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EXISTENCE OF POSITIVE SOLUTIONS TO A BOUNDARY VALUE PROBLEM FOR A DELAYED SINGULAR HIGH ORDER FRACTIONAL DIFFERENTIAL EQUATION WITH SIGN-CHANGING NONLINEARITY*

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Abstract In this paper, we discuss the existence of positive solutions to the boundary value problem for a high order fractional differential equation with delay and singularities including changing sign nonlinearity. By using the properties of the Green function, Guo-krasnosel'skii fixed point theorem, Leray-Schauder's nonlinear alternative theorem, some existence results of positive solutions are obtained, respectively.

Keywords Positive solution, fractional differential equation, delay, cone expansion and cone compression fixed point theorem, changing sign nonlinearity.

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1. Introduction

In this paper, we investigate the existence result of positive solutions for the following high order fractional differential equation with delay and singularities including changing sign nonlinearity:

$$\begin{cases} D_{0^+}^{\alpha} x(t) + f(t, x(t-\tau)) = 0, & t \in (0,1)\backslash\{\tau\}, \\ x(t) = \eta(t), & t \in [-\tau, 0], \\ x'(0) = x''(0) = \cdots = x^{(n-2)}(0) = 0, \ n \geq 3, \\ x^{(n-2)}(1) = 0, \end{cases} \tag{1.1}$$

where $n-1 < \alpha \leq n$, $n = [\alpha]+1$, $D_{0^+}^{\alpha}$ is the standard Riemann-Liouville fractional derivative, $\tau \in (0,1)$, $f(t,x) \in C((0,1) \times R^+, R)$, $f(t,x)$ may change sign and be singular at $t = 0$, $t = 1$ in Theorem 3.1, and be singular at $t = 0$, $t = 1$ and $x = 0$ in Theorem 3.2, and $f(t,x)$ may have negative values, where $R^+ = (0, +\infty)$. $\eta(t) \in C[-\tau, 0]$, and $\eta(t) > 0$ for $t \in [-\tau, 0)$, $\eta(t) = 0$ for $t = 0$. By the Guo-krasnosel'skii fixed point theorem and the Leray-Schauder's nonlinear alternative theorem, we can obtain the existence of the positive solutions.

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When $2 < \alpha \leq 3$, problem (1.1) is reduced to the BVP of fractional differential equation under the special conditions and has been studied by Mu et al. [7]. To the best of our knowledge, very few people have studied the existence of positive solutions for singular boundary value problem (1.1). Key tools used in this paper are the properties of the given Green function, Guo-krasnosel'skii fixed point theorem, and Leray-Schauder's nonlinear alternative theorem, therefore this paper is the extension and supplement of documents [7,9].

Recently, more and more fractional differential equations with all kinds of boundary value conditions have been valued by many people in diverse fields such as science and education. This is because we can use such mathematical models accurately to solve many complex problems in a wide variety of the fields such as chemistry, system physics, economics, aerodynamics, mechanics, polymer rheology, electrodynamic, engineering, and so forth, for the details, see [2,8,16,17]. And it's more difficult to research the fractional differential equations with changing sign nonlinearities and changing sign solutions, the relevant knowledge can be obtained in the references [10,13,14]. Then, the outcomes about singularity problems of fractional differential equations were studied in [1,5,12,15]. From the literature, recent years, there have been more and more papers dealing with the boundary value problems of fractional differential equations with delay, see [3,6,7,9] and the references therein.

In [5], He *et al.* discussed the existence of positive solutions for a high order fractional differential equation with integral boundary condition and changing sign nonlinearity:

$$\begin{cases} D_{0+}^{\alpha}u(t) + \lambda f(t, u(t)) = 0, \ t \in (0, 1), \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ D_{0+}^{\beta}u(1) = \displaystyle\int_0^1 D_{0+}^{\beta}u(t)dA(t), \end{cases}$$

where D_{0+}^{α} is the standard Riemann-Liouville fractional derivative, $n-1 < \alpha \leq n$, $n \geq 3$, $0 < \beta \leq 1$, $\lambda > 0$, and $\int_0^1 D_{0+}^{\beta}u(t)dA(t)$ denotes the Riemann-Stieltjes integral with respect to A, in which $A(t)$ is a monotone increasing function and $f : [0, 1] \times R^+ \to R$ may change sign, $R^+ = (0, +\infty)$. By the Guo-krasnosel'skii fixed point theorem, some positive solutions can be acquired.

In [15], Zhang *et al.* considered the following fractional differential equation

$$\begin{cases} D_{0+}^{\alpha}u(t) + f(t, u(t)) = 0, \ 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \ D_{0+}^{\beta}u(1) = \lambda \displaystyle\int_0^{\eta} h(t)D_{0+}^{\beta}u(t)dt, \end{cases}$$

where D_{0+}^{α} is the standard Riemann-Liouville fractional derivative, $h \in L^1[0, 1]$ is nonnegative and may be singular at $t = 0$ and $t = 1$, $n-1 < \alpha \leq n$, $n \geq 3$, $\beta \geq 1$, $\alpha - \beta - 1 > 0$, $0 < \eta \leq 1$, $0 \leq \lambda \int_0^{\eta} h(t)t^{\alpha-\beta-1}dt < 1$. The nonlinearity $f(t, u)$ permits singularities both on $t = 0, \ 1$ and $u = 0$. By using the Guo-krasnosel'skii fixed theorem, at least three positive solutions can be obtained.

