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GLOBAL STABILITY OF AUTONOMOUS AND NONAUTONOMOUS HEPATITIS B VIRUS MODELS IN PATCHY ENVIRONMENT*

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Abstract Autonomous and nonautonomous hepatitis B virus infection models in patchy environment are investigated respectively to illustrate the influences of population migration and almost periodicity for infection rate on the spread of hepatitis B virus. The basic reproduction number is determined and asymptotic stabilities of disease-free and endemic equilibria are established in case of autonomous system. Moreover, in the nonautonomous system case, existence and global attractivity of almost periodic solution for this system are studied. Finally, feasibility of main theoretical results is showed with the aid of numerical examples for model with two patches.

Keywords HBV infection model, patchy environment, asymptotic stability, almost periodic solution.

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1. Introduction

As a contagious disease triggered by hepatitis B virus (HBV), hepatitis B acutely threatens global public health. According to the latest hepatitis B research report, HBV affects approximately 292 million individuals in 2016, which represents a global prevalence of 3.9% [22]. It is clear that the treatment and prevention for hepatitis B are effective in China and the infection rate is steadily declining by comparing data on HBV infections from then and now. However, there are still nearly 89 million HBV carriers in China, about one third of the world's total, which are the world's largest [22]. It is thus important to monitor HBV infection patterns and predict trends over time.

For the past few years, several mathematics models have been established to analyse the dynamic behaviors of HBV transmission [4, 9, 16, 17, 27–29, 32, 36], especially for the global stabilities of their equilibria. Due to similar main routes of transmission for HBV, HCV (hepatitis C virus) and HIV (human immunodeficiency virus) which include sexual contact, blood transmission and mother-to-child transmission, HBV transmission model is also suitable for describing the spread of HCV and HIV. Dai etc [4] formulated an HBV transmission model to investigate the spread of HBV in mainland China and kinetics of this system. Based on therapy of chronic hepatitis B, Huan etc [9] included antivirus treatment in an HBV

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transmission model, studied the stability of its equilibria. Khan etc [16] considered a transmission model of HBV by taking into account media coverage, analyzed the stability results for the model and presented the optimal control treatment problem with suggested controls. Khan etc [17] incorporated acute-infected and chronic-infected classes into hepatitis B epidemic model and developed the optimal control strategy of HBV transmission. Wang and Tian [32] introduced a CTLs immune response in a time delay HBV infection model and showed that basic reproduction number and basic immune reproduction number determine the asymptotic stabilities of equilibria. Zhang and Zhang [36] formulated a model for HBV to describe how newborn vaccination and treatment influence HBV prevention and got the result the basic reproduction number, as a critical value, determines the stability and persistence of hepatitis B in this model.

Communicable diseases can easily spread between different countries (or regions). For instance, the first case of SARS was found in Guangdong, China in 2002, however, the cumulative cases involved 32 countries and regions as of June 2003 due to the human mobility [23]. The higher interregional mobility may bring about the faster regional and global spread of infectious diseases [14]. Dynamics analysis of epidemic models in patchy environment can show how individual migration among patches affects the dynamic behaviors of epidemic disease transmission, see [5, 21, 33, 35] and the references cited therein.

When modeling the dynamics of population, we usually assumed that coefficients of dynamical models are constant [10, 12]. However, the nonautonomous phenomena are much universal in the real world and nonlinear differential equations can be used to model numerous dynamical problems [11, 13, 20], which could make the model be more realistic than autonomous differential equations. In the case of nonautonomous models, periodic and almost periodic coefficients are taken into consideration in the relevant researches. Moreover, as indicated in [6], almost periodic effects are more approaching to reality in a variety of real world applications than periodic effects. Some recent development on the transmission dynamics of epidemic models with almost periodic coefficients have been discussed in [19, 26, 31, 34] and references therein.

Motivated by the above discussions, we construct the HBV transmission model with almost periodic infection rate in patchy environment based on the model of Kamyad etc [15] and study the stability for this model both in the autonomous and nonautonomous cases. The remaining parts of this paper is organized as follows. An HBV infection model with almost periodic infection rate and patch structure is formulated and some basic properties are deduced in Section 2. In Section 3, the stability analysis of corresponding autonomous system is presented. Section 4 is devoted to existence and global attractivity of almost periodic solution for this system in nonautonomous case. In Section 5, we present numerical examples to demonstrate the effectiveness of established results. Finally, in Section 6, conclusion and discussion for this paper are provided.

2. The model

2.1. System description

Recently, Kamyad etc [15] constructed an HBV transmission model with two controls: vaccination as well as treatment. In this model, two different forms of the infection for HBV, that is, HBV transmits directly from mother to offspring (vertical transmission) and people are infected by contacting with infective individuals (horizontal transmission) were considered. They also accounted for the relapse between recovered people in their paper.

Based on the model of Kamyad etc, we propose a nonautonomous model for the HBV infection which is an extended and improved version of the HBV transmission model in [15] with the inclusion of population travel between n patches and almost periodic infection rate. The total population is divided into five classes in each patch i $(i = 1, 2, ..., n)$: susceptible class S_i; latent class E_i; acute infected class I_i; chronic infected class C_i; and recovered class R_i. Thus, our model is formulated in the following form

$$\begin{cases} \dfrac{\mathrm{d}S_i}{\mathrm{d}t} = \nu_i - \rho_i(t)(I_i + \theta_i C_i)S_i - (\nu_i + \alpha_i)S_i - \nu_i\xi_i C_i + (\gamma_i - \nu_i\eta_i)R_i \\ \qquad + \displaystyle\sum_{j\neq i}(a_{ij}S_j - a_{ji}S_i), \\ \dfrac{\mathrm{d}E_i}{\mathrm{d}t} = \rho_i(t)(I_i + \theta_i C_i)S_i - (\nu_i + \sigma_i)E_i + \displaystyle\sum_{j\neq i}(b_{ij}E_j - b_{ji}E_i), \\ \dfrac{\mathrm{d}I_i}{\mathrm{d}t} = \sigma_i E_i - (\nu_i + \delta_i)I_i + \displaystyle\sum_{j\neq i}(c_{ij}I_j - c_{ji}I_i), \\ \dfrac{\mathrm{d}C_i}{\mathrm{d}t} = \zeta_i\delta_i I_i - (\nu_i(1 - \xi_i) + \varepsilon_i + \lambda_i)C_i + \displaystyle\sum_{j\neq i}(k_{ij}C_j - k_{ji}C_i), \\ \dfrac{\mathrm{d}R_i}{\mathrm{d}t} = \alpha_i S_i + (1 - \zeta_i)\delta_i I_i + (\varepsilon_i + \lambda_i)C_i - (\nu_i(1 - \eta_i) + \gamma_i)R_i \\ \qquad + \displaystyle\sum_{j\neq i}(l_{ij}R_j - l_{ji}R_i), \end{cases} \tag{2.1}$$

where $i, j = 1, 2, ..., n$. a_{ij}, b_{ij}, c_{ij}, k_{ij} and l_{ij} represent the travel rates of susceptible individuals, latent individuals, acute infected people, chronic HBV carriers and recovered (or immune) people from patch (or group) j to patch (or group) i, respectively. The other parameters in patch i are described in Table 1. $\rho_i(t)$ is positive almost periodic with $i = 1, 2, ..., n$.

For the term $\nu_i\xi_i C_i$, it denotes the vertical transmission in patch i. And $\nu_i\eta_i R_i$ denotes immune newborns from recovered class in patch i. Accordingly, the birth flow rate in the susceptible compartment in patch i is denoted by $\nu_i - \nu_i\xi_i C_i - \nu_i\eta_i R_i$. In this case we have $\nu_i - \nu_i\xi_i C_i - \nu_i\eta_i R_i > 0$. For convenience, we denote

$$\begin{aligned} &(S_i, E_i, I_i, C_i, R_i)_n \\ =&(S_1, E_1, I_1, C_1, R_1, S_2, E_2, I_2, C_2, R_2, ..., S_n, E_n, I_n, C_n, R_n) \end{aligned}$$

for a solution of system (2.1).

2.2. Basic properties

2.2.1. Biological feasibility

Lemma 2.1. *The region* Δ_+ *defined by*

$$\Delta_+ = \{(S_i, E_i, I_i, C_i, R_i)_n \mid S_i > 0, E_i, I_i, C_i, R_i \geq 0, i = 1, 2, ..., n\}$$

Table 1. Biological meaning of parameters

Parameter	Meaning
ν_i	Birth (and death) rate
$\nu_i\xi_i$	Rate of newborn from HBV carriers
$\nu_i\eta_i$	Rate of newborn from recovered individuals
ρ_i	Infection rate between susceptible and acute-infected individuals
$\rho_i\theta_i$	Infection rate between susceptible and chronic-infected individuals
α_i	Rate of vaccine-induced immunity
γ_i	Rate of removing from recovered class due to loss of immunity
σ_i	Rate of moving from latent to acute-infected class
δ_i	Removal rate from acute-infected class
$\zeta_i\delta_i$	Rate of moving from acute-infected to chronic-infected class
$(1-\zeta_i)\delta_i$	Recovery rate of acute infection individuals
ε_i	Spontaneous recovery rate of HBV carriers
λ_i	Treatment rate of HBV carriers

is positive invariant for system (2.1).

Proof. From system (2.1), we obtain

$$\begin{cases} \dfrac{\mathrm{d}S_i}{\mathrm{d}t}|_{S_i=0} = \nu_i - \nu_i\xi_i C_i + (\gamma_i - \nu_i\eta_i)R_i + \sum\limits_{j\neq i} a_{ij}S_j > 0, \\ \dfrac{\mathrm{d}E_i}{\mathrm{d}t}|_{E_i=0} = \rho_i(t)(I_i + \theta_i C_i)S_i + \sum\limits_{j\neq i} b_{ij}E_j \geq 0, \\ \dfrac{\mathrm{d}I_i}{\mathrm{d}t}|_{I_i=0} = \sigma_i E_i + \sum\limits_{j\neq i} c_{ij}I_j \geq 0, \\ \dfrac{\mathrm{d}C_i}{\mathrm{d}t}|_{C_i=0} = \zeta_i\delta_i I_i + \sum\limits_{j\neq i} k_{ij}C_j \geq 0, \\ \dfrac{\mathrm{d}R_i}{\mathrm{d}t}|_{R_i=0} = \alpha_i S_i + (1-\zeta_i)\delta_i I_i + (\varepsilon_i + \lambda_i)C_i + \sum\limits_{j\neq i} l_{ij}R_j \geq 0. \end{cases}$$

It is clearly that all the rates are nonnegative on the bounding planes of Δ_+. Now if $(S_i(0), E_i(0), I_i(0), C_i(0), R_i(0))_n \in \Delta_+$, the solution $(S_i, E_i, I_i, C_i, R_i)_n$ cannot escape from the hyperplane of $S_i = E_i = I_i = C_i = R_i = 0$, and the interior of region Δ_+ attracts all solution orbits of (2.1). Thus, all solutions of (2.1) always remain in Δ_+. □

2.2.2. Boundedness of solutions

Let $N = \sum\limits_{i=1}^{n}(S_i + E_i + I_i + C_i + R_i)$ denotes the total population number in all patches.

Lemma 2.2. *The biologically feasible region* Δ *given by*

$$\Delta = \left\{(S_i, E_i, I_i, C_i, R_i)_n \in \Delta_+ |\ N \leq \frac{\sum \nu_i}{\underline{\nu}}\right\},$$

where $\underline{\nu} = \min\{\nu_1, \nu_2, ..., \nu_n\}$, *is positive invariant for system (2.1).*

Proof. Adding all equations of (2.1), together with the fact that

$$\sum_i\sum_j(a_{ij}S_j - a_{ji}S_i) = \sum_i\sum_j(b_{ij}E_j - b_{ji}E_i) = \sum_i\sum_j(c_{ij}I_j - c_{ji}I_i)$$
$$= \sum_i\sum_j(k_{ij}C_j - k_{ji}C_i) = \sum_i\sum_j(l_{ij}R_j - l_{ji}R_i) = 0,$$

we drive that

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} \le \sum_i \nu_i - \underline{\nu}N(t), \tag{2.2}$$

where $\underline{\nu} = \min\{\nu_1, \nu_2, ..., \nu_n\}$.

Thus if $N(t) > \frac{\sum \nu_i}{\underline{\nu}}$, then $\frac{\mathrm{d}N(t)}{\mathrm{d}t} < 0$.

