Journal of Applied Analysis and Computation
Volume 10, Number 3, June 2020, 1094–1106
Website:http://jaac-online.com/
DOI:10.11948/20190192

EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR A QUASILINEAR ELLIPTIC SYSTEM ON UNBOUNDED DOMAINS INVOLVING NONLINEAR BOUNDARY CONDITIONS*

Somayeh Khademloo[1], Ghasem Alizadeh Afrouzi[2] and Jiafa Xu[3,†]

Abstract We prove two existence results for the nonlinear elliptic boundary value system involving p-Laplacian over an unbounded domain in R^N with noncompact boundary. The proofs are based on variational methods applied to weighted spaces.

Keywords Quasilinear elliptic systems, nonlinear boundary conditions, variational methods, weighted function spaces.

MSC(2010) 35J20, 35J60, 35J70.

1. Introduction and main results

The objective of this paper is to study the nonlinear elliptic boundary value system

$$\begin{cases} -\mathrm{div}(a(x)|\nabla u_i|^{p-2}\nabla u_i) = \lambda f(x)u_i|u_i|^{p-2} + F_{u_i}(x,u_1,\cdots,u_n), & x\in\Omega, \\ a(x)|\nabla u_i|^{p-2}\frac{\partial u_i}{\partial n} + b(x)u_i|u_i|^{p-2} = h(x,u_i), & x\in\partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subseteq R^N$ is an unbounded domain with noncompact smooth boundary $\partial\Omega$, the outward unit normal to which is denoted by n with $p>1$ and $i=1,...,n$.

The growing attention for the study of the p-Laplacian operator in the last few decades is motivated by the fact that it arises in various applications. The p-Laplacian operator in (1.1) is a special case of the divergence form operator $-\mathrm{div}(a(x,\nabla u))$, which appears in many nonlinear diffusion problems, in particular in the mathematical modeling of non-Newtonian fluids, for a discussion of some physical background, see [9]. We also refer to Aronsson-Janfalk [1] for the

†the corresponding author. Email address: s.khademloo@nit.ac.ir (S. Khademloo), afrouzi@umz.ac.ir (G. A. Afrouzi), xujiafa292@sina.com (J. Xu)
[1]Department of Mathematics, Faculty of Basic Sciences, Babol (Noushirvani) University of Technology Babol, Iran
[2]Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran
[3]School of Mathematical Sciences, Chongqing Normal University, Chongqing 401331, China
*The authors were supported by Talent Project of Chongqing Normal University (Grant No. 02030307-0040), the National Natural Science Foundation of China(Grant No. 11601048), Natural Science Foundation of Chongqing Normal University (Grant No. 16XYY24).

mathematical treatment of the Hele-Shaw flow of "power-law fluids". The concept of Hele-Shaw flow refers to the flow between two closely-spaced parallel plates, close in the sense that the gap between the plates is small compared to the dimension of the plates. Quasilinear problems with a variable coefficient also appear in the mathematical model of the torsional creep. This study is based on the observation that a prismatic material rod object to a torsional moment, at sufficiently high temperature and for on extended period of time, exhibits a permanent deformation, called creep. The corresponding equations are derived under the assumptions that the components of strain and stress are linked by a power law referred to as the creep-law [12, 15, 16].

The boundary condition of the system (1.1) describes a flux through the boundary which depends in a nonlinear manner on the solution itself, for some physical motivation of such boundary conditions, for example see [11, 19]. Some related the elliptic type equations and p-Laplacian equations results, we refer the reader to [2, 4–8, 10, 13, 14, 17, 22, 25–40] and the references therein.

Let $\Omega \subseteq R^N$ be an unbounded domain with smooth boundary $\partial\Omega$. We assume throughout that $1 < p < N$, $a_0 < a \in L^\infty(\Omega)$, for some positive constant a_0 and $b : \partial\Omega \to R$ is continuous function satisfying

(B_1) $\frac{c}{(1+|x|)^{p-1}} \leq b(x) \leq \frac{C}{(1+|x|)^{p-1}}$, for some constants $0 < c < C$.

Let $C_\delta^\infty(\Omega)$ be the space of $C_0^\infty(R^N)$- functions restricted on Ω. We define the weighted Sobolev space E as the completion of $C_\delta^\infty(\Omega)$ in the norm

$$||u||_E = \left[\int_\Omega (|\nabla u|^p + w(x)|u|^p)dx\right]^{1/p}$$

where $w(x) = \frac{1}{(1+|x|)^p}$, and we denote n times product of this space by $X = E^n$ with respect to the norm

$$||(u_1, ...u_n)||_X = \left(\sum_{i=1}^n ||u_i||_E^p\right)^{1/p}.$$

Denote by $L^p(\Omega, w_1)$, $L^q(\Omega, w_2)$ and $L^m(\partial\Omega, w_3)$ the weighted Lebesgue spaces with weight functions $w_i(x) = (1 + |x|)^{\alpha_i}$ for $i = 1, 2, 3$ and the norms defined by

$$||u||_{p,w_1}^p = \int_\Omega w_1(x)|u|^p dx, \quad ||u||_{q,w_2}^q = \int_\Omega w_2(x)|u|^q dx$$

and

$$||u||_{m,w_3}^m = \int_{\partial\Omega} w_3(x)|u|^m d\sigma,$$

where

$$(H_1) \qquad \begin{aligned} &-N < \alpha_1 \leq -p \ \text{ if } \ p < N, (\alpha_1 < -p \ \text{ when } \ p \geq N),\\ &-N < \alpha_2 \leq q\frac{N-p}{p} - N \ \text{ if } \ p < N, (-N < \alpha_2 < 0 \ \text{ when } \ p \geq N),\\ &-N < \alpha_3 \leq m\frac{N-p}{p} - N + 1 \ \text{ if } \ p < N, (-N < \alpha_3 < 0 \ \text{ when } \ p \geq N).\end{aligned}$$

Then we have the following embedding and trace theorem.

