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ON A *P*-ADIC HILBERT-TYPE INTEGRAL OPERATOR AND ITS APPLICATIONS

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Abstract In this note, we deal with a p-adic Hilbert-type integral operator induced by a symmetric homogeneous kernel of degree -1 and obtain the expression of the norm of this operator. As applications, we establish some new p-adic Hilbert-type inequalities with best constant factors.

Keywords p-adic field, p-adic Hilbert-type integral operator, p-adic Hilbert-type inequalities, norm of operator.

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1. Introduction and main result

Let $q > 1$, $\mathbb{R}_+ = (0, +\infty)$, f be a real-valued function on $\mathbb{R}_+$, then we have

$$\left[\int_0^\infty \left|\int_0^\infty \frac{f(y)}{x+y} dy\right|^q dx\right]^{\frac{1}{q}} \leq \pi \csc(\frac{\pi}{q}) \left[\int_0^\infty |f(x)|^q dx\right]^{\frac{1}{q}}, \tag{1.1}$$

for $f \in L^q(\mathbb{R}_+)$. Here $L^q(\mathbb{R}_+)$ is the usual Lebesgue space on $\mathbb{R}_+$. Inequality (1.1) is well known as Hilbert' s inequality and the constant factor $\pi \csc(\frac{\pi}{q})$ in (1.1) is the best possible, see [4]. Hilbert's inequality can be restated in the language of operator theory. For a measurable kernel $K(x, y)$ on $\mathbb{R}_+^2 = \mathbb{R}_+ \times \mathbb{R}_+$, we define an operator T as: for $f \in L^q(\mathbb{R}_+)$,

$$(Tf)(x) := \int_0^\infty K(x, y) f(y) dy, \quad x \in \mathbb{R}_+. \tag{1.2}$$

Taking the Hilbert kernel $K(x, y) = \frac{1}{x+y}$ in (1.2), we get that T is bounded from $L^q(\mathbb{R}_+)$ to $L^q(\mathbb{R}_+)$ and $\|T\| = \pi \csc(\frac{\pi}{q})$.

If we take $K(x, y) = \frac{1}{\max\{x,y\}}$ in (1.2), then we can show that T is bounded from $L^q(\mathbb{R}_+)$ to $L^q(\mathbb{R}_+)$ and $\|T\| = \frac{q^2}{q-1}$. It follows that the following Hardy-Littlewood-Pólya inequality holds for all $f \in L^q(\mathbb{R}_+)$,

$$\left[\int_0^\infty \left|\int_0^\infty \frac{f(y)}{\max\{x, y\}} dy\right|^q dx\right]^{\frac{1}{q}} \leq \frac{q^2}{q-1} \left[\int_0^\infty |f(x)|^q dx\right]^{\frac{1}{q}}.$$

We can obtain some other inequalities with best constant factors similar to Hilbert's inequality if we take other appropriate kernels, see [4]. We call these

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inequalities Hilbert-type inequalities. Hilbert's inequality and Hilbert-type inequalities are important in analysis and applications, see [4,8] and [9]. In the past two decades, this type of inequalities had been generalized and studied in different directions by many mathematicians and a lot of interesting results had been obtained, see for example [1,3,5–7,12,16], and Yang's books [13–15] and the references cited therein for more details on this topic.

In this note, we introduce and study a p-adic Hilbert-type integral operator induced by a symmetric homogeneous kernel of degree -1. We obtain the expression of the norm of this operator. As applications, we establish some new p-adic Hilbert-type inequalities with best constant factors.

To state our results, we first recall some basic definitions and notations on p-adic analysis.

For a prime number p, let $\mathbb{Q}_p$ be the field of p-adic numbers. It is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the non-Archimedean p-adic norm $|\cdot|_p$. The p-adic norm is defined as follows: $|0|_p = 0$; If any non-zero rational number x is represented as $x = p^\gamma \frac{m}{n}$, where $\gamma \in \mathbb{Z}$, $m \in \mathbb{Z}$, $n \in \mathbb{Z}$, and m and n are not divisible by p, then $|x|_p = p^{-\gamma}$. Any non-zero p-adic number $x \in \mathbb{Q}_p$ can be uniquely represented in the following canonical form $x = p^\gamma \sum_{j=0}^{\infty} a_j p^j, \quad \gamma = \gamma(x) \in \mathbb{Z}$, where a_j are integers with $0 \le a_j \le p-1, a_0 \ne 0$.