In [9], the author Su studied the following boundary value problem for a singular fractional differential equation with delay:

$$\begin{cases} D^{\alpha}x(t) + f(t, x(t-\tau)) = 0, & t \in (0, 1)\backslash\{\tau\}, \\ x(t) = \eta(t), & t \in [-\tau, 0], \\ x(1) = 0, \end{cases}$$

where $1 < \alpha \leq 2$, D^α is a Riemann-Liouville fractional derivative, $\tau \in (0,1)$, $f(t,x) \in C((0,1) \times R^+, R)$ and may be singular at $t = 0,\ t = 1$, and $x = 0$ and may have negative values, where $R^+ = (0,+\infty)$. By the Guo-krasnosel'skii fixed point theorem, the authors got the existence of positive solutions.

In [7], Mu *et al*. investigated the existence of positive solutions for the singular fractional differential equation with delay:

$$\begin{cases} D^\alpha x(t) + \lambda f(t, x(t-\tau)) = 0, & t \in (0,1)\backslash\{\tau\}, \\ x(t) = \eta(t), & t \in [-\tau, 0], \\ x'(1) = x'(0) = 0, \end{cases}$$

where $2 < \alpha \leq 3$, D^α is Riemann-Liouville fractional derivative, λ is a positive parameter, $f(t,x) \in C((0,1) \times R^+, R)$ and may be singular at $t = 0,\ t = 1$, and $x = 0$. By the Guo-krasnosel'skii fixed point theorem, the eigenvalue intervals of the boundary value problem to this nonlinear fractional differential equation were considered, and some positive solutions were obtained, respectively.

To prove our conclusions, we will put forward some necessary definitions and lemmas in Section 2, and give some new properties of the corresponding Green function. In Section 3, by using the Guo-krasnosel'skii fixed point theorems and the Leray-Schauder's nonlinear alternative theorem, the existence of positive solutions to BVP (1.1) will be established finally.

2. Preliminaries and correlative lemmas

In order to expound the main idea of this thesis easily, we first give some essential definitions and lemmas that are significant and used through out this paper. The definitions can also be found in some references such as [1,3,5,6,9].

Definition 2.1. The Riemann-Liouville fractional integral of order α ($\alpha > 0$) of a function f: $(0,+\infty) \to R$ is given by:

$$I_{0^+}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

provided that the right-hand side is pointwise defined on $(0,+\infty)$, where $\Gamma(\alpha)$ is the Gamma function defined by:

$$\Gamma(\alpha) = \int_0^{+\infty} e^{-t} t^{\alpha-1} dt,\ \alpha > 0.$$

Definition 2.2. The Riemann-Liouville fractional derivative of order α ($n - 1 < \alpha < n$) of a function f: $(0,+\infty) \to R$ is given by:

$$D_{0^+}^\alpha f(t) = D^n I_{0^+}^{n-\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s) ds,$$

provided that the right-hand side is pointwise defined on $(0,+\infty)$, where n is the smallest integer than or equal to α and $\Gamma(\cdot)$ is as same as the Gamma function mentioned in the definition above.

Using the definition of the Riemann-Liouville derivative, we can get the following content.

Let $\alpha > 0$. If we assume $u \in C(0,1) \cap L(0,1)$, then the fractional differential equation

$$D_{0^+}^{\alpha} u(t) = 0$$

has the unique solution $u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}$, $c_i \in R,\ i = 1, 2, \cdots, n$.

Assume $f \in C(0,1) \cap L(0,1)$ has the fractional derivative of order $\alpha > 0$. Then

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} f(t) = f(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}$$

for some $c_i \in R,\ i = 1, 2, \cdots, n$.

Next we induce the Green function to solve the boundary value problem of fractional differential equation.

Lemma 2.1. *Let $n - 1 < \alpha \leq n$, and $h \in L^1[0,1]$. The unique solution of the boundary value problem*

$$\begin{cases} D_{0^+}^{\alpha} u(t) + h(t) = 0,\ t \in [0,1], \\ u(0) = u'(0) = u''(0) = \cdots = u^{(n-2)}(0) = u^{(n-2)}(1) = 0 \end{cases} \tag{2.1}$$

is given by

$$u(t) = \int_0^1 G(t,s) h(s) ds, \quad t \in [0,1],$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-n+1} - (t-s)^{\alpha-1}, 0 \leq s \leq t \leq 1, \\ t^{\alpha-1}(1-s)^{\alpha-n+1}, \qquad 0 \leq t \leq s \leq 1. \end{cases} \tag{2.2}$$

Proof. By Definition 2.1, 2.2, we get that

$$\begin{aligned} u(t) &= -I_{0^+}^{\alpha} h(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n} \\ &= -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, \end{aligned}$$

where $c_i \in R,\ i = 1, 2, \cdots, n$.

From the boundary condition

$$u(0) = u'(0) = u''(0) = \cdots = u^{(n-2)}(0),$$

we get

$$c_n = c_{n-1} = c_{n-2} = \cdots = c_2 = 0.$$

Thus,

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds + c_1 t^{\alpha-1}.$$

From the boundary condition $u^{(n-2)}(1) = 0$, and

$$D_{0^+}^{(n-2)} u(t) = -\frac{1}{\Gamma(\alpha-n+2)} \int_0^t (t-s)^{\alpha-n+1} h(s) ds + c_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha-n+2)} t^{\alpha-n+1},$$

we can obtain that

$$u^{(n-2)}(1) = -\frac{1}{\Gamma(\alpha-n+2)}\int_0^1 (1-s)^{\alpha-n+1}h(s)ds + c_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-n+2)} = 0,$$