Lemma 1.1 ([20])**.** *If $p \le q \le \frac{pN}{N-p} = p^*$ and $-N < \alpha_2 \le q\frac{N-p}{p} - N$, then the embedding operator $E^n \hookrightarrow (L^q(\Omega, w_2))^n$ is continuous. If the upper bound for q be strict, then the embedding is compact.*

If $p \le m \le \frac{p(N-1)}{N-p}$ and $-N < \alpha_3 \le m\frac{N-p}{p} - N + 1$, then the trace operator $E^n \hookrightarrow (L^m(\partial\Omega, w_3))^n$ is continuous. If the upper bound for m be strict, then the trace operator is compact.

Furthermore, one can show

Lemma 1.2 ([21])**.** *The quantity*

$$||u||_b = \left[\int_\Omega a(x)|\nabla u|^p dx + \int_{\partial\Omega} b(x)|u|^p d\sigma\right]^{1/p}$$

defines an equivalent norm on E. Moreover

$$||(u_1, ...u_n)||_B = \left(\sum_{i=1}^n ||u_i||_b^p\right)^{1/p}$$

defines an equivalent norm on X.

Because the lack of separability for the functions F and h, we need to restrict the problem (1.1) to the following assumptions on f, F and h:

The function f is nontrivial measurable satisfying

(f_1) $0 \le f(x) \le C(1 + |x|)^{\alpha_1}$ for $a.e.$ $x \in \Omega$.

The mapping $h : \partial\Omega \to R$ is a Caratheodory function which fulfills the assumptions

(f_2) $|h(x, u)| \le h_0(x) + h_1(x)|s|^{m-1}$, where $h_i : \partial\Omega \to R$, $(i = 0, 1)$ are measurable functions satisfying $h_0 \in L^{\frac{m}{m-1}}(\partial\Omega, w_3^{\frac{1}{1-m}})$, $0 \le h_i \le C_h w_3$ $(i = 0, 1)$.

We also assume

(H_2) $\lim_{s\to 0} \frac{h(x,s)}{b(x)|s|^{p-1}} = 0$, uniformly in x.

(H_3) There exists $\mu \in (p, p^*]$ s.t. $\mu H(x, t) \le th(x, t)$ a.e. $x \in \Omega, \forall t \in R$, where $H(x, t) = \int_0^t h(x, s)ds$.

(H_4) There is a nonempty open set $O \subset \partial\Omega$ with $H(x, t) > 0$ for $(x, t) \in O \times (0, \infty)$.

Also we need the following assumptions on F:

(F_1) $F : \overline{\Omega} \times (R^+)^n \to R^+$ is a C^1-function such that $F(x, tu_1, ..., tu_n) = t^{p^*}F(x, u_1, ..., u_n)(t > 0)$ holds for all $(x, u_1, ..., u_n) \in \overline{\Omega} \times (R^+)^n$.

(F_2) $F(x, u_1, ..., u_n) = 0$ if $u_j = 0$ for some $j = 1, ..., n$ and $u_i \in R^+$ for $i = 1, ..., n, i \ne j$.

(F_3) $F_{u_i}(x, u_1, ..., u_n)$ are strictly increasing functions about $(u_1, ..., u_n)$ for all $u_i > 0$, $i = 1, ..., n$.

Moreover, using Homogeneity property in (F_1), we have the so-called Euler identity

$$\begin{cases} (u_1, ..., u_n) \cdot \nabla F(x, u_1, ..., u_n) = p^* F(x, u_1, ..., u_n), \\ F(x, u_1, ..., u_n) \le K\left(\sum_{i=1}^n |u_i|^p\right)^{\frac{p^*}{p}} \quad \text{for some } K > 0. \end{cases} \tag{1.2}$$

We say that $u = (u_1, ..., u_n)$ is a weak solution to the system (1.1) if $u = (u_1, ..., u_n) \in X$ and

$$\sum_{i=1}^{n} \Bigg\{ \int_\Omega a(x)|\nabla u_i|^{p-2}\nabla u_i \nabla v_i dx + \int_{\partial\Omega} b(x)|u_i|^{p-2}u_i v_i d\sigma - \int_{\partial\Omega} h(x,u_i)v_i d\sigma - \lambda \int_\Omega f(x)|u_i|^{p-2}u_i v_i dx - \int_\Omega F_{u_i}(x, u_1, ..., u_n)v_i dx \Bigg\} = 0,$$

for any $(v_1, ..., v_n) \in X$.

The corresponding energy functional of the problem (1.1) is defined by

$$\begin{aligned} J_\lambda(u_1, ..., u_n) =& \frac{1}{p}\left[\int_\Omega \left(a(x)\sum_{i=1}^{n}|\nabla u_i|^p\right)dx + \int_{\partial\Omega}\left(b(x)\sum_{i=1}^{n}|u_i|^p\right)d\sigma\right] \\ &- \frac{\lambda}{p}\int_\Omega\left(f(x)\sum_{i=1}^{n}|u_i|^p\right)dx - \int_{\partial\Omega}\sum_{i=1}^{n}H(x,u_i)d\sigma \\ &- \int_\Omega F(x, u_1, ..., u_n)dx. \end{aligned}$$

Note that using Lemmas 1.1 and 1.2 we deduce that J_λ is well-defined on X.