Also, it is not hard to show that the norm satisfies the following properties:

$$|xy|_p = |x|_p|y|_p, \quad |x+y|_p \le \max\{|x|_p, |y|_p\}.$$

It follows that, if $|x|_p \ne |y|_p$, then $|x+y|_p = \max\{|x|_p, |y|_p\}$. In what follows, we set $\mathbb{Q}_p^* = \mathbb{Q}_p \backslash \{0\}$ and denote by

$$B_\gamma(a) = \{x \in \mathbb{Q}_p : |x-a|_p \le p^\gamma\},$$

the ball with center at $a \in \mathbb{Q}_p$ and radius p^γ, and

$$S_\gamma(a) = \{x \in \mathbb{Q}_p : |x-a|_p = p^\gamma\} = B_\gamma(a) \backslash B_{\gamma-1}(a).$$

For simplicity, we use B_γ and S_γ to denote $B_\gamma(0)$ and $S_\gamma(0)$, respectively.

Since $\mathbb{Q}_p$ is a locally compact Hausdorff space, there exists a Haar measure dx on $\mathbb{Q}_p$, which is unique up to positive constant multiple and is translation invariant. We normalize the measure dx by the equality

$$\int_{B_0} dx = |B_0|_H = 1,$$

where $|E|_H$ denotes the Haar measure of a measurable subset E of $\mathbb{Q}_p$. A simple calculation yields that

$$\int_{B_\gamma} dx = |B_\gamma|_H = p^\gamma, \int_{S_\gamma} dx = |S_\gamma|_H = p^\gamma(1-p^{-1}).$$

We refer the reader to [11] or [10] for a more detailed introduction to the p-adic analysis.

Let $q > 1$, $w(x)$ be a non-negative measurable function on $\mathbb{Q}_p^*$, f be a real-valued measurable function on $\mathbb{Q}_p^*$, we define the weighted Lebesgue space $L_w^q(\mathbb{Q}_p^*)$ on $\mathbb{Q}_p^*$ as

$$L_w^q(\mathbb{Q}_p^*) := \{f(x) : ||f||_{q,w} = \left[\int_{\mathbb{Q}_p^*} |f(x)|^q w(x) dx\right]^{\frac{1}{q}} < \infty\}.$$

We write $L^q(\mathbb{Q}_p^*)$ and $\|f\|_q$ instead of $L_w^q(\mathbb{Q}_p^*)$ and $\|f\|_{q,w}$, respectively, if $w(x) \equiv 1$.

For $r > 1$, let r' be the conjugate of r, i.e., $\frac{1}{r} + \frac{1}{r'} = 1$. Let $K(x,y)$ be non-negative and continuous on $\mathbb{R}_+^2$, and satisfy $K(tx,ty) = t^{-1}K(x,y)$, $K(x,y) = K(y,x)$, for any $t,x,y > 0$. Here we say $K(x,y)$ is a symmetric homogeneous function of degree -1. We assume that

$$0 < k_p(r) := (1-p^{-1}) \sum_{-\infty<\gamma<\infty} K(1,p^\gamma) \cdot p^{\frac{\gamma}{r'}} < \infty.$$

Remark 1.1. Noting that $K(x,y)$ is a symmetric homogeneous function of degree -1, we see that

$$k_p(r) = (1-p^{-1}) \left[K(1,1) + \sum_{\gamma=1}^{\infty} K(1,p^\gamma)(p^{\frac{\gamma}{r'}} + p^{\frac{\gamma}{r}}) \right] = k_p(r').$$

The following theorem is the main result of this paper.

Theorem 1.1. *Let p a prime number, $r > 1, q > 1$, $K(x,y)$ satisfy above conditions. Let $w(x) = |x|_p^{\frac{q}{r}-1}$, we define p-adic Hilbert-type integral operator T^p as: for $f \in L_w^q(\mathbb{Q}_p^*)$,*

$$(T^p f)(y) := \int_{\mathbb{Q}_p^*} K(|x|_p, |y|_p) f(x) dx, \quad y \in \mathbb{Q}_p^*.$$