therefore,

$$\begin{aligned} c_1 &= \frac{1}{\Gamma(\alpha-n+2)}\cdot\int_0^1 (1-s)^{\alpha-n+1}h(s)ds\cdot\frac{\Gamma(\alpha-n+2)}{\Gamma(\alpha)} \\ &= \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-n+1}h(s)ds, \end{aligned}$$

finally,

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds \\ &\quad + \frac{t^{\alpha-1}}{\Gamma(\alpha)}(\int_0^t (1-s)^{\alpha-n+1}h(s)ds + \int_t^1 (1-s)^{\alpha-n+1}h(s)ds) \\ &= \frac{1}{\Gamma(\alpha)}(\int_0^t [t^{\alpha-1}(1-s)^{\alpha-n+1} - (t-s)^{\alpha-1}]h(s)ds \\ &\quad + \int_t^1 t^{\alpha-1}(1-s)^{\alpha-n+1}h(s)ds) \\ &= \int_0^1 G(t,s)h(s)ds,\ t\in[0,1]. \end{aligned}$$

□

The following properties of the Green function play important roles in the whole thesis.

Lemma 2.2. *The Green function $G(t,s)$ given by* (2.2) *has the following properties:*

(1) $0 \le G(t,s) \le \frac{1}{\Gamma(\alpha)}(\alpha-1)s(1-s)^{\alpha-n+1},\ \ \forall\ t,s\in[0,1]$;

(2) $\frac{1}{\Gamma(\alpha)}t^{\alpha-1}s(1-s)^{\alpha-n+1} \le G(t,s) \le \frac{1}{\Gamma(\alpha)}(\alpha-1)t^{\alpha-1}(1-s)^{\alpha-n+1},\ \ \forall\ t,s\in[0,1]$.

Proof. (1) By the definition of the $G(t,s)$, we can get that $G(t,s)\ge 0$ easily.

For $0\le s\le t\le 1$, noticing that $\alpha-n+1>0$, we have

$$\begin{aligned} G(t,s) &= \frac{1}{\Gamma(\alpha)}(t^{\alpha-1}(1-s)^{\alpha-n+1} - (t-s)^{\alpha-1}) \\ &= \frac{1}{\Gamma(\alpha)}(1-s)^{2-n}[(t(1-s))^{\alpha-1} - (1-s)^{n-2}(t-s)^{\alpha-1}] \\ &\le \frac{1}{\Gamma(\alpha)}(1-s)^{2-n}[(t(1-s))^{\alpha-1} - (1-s)^{\alpha-1}(t-s)^{\alpha-1}] \\ &= \frac{1}{\Gamma(\alpha)}(1-s)^{2-n}(\alpha-1)\int_{(1-s)(t-s)}^{t(1-s)} x^{\alpha-2}dx \\ &\le \frac{1}{\Gamma(\alpha)}(1-s)^{2-n}(\alpha-1)t^{\alpha-2}(1-s)^{\alpha-2}[t(1-s)-(1-s)(t-s)] \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(\alpha)}(\alpha-1)(1-s)^{2-n}t^{\alpha-2}(1-s)^{\alpha-2}s(1-s) \\
&\leq \frac{1}{\Gamma(\alpha)}(\alpha-1)s(1-s)^{\alpha-n+1}.
\end{aligned}
$$

For $0 \leq t \leq s \leq 1$, noticing that $\alpha > 2$, we have

$$
\begin{aligned}
G(t,s) &= \frac{1}{\Gamma(\alpha)}t^{\alpha-1}(1-s)^{\alpha-n+1} \\
&\leq \frac{1}{\Gamma(\alpha)}s^{\alpha-1}(1-s)^{\alpha-n+1} \\
&\leq \frac{1}{\Gamma(\alpha)}(\alpha-1)s(1-s)^{\alpha-n+1}.
\end{aligned}
$$

The prove of (1) is completed.

(2) On the one hand, we proof the relationship "$\geq$" holds, $\forall\ t, s \in [0,1]$.

For $0 \leq s \leq t \leq 1$, noticing that $n \geq 3$, we have $(1-s)^{n-2} \leq (1-s)$, therefore,

$$
\begin{aligned}
G(t,s) &= \frac{1}{\Gamma(\alpha)}[t^{\alpha-1}(1-s)^{\alpha-n+1} - (t-s)^{\alpha-1}] \\
&= \frac{1}{\Gamma(\alpha)}[t^{\alpha-1}(1-s)^{\alpha-n+1} - t^{\alpha-1}(1-\frac{s}{t})^{\alpha-1}] \\
&\geq \frac{1}{\Gamma(\alpha)}t^{\alpha-1}[(1-s)^{\alpha-n+1} - (1-s)^{\alpha-(n-2)-1+(n-2)}] \\
&= \frac{1}{\Gamma(\alpha)}t^{\alpha-1}(1-s)^{\alpha-n+1}[1-(1-s)^{n-2}] \\
&\geq \frac{1}{\Gamma(\alpha)}t^{\alpha-1}(1-s)^{\alpha-n+1}[1-(1-s)] \\
&= \frac{1}{\Gamma(\alpha)}t^{\alpha-1}s(1-s)^{\alpha-n+1}.
\end{aligned}
$$

For $0 \leq t \leq s \leq 1$, we have

$$
G(t,s) = \frac{1}{\Gamma(\alpha)}t^{\alpha-1}(1-s)^{\alpha-n+1} \geq \frac{1}{\Gamma(\alpha)}t^{\alpha-1}s(1-s)^{\alpha-n+1}.
$$