Now we state our main results:

Theorem 1.1. *Assume that the conditions* $(f_1), (f2)$, $(H_1)-(H_4)$ *and* $(F_1)-(F_3)$ *hold. Then the problem* (1.1) *has a nontrivial weak solution for every*

$$0 < \lambda < \Lambda = \inf_{(0,...,0)\neq(u_1,...,u_n)\in X} \frac{\int_\Omega (a(x)\sum_{i=1}^{n}|\nabla u_i|^p)dx + \int_{\partial\Omega}(b(x)\sum_{i=1}^{n}|u_i|^p)d\sigma}{\int_\Omega(f(x)\sum_{i=1}^{n}|u_i|^p)dx}.$$

Theorem 1.2. *Assume that* $h(x,s) \equiv 0$. *Then the problem* (1.1) *has infinity many solutions for* $0 < \lambda < \Lambda$.

2. Proof of Theorem 1.1

Let us consider (H_0). We need the following proposition

Proposition 2.1 ([20]). *The corresponding Nemytskii operators*

$$N_h : L^m(\partial\Omega, w_3) \to L^{\frac{m}{m-1}}(\partial\Omega, w_3^{\frac{1}{1-m}}), \quad N_H : L^m(\partial\Omega, w_3) \to L^1(\partial\Omega)$$

are bounded and continuous. Also if we set $\varphi(u) = f(x)u|u|^{p-2}$, *then the operators*

$$N_\varphi : L^p(\Omega, w_1) \to L^{\frac{p}{p-1}}(\Omega, w_1^{\frac{1}{1-p}}), \quad N_\phi : L^p(\partial\Omega, w_1) \to L^1(\Omega)$$

are bounded and continuous, where ϕ *denotes the primitive function of* φ.

Remark 2.1. Note that $\lambda < \Lambda$ implies the existence of some $C_0 > 0$ such that

$$||(u_1, ...u_n)||_B^p - \lambda\int_\Omega\left(f(x)\sum_{i=1}^{n}|u_i|^p\right)dx \geq C_0||(u_1, ...u_n)||_B^p.$$

Lemma 2.1. *Under assumptions* $(H_1)-(H_4)$ *and* $(F_1)-(F_3)$, J_λ *is Ferechet differentiable on* X *and satisfies the Palais-Smale condition.*

Proof. We use the notations

$$I(u_1,...u_n)=\frac{1}{p}||(u_1,...u_n)||_B^p,\quad K_f(u_1,...u_n)=\frac{1}{p}\int_\Omega\left(f(x)\sum_{i=1}^n|u_i|^p\right)dx,$$

$$K_H(u_1,...u_n)=\int_{\partial\Omega}\sum_{i=1}^n H(x,u_i)d\sigma,\quad K_F(u_1,...u_n)=\int_\Omega F(x,u_1,...,u_n)dx.$$

Then the directional derivative of J_λ is

$$\begin{aligned}\langle J'_\lambda(u_1,...u_n),(v_1,...,v_n)\rangle=&\langle I'(u_1,...u_n),(v_1,...,v_n)\rangle-\lambda\langle K'_f(u_1,...u_n),(v_1,...,v_n)\rangle\\&-\langle K'_H(u_1,...u_n),(v_1,...,v_n)\rangle-\langle K'_F(u_1,...u_n),(v_1,...,v_n)\rangle,\end{aligned}$$

where

$$\begin{aligned}&\langle I'(u_1,...u_n),(v_1,...,v_n)\rangle\\&=\int_\Omega\left(a(x)\sum_{i=1}^n|\nabla u_i|^{p-2}\nabla u_i\nabla v_i\right)dx+\int_{\partial\Omega}\left(b(x)\sum_{i=1}^n|u_i|^{p-2}u_iv_i\right)d\sigma,\\&\langle K'_f(u_1,...u_n),(v_1,...,v_n)\rangle=\int_\Omega\left(f(x)\sum_{i=1}^n|u_i|^{p-2}u_iv_i\right)dx,\\&\langle K'_H(u_1,...u_n),(v_1,...,v_n)\rangle=\int_{\partial\Omega}\sum_{i=1}^n h(x,u_i)v_id\sigma,\\&\langle K'_F(u_1,...u_n),(v_1,...,v_n)\rangle=\int_\Omega\sum_{i=1}^n F_{u_i}(x,u_1,...,u_n)v_idx,\end{aligned}$$

for all $(v_1,...,v_n)\in X$.

Clearly $I'_\lambda:X\to X^*$ is continuous. The operator K'_H is a composition of the operators

$$K'_H:X\to(L^m(\partial\Omega,w_3))^n\longrightarrow_{N_1:=(N_H,....,N_H)}(L^{\frac{m}{m-1}}(\partial\Omega,w_3^{\frac{1}{1-m}}))^n\longrightarrow_l X^*$$

where

$$\langle l(u_1,...,u_n),(v_1,...,v_n)\rangle=\int_{\partial\Omega}\sum_{i=1}^n u_iv_id\sigma.$$

Since

$$\sum_{i=1}^n\int_{\partial\Omega}|u_iv_i|d\sigma\le\sum_{i=1}^n\left(\int_{\partial\Omega}|u_i|^{\frac{m}{m-1}}w_3^{\frac{1}{1-m}}d\sigma\right)^{\frac{m-1}{m}}\left(\int_{\partial\Omega}|v_i|^mw_3d\sigma\right)^{\frac{1}{m}},$$

l is continuous by Lemma 1.1.