Then we have T^p is bounded from $L_w^q(\mathbb{Q}_p^*)$ to $L_w^q(\mathbb{Q}_p^*)$ and $||T^p|| = k_p(r)$, where

$$||T^p|| := \sup_{f \in L_w^q(\mathbb{Q}_p^*)} \frac{||T^p f||_{q,w}}{||f||_{q,w}}.$$

It follows that

Corollary 1.1. *Under the assumptions of Theorem 1.1. Let $f \geq 0$, $f \in L_w^q(\mathbb{Q}_p^*)$. Then we have*

$$\left[\int_{\mathbb{Q}_p^*} |y|_p^{\frac{q}{r}-1} \left(\int_{\mathbb{Q}_p^*} K(|x|_p, |y|_p) f(x) dx \right)^q dy \right]^{\frac{1}{q}} \leq k_p(r) ||f||_{q,w},$$

where the constant factor $k_p(r)$ is the best possible.

2. Proof of main result

In this section, we prove Theorem 1.1. The following lemma is needed in our proof.

Lemma 2.1. *Under the assumption of Theorem 1.1 and let r' and q' be the conjugates of r and q, respectively. Denote $W_1(r,q;x)$, $W_2(r',q';y)$ by*

$$W_1(r,q;x) := \int_{\mathbb{Q}_p^*} K(|x|_p, |y|_p) \cdot \frac{|x|_p^{\frac{q-1}{r}}}{|y|_p^{\frac{1}{r'}}} dy, \quad x \in \mathbb{Q}_p^*; \tag{2.1}$$

$$W_2(r',q';y) := \int_{\mathbb{Q}_p^*} K(|y|_p, |x|_p) \cdot \frac{|y|_p^{\frac{q'-1}{r'}}}{|x|_p^{\frac{1}{r}}} dx, \quad y \in \mathbb{Q}_p^*. \tag{2.2}$$

Then we have

$$W_1(r,q;x)=k_p(r)|x|_p^{\frac{q}{r}-1},\quad x\in\mathbb{Q}_p^*;$$

$$W_2(r',q';y)=k_p(r)|y|_p^{\frac{q'}{r'}-1},\quad y\in\mathbb{Q}_p^*.$$

Proof. Let $y=xt$ in (2.1), then, by $dy=|x|_p dt$, we have

$$\begin{aligned}
W_1(r,q;x)&=\int_{\mathbb{Q}_p^*}K(|x|_p,|xt|_p)\cdot\frac{|x|_p^{\frac{q-1}{r}}}{|xt|_p^{\frac{1}{r'}}}\cdot|x|_p dt\\
&=|x|_p^{\frac{q}{r}-1}\int_{\mathbb{Q}_p^*}K(1,|t|_p)\cdot\frac{1}{|t|_p^{\frac{1}{r'}}}dt\\
&=|x|_p^{\frac{q}{r}-1}\sum_{-\infty<\gamma<\infty}\int_{S_\gamma}K(1,|t|_p)|t|_p^{-\frac{1}{r'}}\,dt\\
&=|x|_p^{\frac{q}{r}-1}(1-p^{-1})\sum_{-\infty<\gamma<\infty}K(1,p^\gamma)p^{-\frac{\gamma}{r'}}\cdot p^\gamma\\
&=|x|_p^{\frac{q}{r}-1}(1-p^{-1})\sum_{-\infty<\gamma<\infty}K(1,p^\gamma)p^{\frac{\gamma}{r}}\\
&=k_p(r)|x|_p^{\frac{q}{r}-1}.
\end{aligned}$$

Similarly, we can obtain that $W_2(r',q';y)=k_p(r)|y|_p^{\frac{q'}{r'}-1}$. The lemma is proved. □

Now, we start to prove Theorem 1.1. For $f\in L_w^q(\mathbb{Q}_p^*)$, by using the Hölder's inequality and Lemma 2.1, we get that for $y\in\mathbb{Q}_p^*$,