On the other hand, we prove the relationship "$\leq$" holds, $\forall\ t, s \in [0,1]$, by (1), for $0 \leq s \leq t \leq 1$, we have

$$
\begin{aligned}
G(t,s) &= \frac{1}{\Gamma(\alpha)}(t^{\alpha-1}(1-s)^{\alpha-n+1} - (t-s)^{\alpha-1}) \\
&\leq \frac{1}{\Gamma(\alpha)}(\alpha-1)(1-s)^{2-n}t^{\alpha-2}(1-s)^{\alpha-2}s(1-s) \\
&\leq \frac{1}{\Gamma(\alpha)}(\alpha-1)(1-s)^{2-n}t^{\alpha-2}(1-s)^{\alpha-2}t(1-s) \\
&= \frac{1}{\Gamma(\alpha)}(\alpha-1)t^{\alpha-1}(1-s)^{\alpha-n+1}.
\end{aligned}
$$

For $0 \leq t \leq s \leq 1$, we have

$$
G(t,s) = \frac{1}{\Gamma(\alpha)}t^{\alpha-1}(1-s)^{\alpha-n+1} \leq \frac{1}{\Gamma(\alpha)}(\alpha-1)t^{\alpha-1}(1-s)^{\alpha-n+1}.
$$

The proof of (2) is completed. $\square$

Remark 2.1. The function $G^*(t,s) = t^{2n-1-\alpha}G(t,s)$ satisfies the following condition:

$$\frac{1}{\Gamma(\alpha)}t^{2n-2}s(1-s)^{\alpha-n+1} \le G^*(t,s) \le \frac{\alpha-1}{\Gamma(\alpha)}t^{2n-2}(1-s)^{\alpha-n+1} \text{ for } t,s \in [0,1].$$

Lemma 2.3 ([4])**.** *Let E be a Banach space, and $P \subset E$ be a cone. Assume Ω is a bounded open set in E such that $\theta \in \Omega$. Let operator $T : P \cap \overline{\Omega} \to P$ be completely continuous. Then the following two conclusions are established:*

(1) If $\|Tu\| < \|u\|,\ \forall\ u \in P \cap \partial\Omega$, then $i(T, P \cap \Omega, P) = 1$;

(2) If $\|Tu\| > \|u\|,\ \forall\ u \in P \cap \partial\Omega$, then $i(T, P \cap \Omega, P) = 0$.

Lemma 2.4 (Leray-Schauder's nonlinear alternative theorem)**.** *Let $F : E \to E$ be a completely continuous operator. Let*

$$\sigma(F) = \{x \in E : x = \kappa F(x),\ 0 < \kappa < 1\},$$

then either the set $\sigma(F)$ is unbounded, or F has least one fixed point.

3. Main results

In the section 3, we will try our best to discuss the existence of positive solutions for boundary value problem (1.1). For convenience, we list some preconditions which are significant in this paper.

Throughout this paper, we always suppose that the following condition holds:

(H_1) There exists a nonnegative function $\rho \in C(0,1) \cap L(0,1),\ \rho(t) \not\equiv 0$, such that

$$f(t,x) > -\rho(t),$$

and

$$\varphi_2(t)h_2(x) \le f(t, v(t)x) + \rho(t) \le \varphi_1(t)(g(x) + h_1(x)),$$

for $\forall\ (t,x) \in (0,1) \times R^+$, where $\varphi_1,\ \varphi_2 \in L(0,1)$ are positive, $h_1,\ h_2 \in C(R_0^+, R^+)$ are nondecreasing, $g \in C(R_0^+, R^+)$ is nonincreasing, $R_0^+ = [0, +\infty)$, and

$$v(t) = \begin{cases} 1, & t \in (0,\tau], \\ (t-\tau)^{\alpha-2n+1}, & t \in (\tau,1). \end{cases}$$

When $s \in [0,\tau]$, we have $-\tau \le s-\tau \le 0$, suppose there is a positive number $S > 0$, such taht $\max\limits_{-\tau \le s-\tau \le 0} \eta(s-\tau) = S$, therefore $\eta(s-\tau) \le S$ and $0 < g(S) \le g(\eta(s-\tau)) \le g(0)$.

Let $X = \{x | x \in C[-\tau, 1]\}$, then $(X, \|\cdot\|)$ is a Banach space with the maximum norm

$$\|x\|_{[-\tau,1]} = \max\nolimits_{-\tau \le t \le 1}|x(t)| \text{ for } x \in X.$$

And we set a cone

$$K = \{x \in X | x(t) = 0 \text{ for } t \in [-\tau, 0], \text{ and } x(t) \ge \frac{1}{\alpha-1}t^{2n-2}\|x\| \text{ for } t \in [0,1]\}.$$

Define

$$\overline{\eta}(t) = \begin{cases} \eta(t), & t \in [-\tau, 0], \\ 0, & t \in (0,1], \end{cases}$$

$$\omega(t) = \begin{cases} 0, & t \in [-\tau, 0], \\ \displaystyle\int_0^1 G(t,s)\rho(s)ds, & t \in (0,1], \end{cases}$$

and nonnegative function

$$\begin{aligned} [x(t) + \overline{\eta}(t) - \omega(t)]^+ &= x^*(t) \\ &= \max\{x(t) + \overline{\eta}(t) - \omega(t), 0\} \\ &= \begin{cases} \eta(t), & t \in [-\tau, 0], \\ \max\{x(t) - \omega(t), 0\}, & t \in (0,1] \end{cases} \end{aligned}$$

for any $x \in K$. And we let $f^*(t, x(t)) = f(t, x^*(t)) + \rho(t)$.