As a composition of continuous operators, K'_H is also continuous. Moreover using (H_1), n product of trace operator $X\to(L^m(\partial\Omega,w_3))^n$ is compact and K'_H is also compact.

In a similar way we obtain that the operator K'_F is a composition of the operators

$$K'_f : X \to (L^p(\Omega, w_1))^n \longrightarrow_{N_2:=(N_\phi,....,N_\phi)} (L^{\frac{p}{p-1}}(\Omega, w_1^{\frac{1}{1-p}}))^n \longrightarrow_{l'} X^*$$

where

$$\langle l'(u_1, ..., u_n), (v_1, ..., v_n)\rangle = \int_\Omega \sum_{i=1}^n u_i v_i dx.$$

Since

$$\sum_{i=1}^n \int_\Omega |u_i v_i| dx \le \sum_{i=1}^n \left(\int_\Omega |u_i|^{\frac{p}{p-1}} w_1^{\frac{1}{1-p}} dx\right)^{\frac{p-1}{p}} \left(\int_\Omega |v_i|^p w_1 dx\right)^{\frac{1}{p}},$$

l' is continuous by Lemma 1.1. Again K'_ϕ is also continuous. In a similar way K'_ϕ is also compact.

Since the assumptions (F_1) and (F_3) hold, we get $F_{u_i} \in C(\overline{\Omega} \times (R^+)^n, R^+)$ are positively homogeneous of degree $p^* - 1$. Moreover using the above fact, we get the existence of a positive constant M such that

$$F_{u_i}(x, u_1, ..., u_n) \le M \sum_{i=1}^n |u_i|^{p^*-1}, \quad \forall x \in \overline{\Omega}, \forall (u_1, ..., u_n) \in (R^+)^n. \tag{2.1}$$

By the Sobolev embedding theorem, we derive that K'_F is continuous and compact and the continuous differentiability of J_λ follows.

Now let $U_m = (u_{1_m}, ..., u_{n_m}) \in X$ be a Palais-Smale sequence for the functional J_λ, i.e.,

$$|J'_\lambda(U_m)| \le C, \quad \text{for all } m \tag{2.2}$$

and

$$||J'_\lambda(U_m)||_{X^*} \to 0 \quad \text{as } m \to \infty. \tag{2.3}$$

For m large enough we have

$$|\langle J'_\lambda(U_m), U_m\rangle| \le \mu ||U_m||_B.$$

This implies

$$C + ||U_m||_B \ge J_\lambda(U_m) - \frac{1}{\mu}\langle J'_\lambda(U_m), U_m\rangle. \tag{2.4}$$

Using a direct calculation we have

$$\begin{aligned} J_\lambda(U_m) - \frac{1}{\mu}\langle J'_\lambda(U_m), U_m\rangle = &\left(\frac{1}{p} - \frac{1}{\mu}\right)\left(||U_m||_B^p - \lambda \int_\Omega f(x)(\sum_{i=1}^n |u_{i_m}|^p) dx\right) \\ &- \int_{\partial\Omega} \sum_{i=1}^n \left(H(x, u_{i_m}) - \frac{1}{\mu} h(x, u_{i_m}) u_{i_m}\right) d\sigma \\ &- \int_\Omega (F(x, u_{1_m}, ..., u_{n_m}) \\ &- \frac{1}{\mu} \sum_{i=1}^n F_{u_i}(x, u_{1_m}, ..., u_{n_m}) u_{i_m}) dx. \end{aligned}$$

By (H_3) we deduce that

$$\sum_{i=1}^{n}\int_{\partial\Omega}H(x,u_{i_m})d\sigma\le\frac{1}{\mu}\sum_{i=1}^{n}\int_{\partial\Omega}h(x,u_{i_m})u_{i_m}d\sigma.$$

Also using the property (F_4), we have

$$\int_{\Omega}\left[F(x,u_{1_m},...,u_{n_m})-\frac{1}{\mu}\sum_{i=1}^{n}(u_{1_m},...,u_{n_m})\cdot\nabla F(x,u_{1_m},...,u_{n_m})dx\right]$$
$$=\int_{\Omega}\left[(1-\frac{p^*}{\mu})F(x,u_{1_m},...,u_{n_m})dx\right]<0,$$

since $\mu\in(p,p^*]$. So we deduce that

$$J_\lambda(U_m)-\frac{1}{\mu}\langle J'_\lambda(U_m),U_m\rangle\ge\left(\frac{1}{p}-\frac{1}{\mu}\right)C_0||U_m||_B^p. \tag{2.5}$$

Relations (2.4) and (2.5) yield $C+||U_m||_B\ge\left(\frac{1}{p}-\frac{1}{\mu}\right)C_0||U_m||_B^p$, and hence U_m is bounded.

