y|_p^{\frac{1}{r'} - 1} \left| \int_{\mathbb{Q}_p^\ast} \frac{|x|_p^{\frac{1}{r}}}{\max\{|x|_p, |y|_p\}} f(x) dx \right|^q dy \right]^{\frac{1}{q}} \leq k_p(r) ||f||_{q, w},$$

where the constant factor $k_p(r) = [(1 - p^{-1}) \ln p] \left[ \frac{p^\frac{1}{r}}{(p^\frac{1}{r} - 1)^2} + \frac{p^\frac{1}{r'}}{(p^\frac{1}{r'} - 1)^2} \right]$ is the best possible.

(2) Setting

$$K(x, y) = \frac{|x^\lambda - y^\lambda|}{\max\{x, y\}^{\lambda + 1}}, \quad 0 < \lambda < \infty.$$  

We have

$$k_p(r) = (1 - p^{-1}) \left[ \sum_{\gamma=1}^{\infty} \frac{p^{\gamma \lambda} - 1}{p^{\gamma (\lambda + 1)}} + \frac{p^{\gamma \lambda} - 1}{p^{\gamma (\lambda + 1)}} \right]$$

$$= (1 - p^{-1}) \left[ \frac{1}{p^\frac{1}{r} - 1} + \frac{1}{p^\frac{1}{r'} - 1} - \frac{1}{p^{\lambda + \frac{1}{r} - 1}} - \frac{1}{p^{\lambda + \frac{1}{r'} - 1}} \right].$$

By Theorem 1.1, we have the following inequality holds for all $f \in L^q(\mathbb{Q}_p^\ast)$,

$$\left[ \int_{\mathbb{Q}_p^\ast} |y|_p^{\frac{1}{r'} - 1} \left| \int_{\mathbb{Q}_p^\ast} \frac{|x|_p^{\frac{1}{r}}}{\max\{|x|_p, |y|_p\}^{\lambda + 1}} f(x) dx \right|^q dy \right]^{\frac{1}{q}} \leq k_p(r) ||f||_{q, w},$$

where the constant factor $k_p(r) = (1 - p^{-1}) \left[ \frac{1}{p^\frac{1}{r} - 1} + \frac{1}{p^\frac{1}{r'} - 1} - \frac{1}{p^{\lambda + \frac{1}{r} - 1}} - \frac{1}{p^{\lambda + \frac{1}{r'} - 1}} \right]$ is the best possible.

(3) Setting

$$K(x, y) = \frac{x^\lambda + y^\lambda}{\max\{x, y\}^{\lambda + 1}}, \quad 0 \leq \lambda < \infty.$$  

We have

$$k_p(r) = (1 - p^{-1}) \left[ \sum_{\gamma=1}^{\infty} \frac{p^{\gamma \lambda} - 1}{p^{\gamma (\lambda + 1)}} + \frac{p^{\gamma \lambda} - 1}{p^{\gamma (\lambda + 1)}} \right]$$

$$= (1 - p^{-1}) \left[ \frac{1}{p^\frac{1}{r} - 1} + \frac{1}{p^\frac{1}{r'} - 1} - \frac{1}{p^{\lambda + \frac{1}{r} - 1}} - \frac{1}{p^{\lambda + \frac{1}{r'} - 1}} \right].$$

By Theorem 1.1, we have the following inequality holds for all $f \in L^q(\mathbb{Q}_p^\ast)$,
We have
\[ k_p(r) = (1 - p^{-1}) \left[ 2 + \sum_{\gamma=1}^{\infty} \frac{p^{\gamma\lambda} + 1}{p^{\gamma(\lambda+1)} p^\gamma} + \frac{p^{\gamma\lambda} + 1}{p^{\gamma(\lambda+1)} p^\gamma} \right] \]
\[ = (1 - p^{-1}) \left[ 2 + \frac{1}{p^{\lambda \frac{r}{p} - 1}} + \frac{1}{p^{\lambda \frac{r}{p} - 1}} + \frac{1}{p^{\lambda + \frac{\lambda}{p} - 1}} \right]. \]

By Theorem 1.1, we have the following inequality holds for all \( f \in L^q_w(Q^*_p) \),
\[ \left[ \int_{Q^*_p} |y|_{p}^{\frac{q}{p} - 1} \left| \int_{Q^*_p} \frac{|x|_{p}^{\lambda} + |y|_{p}^{\lambda}}{\max\{|x|_{p}, |y|_{p}\}^{\lambda+1}} f(x)dx \right|^{q} dy \right]^{\frac{1}{q}} \leq k_p(r) \|f\|_{q,w}, \]
where the constant factor \( k_p(r) = (1 - p^{-1}) \left[ 2 + \frac{1}{p^{\lambda \frac{r}{p} - 1}} + \frac{1}{p^{\lambda \frac{r}{p} - 1}} + \frac{1}{p^{\lambda + \frac{\lambda}{p} - 1}} \right] \) is the best possible.