$$\begin{aligned}
&\left|\int_{\mathbb{Q}_p^*}K(|x|_p,|y|_p)f(x)dx\right|\\
&\le\int_{\mathbb{Q}_p^*}\left\{[K(|x|_p,|y|_p)]^{\frac{1}{q}}\frac{|x|_p^{\frac{1}{q'r}}}{|y|_p^{\frac{1}{qr'}}}|f(x)|\right\}\left\{[K(|x|_p,|y|_p)]^{\frac{1}{q'}}\frac{|y|_p^{\frac{1}{qr'}}}{|x|_p^{\frac{1}{q'r}}}\right\}dx\\
&\le W_2^{\frac{1}{q'}}(r',q';y)\left\{\int_{\mathbb{Q}_p^*}K(|x|_p,|y|_p)\frac{|x|_p^{\frac{q-1}{r}}}{|y|_p^{\frac{1}{r'}}}|f(x)|^q\,dx\right\}^{\frac{1}{q}}\\
&=[k_p(r)]^{\frac{1}{q'}}|y|_p^{\frac{1}{r'}-\frac{1}{q'}}\left\{\int_{\mathbb{Q}_p^*}K(|x|_p,|y|_p)\frac{|x|_p^{\frac{q-1}{r}}}{|y|_p^{\frac{1}{r'}}}|f(x)|^q\,dx\right\}^{\frac{1}{q}}.
\end{aligned}$$

Then

$$\begin{aligned}
||T^pf||_{q,w}&=\left\{\int_{\mathbb{Q}_p^*}|y|_p^{\frac{q}{r}-1}\left|\int_{\mathbb{Q}_p^*}K(|x|_p,|y|_p)f(x)dx\right|^q dy\right\}^{\frac{1}{q}}\\
&\le[k_p(r)]^{\frac{1}{q'}}\left\{\int_{\mathbb{Q}_p^*}\int_{\mathbb{Q}_p^*}K(|x|_p,|y|_p)\frac{|x|_p^{\frac{q-1}{r}}}{|y|_p^{\frac{1}{r'}}}|f(x)|^q\,dxdy\right\}^{\frac{1}{q}}
\end{aligned}$$

$$
\begin{aligned}
&= [k_p(r)]^{\frac{1}{q'}} \left\{ \int_{\mathbb{Q}_p^*} W_1(r,q;x)|f(x)|^q \, dx \right\}^{\frac{1}{q}} \\
&= k_p(r)||f||_{q,w}.
\end{aligned}
$$

This proves that T^p is bounded from $L^q_w(\mathbb{Q}_p^*)$ to $L^q_w(\mathbb{Q}_p^*)$ and $||T^p|| \leq k_p(r)$.

We next show that $||T^p|| = k_p(r)$. Let $\varepsilon = p^{-N}, N \in \mathbb{N}$, then $|\varepsilon|_p = p^N$. Set $f_\varepsilon(x) = 0$, when $0 < |x|_p < 1$ and $f_\varepsilon(x) = |x|_p^{-\frac{1}{r}-\frac{\varepsilon}{q}}$, when $|x|_p \geq 1$. Then we get that

$$
\|f_\varepsilon\|_{q,w}^q = \int_{|x|_p \geq 1} |x|_p^{-1-\varepsilon} dx = (1-p^{-1}) \sum_{\gamma=0}^{\infty} p^\gamma p^{\gamma(-1-\varepsilon)} = \frac{1-p^{-1}}{1-p^{-\varepsilon}},
$$

and

$$
T^p f_\varepsilon = \int_{|x|_p \geq 1} K(|x|_p, |y|_p)|x|_p^{-\frac{1}{r}-\frac{\varepsilon}{q}} dx.
$$

Then we have

$$
\begin{aligned}
\|T^p f_\varepsilon\|_{q.w}^q &= \int_{\mathbb{Q}_p^*} |y|_p^{\frac{q}{r}-1} \left(\int_{|x|_p \geq 1} K(|x|_p, |y|_p)|x|_p^{-\frac{1}{r}-\frac{\varepsilon}{q}} dx \right)^q dy \\
&= \int_{\mathbb{Q}_p^*} |y|_p^{-1-\varepsilon} \left(\int_{|t|_p \geq \frac{1}{|y|_p}} K(1, |t|_p)|t|_p^{-\frac{1}{r}-\frac{\varepsilon}{q}} dt \right)^q dy \\
&\geq \int_{|y|_p \geq |\varepsilon|_p} |y|_p^{-1-\varepsilon} \left(\int_{|t|_p \geq \frac{1}{|\varepsilon|_p}} K(1, |t|_p)|t|_p^{-\frac{1}{r}-\frac{\varepsilon}{q}} dt \right)^q dy \\
&= \frac{(1-p^{-1})p^{-N\varepsilon}}{1-p^{-\varepsilon}} \left(\int_{|t|_p \geq \frac{1}{|\varepsilon|_p}} K(1, |t|_p)|t|_p^{-\frac{1}{r}-\frac{\varepsilon}{q}} dt \right)^q.
\end{aligned}
$$