To show that U_m contains a Cauchy sequence we use the following inequalities for $\xi\in R^N$ (see Diaz [9, Lemma 4.10]):

$$|\varepsilon-\xi|^p\le C(|\varepsilon|^{p-2}\varepsilon-|\xi|^{p-2}\xi)(\varepsilon-\xi),\quad\text{for } p\ge2, \tag{2.6}$$
$$|\varepsilon-\xi|^2(|\varepsilon|+|\xi|)^{2-p}\le C(|\varepsilon|^{p-2}\varepsilon-|\xi|^{p-2}\xi)(\varepsilon-\xi),\quad\text{for } 1<p<2. \tag{2.7}$$

In the case $p\ge2$:

$$\begin{aligned}
&||U_m-U_k||_B^p\\
=&||(u_{1_m}-u_{1_k},...,u_{n_m}-u_{n_k})||_B^p\\
=&\sum_{i=1}^{n}||u_{i_m}-u_{i_k}||_b^p\sum_{i=1}^{n}\left[\int_\Omega a(x)|\nabla u_{i_m}-\nabla u_{i_k}|^pdx+\int_{\partial\Omega}b(x)|u_{i_m}-u_{i_k}|^pd\sigma\right]\\
\le&C\sum_{i=1}^{n}\left[\int_\Omega a(x)(|\nabla u_{i_m}|^{p-2}\nabla u_{i_m}\nabla(u_{i_m}-u_{i_k})-|\nabla u_{i_k}|^{p-2}\nabla u_{i_k}\nabla(u_{i_m}-u_{i_k}))dx\right.\\
&\left.+\int_{\partial\Omega}b(x)(|u_{i_m}|^{p-2}u_{i_m}(u_{i_m}-u_{i_k})-|u_{i_k}|^{p-2}u_{i_k}(u_{i_m}-u_{i_k}))d\sigma\right]\\
=&C\langle I'(U_m),(U_m-U_k)\rangle-\langle I'(U_k),(U_m-U_k)\rangle\\
=&C[\langle J'_\lambda(U_m),(U_m-U_k)\rangle-\langle J'_\lambda(U_k),(U_m-U_k)\rangle\\
&+\lambda\langle K'_f(U_m),(U_m-U_k)\rangle-\lambda\langle K'_f(U_k),(U_m-U_k)\rangle\\
&+\langle K'_H(U_m),(U_m-U_k)\rangle-\langle K'_H(U_k),(U_m-U_k)\rangle\\
&+\langle K'_F(U_m),(U_m-U_k)\rangle-\langle K'_F(U_k),(U_m-U_k)\rangle]\\
\le&C(||J'_\lambda(U_m)-J'_\lambda(U_k)||_{X^*}+|\lambda|||K'_f(U_m)-K'_f(U_k)||_{X^*}\\
&+||K'_H(U_m)-K'_H(U_k)||_{X^*}+||K'_F(U_m)-K'_F(U_k)||_{X^*})||U_m-U_k||_B
\end{aligned}$$

$$\le C(||J'_\lambda(U_m)||_{X^*} + ||J'_\lambda(U_k)||_{X^*} + |\lambda|(||K'_f(U_m) - K'_f(U_k)||_{X^*})$$
$$+ ||K'_H(U_m) - K'_H(U_k)||_{X^*} + ||K'_F(U_m) - K'_F(U_k)||_{X^*})||U_m - U_k||_B.$$

This concludes that there exists a subsequence of U_m which converges in X because of $J'_\lambda(U_m) \to 0$ and K'_γ is compact for $\gamma \in \{f, H, F\}$.

If $1 < p < 2$, modifying the proof of [18, Lemma 3], we can easily deduce that

$$||U_m - U_k||_B^2 \le C|\langle I'(U_m), (U_m - U_k)\rangle - \langle I'(U_k), (U_m - U_k)\rangle|(||U_m||_B^{2-p} + ||U_k||_B^{2-p}).$$

Since $||U_m||_B$ is bounded, the same arguments as the case $p \ge 2$, lead to a convergent subsequence. □

Proof of Theorem 1.1. We shall use the mountain pass lemma to obtain a solution. In what follows, we notice two points to verify the geometric assumptions of the mountain pass theorem. From assumptions (f_2) and (H_2), for every $\epsilon_i > 0$ there is a $C_{\epsilon_i} > 0$ such that

$$|H(x, u_i)| \le \epsilon_i b(x)|u_i|^p + C_{\epsilon_i} w_3(x)|u_i|^m.$$

Thus using (B_1) and Lemma 1.1, we have

$$\begin{aligned}\sum_{i=1}^n \int_{\partial\Omega} H(x, u_i) d\sigma &\le \sum_{i=1}^n \epsilon_i \int_{\partial\Omega} b(x)|u_i|^p d\sigma + \sum_{i=1}^n C_{\epsilon_i} \int_{\partial\Omega} w_3(x)|u_i|^m d\sigma \\ &\le \epsilon C_1||(u_1, ...u_n)||_B^p + C_\epsilon C_2||(u_1, ...u_n)||_B^m,\end{aligned}$$

where $\epsilon = \max\{\epsilon_i; i = 1, ..., n\}$ and $C_\epsilon = \max\{C_{\epsilon_i}; i = 1, ..., n\}$.