(4) Setting
\[ K(x, y) = \frac{(xy)^{\frac{\lambda}{p}}}{\max\{x, y\}^{\lambda+1}}, \quad 0 \leq \lambda < \infty. \]

We have
\[ k_p(r) = (1 - p^{-1}) \left[ 2 + \sum_{\gamma=1}^{\infty} \frac{p^{\gamma\lambda} + 1}{p^{\gamma(\lambda+1)} p^\gamma} + \frac{p^{\gamma\lambda} + 1}{p^{\gamma(\lambda+1)} p^\gamma} \right] \]
\[ = (1 - p^{-1}) \left[ 2 + \frac{1}{p^{\lambda \frac{r}{p} - 1}} + \frac{1}{p^{\lambda \frac{r}{p} - 1}} + \frac{1}{p^{\lambda + \frac{\lambda}{p} - 1}} \right]. \]

By Theorem 1.1, we have the following inequality holds for all \( f \in L^q_w(Q^*_p) \),
\[ \left[ \int_{Q^*_p} |y|_{p}^{\frac{q}{p} - 1} \left| \int_{Q^*_p} \frac{|x|_{p}^{\lambda}}{\max\{|x|_{p}, |y|_{p}\}^{\lambda+1}} f(x)dx \right|^{q} dy \right]^{\frac{1}{q}} \leq k_p(r) \|f\|_{q,w}, \]
where the constant factor \( k_p(r) = (1 - p^{-1}) \left[ 2 + \frac{1}{p^{\lambda \frac{r}{p} - 1}} + \frac{1}{p^{\lambda \frac{r}{p} - 1}} + \frac{1}{p^{\lambda + \frac{\lambda}{p} - 1}} \right] \) is the best possible.

(5) Setting
\[ K(x, y) = \min\{x, y\}^{\frac{\lambda}{p}} \max\{x, y\}^{\lambda+1}, \quad 0 \leq \lambda < \infty. \]

We have
\[ k_p(r) = (1 - p^{-1}) \left[ 2 + \sum_{\gamma=1}^{\infty} \frac{1}{p^{\gamma(\lambda+1)} p^\gamma} + \frac{1}{p^{\gamma(\lambda+1)} p^\gamma} \right] \]
\[ = (1 - p^{-1}) \left[ 2 + \frac{1}{p^{\lambda \frac{r}{p} - 1}} + \frac{1}{p^{\lambda \frac{r}{p} - 1}} + \frac{1}{p^{\lambda + \frac{\lambda}{p} - 1}} \right]. \]
By Theorem 1.1, we have the following inequality holds for all \( f \in L^q_{w}(\mathbb{Q}_p^*) \),

\[
\left[ \int_{\mathbb{Q}_p^*} |y|_{p}^{\frac{r-1}{q}} \left| \int_{\mathbb{Q}_p^*} \frac{\min\{|x|_p, |y|_p\}}{\max\{|x|_p, |y|_p\}} f(x)dx \right|^q dy \right]^{\frac{1}{q}} \leq k_p(r)||f||_{q,w},
\]

where the constant factor \( k_p(r) = (1 - p^{-1}) \left[ 1 + \frac{1}{p^\lambda - 1} + \frac{1}{p^{\lambda^*} - 1} \right] \) is the best possible.

**Remark 3.1.** Taking \( \lambda = 0 \) in kernel (4) or (5), we get the \( p \)-adic Hardy-Littlewood-Pólya inequality as follows:

\[
\left[ \int_{\mathbb{Q}_p^*} |y|_{p}^{\frac{r-1}{q}} \left| \int_{\mathbb{Q}_p^*} \frac{1}{\max\{|x|_p, |y|_p\}} f(x)dx \right|^q dy \right]^{\frac{1}{q}} \leq k_p(r)||f||_{q,w}, \tag{3.1}
\]

where the constant factor \( k_p(r) = (1 - p^{-1}) \left[ 1 + \frac{1}{p^\lambda - 1} + \frac{1}{p^{\lambda^*} - 1} \right] \) is the best possible.

**Remark 3.2.** Recently, the equivalent form of (3.1) has been obtained in [2].

In particular, (i) when \( r = q \) in (3.1), we get that

\[
\left[ \int_{\mathbb{Q}_p^*} \left| \int_{\mathbb{Q}_p^*} \frac{1}{\max\{|x|_p, |y|_p\}} f(x)dx \right|^q \frac{dy}{q} \right]^{\frac{1}{q}} \leq (1 - p^{-1}) \left[ 1 + \frac{1}{p^\lambda - 1} + \frac{1}{p^{\lambda^*} - 1} \right] ||f||_q,
\]

holds for all \( f \in L^q_{w}(\mathbb{Q}_p^*) \).

(ii) When \( r = q^* \) in (3.1), we get that

\[
\left[ \int_{\mathbb{Q}_p^*} |y|_{p}^{\frac{2}{q^*}} \left| \int_{\mathbb{Q}_p^*} \frac{1}{\max\{|x|_p, |y|_p\}} f(x)dx \right|^q dy \right]^{\frac{1}{q^*}} \leq (1 - p^{-1}) \left[ 1 + \frac{1}{p^\lambda - 1} + \frac{1}{p^{\lambda^*} - 1} \right] ||f||_{q,w},
\]

holds for all \( f \in L^q_{w}(\mathbb{Q}_p^*) \), where \( w(x) = |x|^p_q - 2 \).

**References**