Moreover, we observe the ordinary differential equation

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = \sum_i \nu_i - \underline{\nu}N(t),$$

with general solution

$$N(t) = \frac{\sum \nu_i}{\underline{\nu}} + (N(0) - \frac{\sum \nu_i}{\underline{\nu}})e^{-\underline{\nu}t},$$

where $N(0)$ means the initial value of total population. By applying the standard comparison theorem, we have for all $t \ge 0$,

$$N(t) \le \frac{\sum \nu_i}{\underline{\nu}}, \quad \text{if } N(0) \le \frac{\sum \nu_i}{\underline{\nu}}.$$

Hence, Δ is positive invariant for system (2.1). □

3. Autonomous system case

In this section, the global stability are studied for autonomous system corresponding to (2.1) by taking infection rate ρ_i as a constant for $i = 1, 2, ..., n$. Thus, system (2.1) could be given in the following form

$$\begin{cases} \frac{\mathrm{d}S_i}{\mathrm{d}t} = \nu_i - \rho_i(I_i + \theta_i C_i)S_i - (\nu_i + \alpha_i)S_i - \nu_i\xi_i C_i \\ \qquad + (\gamma_i - \nu_i\eta_i)R_i + \sum_{j\ne i}(a_{ij}S_j - a_{ji}S_i), \\ \frac{\mathrm{d}E_i}{\mathrm{d}t} = \rho_i(I_i + \theta_i C_i)S_i - (\nu_i + \sigma_i)E_i + \sum_{j\ne i}(b_{ij}E_j - b_{ji}E_i), \\ \frac{\mathrm{d}I_i}{\mathrm{d}t} = \sigma_i E_i - (\nu_i + \delta_i)I_i + \sum_{j\ne i}(c_{ij}I_j - c_{ji}I_i), \\ \frac{\mathrm{d}C_i}{\mathrm{d}t} = \zeta_i\delta_i I_i - (\nu_i(1-\xi_i) + \varepsilon_i + \lambda_i)C_i + \sum_{j\ne i}(k_{ij}C_j - k_{ji}C_i), \\ \frac{\mathrm{d}R_i}{\mathrm{d}t} = \alpha_i S_i + (1-\zeta_i)\delta_i I_i + (\varepsilon_i + \lambda_i)C_i - (\nu_i(1-\eta_i) + \gamma_i)R_i \\ \qquad + \sum_{j\ne i}(l_{ij}R_j - l_{ji}R_i), \end{cases} \tag{3.1}$$

where $i, j = 1, 2, ..., n$.

3.1. Global dynamics for disease-free equilibrium

Noting the fact that $E_i = I_i = C_i = 0$, $i = 1, 2, \cdots, n$ at disease-free equilibrium of system (3.1), substituting it into (3.1), we drive

$$\begin{cases} \nu_i - (\nu_i + \alpha_i)S_i + (\gamma_i - \nu_i\eta_i)R_i + \sum_{j\neq i}(a_{ij}S_j - a_{ji}S_i) = 0, \\ \alpha_i S_i - (\nu_i(1-\eta_i) + \gamma_i)R_i + \sum_{j\neq i}(l_{ij}R_j - l_{ji}R_i) = 0, \end{cases}$$

which could be rewritten in form of matrix equation

$$\begin{cases} A_1 S + B_1 R = D, \\ A_2 S = B_2 R, \end{cases} \tag{3.2}$$

where

$$A_1 = \begin{pmatrix} \nu_1+\alpha_1+\sum_{j\neq 1} a_{j1} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \nu_2+\alpha_2+\sum_{j\neq 2} a_{j2} & \cdots & -a_{2n} \\ \vdots & \vdots & & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \nu_n+\alpha_n+\sum_{j\neq n} a_{jn} \end{pmatrix},$$

$$B_2 = \begin{pmatrix} \nu_1(1-\eta_1)+\gamma_1+\sum_{j\neq 1} l_{j1} & -l_{12} & \cdots & -l_{1n} \\ -l_{21} & \nu_2(1-\eta_2)+\gamma_2+\sum_{j\neq 2} l_{j2} & \cdots & -l_{2n} \\ \vdots & \vdots & & \vdots \\ -l_{n1} & -l_{n2} & \cdots & \nu_n(1-\eta_n)+\gamma_n+\sum_{j\neq n} l_{jn} \end{pmatrix},$$

$$B_1 = \text{diag}(\nu_1\eta_1 - \gamma_1, \nu_2\eta_2 - \gamma_2, ..., \nu_n\eta_n - \gamma_n), \quad A_2 = \text{diag}(\alpha_1, \alpha_2, ..., \alpha_n),$$
$$S = (S_1, S_2, ..., S_n)^T, \quad R = (R_1, R_2, ..., R_n)^T, \quad D = (\nu_1, \nu_2, ..., \nu_n)^T.$$

Assume that the following hypotheses hold:
(H1) $\nu_i\eta_i > \gamma_i$ for all $i = 1, 2, ..., n$;
(H2) $a_{ij} = a_{ji}$ and $l_{ij} = l_{ji}$ for all $i \neq j$;
(H3) $(\nu_i\eta_i - \gamma_i)\alpha_j = (\nu_j\eta_j - \gamma_j)\alpha_i$ for all $i \neq j$.

It is obvious that all off-diagonal elements of A_1 and B_2 are nonpositive, and column sums of A_1 and B_2 are positive, respectively. Thus, A_1 and B_2 are nonsingular M-matrices (M_{35} in [1, p137]). From **(H2)**, A_1 and B_2 are symmetric matrices, then we have A_1 and B_2 are also positive definite matrices (E_{17} in [1, p135]).

It follows from the converse of positive definite matrix is still positive definite and the converse of symmetric matrix is symmetric, we have B_2^{-1} is positive definite and symmetric. Furthermore, **(H1)** shows all diagonal entries of B_1 are positive. By matrix operation, all leading principal minors of $B_1B_2^{-1}A_2$ are positive. Moreover, $B_1B_2^{-1}A_2$ is symmetric according to **(H3)**, which means $B_1B_2^{-1}A_2$ is a positive definite matrix. We have $A_1+B_1B_2^{-1}A_2$ is also positive definite according to the fact that sum of positive definite matrixes is positive definite. Then, $(A_1+B_1B_2^{-1}A_2)^{-1}$ exists by another property of positive definite matrix: positive definite matrix is invertible. Hence, system (3.2) admits unique positive solution

$$\begin{cases} S^0 = (S_1^0, S_2^0, ..., S_n^0)^T = (A_1 + B_1 B_2^{-1} A_2)^{-1} D > 0, \\ R^0 = (R_1^0, R_2^0, ..., R_n^0)^T = B_2^{-1} A_2 S^0 > 0. \end{cases}$$

Accordingly, system (3.1) admits a unique disease-free equilibrium

$$\begin{aligned} P_0 &= (S_i^0, 0, 0, 0, R_i^0)_n \\ &= (S_1^0, 0, 0, 0, R_1^0, S_2^0, 0, 0, 0, R_2^0, ..., S_n^0, 0, 0, 0, R_n^0). \end{aligned}$$

Naturally, we can draw the following conclusion.

Theorem 3.1. *Suppose hypotheses* **(H1)**-**(H3)** *hold, then system (3.1) has a unique disease-free equilibrium.*

We utilize next generation matrix approach [30] as follows so as to derive the basic reproduction number of system (3.1). For simplicity, we rearrange (3.1) as following

$$\begin{cases} \dfrac{\mathrm{d}E_i}{\mathrm{d}t} = \rho_i(I_i + \theta_i C_i)S_i - (\nu_i + \sigma_i)E_i + \sum\limits_{j \neq i}(b_{ij}E_j - b_{ji}E_i), \\ \dfrac{\mathrm{d}I_i}{\mathrm{d}t} = \sigma_i E_i - (\nu_i + \delta_i)I_i + \sum\limits_{j \neq i}(c_{ij}I_j - c_{ji}I_i), \\ \dfrac{\mathrm{d}C_i}{\mathrm{d}t} = \zeta_i \delta_i I_i - (\nu_i(1 - \xi_i) + \varepsilon_i + \lambda_i)C_i + \sum\limits_{j \neq i}(k_{ij}C_j - k_{ji}C_i), \\ \dfrac{\mathrm{d}R_i}{\mathrm{d}t} = \alpha_i S_i + (1 - \zeta_i)\delta_i I_i + (\varepsilon_i + \lambda_i)C_i - (\nu_i(1 - \eta_i) + \gamma_i)R_i \\ \qquad + \sum\limits_{j \neq i}(l_{ij}R_j - l_{ji}R_i), \\ \dfrac{\mathrm{d}S_i}{\mathrm{d}t} = \nu_i - \rho_i(I_i + \theta_i C_i)S_i - (\nu_i + \alpha_i)S_i - \nu_i \xi_i C_i + (\gamma_i - \nu_i \eta_i)R_i \\ \qquad + \sum\limits_{j \neq i}(a_{ij}S_j - a_{ji}S_i), \end{cases} \tag{3.3}$$

Let $\widetilde{x} = (E_1, ..., E_n, I_1, ..., I_n, C_1, ..., C_n, R_1, ..., R_n, S_1, ..., S_n)^T$, then system (3.3) could be rewritten in form of matrix equation

$$\frac{\mathrm{d}\widetilde{x}}{\mathrm{d}t} = \mathscr{F}(\widetilde{x}) - \mathscr{V}(\widetilde{x}),$$

where

$$\mathscr{F}(\widetilde{x}) = \begin{pmatrix} \rho_1(I_1+\theta_1 C_1)S_1 \\ \vdots \\ \rho_n(I_n+\theta_n C_n)S_n \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

$$\mathscr{V}(\widetilde{x}) = \begin{pmatrix} (\nu_1+\sigma_1)E_1 - \sum_{j\neq 1}(b_{1j}E_j - b_{j1}E_1) \\ \vdots \\ (\nu_n+\sigma_n)E_n - \sum_{j\neq n}(b_{nj}E_j - b_{jn}E_n) \\ -\sigma_1 E_1 + (\nu_1+\delta_1)I_1 - \sum_{j\neq 1}(c_{1j}I_j - c_{j1}I_1) \\ \vdots \\ -\sigma_n E_n + (\nu_n+\delta_n)I_n - \sum_{j\neq n}(c_{nj}I_j - c_{jn}I_n) \\ -\zeta_1\delta_1 I_1 + (\nu_1(1-\xi_1)+\varepsilon_1+\lambda_1)C_1 - \sum_{j\neq 1}(k_{1j}C_j - k_{j1}C_1) \\ \vdots \\ -\zeta_n\delta_n I_n + (\nu_n(1-\xi_n)+\varepsilon_n+\lambda_n)C_n - \sum_{j\neq n}(k_{nj}C_j - k_{jn}C_n) \\ -\alpha_1 S_1 - (1-\zeta_1)\delta_1 I_1 - (\varepsilon_1+\lambda_1)C_1 + (\nu_1(1-\eta_1)+\gamma_1)R_1 - \sum_{j\neq 1}(l_{1j}R_j - l_{j1}R_1) \\ \vdots \\ -\alpha_n S_n - (1-\zeta_n)\delta_n I_n - (\varepsilon_n+\lambda_n)C_n + (\nu_n(1-\eta_n)+\gamma_n)R_n - \sum_{j\neq n}(l_{nj}R_j - l_{jn}R_n) \\ -\nu_1 + \rho_1(I_1+\theta_1 C_1)S_1 + (\nu_1+\alpha_1)S_1 + \nu_1\xi_1 C_1 - (\gamma_1 - \nu_1\eta_1)R_1 - \sum_{j\neq 1}(a_{1j}S_j - a_{j1}S_1) \\ \vdots \\ -\nu_n + \rho_n(I_n+\theta_n C_n)S_n + (\nu_n+\alpha_n)S_n + \nu_n\xi_n C_n - (\gamma_n - \nu_n\eta_n)R_n - \sum_{j\neq n}(a_{nj}S_j - a_{jn}S_n) \end{pmatrix}.$$

The Jacobian matrices of $\mathscr{F}(\widetilde{x})$ and $\mathscr{V}(\widetilde{x})$ at the disease-free equilibrium P_0 are, respectively,

$$D\mathscr{F}(P_0) = \begin{pmatrix} F_{3n\times 3n} & 0 \\ 0 & 0 \end{pmatrix}, \quad D\mathscr{V}(P_0) = \begin{pmatrix} V_{3n\times 3n} & 0 \\ A & B \end{pmatrix},$$

where

$$F_{3n\times 3n} = \begin{pmatrix} 0 & F_{12} & F_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad V_{3n\times 3n} = \begin{pmatrix} V_{11} & 0 & 0 \\ V_{21} & V_{22} & 0 \\ 0 & V_{32} & V_{33} \end{pmatrix},$$

$$F_{12} = \mathrm{diag}(\rho_1 S_1^0, \rho_2 S_2^0, ..., \rho_n S_n^0),$$
$$F_{13} = \mathrm{diag}(\rho_1\theta_1 S_1^0, \rho_2\theta_2 S_2^0, ..., \rho_n\theta_n S_n^0),$$

$$V_{11}=\begin{pmatrix}\nu_1+\sigma_1+\sum\limits_{j\neq 1}b_{j1} & -b_{12} & \cdots & -b_{1n}\\ -b_{21} & \nu_2+\sigma_2+\sum\limits_{j\neq 2}b_{j2} & \cdots & -b_{2n}\\ \vdots & \vdots & & \vdots\\ -b_{n1} & -b_{n2} & \cdots & \nu_n+\sigma_n+\sum\limits_{j\neq n}b_{jn}\end{pmatrix},$$

$$V_{22}=\begin{pmatrix}\nu_1+\delta_1+\sum\limits_{j\neq 1}c_{j1} & -c_{12} & \cdots & -c_{1n}\\ -c_{21} & \nu_2+\delta_2+\sum\limits_{j\neq 2}c_{j2} & \cdots & -c_{2n}\\ \vdots & \vdots & & \vdots\\ -c_{n1} & -c_{n2} & \cdots & \nu_n+\delta_n+\sum\limits_{j\neq n}c_{jn}\end{pmatrix},$$

$$V_{33}=\begin{pmatrix}\nu_1(1-\xi_1)+\varepsilon_1+\lambda_1+\sum\limits_{j\neq 1}k_{j1} & -k_{12} & \cdots & -k_{1n}\\ -k_{21} & \nu_2(1-\xi_2)+\varepsilon_2+\lambda_2+\sum\limits_{j\neq 2}k_{j2} & \cdots & -k_{2n}\\ \vdots & \vdots & & \vdots\\ -k_{n1} & -k_{n2} & \cdots & \nu_n(1-\xi_n)+\varepsilon_n+\lambda_n+\sum\limits_{j\neq n}k_{jn}\end{pmatrix},$$

$$V_{21}=\text{diag}(-\sigma_1,-\sigma_2,...,\sigma_n),$$
$$V_{32}=\text{diag}(-\zeta_1\delta_1,-\zeta_2\delta_2,...,-\zeta_n\delta_n).$$