Define $\omega|_{[0,1]}$ is the solution of

$$\begin{cases} D^\alpha x(t) + \rho(t) = 0, & t \in (0,1), \\ x^{(n-2)}(1) = x^{(n-2)}(0) = \cdots = x''(0) = x'(0) = x(0) = 0. \end{cases}$$

As $f : (0,1) \times R^+ \to R$ is a continuous function, we can know that function x is a solution of boundary value problem (1.1) if and only if it satisfies

$$x(t) = \begin{cases} \displaystyle\int_0^1 G(t,s) f(s, x(s-\tau))ds, & t \in (0,1), \\ \eta(t), & t \in [-\tau, 0]. \end{cases}$$

Considering the following operator:

$$(Ax)(t) = \begin{cases} \displaystyle\int_0^1 G(t,s)(f(s, x^*(s-\tau)) + \rho(s))ds, & t \in (0,1], \\ 0, & t \in [-\tau, 0]. \end{cases} \tag{3.1}$$

Let

$$y(t) = \begin{cases} t^{2n-1-\alpha} x(t), & t \in (0,1), \\ 0, & t \in [-\tau, 0], \end{cases}$$

and

$$y^*(t) = \begin{cases} \max\{t^{\alpha-2n+1} y(t) - \omega(t), 0\}, & t \in (0,1], \\ \eta(t), & t \in [-\tau, 0]. \end{cases}$$

Next (3.1) is equivalent to

$$(Ty)(t) = \begin{cases} \displaystyle\int_0^1 G^*(t,s)(f(s, y^*(s-\tau)) + \rho(s))ds, & t \in (0,1], \\ 0, & t \in [-\tau, 0]. \end{cases} \tag{3.2}$$

Clearly, if $\widetilde{y}$ is a fixed point of operator T in (3.2), then

$$\widetilde{x}(t) = \begin{cases} t^{\alpha-2n+1}\widetilde{y}(t), & t \in (0,1], \\ 0, & t \in [-\tau, 0] \end{cases}$$

is a fixed point of operator A defined in (3.1). By Lemma 2.1 we can obtain that

$$\begin{cases} D^{\alpha}\widetilde{x}(t) + (f(t, \widetilde{x}^*(t-\tau)) + \rho(t)) = 0, \ t \in (0,1)\backslash\{\tau\}, \\ \widetilde{x}(t) = 0, \qquad t \in [-\tau, 0], \\ \widetilde{x}^{(n-2)}(1) = \widetilde{x}^{(n-2)}(0) = \cdots = \widetilde{x}''(0) = \widetilde{x}'(0) = 0. \end{cases} \tag{3.3}$$

If

$$\widetilde{x}(t-\tau) + \overline{\eta}(t-\tau) - \omega(t-\tau) \geq 0 \ for \ t \in [0,1], \tag{3.4}$$

then

$$\widetilde{x}^*(t-\tau) = \widetilde{x}(t-\tau) + \overline{\eta}(t-\tau) - \omega(t-\tau).$$

Let

$$x(t) = \widetilde{x}(t) + \overline{\eta}(t) - \omega(t). \tag{3.5}$$

Then some conclusion will be verified below.

Lemma 3.1. *x is a positive solution of boundary value problem* (1.1) *if and only if $\widetilde{x}(t) = x(t) + \omega(t) - \overline{\eta}(t)$ is a positive solution of boundary value problem* (3.3) *and inequality $\widetilde{x}(t) + \overline{\eta}(t) - \omega(t) \geq 0$ holds up when $t \in (0,1)\backslash\tau$.*

Proof. If x is a positive solution of boundary value problem (1.1), we shall prove it in two cases.

For $t \in [-\tau, 0]$,

$$\begin{aligned} \widetilde{x}(t) &= x(t) + \omega(t) - \overline{\eta}(t) \\ &= x(t) - \overline{\eta}(t) \\ &= \eta(t) - \eta(t) \\ &= 0, \end{aligned}$$

which implies that $\widetilde{x}(t) = 0$.

It is easy to show that $\widetilde{x}(t)$ satisfies the rest boundary conditions in (3.3) when $t \in [-\tau, 0]$.

For $t \in (0,1)\backslash\{\tau\}$,

$$\begin{aligned} & D_{0+}^{\alpha}(x(t) + \omega(t) - \overline{\eta}(t)) \\ &= D_{0+}^{\alpha}x(t) + D_{0+}^{\alpha}\omega(t) - D_{0+}^{\alpha}\overline{\eta}(t) \\ &= D_{0+}^{\alpha}x(t) + D_{0+}^{\alpha}\omega(t) \\ &= -f(t, x(t-\tau)) - \rho(t) \\ &= -(f(t, x(t-\tau)) + \rho(t)) \\ &= -(f(t, \widetilde{x}^*(t-\tau)) + \rho(t)), \end{aligned}$$

which implies that

$$D_{0+}^{\alpha}\widetilde{x}(t) = -(f(t, \widetilde{x}^*(t-\tau)) + \rho(t)).$$

Since $x(t)$ is a positive solution, then $\widetilde{x}(t)+\overline{\eta}(t)-\omega(t)\geq 0$ holds when $t\in(0,1)\backslash\tau$. It is easy to show that $\widetilde{x}(t)$ satisfies the boundary conditions in (3.3).

Therefore, $\widetilde{x}(t)$ is a positive solution of boundary value problem (3.3).