Additionally, we recall the following result:

For all $s \in (0, \infty)$ there is a constant $C_s > 0$ such that

$$(x + y)^s \le C_s(x^s + y^s) \quad \text{for all} \quad x, y \in (0, \infty).$$

Now using the estimate (1.2) and Lemma 1.1 we get

$$\begin{aligned}\int_\Omega F(x, u_1, ..., u_n) dx &\le K \int_\Omega (\sum_{i=1}^n |u_i|^p)^{\frac{p^*}{p}} dx \\ &= K \int_\Omega (|u_1|^p + ... + |u_n|^p)^{\frac{p^*}{p}} dx \\ &\le KC_p \int_\Omega (|u_1|^{p(p^*/p)} + ... + |u_n|^{p(p^*/p)}) dx \\ &\le KC_pC_3||(u_1, ...u_n)||_B^{p^*}.\end{aligned}$$

Consequently this two facts and Remark 2.1 imply that

$$\begin{aligned}J_\lambda(u_1, ...u_n) &= \frac{1}{p}||(u_1, ..., u_n)||_B^p - \frac{\lambda}{p}\sum_{i=1}^n \int_\Omega f(x)|u_i|^p dx \\ &\quad - \sum_{i=1}^n \int_{\partial\Omega} H(x, u_i) d\sigma - \int_\Omega F(x, u_1, ..., u_n) dx \\ &\ge \frac{1}{p}C_0||(u_1, ..., u_n)||_B^p - \lambda\epsilon C_1||(u_1, ..., u_n)||_B^p \\ &\quad - C_\epsilon C_2||(u_1, ..., u_n)||_B^m - KC_pC_3||(u_1, ...u_n)||_B^{p^*}.\end{aligned}$$

For $\epsilon > 0$ and $R > 0$ small enough, we deduce that for every $(u_1, ..., u_n) \in X$ with $||(u_1, ...u_n)||_B = R$, the righthand side is strictly greater than 0.

It remains to show that there exists $V = (v_1, ..., v_n) \in X$ with $||(v_1, ...v_n)||_B > R$ such that $J_\lambda(v_1, ...v_n) \leq 0$. Choose $\psi \in C_\delta^\infty(\Omega), \psi \geq 0$ such that $\mathrm{Supp}\psi \cap \partial\Omega \subset O$. From (H_3) we see that $H(x,t) \geq C_4 t^\mu - C_5$ on $O \times (0, \infty)$. Then using (F_2), for $t > 0$, we have

$$\begin{aligned} J_\lambda(t\psi, 0, ..., 0) &= \frac{t^p}{p}(||(t\psi, 0, ..., 0)||_B^p - \lambda \int_\Omega f(x)\psi^p dx) \\ &\quad - \int_{\partial\Omega} H(x, t\psi) d\sigma - \int_\Omega F(x, t\psi, 0, ..., 0) dx \\ &\geq \frac{t^p}{p}||(\psi, 0, ..., 0)||_B^p - C_4 t^\mu \int_O \psi^\mu d\sigma + C_5|O|. \end{aligned}$$

Since $\mu > p$ the righthand side tends to $-\infty$ as $t \to \infty$ and for sufficiently large t_0, $V = (t\psi, 0, ..., 0)$ has the desired property.

Since J_λ satisfies the Palais-Smale condition and $J_\lambda(0, ..., 0) = 0$, the mountain pass lemma shows that there is a nontrivial critical point of J_λ in X with critical value

$$c = \inf_{g \in G} \max_{t \in [0,1]} J_\lambda(g(t)) > 0$$

where $G = \{g \in C([0, 1], X); g(0) = (0, ..., 0), g(1) = V\}$. □

3. Proof of Theorem 1.2

We recall here a version of the Ljusternik-Schnirelman principle in Banach spaces which was discussed by Browder [3], Zeidler [41], Rabinowitz [23] and Szulkin [24]. We then shall apply the principle to establish the existence of a sequence of solutions for the problem (1.1).

Let Y be a real reflexive Banach space and Σ the collection of all symmetric subsets of $Y - \{0\}$ which are closed in X (A is symmetric if $A = -A$). A nonempty set $A \in \Sigma$ is said to be of genus k (denoted by $\gamma(A) = k$) if k is the smallest integer with the property that there exists an odd continuous mapping from A to $R^k - \{0\}$. If there is no such k, $\gamma(A) = \infty$, and if $A = \emptyset, \gamma(A) = 0$.

In order to continue the proof we shall need the following proposition.

Proposition 3.1 ([23, Corollary 4.1]). *Suppose that M is a closed symmetric C^1-submanifold of a real Banach space Y and $0 \notin M$. Suppose also that $\mathcal{J} \in C^1(M, R)$ is even and bounded below. Define*

$$c_j = \inf_{A \in \Gamma_j} \sup_{x \in A} \mathcal{J}(x),$$

where $\Gamma_j = \{A \subset M : A \in \Sigma, \gamma(A) \geq j$ and A is compact$\}$. If $\Gamma_k \neq \emptyset$ for some $k \geq 1$ and if $\mathcal{J}$ satisfies $(PS)_c$ for all $c = c_j, j = 1, ..., k$, then $\mathcal{J}$ has at least k distinct pairs of critical points.