Since column sums of V are positive and all off-diagonal elements of V are non-positive, then V is a nonsingular M-matrix (M_{35} in [1, p137]). Furthermore, we get $V^{-1}\geq 0$ (N_{38} in [1, p137]). Consequently, FV^{-1} is non-negative. Applying the approach in [30], the basic reproductive number is shown by

$$R_0=\widetilde{\rho}(FV^{-1})=\widetilde{\rho}(F_{13}V_{33}^{-1}V_{32}V_{22}^{-1}V_{21}V_{11}^{-1}-F_{12}V_{22}^{-1}V_{21}V_{11}^{-1}),$$

where $\widetilde{\rho}(\cdot)$ represents the spectral radius of matrix.

The following result follows by applying Theorem 2 of [30].

Theorem 3.2. *The disease-free equilibrium, P_0, is local asymptotical stable as $R_0<1$ and unstable as $R_0>1$.*

Theorem 3.3. *Suppose the assumptions* **(H1)**-**(H3)** *hold, $\mathfrak{B}=(b_{ij})$, $\mathfrak{C}=(c_{ij})$ and $\mathfrak{K}=(k_{ij})$ are irreducible and there exist $i\neq j$ satisfying $\nu_i\neq\nu_j$, then no other equilibrium exists except for P_0 on $\partial\Delta$ (boundary of Δ).*

Proof. Firstly, we show that $E_i=0$ or $I_i=0$ or $C_i=0$ for certain i means $E_j=I_j=C_j=0$ for all j.

Let $E_i=0$, invoking the second equation of (3.1), we have

$$\rho_i(I_i+\theta_iC_i)S_i+\sum_{j\neq i}b_{ij}E_j=0.$$

From Lemma 2.1, it follows that $I_i=0$, $C_i=0$ and $E_j=0$ if $b_{ij}>0$. For $E_i=I_i=C_i=0$, according to the third and forth equations of (3.1), it holds that

$$\begin{cases}\sum\limits_{j\neq i}c_{ij}I_j=0,\\ \sum\limits_{j\neq i}k_{ij}C_j=0.\end{cases}$$

Thus, for any $i,j=1,2,...,n$,

$$E_i=0,\ b_{ij}>0,\ c_{ij}>0 \text{ and } k_{ij}>0\Rightarrow E_j=I_j=C_j=0. \tag{3.4}$$

Since $\mathfrak{B} = (b_{ij})$, $\mathfrak{C} = (c_{ij})$ and $\mathfrak{K} = (k_{ij})$ are irreducible and directed graph of irreducible matrix is strongly connected (Theorem 2.7 in [1, p30]), there exist sequences of ordered pairs

$$\{(i, s_1), (s_1, s_2), \cdots, (s_{n-2}, j)\},$$

$$\{(i, q_1), (q_1, q_2), \cdots, (q_{n-2}, j)\},$$

$$\{(i, r_1), (r_1, r_2), \cdots, (r_{n-2}, j)\}$$

such that

$$b_{is_1} > 0,\ b_{s_1 s_2} > 0,\ \cdots,\ b_{s_{n-2}j} > 0,$$

$$c_{iq_1} > 0,\ c_{q_1 q_2} > 0,\ \cdots,\ c_{q_{n-2}j} > 0,$$

$$k_{ir_1} > 0,\ k_{r_1 r_2} > 0,\ \cdots,\ k_{r_{n-2}j} > 0,$$

where $\{i, s_1, s_2, ..., s_{n-2}, j\} = \{i, q_1, q_2, ..., q_{n-2}, j\} = \{i, r_1, r_2, ..., r_{n-2}, j\} = \{1, 2, ..., n\}$. Application of (3.4) to the above sequences, combining with $E_i = 0$, yields

$$E_{s_1} = 0,\ E_{s_2} = 0,\ ...,\ E_{s_{n-2}} = 0,\ E_j = 0,$$

$$I_{q_1} = 0,\ I_{q_2} = 0,\ ...,\ I_{q_{n-2}} = 0,\ I_j = 0,$$

$$C_{r_1} = 0,\ C_{r_2} = 0,\ ...,\ C_{r_{n-2}} = 0,\ C_j = 0.$$

From $\{i, s_1, s_2, ..., s_{n-2}, j\} = \{i, q_1, q_2, ..., q_{n-2}, j\} = \{i, r_1, r_2, ..., r_{n-2}, j\} = \{1, 2, ..., n\}$, we have $E_j = I_j = C_j = 0$ for all $j = 1, 2, ..., n$. Similarly, $I_i = 0$ for certain i indicates $E_j = I_j = C_j = 0$ for all j and $C_i = 0$ for certain i indicates $E_j = I_j = C_j = 0$ for all j.

We next show that there exist no equilibria on the boundary $\partial\Delta$ of non zero elements. It is obvious that (2.2) takes the equal sign if and only if $\nu_1 = \nu_2 = \cdots = \nu_n$. Thus if there exist $i \neq j$ such that $\nu_i \neq \nu_j$, we have $\frac{\mathrm{d}N(t)}{\mathrm{d}t} < -\underline{\nu}N(t) + \sum_{i=1}^{n} \nu_i$. To find the equilibrium of system (3.1), we set the right side of (3.1) equal to 0. Thus, $0 = \frac{\mathrm{d}N(t)}{\mathrm{d}t} < -\underline{\nu}N(t) + \sum_{i=1}^{n} \nu_i$, which means $N(t) < \frac{\sum \nu_i}{\underline{\nu}}$. Therefore, no non zero equilibrium lies on the boundary $\partial\Delta$ when there exist $i \neq j$ such that $\nu_i \neq \nu_j$.

Hence, owing to Theorem 3.1, P_0 is the unique equilibrium lies on $\partial\Delta$. □

Theorem 3.4. *Suppose assumptions in Theorem 3.3 hold,* $\alpha_i = 0$ *for all* $i = 1, 2, ..., n$ *and* $S(0) \leq S^0$. *If* $R_0 < 1$, *then* P_0 *is global asymptotical stable in* Δ.

Proof. For convenience, we denote $(y_1, y_2, ..., y_n)^T > (z_1, z_2, ..., z_n)^T$ as $y_i > z_i$ for all $i = 1, 2, ..., n$, and the same for $(y_1, y_2, ..., y_n)^T \leq (z_1, z_2, ..., z_n)^T$. Suppose that there is no vaccination in system (3.1), that is, $\alpha_i = 0$ for all $i = 1, 2, ..., n$, then $R^0 = 0$. Combining M_{35} and N_{38} in Theorem 2.3 of [1] gives that ${A_1}^{-1} > 0$. Hence, $S^0 = {A_1}^{-1}D > 0$.

In views of (3.1), we get $\frac{dS}{dt} \leq D - A_1 S = A_1 S^0 - A_1 S$. Using Laplace transform to this inequality, we drive

$$s\mathcal{L}(S) - S(0) \leq A_1 S^0 s^{-1} - A_1 \mathcal{L}(S),$$

which yields

$$\mathcal{L}(S) \leq s^{-1}(sI + A_1)^{-1} A_1 S^0 + (sI + A_1)^{-1} S(0). \tag{3.5}$$

The Laplace transform of $t^{l-1}E_{m,l}(\pm\lambda t^m)$ is

$$\mathcal{L}[t^{l-1}E_{m,l}(\pm\lambda t^m)] = \frac{s^{m-l}}{s^m \mp \lambda},$$

where $E_{m,l}(s)$, $m,l>0$ is the Mittag-Leffler function [8] defined as follows

$$E_{m,l}(s) = \sum_{i=0}^{\infty} \frac{s^i}{\Gamma(mi+l)}.$$

Using the property of Mittage-Leffler function given in [8]

$$E_{m,l}(s) = sE_{m,m+l}(s) + \frac{1}{\Gamma(l)},$$

we infer that if $S(0) \le S^0$, (3.5) implies

$$\begin{aligned} S &\le tE_{1,2}(-A_1t)A_1S^0 + E_{1,1}(-A_1t)S(0) \\ &\le [tE_{1,2}(-A_1t)A_1 + E_{1,1}(-A_1t)]S^0 \\ &= \frac{1}{\Gamma(1)}S^0 \\ &= S^0. \end{aligned}$$

Since $S_i \le S_i^0$, system (3.1) gives the inequality

$$\frac{\mathrm{d}E_i}{\mathrm{d}t} \le \rho_i(I_i+\theta_iC_i)S_i^0 - (\nu_i+\sigma_i)E_i + \sum_{j\ne i}(b_{ij}E_j - b_{ji}E_i). \tag{3.6}$$

Thus, we define the following auxiliary linear system

$$\begin{cases} \dfrac{\mathrm{d}\bar{E}_i}{\mathrm{d}t} =\rho_i(I_i+\theta_iC_i)S_i^0 - (\nu_i+\sigma_i)\bar{E}_i + \displaystyle\sum_{j\ne i}(b_{ij}\bar{E}_j - b_{ji}\bar{E}_i), \\ \dfrac{\mathrm{d}I_i}{\mathrm{d}t} =\sigma_iE_i - (\nu_i+\delta_i)I_i + \displaystyle\sum_{j\ne i}(c_{ij}I_j - c_{ji}I_i), \\ \dfrac{\mathrm{d}C_i}{\mathrm{d}t} =\zeta_i\delta_iI_i - (\nu_i(1-\xi_i)+\varepsilon_i+\lambda_i)C_i + \displaystyle\sum_{j\ne i}(k_{ij}C_j - k_{ji}C_i). \end{cases} \tag{3.7}$$

According to [30], we have

$$\widetilde{\rho}(FV^{-1}) < 1 \;\Leftrightarrow\; s(F-V) < 0,$$

where $s(F-V) = \max_i\{\mathrm{Re}(z_i)\}$ denotes the spectral abscissa of $F-V$, $z_1, z_2, ..., z_s$ are eigenvalues of $F-V$. Consequently, all eigenvalues of $F-V$ possess negative real parts if $R_0 = \widetilde{\rho}(FV^{-1}) < 1$. It is evident the right side of (3.7) has efficient matrix $F-V$, then all non-negative solutions of (3.7) satisfies that $\lim_{t\to\infty}\bar{E}_i = \lim_{t\to\infty} I_i = \lim_{t\to\infty} C_i = 0$. From the basic comparison theorem (Theorem B.1 in [24]), together with the fact that all the variables in system (3.1) are nonnegative, it follows $\lim_{t\to\infty} E_i = \lim_{t\to\infty} I_i = \lim_{t\to\infty} C_i = 0$. This implies that

$$\begin{cases} \dfrac{\mathrm{d}S}{\mathrm{d}t} = D - A_1S - B_1R, \\ \dfrac{\mathrm{d}R}{\mathrm{d}t} = -B_2R, \end{cases}$$

performs the limiting system of $\frac{\mathrm{d}S}{\mathrm{d}t}$, $\frac{\mathrm{d}R}{\mathrm{d}t}$ terms of (3.1). From [25] and Theorem 2.3 in [3], we conclude that all the solutions to (3.1) satisfy that $\lim_{t\to\infty} S_i(t) = S_i^0$, $\lim_{t\to\infty} R_i(t) = R_i^0$. By (3.3), P_0 is the unique equilibrium lies on $\partial\Delta$. Hence, equilibrium P_0 is global asymptotic stable whenever $R_0 < 1$. □

Theorem 3.5. *Suppose assumptions in Theorem 3.3 hold. If $R_0 > 1$, then system (3.1) is uniform persistent and admits an endemic equilibrium in $\mathring{\Delta}$ (interior of Δ).*

Proof. If $R_0 > 1$, according to Theorem 3.2, P_0 is unstable. Choose $X = \mathbb{R}^{5n}$ and $E = \Delta$ for Theorem 4.3 in [7]. When $\mathfrak{B}$, $\mathfrak{C}$ and $\mathfrak{K}$ are irreducible, by Theorem 3.3, singleton $\{P_0\}$ is isolated as the maximal invariant set on $\partial\Delta$. Accordingly, hypothesis (H) in [7] is valid for system (3.1). Note that the instability of P_0 is equivalent to the necessary and sufficient condition of Theorem 4.3 in [7], which indicates the uniformly persistence of system (3.1).