On the other hand, if $\widetilde{x}(t)=x(t)+\omega(t)-\overline{\eta}(t)$ is a positive solution of boundary value problem (3.3) and $\widetilde{x}(t)+\overline{\eta}(t)-\omega(t)\geq 0$ holds when $t\in(0,1)\backslash\tau$, as similar as the proof above, we can easily prove that $x(t)$ is a positive solution of boundary value problem (1.1). □

As a result, in the following paper we will concentrate our mind on finding the fixed points of operator T defined by (3.2).

In the following content, we give other three conditions:

(H_2)

$$\int_0^1 (1-s)^{\alpha-n+1}\varphi_1(s)ds>0.$$

(H_3) Let

$$\limsup_{y\to+\infty}\frac{h_1(y)}{y}\leq e,\ \ e>0, \tag{3.6}$$

such that e satisfies $\frac{\Gamma(\alpha)}{\alpha-1}>\int_\tau^1 e(1-s)^{\alpha-n+1}\varphi_1(s)ds$.

In view of (3.6), there exists a $M>0$ such that

$$h_1(y)\leq ey \text{ for } y>M. \tag{3.7}$$

(H_4) Suppose there exists a subinterval $[a,b]\subset(\tau,1)$, such that

$$\zeta_1=\min_{t\in[a,b]}\frac{(t-\tau)^{2n-2}}{\alpha-1}=\frac{(a-\tau)^{2n-2}}{\alpha-1},\quad \zeta_2=\min_{t\in[a,b]}t^{2n-2}=a^{2n-2}.$$

And there exists a $r_1\geq\max\{2,2c\}$, where

$$c=\frac{(\alpha-1)^2}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-n+1}\rho(s)ds<+\infty. \tag{3.8}$$

Let

$$\begin{aligned}\xi_1&=\tfrac{\zeta_2}{\Gamma(\alpha)}h_2(\tfrac{r_1\zeta_1}{2})\textstyle\int_a^b s(1-s)^{\alpha-n+1}\varphi_2(s)ds\\&>r_1.\end{aligned}$$

By the above conditions, we denote:

$$\begin{aligned}\xi_2&=\frac{\int_0^\tau(1-s)^{\alpha-n+1}\varphi_1(s)(g(0)+h_1(S))ds+\int_\tau^1(1-s)^{\alpha-n+1}\varphi_1(s)g(\frac{1}{\alpha-1}(s-\tau)^{2n-2})ds}{\frac{\Gamma(\alpha)}{\alpha-1}-\int_\tau^1 e(1-s)^{\alpha-n+1}\varphi_1(s)ds}\\&>0.\end{aligned}$$

Next we choose a $r_2>\max\{M+1,\ r_1+1,\ \xi_2\}$. Define

$$\Omega_1=\{y\in K:\|y\|<r_1\},\ \Omega_2=\{y\in K:\|y\|<r_2\}.$$

And when $s\in[\tau,1]$, because $0\leq\frac{1}{\alpha-1}\frac{r_1}{2}(s-\tau)^{2n-2}<\frac{1}{\alpha-1}\frac{r_1}{2}$, thus,

$$0<g(\frac{1}{\alpha-1}\frac{r_1}{2})<g(\frac{1}{\alpha-1}\frac{r_1}{2}(s-\tau)^{2n-2})\leq g(0).$$

Then $\forall\ y \in \overline{\Omega_2}\backslash\Omega_1$,

$$\begin{aligned} t^{2n-1-\alpha}\omega(t) &= t^{2n-1-\alpha}\int_0^1 G(t,s)\rho(s)ds \\ &\leq \frac{(\alpha-1)t^{2n-2}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-n+1}\rho(s)ds \\ &= \frac{1}{\alpha-1}t^{2n-2}c, \end{aligned} \tag{3.9}$$

where c is defined as (3.8). Thus, for $t \in (0,1)$,

$$\begin{aligned} y(t) - t^{2n-1-\alpha}\omega(t) &\geq \frac{1}{\alpha-1}t^{2n-2}r_1 - \frac{1}{\alpha-1}t^{2n-2}c \\ &\geq \frac{1}{\alpha-1}t^{2n-2}(r_1 - \frac{1}{2}r_1) \\ &= \frac{1}{\alpha-1}\frac{r_1}{2}t^{2n-2}. \end{aligned} \tag{3.10}$$

Then

$$\begin{aligned} (Ty)(t) &= \int_0^\tau G^*(t,s)(f(s,\eta(s-\tau)) + \rho(s))ds \\ &\quad + \int_\tau^1 G^*(t,s)(f(s,(s-\tau)^{\alpha-2n+1}y(s-\tau) - \omega(s-\tau)) + \rho(s))ds \\ &\leq \frac{(\alpha-1)t^{2n-2}}{\Gamma(\alpha)}\int_0^\tau (1-s)^{\alpha-n+1}\varphi_1(s)(g(\eta(s-\tau)) + h_1(\eta(s-\tau)))ds \\ &\quad + \frac{(\alpha-1)t^{2n-2}}{\Gamma(\alpha)}\int_\tau^1 (1-s)^{\alpha-n+1}\varphi_1(s) \\ &\quad \times (g(\frac{1}{(\alpha-1)}\frac{r_1}{2}(s-\tau)^{2n-2}) + h_1(y(s-\tau) - (s-\tau)^{2n-1-\alpha}\omega(s-\tau)))ds \\ &\leq \frac{\alpha-1}{\Gamma(\alpha)}\int_0^\tau (1-s)^{\alpha-n+1}\varphi_1(s)(g(0) + h_1(S))ds \\ &\quad + \frac{\alpha-1}{\Gamma(\alpha)}\int_\tau^1 (1-s)^{\alpha-n+1}\varphi_1(s)(g(\frac{1}{\alpha-1}\frac{r_1}{2}(s-\tau)^{2n-2}) + h_1(\|y\|))ds \\ &\leq \frac{\alpha-1}{\Gamma(\alpha)}\int_0^\tau (1-s)^{\alpha-n+1}\varphi_1(s)(g(0) + h_1(S))ds \\ &\quad + \frac{\alpha-1}{\Gamma(\alpha)}\int_\tau^1 (1-s)^{\alpha-n+1}\varphi_1(s)(g(0) + h_1(\|y\|))ds \\ &< +\infty. \end{aligned}$$

Therefore, T is well-defined on $\overline{\Omega_2}\backslash\Omega_1$.