It follows that

$$
\|T^p\| \geq \frac{||T^p f_\varepsilon||_{q,w}}{\|f_\varepsilon\|_{q,w}} \geq \sqrt[q]{\varepsilon^\varepsilon} \int_{|t|_p \geq \frac{1}{|\varepsilon|_p}} K(1, |t|_p)|t|_p^{-\frac{1}{r}-\frac{\varepsilon}{q}} dt. \tag{2.3}
$$

Let $A_N = \{t \in \mathbb{Q}_p^* : |t|_p \geq \frac{1}{|\varepsilon|_p}\} = \{t \in \mathbb{Q}_p^* : |t|_p \geq \frac{1}{p^N}\}$, then

$$
\int_{|t|_p \geq \frac{1}{|\varepsilon|_p}} K(1, |t|_p)|t|_p^{-\frac{1}{r}-\frac{\varepsilon}{q}} dt = \int_{\mathbb{Q}_p^*} K(1, |t|_p)\chi_{A_N}(t)|t|_p^{-\frac{1}{r}-\frac{1}{qp^N}} dt.
$$

On the other hand, it is clean that for any $t \in \mathbb{Q}_p^*$,

$$
K(1, |t|_p)\chi_{A_N}(t)|t|_p^{-\frac{1}{r}-\frac{1}{qp^N}} \to K(1, |t|_p)|t|_p^{-\frac{1}{r}}, \quad N \to \infty.
$$

and $\sqrt[q]{\varepsilon^\varepsilon} \to 1, \quad N \to \infty$.

Thus, by Fatou's lemma and (2.3), we obtain that

$$
\|T^p\| \geq \int_{\mathbb{Q}_p^*} K(1, |t|_p)|t|_p^{-\frac{1}{r}} \, dt = k_p(r).
$$

Hence we have $||T^p|| = k_p(r)$. Theorem 1.1 is proved.

3. Some new p-adic Hilbert-type inequalities

In this section, we establish some p-adic Hilbert-type inequalities with the best constant factors. For $r, q > 1$, let r' and q' be the conjugates of r and q, respectively.

(1) Setting

$$K(x,y) = \frac{\left|\ln \frac{y}{x}\right|}{\max\{x,y\}}.$$

We have

$$k_p(r) = (1-p^{-1})\left[\sum_{\gamma=1}^{\infty} \frac{\gamma \ln p}{p^\gamma} p^{\frac{\gamma}{r'}} + \frac{\gamma \ln p}{p^\gamma} p^{\frac{\gamma}{r}}\right]$$
$$= [(1-p^{-1})\ln p] \cdot \left[\frac{p^{\frac{1}{r}}}{(p^{\frac{1}{r}}-1)^2} + \frac{p^{\frac{1}{r'}}}{(p^{\frac{1}{r'}}-1)^2}\right].$$

By Theorem 1.1, we have the following inequality holds for all $f \in L^q_w(\mathbb{Q}_p^*)$,

$$\left[\int_{\mathbb{Q}_p^*} |y|_p^{\frac{q}{r}-1} \left|\int_{\mathbb{Q}_p^*} \frac{\left|\ln \frac{|y|_p}{|x|_p}\right|}{\max\{|x|_p, |y|_p\}} f(x)dx\right|^q dy\right]^{\frac{1}{q}} \leq k_p(r)||f||_{q,w},$$

where the constant factor $k_p(r) = [(1-p^{-1})\ln p] \cdot \left[\frac{p^{\frac{1}{r}}}{(p^{\frac{1}{r}}-1)^2} + \frac{p^{\frac{1}{r'}}}{(p^{\frac{1}{r'}}-1)^2}\right]$ is the best possible.