From the positive invariance of Δ, we get that solutions in $\mathring{\Delta}$ are uniform bounded. Then, by Theorem 2.8.6 in [2], and according to the uniform persistence of system (3.1), we draw the conclusion that there exists an equilibrium in $\mathring{\Delta}$. □

3.2. Local dynamics for endemic equilibrium

Let $R_0^{(i)}$ denotes the basic reproductive number in patch i.

Theorem 3.6. *Suppose $a_{ij} = b_{ij} = c_{ij} = k_{ij} = l_{ij} = 0$ for all $i, j = 1, 2, ..., n$. If $R_0^{(i)} > 1$ for all $i = 1, 2, ..., n$, then the endemic equilibrium P^* for system (3.1) is local asymptotical stable.*

Proof. Suppose that there is no population movement among any patches (i.e. each patch is isolated from others), then A_1, B_2, V_{11}, V_{22} and V_{33} are diagonal matrices. The basic reproductive number in patch i takes now the form

$$R_0^{(i)} = \frac{\sigma_i\rho_i S_i^0(\theta_i\zeta_i\delta_i + (1-\xi_i)\nu_i + \varepsilon_i + \lambda_i)}{(\nu_i+\sigma_i)(\nu_i+\delta_i)((1-\xi_i)\nu_i + \varepsilon_i + \lambda_i)}, \tag{3.8}$$

where $S_i^0 = \frac{\nu_i(1-\eta_i)+\gamma_i}{\nu_i(1-\eta_i)+\gamma_i+\alpha_i}$.

To find the endemic equilibrium

$$\begin{aligned} P^* &= (S_i^*, E_i^*, I_i^*, C_i^*, R_i^*)_n \\ &= (S_1^*, E_1^*, I_1^*, C_1^*, R_1^*, S_2^*, E_2^*, I_2^*, C_2^*, R_2^*, ..., S_n^*, E_n^*, I_n^*, C_n^*, R_n^*), \end{aligned}$$

let the right side of system (3.1) with $a_{ij} = b_{ij} = c_{ij} = k_{ij} = l_{ij} = 0$ be equal to zero. Then, we get

$$\begin{aligned} S_i^* =& \frac{(\nu_i+\sigma_i)(\nu_i+\delta_i)[\nu_i(1-\xi_i)+\varepsilon_i+\lambda_i]}{\sigma_i\rho_i[\nu_i(1-\xi_i)+\varepsilon_i+\lambda_i+\theta_i\zeta_i\delta_i]}, \\ E_i^* =& \frac{\nu_i+\delta_i}{\sigma} I_i^*, \\ I_i^* =& \left\{\frac{\sigma_i+\nu_i+\delta_i}{\sigma_i} + \frac{\zeta_i\delta_i+\nu_i\delta_i(1-\zeta_i)(1-\xi_i)+\delta_i(\varepsilon_i+\lambda_i)}{[\nu_i(1-\eta_i)+\gamma_i][\nu_i(1-\xi_i)+\varepsilon_i+\lambda_i]}\right\} S_i^* \\ &\times (R_0^{(i)}-1), \end{aligned} \tag{3.9}$$

$$\begin{aligned} C_i^* =& \frac{\zeta_i \delta_i}{\nu_i(1-\xi_i)+\varepsilon_i+\lambda_i} I_i^*, \\ R_i^* =& \frac{\alpha_i}{\nu_i(1-\eta_i)+\gamma_i} S_i^* + \frac{\nu_i\delta_i(1-\zeta_i)(1-\xi_i)+\delta_i(\varepsilon_i+\lambda_i)}{[\nu_i(1-\eta_i)+\gamma_i][\nu_i(1-\xi_i)+\varepsilon_i+\lambda_i]} I_i^*. \end{aligned}$$

Clearly P^* is feasible if $I_i^* > 0$, that is, $R_0^{(i)} > 1$ for $i = 1, 2, ..., n$.

The Jacobian matrix of model (3.1) with $a_{ij} = b_{ij} = c_{ij} = k_{ij} = l_{ij} = 0$ around the endemic equilibrium point P^* is

$$J_{5n\times 5n} = \begin{pmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & J_n \end{pmatrix},$$

where J_i for $i = 1, 2, ..., n$ is 5×5 matrix and takes the following form

$$J_i = \begin{pmatrix} -\rho_i(I_i^*+\theta_i C_i^*)-(\nu_i+\alpha_i) & 0 & -\rho_i S_i^* & -\rho_i\theta_i S_i^*-\nu_i\xi_i & -(\nu_i\eta_i-\gamma_i) \\ \rho_i(I_i^*+\theta_i C_i^*) & -(\nu_i+\sigma_i) & \rho_i S_i^* & \rho_i\theta_i S_i^* & 0 \\ 0 & \sigma_i & -(\nu_i+\delta_i) & 0 & 0 \\ 0 & 0 & \zeta_i\delta_i & -[\nu_i(1-\xi_i)+\varepsilon_i+\lambda_i] & 0 \\ \alpha_i & 0 & (1-\zeta_i)\delta_i & (\varepsilon_i+\lambda_i) & -[\nu_i(1-\eta_i)+\gamma_i] \end{pmatrix}.$$

The characteristic equation of J_i is given by

$$(\overline{\lambda}+\nu_i)(\overline{\lambda}^4 + f_1\overline{\lambda}^3 + f_2\overline{\lambda}^2 + f_3\overline{\lambda} + f_4) = 0,$$

where

$$\begin{aligned} f_1 &= d_0 + d_1 + d_2 + d_4 + d_5 + \alpha_i > 0, \\ f_2 &= d_1d_2 + (d_1+d_2)(d_0+d_4+d_5) + d_4(d_0+d_5) + \sigma_i(d_0-d_6) + \alpha_i(d_1+d_4 \\ &\quad +d_5), \\ f_3 &= d_1d_2(d_0+d_4+d_5) + (d_1+d_2)[d_4(d_0+d_5)+\sigma_i(d_0-d_6)] + \sigma_i\delta_i d_0 \\ &\quad -\alpha_i\sigma_i d_6 + \alpha_i(d_1d_4 + d_1d_5 + d_4d_5), \\ f_4 &= d_1d_2d_4d_0 + \sigma_i d_1d_2d_0 + \sigma_i\zeta_i\delta_i d_2d_0 - \sigma_i\zeta_i\delta_i d_0d_3 + \sigma_i\delta_i d_0 d_1, \end{aligned}$$

and

$$\begin{aligned} &d_0 = \rho_i(I_i^*+\theta_i C_i^*), \quad d_1 = \nu_i(1-\xi_i)+\varepsilon_i+\lambda_i, \quad d_2 = \nu_i(1-\eta_i)+\gamma_i, \\ &d_3 = \nu_i(1-\xi_i), \quad d_4 = \nu_i+\delta_i, \quad d_5 = \nu_i+\sigma_i, \quad d_6 = \rho_i S_i^*. \end{aligned}$$

It is obvious one of the eigenvalues for J_i, $-\nu_i$, is negative. To proceed, we consider the following equation

$$\overline{\lambda}^4 + f_1\overline{\lambda}^3 + f_2\overline{\lambda}^2 + f_3\overline{\lambda} + f_4 = 0. \tag{3.10}$$

In view of the Routh-Hurwitz criteria [18], all roots of (3.10) possess negative real parts iff $f_i > 0$ for $i = 1, 2, 3, 4$ and $f_1f_2f_3 > {f_3}^2 + {f_1}^2 f_4$.

From (3.9), we obtain

$$\sigma_i d_6 = \sigma_i(\rho_i S_i^*) = \frac{(\nu_i+\sigma_i)(\nu_i+\delta_i)[\nu_i(1-\xi_i)+\varepsilon_i+\lambda_i]}{\nu_i(1-\xi_i)+\varepsilon_i+\lambda_i+\theta_i\zeta_i\delta_i}$$

$$= \frac{d_1 d_4 d_5}{d_1 + \theta_i \zeta_i \delta_i} < d_4 d_5. \tag{3.11}$$

Thus, we have

$$f_2 > d_1 d_2 + (d_1 + d_2)(d_0 + d_4 + d_5) + d_4 d_0 + \sigma_i d_0 + \alpha_i (d_1 + d_4 + d_5) > 0,$$

and

$$\begin{aligned} f_3 &> d_1 d_2 (d_4 + d_5 + d_0) + d_0 (d_1 + d_2)(\sigma_i + d_4) + \sigma_i \delta_i d_0 + \alpha_i d_1 (d_4 + d_5) \\ &> 0. \end{aligned} \tag{3.12}$$

Furthermore, by (3.8) and (3.9), we get

$$\begin{aligned} f_4 &= d_1 d_2 d_4 d_0 + \sigma_i d_1 d_2 d_0 + \sigma_i \zeta_i \delta_i d_2 d_0 - \sigma_i \zeta_i \delta_i d_0 d_3 + \sigma_i \delta_i d_0 d_1 \\ &= d_1 d_4 d_5 (\alpha_i + d_2)(R_0^{(i)} - 1). \end{aligned}$$

Clearly, $f_4 > 0$ if $R_0^{(i)} > 1$. Since

$$\begin{aligned} f_4 &= d_1 d_2 d_4 d_0 + \sigma_i d_1 d_2 d_0 + \sigma_i \zeta_i \delta_i d_2 d_0 - \sigma_i \zeta_i \delta_i d_0 d_3 + \sigma_i \delta_i d_0 d_1 \\ &< d_1 d_2 d_4 d_0 + \sigma_i d_1 d_2 d_0 + \sigma_i \delta_i d_0 (d_1 + d_2), \end{aligned}$$

and $d_4 > \delta_i$, $d_5 > \sigma_i$, (3.11) and (3.12) imply that

$$\begin{aligned} & f_3(f_1 f_2 - f_3) - {f_1}^2 f_4 \\ &> [d_1 d_2 h_2 + h_1 (d_4 + \sigma_i) d_0 + \sigma_i \delta_i d_0]\{h_1 h_2 (h_1 + h_2 + \alpha_i) + d_1 d_2 (h_1 + 2\alpha_i) \\ &\quad + \alpha_i [(d_1 + d_5)\alpha_i + (d_4 + d_5) d_0 + d_4 h_2]\} - (h_1 + h_2 + \alpha_i)^2 (d_1 d_2 d_4 d_0 \\ &\quad + \sigma_i d_1 d_2 d_0 + \sigma_i \delta_i d_0 h_1) \\ &> d_0 d_4 h_2 [({d_1}^2 + {d_2}^2) h_2 + {d_1}^3 + {d_2}^3] + d_0 d_1 {\alpha_i}^2 [(d_4 + \sigma_i) d_1 + d_4 d_5] + {d_0}^2 d_4 \\ &\quad \times h_1 \alpha_i (d_4 + d_5) + d_1 d_2 h_2 [h_1 h_2 (h_1 + h_2) + d_1 d_2 (h_1 + 2\alpha_i) + {\alpha_i}^2 (d_1 + d_5) \\ &\quad + h_2 \alpha_i (h_1 + d_4)] + \alpha_i \sigma_i d_0 (h_1 + \delta_i)[(d_4 + d_5) d_0 + d_5 \alpha_i + d_4 h_2] \\ &> 0, \end{aligned}$$

where $h_1 = d_1 + d_2$ and $h_2 = d_4 + d_5 + d_0$. Thus, J_i only has eigenvalues with negative real part as $R_0^{(i)} > 1$ for all $i = 1, 2, ..., n$, and it is an immediate consequence the endemic equilibrium P^* is locally asymptotically stable. □

4. Nonautonomous system case

Since almost periodic functions are bounded, we define

$$\rho_i^* = \max_{t \in R} \rho_i(t).$$

Theorem 4.1. *Suppose that for all $i = 1, 2, ..., n$,*

$$\begin{cases} \nu_i > 2\rho_i^* M, \\ \nu_i > 2\nu_i \xi_i + 2\rho_i^* \theta_i M, \\ \nu_i > 2\nu_i \eta_i - 2\gamma_i, \end{cases} \tag{4.1}$$

holds, where $M = \frac{\sum \nu_i}{\underline{\nu}}$, $\underline{\nu} = \min\{\nu_1, \nu_2, ..., \nu_n\}$. *Then, two arbitrary solutions of system (2.1)*

$$X = (S_i^1, E_i^1, I_i^1, C_i^1, R_i^1)_n$$

and

$$Y = (S_i^2, E_i^2, I_i^2, C_i^2, R_i^2)_n$$

satisfies

$$\lim_{t \to +\infty} |X(t) - Y(t)| = 0.$$

Proof. From Lemma 2.2, we have

$$0 \le S_i^j, E_i^j, I_i^j, C_i^j, R_i^j \le \frac{\sum \nu_i}{\underline{\nu}} = M, \tag{4.2}$$

for all $t > 0$ and $j = 1, 2$.