Lemma 3.2. *Suppose* (H_1) *and* (H_2) *hold. Then the operator* $T:\ \overline{\Omega_2}\backslash\Omega_1 \to K$ *is completely continuous.*

Proof. Step 1: First we show that $T:\ \overline{\Omega_2}\backslash\Omega_1 \to K$. In fact, for $y \in \overline{\Omega_2}\backslash\Omega_1,\ t \in (0,1)$, in view of Remark 2.1, denote

$$B(\alpha) = \frac{1}{\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-n+1}(f(s,y^*(s-\tau)) + \rho(s))ds.$$

On the one hand,

$$(Ty)(t) \leq \frac{(\alpha-1)t^{2n-1-\alpha}}{\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-n+1}(f(s,y^*(s-\tau))+\rho(s))ds$$
$$=(\alpha-1)t^{2n-1-\alpha}B(\alpha),$$

therefore, $\|Ty\| = \max_{t\in[0,1]}|(Ty)(t)| \leq (\alpha-1)B(\alpha)$. On the other hand,

$$(Ty)(t) \geq \frac{t^{2n-2}}{\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-n+1}(f(s,y^*(s-\tau))+\rho(s))ds$$
$$=t^{2n-2}B(\alpha)$$
$$\geq \frac{1}{\alpha-1}t^{2n-2}\|Ty\|.$$

Hence, $T:\ \overline{\Omega_2}\backslash\Omega_1 \to K$.

Step 2: we prove that $T:\ \overline{\Omega_2}\backslash\Omega_1 \to K$ is a continuous operator.

For any $y_m,\ y\in\overline{\Omega_2}\backslash\Omega_1,\ m=1,2,\cdots$ with $\|y_m-y\|_{[-\tau,1]}\to 0$ as $m\to\infty$. We know that $r_1\leq\|y_m\|\leq r_2$, $r_1\leq\|y\|\leq r_2$ and $y(t)\geq\frac{1}{\alpha-1}t^{2n-2}\|y\|\geq\frac{1}{\alpha-1}t^{2n-2}r_1$, $y_m(t)\geq\frac{1}{\alpha-1}t^{2n-2}\|y_m\|\geq\frac{1}{\alpha-1}t^{2n-2}r_1$, for $t\in(0,1)$. Then, for $t\in(0,1)$, by (3.10), we get

$$y_m(t)-t^{2n-1-\alpha}\omega(t)\geq\frac{1}{\alpha-1}\frac{r_1}{2}t^{2n-2},$$

and

$$y(t)-t^{2n-1-\alpha}\omega(t)\geq\frac{1}{\alpha-1}\frac{r_1}{2}t^{2n-2}.$$

By (H_1), we have

$$\begin{aligned}
&f(s,(s-\tau)^{\alpha-2n+1}y_m(s-\tau)-\omega(s-\tau))+\rho(s)\\
=&f(s,(s-\tau)^{\alpha-2n+1}(y_m(s-\tau)-(s-\tau)^{2n-\alpha-1}\omega(s-\tau)))+\rho(s)\\
\leq&\varphi_1(s)(g(y_m(s-\tau)-(s-\tau)^{2n-\alpha-1}\omega(s-\tau))\\
&+h_1(y_m(s-\tau)-(s-\tau)^{2n-\alpha-1}\omega(s-\tau)))\\
\leq&\varphi_1(s)(g(\frac{1}{\alpha-1}(s-\tau)^{2n-2})+h_1(r_2)),
\end{aligned}$$

and similarly,

$$f(s,(-\tau)^{\alpha-2n+1}y(s-\tau)-\omega(s-\tau))+\rho(s)\leq\varphi_1(s)(g(\frac{1}{\alpha-1}(s-\tau)^{2n-2})+h_1(r_2)).$$

Because

$$\varphi_1(s)(g(0)+h_1(r_2))\in L^1(0,1),$$

and

$$\begin{aligned}
&|f(s,(s-\tau)^{\alpha-2n+1}y_m(s-\tau)-\omega(s-\tau))+\rho(s)\\
&-(f(s,(s-\tau)^{\alpha-2n+1}y(s-\tau)-\omega(s-\tau))+\rho(s))|\\
\leq&2\varphi_1(s)(g(\frac{1}{\alpha-1}(s-\tau)^{2n-2})+h_1(r_2))\\
\leq&2\varphi_1(s)(g(0)+h_1(r_2)).
\end{aligned}$$