(2) Setting

$$K(x,y) = \frac{\left|x^\lambda - y^\lambda\right|}{\max\{x,y\}^{\lambda+1}}, \quad 0 < \lambda < \infty.$$

We have

$$k_p(r) = (1-p^{-1})\left[\sum_{\gamma=1}^{\infty} \frac{p^{\gamma\lambda}-1}{p^{\gamma(\lambda+1)}} p^{\frac{\gamma}{r'}} + \frac{p^{\gamma\lambda}-1}{p^{\gamma(\lambda+1)}} p^{\frac{\gamma}{r}}\right]$$
$$= (1-p^{-1})\left[\frac{1}{p^{\frac{1}{r}}-1} + \frac{1}{p^{\frac{1}{r'}}-1} - \frac{1}{p^{\lambda+\frac{1}{r}}-1} - \frac{1}{p^{\lambda+\frac{1}{r'}}-1}\right].$$

By Theorem 1.1, we have the following inequality holds for all $f \in L^q_w(\mathbb{Q}_p^*)$,

$$\left[\int_{\mathbb{Q}_p^*} |y|_p^{\frac{q}{r}-1} \left|\int_{\mathbb{Q}_p^*} \frac{\left||x|_p^\lambda - |y|_p^\lambda\right|}{\max\{|x|_p, |y|_p\}^{\lambda+1}} f(x)dx\right|^q dy\right]^{\frac{1}{q}} \leq k_p(r)||f||_{q,w},$$

where the constant factor $k_p(r) = (1-p^{-1})\left[\frac{1}{p^{\frac{1}{r}}-1} + \frac{1}{p^{\frac{1}{r'}}-1} - \frac{1}{p^{\lambda+\frac{1}{r}}-1} - \frac{1}{p^{\lambda+\frac{1}{r'}}-1}\right]$ is the best possible.

(3) Setting

$$K(x,y) = \frac{x^\lambda + y^\lambda}{\max\{x,y\}^{\lambda+1}}, \quad 0 \leq \lambda < \infty.$$

We have

$$k_p(r) = (1-p^{-1})\left[2+\sum_{\gamma=1}^{\infty}\frac{p^{\gamma\lambda}+1}{p^{\gamma(\lambda+1)}}p^{\frac{\gamma}{r'}}+\frac{p^{\gamma\lambda}+1}{p^{\gamma(\lambda+1)}}p^{\frac{\gamma}{r}}\right]$$
$$= (1-p^{-1})\left[2+\frac{1}{p^{\frac{1}{r}}-1}+\frac{1}{p^{\frac{1}{r'}}-1}+\frac{1}{p^{\lambda+\frac{1}{r}}-1}+\frac{1}{p^{\lambda+\frac{1}{r'}}-1}\right].$$

By Theorem 1.1, we have the following inequality holds for all $f \in L^q_w(\mathbb{Q}_p^*)$,

$$\left[\int_{\mathbb{Q}_p^*}|y|_p^{\frac{q}{r}-1}\left|\int_{\mathbb{Q}_p^*}\frac{|x|_p^\lambda+|y|_p^\lambda}{\max\{|x|_p,|y|_p\}^{\lambda+1}}f(x)dx\right|^q dy\right]^{\frac{1}{q}} \le k_p(r)||f||_{q,w},$$

where the constant factor $k_p(r)=(1-p^{-1})\left[2+\frac{1}{p^{\frac{1}{r}}-1}+\frac{1}{p^{\frac{1}{r'}}-1}+\frac{1}{p^{\lambda+\frac{1}{r}}-1}+!\frac{1}{p^{\lambda+\frac{1}{r'}}-1}\right]$ is the best possible.

(4) Setting

$$K(x,y)=\frac{(xy)^{\frac{\lambda}{2}}}{\max\{x,y\}^{\lambda+1}}, \quad 0\le\lambda<\infty.$$

We have

$$k_p(r) = (1-p^{-1})\left[1+\sum_{\gamma=1}^{\infty}\frac{p^{\frac{\gamma\lambda}{2}}}{p^{\gamma(\lambda+1)}}p^{\frac{\gamma}{r'}}+\frac{p^{\frac{\gamma\lambda}{2}}}{p^{\gamma(\lambda+1)}}p^{\frac{\gamma}{r}}\right]$$
$$= (1-p^{-1})\left[1+\frac{1}{p^{\frac{\lambda}{2}+\frac{1}{r}}-1}+\frac{1}{p^{\frac{\lambda}{2}+\frac{1}{r'}}-1}\right].$$

By Theorem 1.1, we have the following inequality holds for all $f \in L^q_w(\mathbb{Q}_p^*)$,

$$\left[\int_{\mathbb{Q}_p^*}|y|_p^{\frac{q}{r}-1}\left|\int_{\mathbb{Q}_p^*}\frac{|xy|_p^\lambda}{\max\{|x|_p,|y|_p\}^{\lambda+1}}f(x)dx\right|^q dy\right]^{\frac{1}{q}} \le k_p(r)||f||_{q,w},$$

where the constant factor $k_p(r) = (1-p^{-1})\left[1+\frac{1}{p^{\frac{\lambda}{2}+\frac{1}{r}}-1}+\frac{1}{p^{\frac{\lambda}{2}+\frac{1}{r'}}-1}\right]$ is the best possible.