We consider the Lyapunov function

$$V(t) = \sum_{i=1}^{n} V_i(t), \tag{4.3}$$

where

$$V_i(t) = \sum_{k=1}^{5} V_{ki}(t),$$

and

$$V_{1i} = |S_i^1 - S_i^2|, \quad V_{2i} = |E_i^1 - E_i^2|, \quad V_{3i} = |I_i^1 - I_i^2|,$$
$$V_{4i} = |C_i^1 - C_i^2|, \quad V_{5i} = |R_i^1 - R_i^2|.$$

Define

$$o(\psi(t)) = \begin{cases} 1, & \text{if } \psi(t) > 0, \\ -1, & \text{if } \psi(t) < 0, \\ 0, & \text{if } \psi(t) = 0,\ \psi'(t) = 0, \\ 1, & \text{if } \psi(t) = 0,\ \psi'(t) > 0, \\ -1, & \text{if } \psi(t) = 0,\ \psi'(t) < 0. \end{cases}$$

Thus, $|\psi(t)| = o(\psi(t))\psi(t)$ and $D^+|\psi(t)| = o(\psi(t))\psi'(t)$. Combining this property of upper right-hand Dini derivative with (4.2) gives that

$$\begin{aligned} & D^+(V_{1i}(t)) \\ &= o(S_i^1 - S_i^2)(\dot{S}_i^1 - \dot{S}_i^2) \\ &= o(S_i^1 - S_i^2)\big\{ - \rho_i(t)(I_i^1 + \theta_i C_i^1)S_i^1 + \rho_i(t)(I_i^2 + \theta_i C_i^2)S_i^2 - (\nu_i + \alpha_i) \\ &\quad \times (S_i^1 - S_i^2) - \nu_i \xi_i (C_i^1 - C_i^2) + (\gamma_i - \nu_i \eta_i)(R_i^1 - R_i^2) + \sum_{j \ne i} \{[a_{ij}(S_j^1 - S_j^2) \\ &\quad - a_{ji}(S_i^1 - S_i^2)]\}\big\} \\ &= o(S_i^1 - S_i^2)\big\{ - \rho_i(t)(I_i^1 S_i^1 + \theta_i C_i^1 S_i^1 - I_i^2 S_i^2 - \theta_i C_i^2 S_i^2) - (\nu_i + \alpha_i) \end{aligned}$$

$$\times(S_i^1 - S_i^2) - \nu_i\xi_i(C_i^1 - C_i^2) + (\gamma_i - \nu_i\eta_i)(R_i^1 - R_i^2) + \sum_{j\neq i}\{[a_{ij}(S_j^1 - S_j^2)$$
$$-a_{ji}(S_i^1 - S_i^2)]\}\}$$
$$= o(S_i^1 - S_i^2)\big\{-\rho_i(t)[I_i^1(S_i^1 - S_i^2) + S_i^2(I_i^1 - I_i^2) + \theta_i C_i^1(S_i^1 - S_i^2) + \theta_i S_i^2$$
$$(C_i^1 - C_i^2)] - (\nu_i + \alpha_i)(S_i^1 - S_i^2) - \nu_i\xi_i(C_i^1 - C_i^2) + (\gamma_i - \nu_i\eta_i)(R_i^1 - R_i^2)$$
$$+\sum_{j\neq i}[a_{ij}(S_j^1 - S_j^2) - a_{ji}(S_i^1 - S_i^2)]\big\}$$
$$\leq -\rho_i(t)I_i^1|S_i^1 - S_i^2| + \rho_i(t)S_i^2|I_i^1 - I_i^2| - \rho_i(t)\theta_i C_i^1|S_i^1 - S_i^2| + \rho_i(t)\theta_i S_i^2$$
$$\times|C_i^1 - C_i^2| - (\nu_i + \alpha_i)|S_i^1 - S_i^2| + \nu_i\xi_i|C_i^1 - C_i^2| + (\gamma_i - \nu_i\eta_i)|R_i^1 - R_i^2|$$
$$+\sum_{j\neq i}[a_{ij}|S_j^1 - S_j^2| - a_{ji}|S_i^1 - S_i^2|].$$

Similarly,

$$D^+(V_{2i}(t))$$
$$\leq \rho_i(t)I_i^1|S_i^1 - S_i^2| + \rho_i(t)S_i^2|I_i^1 - I_i^2| + \rho_i(t)\theta_i C_i^1|S_i^1 - S_i^2| + \rho_i(t)\theta_i S_i^2$$
$$\times|C_i^1 - C_i^2| - (\nu_i + \sigma_i)|E_i^1 - E_i^2| + \sum_{j\neq i}[b_{ij}|E_j^1 - E_j^2| - b_{ji}|E_i^1 - E_i^2|],$$

$$D^+(V_{3i}(t))$$
$$\leq \sigma_i|E_i^1 - E_i^2| - (\nu_i + \delta_i)|I_i^1 - I_i^2| + \sum_{j\neq i}[c_{ij}|I_j^1 - I_j^2| - c_{ji}|I_i^1 - I_i^2|],$$

$$D^+(V_{4i}(t))$$
$$\leq \zeta_i\delta_i|I_i^1 - I_i^2| - (\nu_i(1-\xi_i) + \varepsilon_i + \lambda_i)|C_i^1 - C_i^2| + \sum_{j\neq i}\{[k_{ij}|C_j^1 - C_j^2| - k_{ji}$$
$$\times|C_i^1 - C_i^2|]\},$$

$$D^+(V_{5i}(t))$$
$$\leq \alpha_i|S_i^1 - S_i^2| + (1-\zeta_i)\delta_i|I_i^1 - I_i^2| + (\varepsilon_i + \lambda_i)|C_i^1 - C_i^2| - (\nu_i(1-\eta_i) + \gamma_i)$$
$$\times|R_i^1 - R_i^2| + \sum_{j\neq i}[l_{ij}|R_j^1 - R_j^2| - l_{ji}|R_i^1 - R_i^2|].$$

For $t > 0$, we have

$$D^+(V_i(t))$$
$$\leq -\nu_i|S_i^1 - S_i^2| - \nu_i|E_i^1 - E_i^2| - (\nu_i - 2\rho_i^* M)|I_i^1 - I_i^2| - (\nu_i - 2\nu_i\xi_i - 2\rho_i^*\theta_i$$
$$\times M)|C_i^1 - C_i^2| - (\nu_i + 2\gamma_i - 2\nu_i\eta_i)|R_i^1 - R_i^2| + \sum_{j\neq i}\{[a_{ij}|S_j^1 - S_j^2| - a_{ji}$$
$$\times|S_i^1 - S_i^2|]\} + \sum_{j\neq i}[b_{ij}|E_j^1 - E_j^2| - b_{ji}|E_i^1 - E_i^2|] + \sum_{j\neq i}\{[c_{ij}|I_j^1 - I_j^2| - c_{ji}$$
$$\times|I_i^1 - I_i^2|]\} + \sum_{j\neq i}[k_{ij}|C_j^1 - C_j^2| - k_{ji}|C_i^1 - C_i^2|] + \sum_{j\neq i}\{[l_{ij}|R_j^1 - R_j^2| - l_{ji}$$
$$\times|R_i^1 - R_i^2|]\}.$$

Let $\beta_1 = \min_i\{\nu_i - 2\rho_i^* M\}$, $\beta_2 = \min_i\{\nu_i - 2\nu_i\xi_i - 2\rho_i^*\theta_i M\}$ and $\beta_3 = \min_i\{\nu_i + 2\gamma_i - 2\nu_i\eta_i\}$. From condition (4.1), we have $\beta_1, \beta_2, \beta_3 > 0$ which means $\beta = \min\{\underline{\nu}, \beta_1, \beta_2, \beta_3\} > 0$. It follows that

$$\begin{aligned}
&D^+(V(t)) \\
\le& -\underline{\nu}\sum_i |S_i^1 - S_i^2| - \underline{\nu}\sum_i |E_i^1 - E_i^2| - \beta_1\sum_i |I_i^1 - I_i^2| - \beta_2\sum_i |C_i^1 - C_i^2| \\
&-\beta_3\sum_i |R_i^1 - R_i^2| \\
\le& -\beta V(t). \qquad (4.4)
\end{aligned}$$

For $t > 0$, (4.4) gives

$$V(t) + \beta\int_0^t V(s)ds \le V(0). \qquad (4.5)$$

It follows from (2.1), (4.2) and (4.3) that $V(t)$ is uniformly continuous on $(0, +\infty)$. Consequently,

$$V(t) \to 0,$$

as $t \to +\infty$. Otherwise, $\beta\int_0^t V(s)ds \to +\infty$ when $t \to +\infty$ which contradicts (4.5). Thus, we conclude that

$$\begin{aligned}
\lim_{t\to+\infty}|S_i^1 - S_i^2| &= \lim_{t\to+\infty}|E_i^1 - E_i^2| = \lim_{t\to+\infty}|I_i^1 - I_i^2| \\
&= \lim_{t\to+\infty}|C_i^1 - C_i^2| = \lim_{t\to+\infty}|R_i^1 - R_i^2| = 0,
\end{aligned}$$

for all $i = 1, 2, ..., n$. □

Theorem 4.1 implies any solution of (2.1) converges to X, which reveals the global attractivity of this system.

Theorem 4.2. *Suppose condition (4.1) of Theorem 4.1 holds, then system (2.1) admits a global attractive almost periodic solution.*

Proof. We only need to verify the existence of almost periodic solution for (2.1). The almost periodicity of function $\rho_i(t)$, $i = 1, 2, ..., n$ allows us to find sequence $\{t_m\}$ satisfying

$$t_m \to +\infty,\ \rho_i(t + t_m) \to \rho_i(t), \qquad (4.6)$$

for $i = 1, 2, ..., n$ as $m \to \infty$, where the convergence of $\{\rho_i(t + t_m)\}$ is uniform. Then, $u(t + t_m) = (u_{1i}(t + t_m), u_{2i}(t + t_m), u_{3i}(t + t_m), u_{4i}(t + t_m), u_{5i}(t + t_m))_n$ is a solution of

$$\begin{cases} \dfrac{\mathrm{d}S_i}{\mathrm{d}t} =& \nu_i - \rho_i(t+t_m)(I_i+\theta_i C_i)S_i - (\nu_i+\alpha_i)S_i - \nu_i\xi_i C_i + (\gamma_i - \nu_i\eta_i)R_i \\ & + \sum_{j\neq i}(a_{ij}S_j - a_{ji}S_i), \\ \dfrac{\mathrm{d}E_i}{\mathrm{d}t} =& \rho_i(t+t_m)(I_i+\theta_i C_i)S_i - (\nu_i+\sigma_i)E_i + \sum_{j\neq i}(b_{ij}E_j - b_{ji}E_i), \\ \dfrac{\mathrm{d}I_i}{\mathrm{d}t} =& \sigma_i E_i - (\nu_i+\delta_i)I_i + \sum_{j\neq i}(c_{ij}I_j - c_{ji}I_i), \\ \dfrac{\mathrm{d}C_i}{\mathrm{d}t} =& \zeta_i\delta_i I_i - (\nu_i(1-\xi_i)+\varepsilon_i+\lambda_i)C_i + \sum_{j\neq i}(k_{ij}C_j - k_{ji}C_i), \\ \dfrac{\mathrm{d}R_i}{\mathrm{d}t} =& \alpha_i S_i + (1-\zeta_i)\delta_i I_i + (\varepsilon_i+\lambda_i)C_i - (\nu_i(1-\eta_i)+\gamma_i)R_i + \sum_{j\neq i}(l_{ij}R_j \\ & - l_{ji}R_i). \end{cases}$$

Since $\{u_{ji}(t+t_m)\}$ and also $\{\dot{u}_{ji}(t+t_m)\}$ for $i=1,2,...,n,\ j=1,2,...,5$ are uniformly bounded and equicontinuous, then $\{u_{ji}(t+t_m)\}$ has a uniformly convergent subsequence $\{u_{ji}(t+t_{m_k})\}$ for $t\in[a,b]$ resulting from Arzelà-Ascoli theorem. We assume

$$\lim_{k\to\infty} u_{ji}(t+t_{m_k}) = v_{ji}(t) \tag{4.7}$$

uniformly. Then, from $t_m\to+\infty$ as $m\to\infty$, we have $v_{ji}\in C[t_0,+\infty]$. Since $\rho_i(t)$ is uniformly continuous for $i=1,2,...,n$ (Corollary 1.15 in [6, p10]), from (4.6) and (4.7), we conclude that $v=(v_{1i},v_{2i},v_{3i},v_{4i},v_{5i})_n$ is the solution of the system (2.1).