Define on X the even functional

$$\tilde{J}_\lambda(u_1, ...u_n) = \frac{1}{p}\left(||(u_1, ..., u_n)||_B^p - \lambda \sum_{i=1}^n \int_\Omega f(x)|u_i|^p dx\right),$$

on the closed symmetric C^1-manifold

$$S_F = \{(u_1, ...u_n) \in X; K_F(u_1, ..., u_n) = 1\}.$$

By our hypotheses of f, F and h, Lemma 2.2 and Proposition 3.1, we claim that $\tilde{J}_\lambda|_{S_F}$ possesses at least $\gamma(S_F)$ pairs of distinct critical points. Since $F : \overline{\Omega} \times (R^+)^n \to R^+$ is a C^1-function, there exists a nonempty open set $\widetilde{O} \subset \Omega$ such that $F(x, t_1, ..., t_n) > 0$ for all $(x, t_1, ..., t_n) \in \widetilde{O} \times (R^+)^n$. Using the properties of the genus it follows that $\gamma(\widetilde{O}) \geq \gamma(B_{\widetilde{O}})$, where $B_{\widetilde{O}}$ is the unit ball of $W_0^{1,p}(\widetilde{O}) \subset X$. On the other hand it is well known that the genus of the unit ball of an infinite dimensional Banach space is infinity, so $\gamma(S_F) = \infty$. Therefore we conclude that there exists a sequence $\{(u_{1_m}, ..., u_{n_m})\} \subset X$ such that any $(u_{1_m}, ..., u_{n_m})$ is a constrained critical point of $\tilde{J}_\lambda$ on S_F.

By the Lagrange multipliers rule, there exists a sequence $\{\lambda_m\} \subset R$ such that

$$||(u_{1_m}, ..., u_{n_m})||_B^p - \lambda \sum_{i=1}^n \int_\Omega f(x)|u_{i_m}|^p dx = \lambda_m K_F(u_{1_m}, ..., u_{n_m}). \tag{3.1}$$

Since $(u_{1_m}, ..., u_{n_m}) \in S_F$ and $0 < \lambda < \Lambda$, so the right hand side of (3.1) is positive and so $\lambda_m > 0$. Setting

$$v_{i_m} = \lambda_m^{\frac{1}{p^*-p}} u_{i_m},$$

we have the following equation

$$\lambda_m^{\frac{p}{p-p^*}} ||(v_{1_m}, ..., v_{n_m})||_B^p - \lambda\lambda_m^{\frac{p}{p-p^*}} \sum_{i=1}^n \int_\Omega f(x)|v_{i_m}|^p dx = \lambda_m \lambda_m^{\frac{p^*}{p-p^*}} K_F(v_{1_m}, ..., v_{n_m}).$$

Since $\lambda_m \neq 0$, we derive

$$||(v_{1_m}, ..., v_{n_m})||_B^p - \lambda \sum_{i=1}^n \int_\Omega f(x)|v_{i_m}|^p dx = K_F(v_{1_m}, ..., v_{n_m}).$$

This proves the theorem. □

Acknowledgements

We would like to express our thanks to the anonymous referees and the editor for their constructive comments and suggestions, which greatly improved this article.

References

[1] G. Aronsson and U. Janfalk, *On Hele-Shaw flow of power-law fluids*, Eur. J. Appl. Math., 1992, 3(4), 343–366.

[2] Z. Bai, Z. Du and S. Zhang, *Iterative method for a class of fourth-order p-Laplacian beam equation*, Journal of Applied Analysis and Computation, 2019, 9(4), 1443–1453.

[3] F. Browder, *Existence theorems for nonlinear partial differential equations*, Proc. Sympos. Pure Math., Vol. 16, Amer. Math. Sot., Providence, RI, 1970.

[4] L. Chen, C. Chen, H. Yang and H. Song, *Infinite radial solutions for the fractional Kirchhoff equation*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat., 2019, 113(3), 2309–2318.

[5] C. Chen and H. Yang, *Multiple solutions for a class of quasilinear Schrödinger systems in* $\mathbb{R}^N$, Bull. Malays. Math. Sci. Soc., 2019, 42(2), 611–636.

[6] L. Chen, C. Chen, H. Yang and H. Song, *Nonexistence of stable solutions for quasilinear Schrödinger equation*, Bound. Value Probl., 2018, 168, 11.

[7] C. Chen, H. Song and H. Yang, *Liouville-type theorems for stable solutions of singular quasilinear elliptic equations in* $\mathbb{R}^N$, Electron. J. Differ. Equ., 2018, 2018(81), 1–11.

[8] C. Chen, H. Song and H. Yang, *Liouville type theorems for stable solutions of p-Laplace equation in* $\mathbb{R}^N$, Nonlinear Anal., 2017, 160, 44–52.

[9] I. J. Díaz, *Nonlinear partial differential equations and free boundaries*, Vol. I. Elliptic equations, Research Notes in Mathematics, 106. Pitman, Boston, MA, 1985.

[10] X. Dong, Z. Bai and S. Zhang, *Positive solutions to boundary value problems of p-Laplacian with fractional derivative*, Bound. Value Probl., 2017, 5, 15.

[11] R. Dautray and L. J. Lions, *Mathematical analysis and numerical methods for science and technology, Vol. 1: Physical origins and classical methods*, Springer-Verlag, Berlin, 1985.

[12] L. W. Findley, S. Lai and K. Onaran, *Creep and relaxation of nonlinear viscoelastic materials*, North Holland Publ. House, Amsterdam-New York-Oxford, 1976.

[13] X. Hao, H. Wang, L. Liu and Y. Cui, *Positive solutions for a system of nonlinear fractional nonlocal boundary value problems with parameters and p-Laplacian operator*, Bound. Value Probl., 2017, 182, 18.

[14] J. Jiang, D. O'Regan, J. Xu and Y. Cui, *Positive solutions for a Hadamard fractional p-Laplacian three-point boundary value problem*, Mathematics, 2019, 7(5), 439.