(5) Setting

$$K(x,y)=\frac{\min\{\frac{x}{y},\frac{y}{x}\}^{\lambda}}{\max\{x,y\}^{\lambda+1}}, \quad 0\le\lambda<\infty.$$

We have

$$k_p(r) = (1-p^{-1})\left[1+\sum_{\gamma=1}^{\infty}\frac{1}{p^{\gamma(\lambda+1)}}p^{\frac{\gamma}{r'}}+\frac{1}{p^{\gamma(\lambda+1)}}p^{\frac{\gamma}{r}}\right]$$
$$= (1-p^{-1})\left[1+\frac{1}{p^{\lambda+\frac{1}{r}}-1}+\frac{1}{p^{\lambda+\frac{1}{r'}}-1}\right].$$

By Theorem 1.1, we have the following inequality holds for all $f \in L^q_w(\mathbb{Q}_p^*)$,

$$\left[\int_{\mathbb{Q}_p^*} |y|_p^{\frac{q}{r}-1}\left|\int_{\mathbb{Q}_p^*} \frac{\min\{\frac{|x|_p}{|y|_p}, \frac{|y|_p}{|x|_p}\}^\lambda}{\max\{|x|_p, |y|_p\}^{\lambda+1}} f(x)dx\right|^q dy\right]^{\frac{1}{q}} \leq k_p(r)||f||_{q,w},$$

where the constant factor $k_p(r) = (1-p^{-1})\left[1+\frac{1}{p^{\lambda+\frac{1}{r}}-1}+\frac{1}{p^{\lambda+\frac{1}{r'}}-1}\right]$ is the best possible.

Remark 3.1. Taking $\lambda = 0$ in kernel (4) or (5), we get the p-adic Hardy-Littlewood-Pólya inequality as follows:

$$\left[\int_{\mathbb{Q}_p^*} |y|_p^{\frac{q}{r}-1}\left|\int_{\mathbb{Q}_p^*} \frac{1}{\max\{|x|_p, |y|_p\}} f(x)dx\right|^q dy\right]^{\frac{1}{q}} \leq k_p(r)||f||_{q,w}, \tag{3.1}$$

where the constant factor $k_p(r) = (1-p^{-1})\left[1+\frac{1}{p^{\frac{1}{r}}-1}+\frac{1}{p^{\frac{1}{r'}}-1}\right]$ is the best possible.

Remark 3.2. Recently, the equivalent form of (3.1) has been obtained in [2].

In particular, (i) when $r = q$ in (3.1), we get that

$$\left[\int_{\mathbb{Q}_p^*}\left|\int_{\mathbb{Q}_p^*} \frac{1}{\max\{|x|_p, |y|_p\}} f(x)dx\right|^q dy\right]^{\frac{1}{q}}$$
$$\leq (1-p^{-1})\left[1+\frac{1}{p^{\frac{1}{q}}-1}+\frac{1}{p^{\frac{1}{q'}}-1}\right]||f||_q,$$

holds for all $f \in L^q(\mathbb{Q}_p^*)$.

(ii) When $r = q'$ in (3.1), we get that

$$\left[\int_{\mathbb{Q}_p^*} |y|_p^{q-2}\left|\int_{\mathbb{Q}_p^*} \frac{1}{\max\{|x|_p, |y|_p\}} f(x)dx\right|^q dy\right]^{\frac{1}{q}}$$
$$\leq (1-p^{-1})\left[1+\frac{1}{p^{\frac{1}{q}}-1}+\frac{1}{p^{\frac{1}{q'}}-1}\right]||f||_{q,w},$$

holds for all $f \in L^q_w(\mathbb{Q}_p^*)$, where $w(x) = |x|_p^{q-2}$.

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