Next we demonstrate the almost periodicity of $v(t)$. For any $\varepsilon>0$, let

$$\widetilde{\varepsilon} = \frac{\beta}{8(1+\overline{\theta})M^2}\varepsilon, \tag{4.8}$$

where $\overline{\theta}=\max\{\theta_1,\theta_2,...,\theta_n\}$, $M=\frac{\sum\nu_i}{\underline{\nu}}$, $\beta=\min\{\underline{\nu},\beta_1,\beta_2,\beta_3\}$, and $\beta_1=\min_i\{\nu_i-2\rho_i^*M\}$, $\beta_2=\min_i\{\nu_i-2\nu_i\xi_i-2\rho_i^*\theta_i M\}$, $\beta_3=\min_i\{\nu_i+2\gamma_i-2\nu_i\eta_i\}$. It is obvious that $\widetilde{\varepsilon}>0$ in view of condition (4.1). Furthermore, we let $\varpi\in\bigcap_{i=1}^n T(\rho_i,\widetilde{\varepsilon})$, where $T(\rho_i,\widetilde{\varepsilon})=\{\kappa\mid|\rho_i(t+\kappa)-\rho_i(t)|<\widetilde{\varepsilon},$ for all $t\}$ (Definition 1.11 in [6, p7]). Thus,

$$|\rho_i(t+\varpi)-\rho_i(t)|<\widetilde{\varepsilon}. \tag{4.9}$$

Define function $W(t)$ as follows

$$W(t)=\sum_{i=1}^n W_i(t), \tag{4.10}$$

where

$$W_i(t)=\sum_{k=1}^5 W_{ki}(t),$$

and

$$W_{1i}(t) = |v_{1i}(t+\varpi) - v_{1i}(t)|, \quad W_{2i}(t) = |v_{2i}(t+\varpi) - v_{2i}(t)|,$$
$$W_{3i}(t) = |v_{3i}(t+\varpi) - v_{3i}(t)|, \quad W_{4i}(t) = |v_{4i}(t+\varpi) - v_{4i}(t)|,$$
$$W_{5i}(t) = |v_{5i}(t+\varpi) - v_{5i}(t)|.$$

Since $v = (v_{1i}, v_{2i}, v_{3i}, v_{4i}, v_{5i})_n$ is the solution of system (2.1), by using the method similar to $V(t)$ and (4.9), we arrive at

$$\begin{aligned}
&D^+(W_{1i}(t)) \\
&= o(v_{1i}(t+\varpi) - v_{1i}(t)) \left\{ \frac{dv_{1i}(t+\varpi)}{dt} - \frac{dv_{1i}(t)}{dt} \right\} \\
&= o(v_{1i}(t+\varpi) - v_{1i}(t))\big\{ - \rho_i(t+\varpi)[v_{3i}(t+\varpi) + \theta_i v_{4i}(t+\varpi)]v_{1i}(t+\varpi) \\
&\quad +\rho_i(t)[v_{3i}(t) + \theta_i v_{4i}(t)]v_{1i}(t) - (\nu_i + \alpha_i)(v_{1i}(t+\varpi) - v_{1i}(t)) - \nu_i\xi_i \\
&\quad \times(v_{4i}(t+\varpi) - v_{4i}(t)) + (\gamma_i - \nu_i\eta_i)(v_{5i}(t+\varpi) - v_{5i}(t)) + \sum_{j\neq i} a_{ij} \\
&\quad \times(v_{1j}(t+\varpi) - v_{1j}(t)) - \sum_{j\neq i} a_{ji}(v_{1i}(t+\varpi) - v_{1i}(t))\big\} \\
&= o(v_{1i}(t+\varpi) - v_{1i}(t))\big\{ - \rho_i(t+\varpi)[v_{3i}(t+\varpi) + \theta_i v_{4i}(t+\varpi)]v_{1i}(t+\varpi) \\
&\quad +\rho_i(t+\varpi)[v_{3i}(t) + \theta_i v_{4i}(t)]v_{1i}(t) - (\nu_i + \alpha_i)(v_{1i}(t+\varpi) - v_{1i}(t)) - \nu_i\xi_i \\
&\quad \times(v_{4i}(t+\varpi) - v_{4i}(t)) + (\gamma_i - \nu_i\eta_i)(v_{5i}(t+\varpi) - v_{5i}(t)) + \sum_{j\neq i}\{[a_{ij} \\
&\quad (v_{1j}(t+\varpi) - v_{1j}(t)) - a_{ji}(v_{1i}(t+\varpi) - v_{1i}(t))]\}\big\} + o(v_{1i}(t+\varpi) - v_{1i}(t)) \\
&\quad \times(\rho_i(t) - \rho_i(t+\varpi))[v_{3i}(t) + \theta_i v_{4i}(t)]v_{1i}(t) \\
&\leq o(v_{1i}(t+\varpi) - v_{1i}(t))\big\{ - \rho_i(t+\varpi)[v_{3i}(t+\varpi) + \theta_i v_{4i}(t+\varpi)]v_{1i}(t+\varpi) \\
&\quad +\rho_i(t+\varpi)[v_{3i}(t) + \theta_i v_{4i}(t)]v_{1i}(t) - (\nu_i + \alpha_i)(v_{1i}(t+\varpi) - v_{1i}(t)) - \nu_i\xi_i \\
&\quad \times(v_{4i}(t+\varpi) - v_{4i}(t)) + (\gamma_i - \nu_i\eta_i)(v_{5i}(t+\varpi) - v_{5i}(t)) + \sum_{j\neq i}\{[a_{ij} \\
&\quad \times(v_{1j}(t+\varpi) - v_{1j}(t)) - a_{ji}(v_{1i}(t+\varpi) - v_{1i}(t))]\}\big\} + |\rho_i(t+\varpi) - \rho_i(t)| \\
&\quad \times[v_{3i}(t) + \theta_i v_{4i}(t)]v_{1i}(t) \\
&< o(v_{1i}(t+\varpi) - v_{1i}(t))\big\{ - \rho_i(t+\varpi)[v_{3i}(t+\varpi) + \theta_i v_{4i}(t+\varpi)]v_{1i}(t+\varpi) \\
&\quad +\rho_i(t+\varpi)[v_{3i}(t) + \theta_i v_{4i}(t)]v_{1i}(t) - (\nu_i + \alpha_i)(v_{1i}(t+\varpi) - v_{1i}(t)) - \nu_i\xi_i \\
&\quad \times(v_{4i}(t+\varpi) - v_{4i}(t)) + (\gamma_i - \nu_i\eta_i)(v_{5i}(t+\varpi) - v_{5i}(t)) + \sum_{j\neq i}\{[a_{ij} \\
&\quad \times(v_{1j}(t+\varpi) - v_{1j}(t)) - a_{ji}(v_{1i}(t+\varpi) - v_{1i}(t))]\}\big\} \\
&\quad +\widetilde{\varepsilon}[v_{3i}(t) + \theta_i v_{4i}(t)]v_{1i}(t) \\
&= o(v_{1i}(t+\varpi) - v_{1i}(t))\big\{ - \rho_i(t+\varpi)[v_{3i}(t+\varpi)(v_{1i}(t+\varpi) - v_{1i}(t)) + v_{1i}(t) \\
&\quad \times(v_{3i}(t+\varpi) - v_{3i}(t)) + \theta_i v_{4i}(t+\varpi)(v_{1i}(t+\varpi) - v_{1i}(t)) + \theta_i v_{1i}(t) \\
&\quad \times(v_{4i}(t+\varpi) - v_{4i}(t))] - (\nu_i + \alpha_i)(v_{1i}(t+\varpi) - v_{1i}(t)) - \nu_i\xi_i(v_{4i}(t+\varpi) \\
&\quad \times - v_{4i}(t)) + (\gamma_i - \nu_i\eta_i)(v_{5i}(t+\varpi) - v_{5i}(t)) + \sum_{j\neq i}\{[a_{ij}(v_{1j}(t+\varpi) \\
&\quad -v_{1j}(t)) - a_{ji}(v_{1i}(t+\varpi) - v_{1i}(t))]\}\big\} + \widetilde{\varepsilon}[v_{3i}(t) + \theta_i v_{4i}(t)]v_{1i}(t) \\
&\leq -\rho_i(t+\varpi)v_{3i}(t+\varpi)|v_{1i}(t+\varpi) - v_{1i}(t)| + \rho_i(t+\varpi)v_{1i}(t)|v_{3i}(t+\varpi)
\end{aligned}$$

$$-v_{3i}(t)| - \rho_i(t+\varpi)\theta_i v_{4i}(t+\varpi)|v_{1i}(t+\varpi) - v_{1i}(t)| + \rho_i(t+\varpi)\theta_i v_{1i}(t)| \times v_{4i}(t+\varpi) - v_{4i}(t)| - (\nu_i + \alpha_i)|v_{1i}(t+\varpi) - v_{1i}(t)| + \nu_i \xi_i |v_{4i}(t+\varpi) - v_{4i}(t)| + (\gamma_i - \nu_i \eta_i)|v_{5i}(t+\varpi) - v_{5i}(t)| + \sum_{j\neq i}\{[a_{ij}|v_{1j}(t+\varpi) - v_{1j}(t)| - a_{ji}|v_{1i}(t+\varpi) - v_{1i}(t)|]\} + \widetilde{\varepsilon}[v_{3i}(t) + \theta_i v_{4i}(t)]v_{1i}(t).$$

Similarly, combining with (4.9), the upper right-hand derivative of $W_{2i}(t)$ satisfies

$$\begin{aligned} &D^+(W_{2i}(t)) \\ &< \rho_i(t)v_{3i}(t+\varpi)|v_{1i}(t+\varpi) - v_{1i}(t)| + \rho_i(t)v_{1i}(t)|v_{3i}(t+\varpi) - v_{3i}(t)| + \rho_i(t) \\ &\quad \times \theta_i v_{4i}(t+\varpi)|v_{1i}(t+\varpi) - v_{1i}(t)| + \rho_i(t)\theta_i v_{1i}(t)|v_{4i}(t+\varpi) - v_{4i}(t)| \\ &\quad -(\nu_i + \sigma_i)|v_{2i}(t+\varpi) - v_{2i}(t)| + \sum_{j\neq i}\{[b_{ij}|v_{2j}(t+\varpi) - v_{2j}(t)| - b_{ji} \\ &\quad \times |v_{2i}(t+\varpi) - v_{2i}(t)|]\} + \widetilde{\varepsilon}[v_{3i}(t+\varpi) + \theta_i v_{4i}(t+\varpi)]v_{1i}(t+\varpi), \end{aligned}$$

Similarly,

$$\begin{aligned} &D^+(W_{3i}(t)) \\ &\le \sigma_i|v_{2i}(t+\varpi) - v_{2i}(t)| - (\nu_i + \delta_i)|v_{3i}(t+\varpi) - v_{3i}(t)| \\ &\quad + \sum_{j\neq i}\{[c_{ij}|v_{3j}(t+\varpi) - v_{3j}(t)| - c_{ji}|v_{3i}(t+\varpi) - v_{3i}(t)|]\}, \end{aligned}$$

$$\begin{aligned} &D^+(W_{4i}(t)) \\ &\le \zeta_i\delta_i|v_{3i}(t+\varpi) - v_{3i}(t)| - [\nu_i(1-\xi_i) + \widetilde{\varepsilon}_i + \lambda_i]|v_{4i}(t+\varpi) - v_{4i}(t)| \\ &\quad + \sum_{j\neq i}\{[k_{ij}|v_{4j}(t+\varpi) - v_{4j}(t)| - k_{ji}|v_{4i}(t+\varpi) - v_{4i}(t)|]\}, \end{aligned}$$

$$\begin{aligned} &D^+(W_{5i}(t)) \\ &\le \alpha_i|v_{1i}(t+\varpi) - v_{1i}(t)| + (1-\zeta_i)\delta_i|v_{3i}(t+\varpi) - v_{3i}(t)| \\ &\quad + (\widetilde{\varepsilon}_i + \lambda_i)|v_{4i}(t+\varpi) - v_{4i}(t)| - [\nu_i(1-\eta_i) + \gamma_i]|v_{5i}(t+\varpi) - v_{5i}(t)| \\ &\quad + \sum_{j\neq i}\{[l_{ij}|v_{5j}(t+\varpi) - v_{5j}(t)| - l_{ji}|v_{5i}(t+\varpi) - v_{5i}(t)|]\}. \end{aligned}$$