[15] M. L. Kachanov, *The theory of creep, National Lending Liberary for science and technology*, Boston Spa, Yorkshire, England, 1967.

[16] M. L. Kachanov, *Foundations of the theory of plasticity*, North Holland Publ. House, Amsterdam-London, 1971.

[17] H. Lian, D. Wang, Z. Bai and R. P. Agarwal, *Periodic and subharmonic solutions for a class of second-order p-Laplacian Hamiltonian systems*, Bound. Value Probl., 2014, 260, 15.

[18] E. Montefusco and V. Rădulescu, *Nonlinear eigenvalue problems for quasilinear operators on unbounded domains*, Nonlinear Differ. Eua. Appl., 2001, 8(4), 481–497.

[19] V. Pao, *Nonlinear parabolic and elliptic equations*, Plenum press, New York, London, 1992.

[20] K. Pfluger, *Existence and multiplicity of solutions to a p-Laplacian equation with nonlinear boundary condition*, Electron. J. Differ. Equ., 1998, 1998(10), 1–13.

[21] K. Pfluger, *Compact traces in weighted Sobolev spaces*, Analysis, 1998, 18, 65–83.

[22] T. Ren, S. Li, X. Zhang and L. Liu, *Maximum and minimum solutions for a nonlocal p-Laplacian fractional differential system from eco-economical processes*, Bound. Value Probl., 2017, 118, 15.

[23] H. P. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, CMBS Reg. Conf. Ser. Math., 1986, 65.

[24] A. Szulkin, *Ljusternik-Schnirelmann theory on C^1-manifold*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 1988, 5(2), 119–139.

[25] J. Sun, J. Chu and T. Wu, *Existence and multiplicity of nontrivial solutions for some biharmonic equations with p-Laplacian*, J. Differ. Equ., 2017, 262(2), 945–977.

[26] K. Sheng, W. Zhang and Z. Bai, *Positive solutions to fractional boundary-value problems with p-Laplacian on time scales*, Bound. Value Probl., 2018, 70, 15.

[27] Y. Tian, S. Sun and Z. Bai, *Positive solutions of fractional differential equations with p-Laplacian*, J. Funct. Spaces., 2017, 3187492, 9.

[28] J. Wu, X. Zhang, L. Liu, Y. Wu and Y. Cui, *The convergence analysis and error estimation for unique solution of a p-Laplacian fractional differential equation with singular decreasing nonlinearity*, Bound. Value Probl., 2018, 82, 15.

[29] Y. Wang, Y. Liu and Y. Cui, *Infinitely many solutions for impulsive fractional boundary value problem with p-Laplacian*, Bound. Value Probl., 2018, 94, 16.

[30] Y. Wei, C. Chen, Q. Chen and H. Yang, *Liouville-type theorem for nonlinear elliptic equations involving p-Laplace-type Grushin operators*, Math. MethodsAppl. Sci., https://doi.org/10.1002/mma.5886.

[31] Y. Wei, C. Chen, H. Song and H. Yang, *Liouville-type theorems for stable solutions of Kirchhoff equations with exponential and superlinear nonlinearities*, Complex Var. Elliptic Equ., 2019, 64(8), 1297–1309.

[32] Y. Wei, C. Chen, H. Yang and H. Song, *Multiplicity of solutions for a class of fractional p-Kirchhoff system with sign-changing weight functions*, Bound. Value Probl., 2018, 78, 18.

[33] Q. Yuan, C. Chen and H. Yang, *Existence of positive solutions for a Schrödinger-Poisson system with bounded potential and weighted functions in* $\mathbb{R}^3$, Bound. Value Probl., 2017, 151, 17.

[34] X. Zhang, L. Liu, Y. Wu and Y. Cui, *Entire blow-up solutions for a quasilinear p-Laplacian Schrödinger equation with a non-square diffusion term*, Appl. Math. Lett., 2017, 74, 85–93.

[35] X. Zhang, L. Liu, Y. Wu and Y. Cui, *The existence and nonexistence of entire large solutions for a quasilinear Schrödinger elliptic system by dual approach*, J. Math. Anal. Appl., 2018, 464(2), 1089–1106.

[36] X. Zhang, L. Liu, Y. Wu and Y. Cui, *Existence of infinitely solutions for a modified nonlinear Schrödinger equation via dual approach*, Electron. J. Differential Equ., 2018, 2018(147), 1–15.

[37] X. Zhang, J. Jiang, Y. Wu and Y. Cui, *Existence and asymptotic properties of solutions for a nonlinear Schrödinger elliptic equation from geophysical fluid flows*, Appl. Math. Lett., 2019, 90, 229–237.

[38] X. Zhang, L. Liu, Y. Wu and Y. Cui, *New result on the critical exponent for solution of an ordinary fractional differential problem*, J. Funct. Spaces, 2017, 3976469, 4.

[39] X. Zhang, L. Liu and Y. Wu, *The entire large solutions for a quasilinear Schrödinger elliptic equation by the dual approach*, Appl. Math. Lett., 2016, 55, 1–9.

[40] X. Zhang, L. Liu, Y. Wu and L. Caccetta, *Entire large solutions for a class of Schrödinger systems with a nonlinear random operator*, J. Math. Anal. Appl., 2015, 423(2), 1650–1659.

[41] E. Zeidler, *The Ljusternik-Schnirelmann theory for indefinite and not necessarily odd nonlinear operators and its applications*, Nonlinear Anal. 1980, 4(3), 451–489.