Making use of (4.9), we have

$$\begin{aligned} &D^+(W_i(t)) \\ &< -\nu_i|v_{1i}(t+\varpi) - v_{1i}(t)| - \nu_i|v_{2i}(t+\varpi) - v_{2i}(t)| - (\nu_i - 2\rho_i^* M)|v_{3i}(t+\varpi) \\ &\quad -v_{3i}(t)| - (\nu_i - 2\nu_i\xi_i - 2\rho_i^*\theta_i M)|v_{4i}(t+\varpi) - v_{4i}(t)| - (\nu_i + 2\gamma_i - 2\nu_i\eta_i) \\ &\quad \times |v_{5i}(t+\varpi) - v_{5i}(t)| + \widetilde{\varepsilon}v_{3i}(t+\varpi)|v_{1i}(t+\varpi) - v_{1i}(t)| + \widetilde{\varepsilon}\theta_i v_{4i}(t+\varpi)| \\ &\quad v_{1i}(t+\varpi) - v_{1i}(t)| + \widetilde{\varepsilon}[v_{3i}(t) + \theta_i v_{4i}(t)]v_{1i}(t) + \widetilde{\varepsilon}[v_{3i}(t+\varpi) + \theta_i \\ &\quad \times v_{4i}(t+\varpi)]v_{1i}(t+\varpi) + \sum_{j\neq i}\{[a_{ij}|v_{1j}(t+\varpi) - v_{1j}(t)| - a_{ji}|v_{1i}(t+\varpi) \\ &\quad -v_{1i}(t)|]\} + \sum_{j\neq i}[b_{ij}|v_{2j}(t+\varpi) - v_{2j}(t)| - b_{ji}|v_{2i}(t+\varpi) - v_{2i}(t)|] \\ &\quad + \sum_{j\neq i}[c_{ij}|v_{3j}(t+\varpi) - v_{3j}(t)| - c_{ji}|v_{3i}(t+\varpi) - v_{3i}(t)|] \end{aligned}$$

$$+\sum_{j\neq i}[k_{ij}|v_{4j}(t+\varpi)-v_{4j}(t)|-k_{ji}|v_{4i}(t+\varpi)-v_{4i}(t)|]$$
$$+\sum_{j\neq i}[l_{ij}|v_{5j}(t+\varpi)-v_{5j}(t)|-l_{ji}|v_{5i}(t+\varpi)-v_{5i}(t)|].$$

Then,

$$\begin{aligned}
&D^+(W(t))\\
<&-\underline{\nu}\sum_i|v_{1i}(t+\varpi)-v_{1i}(t)|-\underline{\nu}\sum_i|v_{2i}(t+\varpi)-v_{2i}(t)|-\beta_1\\
&\times\sum_i|v_{3i}(t+\varpi)-v_{3i}(t)|-\beta_2\sum_i|v_{4i}(t+\varpi)-v_{4i}(t)|-\beta_3\\
&\times\sum_i|v_{5i}(t+\varpi)-v_{5i}(t)|+\widetilde{\varepsilon}\sum_i[v_{3i}(t+\varpi)+\theta_i v_{4i}(t+\varpi)]\\
&\times|v_{1i}(t+\varpi)-v_{1i}(t)|+\widetilde{\varepsilon}\sum_i[(v_{3i}(t)+\theta_i v_{4i}(t))v_{1i}(t)\\
&+(v_{3i}(t+\varpi)+\theta_i v_{4i}(t+\varpi))v_{1i}(t+\varpi)]\\
\leq&-\beta W(t)+4\widetilde{\varepsilon}(1+\overline{\theta})M^2.
\end{aligned}\tag{4.11}$$

Now integrating both sides of (4.11), together with (4.8), we obtain

$$\begin{aligned}
W(t)&<\frac{4\widetilde{\varepsilon}(1+\overline{\theta})M^2}{\beta}+(W(0)-\frac{4\widetilde{\varepsilon}(1+\overline{\theta})M^2}{\beta})e^{-\beta t}\\
&=\frac{1}{2}\varepsilon+(W(0)-\frac{4\widetilde{\varepsilon}(1+\overline{\theta})M^2}{\beta})e^{-\beta t}.
\end{aligned}$$

Clearly, $(W(0)-\frac{4\widetilde{\varepsilon}(1+\overline{\theta})M^2}{\beta})e^{-\beta t}\to 0$ as $t\to+\infty$. Then there exists $\widetilde{t}>0$ such that for all $t>\widetilde{t}$,

$$\left|(W(0)-\frac{4\widetilde{\varepsilon}(1+\overline{\theta})M^2}{\beta})e^{-\beta t}\right|<\frac{1}{2}\varepsilon.$$

Thus, we conclude that

$$W(t)<\varepsilon.$$

Then, in views of (4.10), for all $t>\widetilde{t}$,

$$|v_{ji}(t+\varpi)-v_{ji}(t)|<\varepsilon,\quad j=1,2,3,4,5,\ i=1,2,...,n,$$

that is, $|v(t+\varpi)-v(t)|<\varepsilon$. Therefore, $\varpi\in T(v,\varepsilon)$ which means

$$\bigcap_{i=1}^{n}T(\rho_i,\widetilde{\varepsilon})\subset T(v,\varepsilon).$$

Thus, $T(v,\varepsilon)$ is relatively dense and $v(t)$ is the almost periodic solution of (2.1). □

5. Numerical simulations

To proceed, several numerical examples are provided to validate the theoretical findings derived in previous sections. We focus on the following HBV transmission model with two patches which is a special case of system (2.1)

$$\begin{cases} \frac{\mathrm{d}S_1}{\mathrm{d}t} =\nu_1 - \rho_1(t)(I_1+\theta_1 C_1)S_1 - (\nu_1+\alpha_1)S_1 - \nu_1\xi_1 C_1 + (\gamma_1-\nu_1\eta_1)R_1 \\ \qquad + a_{12}S_2 - a_{21}S_1, \\ \frac{\mathrm{d}E_1}{\mathrm{d}t} =\rho_1(t)(I_1+\theta_1 C_1)S_1 - (\nu_1+\sigma_1)E_1 + b_{12}E_2 - b_{21}E_1, \\ \frac{\mathrm{d}I_1}{\mathrm{d}t} =\sigma_1 E_1 - (\nu_1+\delta_1)I_1 + c_{12}I_2 - c_{21}I_1, \\ \frac{\mathrm{d}C_1}{\mathrm{d}t} =\zeta_1\delta_1 I_1 - (\nu_1(1-\xi_1)+\varepsilon_1+\lambda_1)C_1 + k_{12}C_2 - k_{21}C_1, \\ \frac{\mathrm{d}R_1}{\mathrm{d}t} =\alpha_1 S_1 + (1-\zeta_1)\delta_1 I_1 + (\varepsilon_1+\lambda_1)C_1 - (\nu_1(1-\eta_1)+\gamma_1)R_1 + l_{12}R_2 \\ \qquad - l_{21}R_1, \\ \frac{\mathrm{d}S_2}{\mathrm{d}t} =\nu_2 - \rho_2(t)(I_2+\theta_2 C_2)S_2 - (\nu_2+\alpha_2)S_2 - \nu_2\xi_2 C_2 + (\gamma_2-\nu_2\eta_2)R_2 \\ \qquad + a_{21}S_1 - a_{12}S_2, \\ \frac{\mathrm{d}E_2}{\mathrm{d}t} =\rho_2(t)(I_2+\theta_2 C_2)S_2 - (\nu_2+\sigma_2)E_2 + b_{21}E_1 - b_{12}E_2, \\ \frac{\mathrm{d}I_2}{\mathrm{d}t} =\sigma_2 E_2 - (\nu_2+\delta_2)I_2 + c_{21}I_1 - c_{12}I_2, \\ \frac{\mathrm{d}C_2}{\mathrm{d}t} =\zeta_2\delta_2 I_2 - (\nu_2(1-\xi_2)+\varepsilon_2+\lambda_2)C_2 + k_{21}C_1 - k_{12}C_2, \\ \frac{\mathrm{d}R_2}{\mathrm{d}t} =\alpha_2 S_2 + (1-\zeta_2)\delta_2 I_2 + (\varepsilon_2+\lambda_2)C_2 - (\nu_2(1-\eta_2)+\gamma_2)R_2 + l_{21}R_1 \\ \qquad - l_{12}R_2, \end{cases} \tag{5.1}$$

Example 5.1. We take parameter values as follows

$$\begin{aligned} &\nu_1 = 0.4,\ \rho_1(t) = 0.03,\ \theta_1 = 5,\ \sigma_1 = 0.2,\ \delta_1 = 0.05,\ \varepsilon_1 = 0.0025, \\ &\gamma_1 = 0.06,\ \xi_1 = 0.11,\ \eta_1 = 0.4,\ \zeta_1 = 0.2,\ \alpha_1 = 0.3,\ \lambda_1 = 0.1, \\ &\nu_2 = 0.5,\ \rho_2(t) = 0.04,\ \theta_2 = 4,\ \sigma_2 = 0.3,\ \delta_2 = 0.07,\ \varepsilon_2 = 0.002, \\ &\gamma_2 = 0.05,\ \xi_2 = 0.15,\ \eta_2 = 0.3,\ \zeta_2 = 0.3,\ \alpha_2 = 0.3,\ \lambda_2 = 0.2, \\ &a_{12} = 0.2,\ b_{12} = 0.2,\ c_{12} = 0.09,\ k_{12} = 0.02,\ l_{12} = 0.2, \\ &a_{21} = 0.2,\ b_{21} = 0.1,\ c_{21} = 0.08,\ k_{21} = 0.01,\ l_{21} = 0.2. \end{aligned}$$

For the model system (5.1), the assumptions **(H1)-(H3)** hold and 2-order square matrices $\mathfrak{B} = (b_{ij})$, $\mathfrak{C} = (c_{ij})$ and $\mathfrak{K} = (k_{ij})$ are irreducible. We also have $R_0 \approx 0.0124 < 1$. Hence, system (5.1) has a disease-free equilibrium which is local asymptotic stable from Theorem 3.2, see Fig. 1.

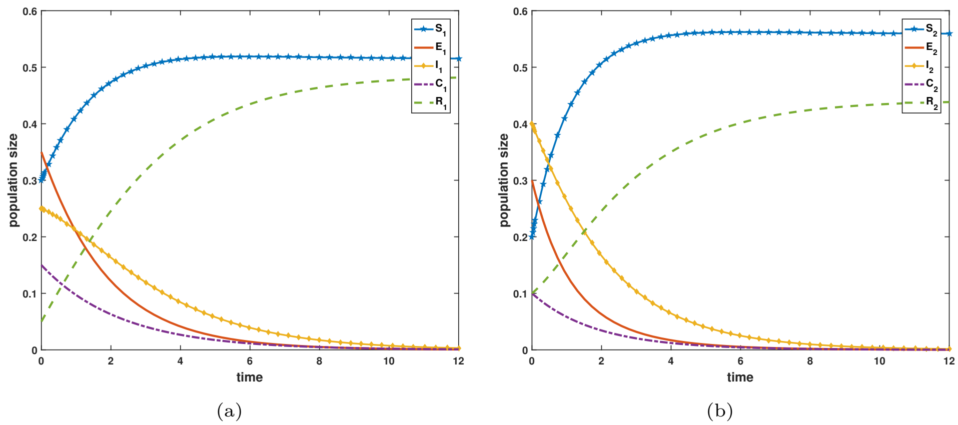


Figure 1. State trajectories for populations with $R_0 < 1$. (a) State trajectories for populations in patch 1. (b) State trajectories for populations in patch 2

Example 5.2. We suppose that $\alpha_1 = \alpha_2 = 0$ and take the other parameter values as follows

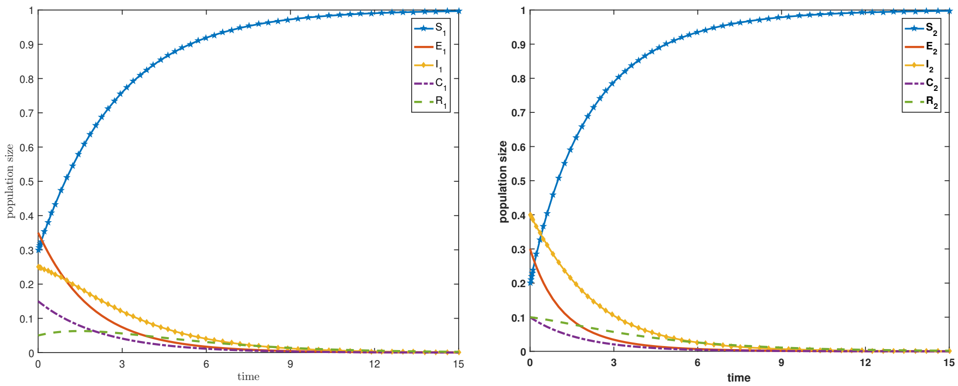
$$\nu_1 = 0.4,\ \rho_1(t) = 0.03,\ \theta_1 = 5,\ \sigma_1 = 0.2,\ \delta_1 = 0.05,\ \varepsilon_1 = 0.0025,$$
$$\gamma_1 = 0.06,\ \xi_1 = 0.11,\ \eta_1 = 0.4,\ \zeta_1 = 0.2,\ \lambda_1 = 0.1,$$
$$\nu_2 = 0.5,\ \rho_2(t) = 0.04,\ \theta_2 = 4,\ \sigma_2 = 0.3,\ \delta_2 = 0.07,\ \varepsilon_2 = 0.002,$$
$$\gamma_2 = 0.05,\ \xi_2 = 0.15,\ \eta_2 = 0.3,\ \zeta_2 = 0.3,\ \lambda_2 = 0.2,$$
$$a_{12} = 0.1,\ b_{12} = 0.2,\ c_{12} = 0.09,\ k_{12} = 0.02,\ l_{12} = 0.2,$$
$$a_{21} = 0.2,\ b_{21} = 0.1,\ c_{21} = 0.08,\ k_{21} = 0.01,\ l_{21} = 0.1.$$

For the model system (5.1), the dispersal matrices $\mathfrak{B} = (b_{ij})$, $\mathfrak{C} = (c_{ij})$ and $\mathfrak{K} = (k_{ij})$ are irreducible. We also have $R_0 \approx 0.0135 < 1$. And we let $S_1(0) = 0.3 < S_1^0 = 1$, $S_2(0) = 0.2 < S_2^0 = 1$. Hence, system (5.1) has a global asymptotic stable disease-free equilibrium P_0 from Theorem 3.4, see Fig. 2.

Example 5.3. We take parameter values as follows

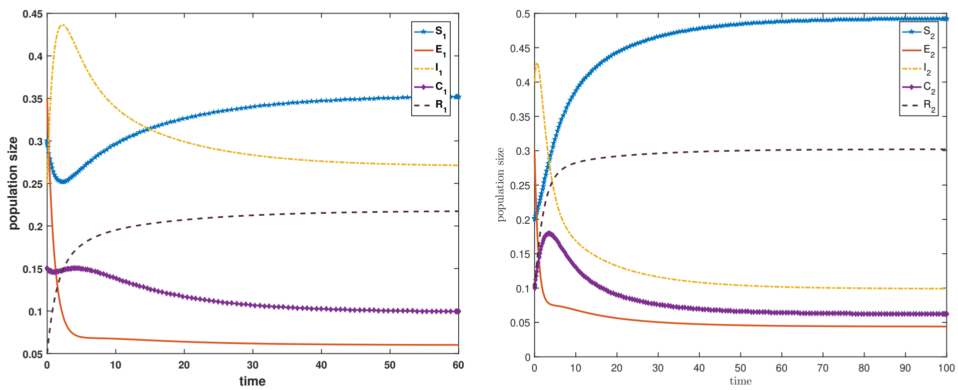
$$\nu_1 = 0.1,\ \rho_1(t) = 0.3,\ \theta_1 = 3,\ \sigma_1 = 0.9,\ \delta_1 = 0.1,\ \varepsilon_1 = 0.0025,$$
$$\gamma_1 = 0.5,\ \xi_1 = 0.11,\ \eta_1 = 0.3,\ \zeta_1 = 0.7,\ \alpha_1 = 0.3,\ \lambda_1 = 0.1,$$
$$\nu_2 = 0.1,\ \rho_2(t) = 0.4,\ \theta_2 = 2,\ \sigma_2 = 0.9,\ \delta_2 = 0.3,\ \varepsilon_2 = 0.002,$$
$$\gamma_2 = 0.5,\ \xi_2 = 0.15,\ \eta_2 = 0.3,\ \zeta_2 = 0.6,\ \alpha_2 = 0.3,\ \lambda_2 = 0.2,$$
$$a_{12} = a_{21} = b_{21} = b_{12} = c_{21} = c_{12} = k_{21} = k_{12} = l_{12} = l_{21} = 0.$$

For the model system (5.1), we have $R_0^{(1)} \approx 1.8544 > 1$, $R_0^{(2)} \approx 1.3293 > 1$. Hence, system (5.1) admits unique endemic equilibrium which is local asymptotic stable from Theorem 3.6, see Fig. 3.



(a) (b)

Figure 2. State trajectories for populations with $R_0 < 1$. (a) State trajectories for populations in patch 1. (b) State trajectories for populations in patch 2



(a) (b)

Figure 3. State trajectories for populations with $R_0 > 1$. (a) State trajectories for populations in patch 1. (b) State trajectories for populations in patch 2

Example 5.4. Let

$$\rho_1(t) = \frac{3}{100} \times (\cos\sqrt{3}t + \sin t)^2,\ \rho_2(t) = \frac{1}{40} \times (\cos\sqrt{2}t + \sin t)^2.$$

Then, we take other parameter values as follows

$$\nu_1 = 0.5,\ \theta_1 = 0.7,\ \sigma_1 = 0.9,\ \delta_1 = 0.1,\ \varepsilon_1 = 0.0025,$$

$$\gamma_1 = 0.1,\ \xi_1 = 0.15,\ \eta_1 = 0.1,\ \zeta_1 = 0.7,\ \alpha_1 = 0.3,\ \lambda_1 = 0.1,$$

$$\nu_2 = 0.5,\ \theta_2 = 0.8,\ \sigma_2 = 0.9,\ \delta_2 = 0.3,\ \varepsilon_2 = 0.002,$$

$$\gamma_2 = 0.1,\ \xi_2 = 0.17,\ \eta_2 = 0.1,\ \zeta_2 = 0.6,\ \alpha_2 = 0.3,\ \lambda_2 = 0.2,$$

$$a_{12} = 0.5,\ b_{12} = 0.02,\ c_{12} = 0.01,\ k_{12} = 0.02,\ l_{21} = 0.1,$$

$$a_{21} = 0.5,\ b_{21} = 0.01,\ c_{21} = 0.02,\ k_{21} = 0.01,\ l_{12} = 0.1.$$

All the sufficient conditions given in Theorems 4.1 and 4.2 for system (5.1) are well satisfied as

$$\begin{cases} \nu_1 - 2\rho_1^* M = 0.02 > 0, \ \nu_1 - 2\nu_1\xi_1 + 2\rho_1^*\theta_1 M = 0.014 > 0, \\ \nu_1 - 2\nu_1\eta_1 - 2\gamma_1 = 0.2 > 0, \\ \nu_2 - 2\rho_2^* M = 0.1 > 0, \ \nu_2 - 2\nu_2\xi_2 + 2\rho_2^*\theta_2 M = 0.01 > 0, \\ \nu_2 - 2\nu_2\eta_2 - 2\gamma_2 = 0.2 > 0. \end{cases}$$

The model system admits a globally attractive positive almost periodic solution, see Fig. 4.

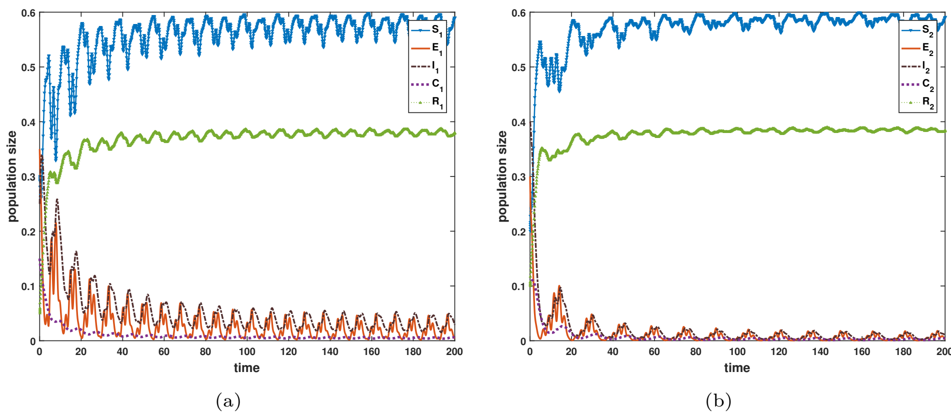


Figure 4. Almost periodic solution of nonautonomous epidemic system (5.1). (a) State trajectories for populations in patch 1. (b) State trajectories for populations in patch 2

Example 5.5. Let

$$\rho_1(t) = \tfrac{1}{4} \times (\cos\sqrt{3}t + \sin t)^2, \ \rho_2(t) = \tfrac{1}{8} \times (\cos\sqrt{2}t + \sin t)^2,$$

$$\nu_1 = 0.1, \ \theta_1 = 1, \ \sigma_1 = 0.8, \ \delta_1 = 0.1, \ \varepsilon_1 = 0.0025,$$

$$\gamma_1 = 0.5, \ \xi_1 = 0.11, \ \eta_1 = 0.2, \ \zeta_1 = 0.7, \ \alpha_1 = 0.3, \ \lambda_1 = 0.1,$$

$$\nu_2 = 0.2, \ \theta_2 = 2, \ \sigma_2 = 0.9, \ \delta_2 = 0.3, \ \varepsilon_2 = 0.002,$$

$$\gamma_2 = 0.5, \ \xi_2 = 0.15, \ \eta_2 = 0.1, \ \zeta_2 = 0.6, \ \alpha_2 = 0.3, \ \lambda_2 = 0.2,$$

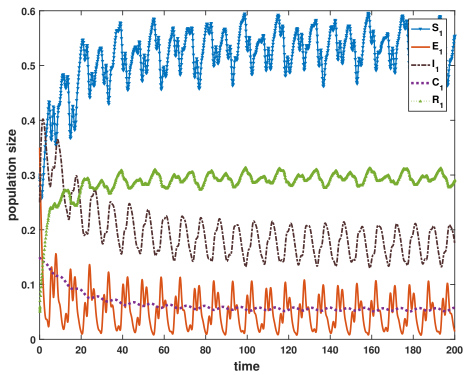
$$a_{12} = 0.5, \ b_{12} = 0.02, \ c_{12} = 0.01, \ k_{12} = 0.02, \ l_{12} = 0.1,$$

$$a_{21} = 0.5, \ b_{21} = 0.01, \ c_{21} = 0.02, \ k_{21} = 0.01, \ l_{21} = 0.1.$$

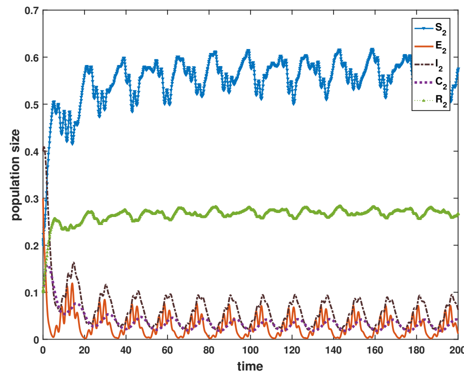
Then

$$\begin{cases} \nu_1 - 2\rho_1^* M = -5.9 < 0, \ \nu_1 - 2\nu_1\xi_1 + 2\rho_1^*\theta_1 M = 6.078 > 0, \\ \nu_1 - 2\nu_1\eta_1 - 2\gamma_1 = -0.94 < 0, \\ \nu_2 - 2\rho_2^* M = -2.8 < 0, \ \nu_2 - 2\nu_2\xi_2 + 2\rho_2^*\theta_2 M = 6.14 > 0, \\ \nu_2 - 2\nu_2\eta_2 - 2\gamma_2 = -0.84 < 0. \end{cases}$$

The model system admits a global attractive positive almost periodic solution, see Fig. 5.



(a)



(b)

Figure 5. Almost periodic solution of nonautonomous epidemic system (5.1). (a) State trajectories for populations in patch 1. (b) State trajectories for populations in patch 2

Remark 5.1. From Theorem 4.1 and Theorem 4.2, almost periodic solution of system (2.1) exists when condition (4.1) holds. But through the simulation of Example 5.5, it seems that the existence of almost periodic solution are ensured whenever parameters of model satisfy condition (4.1), which means condition (4.1) is not necessary for Theorem 4.1 and Theorem 4.2.

6. Conclusion

A nonautonomous model for HBV infection in a patchy environment has been constructed to reveal the influences of population migration and almost periodicity for infection rate on the spread of HBV in this paper. Compared with the HBV transmission model presented by Kamyad etc [15], we have taken into account the population travel between n patches and almost periodic infection rate.

Firstly, the qualitative behaviour of autonomous model (3.1) associated with model system (2.1) has been carried out. The basic reproduction number has been determined and sufficient conditions guaranteeing the global stability for disease-free equilibrium have been derived by combining the stability theory of asymptotically autonomous systems with basic comparison theorem of differential equations. Furthermore, conditions under which system admits unique and locally asymptotically stable endemic equilibrium have been obtained, respectively. Secondly, we have studied the existence and global attractivity for almost periodic solution of system in nonautonomous case. Moreover, we have deduced that the almost periodicity of time evolution for all the populations is ensured when model parameters satisfy the conditions of Theorem 4.2. Finally, to illustrate the analytical findings, numerical simulations of the model with two patches has been done in cases of autonomous and nonautonomous system.

There yet have many challenging and interesting issues remain to be investigated in future work. From Example 5.5, we find the existence of almost periodic solution are ensured though condition (4.1) of Theorem 4.1 is dissatisfied. Nevertheless, we are unable to prove it at present. In addition, it is known that there may exist time-lag when susceptible individual to be immune after vaccination and to be infected after contacting with HBV carriers. We leave these issues for future research.

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