By using the Lebesgue dominated convergence theorem, for $t \in [0,1]$, we have

$$\begin{aligned}
&|(Ty_m)(t) - (Ty)(t)|\\
=&|\int_\tau^1 G^*(t,s)[f(s,(s-\tau)^{\alpha-2n+1}y_m(s-\tau) - \omega(s-\tau)) + \rho(s)\\
&- (f(s,(s-\tau)^{\alpha-2n+1}y(s-\tau) - \omega(s-\tau)) + \rho(s))]ds|\\
\leq&\int_\tau^1 G^*(t,s)|f(s,(s-\tau)^{\alpha-2n+1}y_m(s-\tau) - \omega(s-\tau)) + \rho(s)\\
&- (f(s,(s-\tau)^{\alpha-2n+1}y(s-\tau) - \omega(s-\tau)) + \rho(s))|ds\\
\leq&\frac{2(\alpha-1)}{\Gamma(\alpha)}\int_\tau^1 (1-s)^{\alpha-n+1}\varphi_1(s)(g(0) + h_1(r_2))ds,
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{m\to+\infty}|\int_\tau^1 G^*(t,s)[f(s,(s-\tau)^{\alpha-2n+1}y_m(s-\tau) - \omega(s-\tau)) + \rho(s)\\
&- (f(s,(s-\tau)^{\alpha-2n+1}y(s-\tau) - \omega(s-\tau)) + \rho(s))]ds|\\
\leq&\int_\tau^1 G^*(t,s)\lim_{m\to+\infty}|f(s,(s-\tau)^{\alpha-2n+1}y_m(s-\tau) - \omega(s-\tau)) + \rho(s)\\
&- (f(s,(s-\tau)^{\alpha-2n+1}y(s-\tau) - \omega(s-\tau)) + \rho(s))|ds\\
=&0.
\end{aligned}$$

This implies that $\|Ty_m - Ty\|_{[-\tau,1]} \to 0$ as $m \to +\infty$. Hence T is continuous.

Step 3: T is a compact operator.

Let $\Omega \subset \overline{\Omega_2}\backslash\Omega_1$ be any nonempty bounded set.

(1) First we show that $T(\Omega)$ is uniformly bounded.

For any $y \in \Omega$, in view of (H_1), (H_2) and Remark 2.1, we show

$$\begin{aligned}
(Ty)(t) =& \int_0^\tau G^*(t,s)(f(s,\eta(s-\tau)) + \rho(s))ds\\
&+ \int_\tau^1 G^*(t,s)(f(s,(s-\tau)^{\alpha-2n+1}y(s-\tau) - \omega(s-\tau)) + \rho(s))ds\\
\leq&\frac{(\alpha-1)t^{2n-2}}{\Gamma(\alpha)}\int_0^\tau (1-s)^{\alpha-n+1}\varphi_1(s)(g(\eta(s-\tau)) + h_1(\eta(s-\tau)))ds\\
&+ \frac{(\alpha-1)t^{2n-2}}{\Gamma(\alpha)}\int_\tau^1 (1-s)^{\alpha-n+1}\varphi_1(s)\\
&\times (g(\frac{1}{\alpha-1}\frac{r_1}{2}(s-\tau)^{2n-2}) + h_1(y(s-\tau) - (s-\tau)^{2n-1-\alpha}\omega(s-\tau)))ds\\
\leq&\frac{\alpha-1}{\Gamma(\alpha)}\int_0^\tau (1-s)^{\alpha-n+1}\varphi_1(s)(g(\eta(s-\tau)) + h_1(\eta(s-\tau)))ds\\
&+ \frac{\alpha-1}{\Gamma(\alpha)}\int_\tau^1 (1-s)^{\alpha-n+1}\varphi_1(s)(g(\frac{1}{\alpha-1}\frac{r_1}{2}(s-\tau)^{2n-2}) + h_1(y(s-\tau)))ds\\
\leq&\frac{\alpha-1}{\Gamma(\alpha)}\int_0^\tau (1-s)^{\alpha-n+1}\varphi_1(s)(g(\eta(s-\tau)) + h_1(\eta(s-\tau)))ds\\
&+ \frac{\alpha-1}{\Gamma(\alpha)}\int_\tau^1 (1-s)^{\alpha-n+1}\varphi_1(s)(g(\frac{1}{\alpha-1}(s-\tau)^{2n-2}) + h_1(r_2))ds
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{\alpha-1}{\Gamma(\alpha)}\int_0^{\tau}(1-s)^{\alpha-n+1}\varphi_1(s)(g(0)+h_1(S))ds\\
&\quad+\frac{\alpha-1}{\Gamma(\alpha)}\int_{\tau}^{1}(1-s)^{\alpha-n+1}\varphi_1(s)(g(0)+h_1(r_2))ds\\
&<+\infty.
\end{aligned}$$

Denote

$$\begin{aligned}
J(\alpha)=&\frac{\alpha-1}{\Gamma(\alpha)}\int_0^{\tau}(1-s)^{\alpha-n+1}\varphi_1(s)(g(0)+h_1(S))ds\\
&+\frac{\alpha-1}{\Gamma(\alpha)}\int_{\tau}^{1}(1-s)^{\alpha-n+1}\varphi_1(s)(g(0)+h_1(r_2))ds.
\end{aligned}$$

By the proof above, we can get that $\|Ty\| \leq J(\alpha)$. Hence, $T(\Omega)$ is uniformly bounded.

(2) Next we prove that $T(\Omega)$ is equicontinuous.

Since G^* is uniformly continuous for $(t,s)\in[0,1]\times[0,1]$, for any $\epsilon>0$,there exists $\delta_0>0$, when $t_1,\ t_2,\ s\in[0,1]$ and $|t_1-t_2|<\delta_0$, we have

$$\begin{aligned}
|G^*(t_1,s)-G^*(t_2,s)|<&\epsilon\Big(\int_0^{\tau}\varphi_1(s)(g(0)+h_1(S))ds\\
&+\int_{\tau}^{1}\varphi_1(s)(g(\frac{1}{\alpha-1}(s-\tau)^{2n-2})+h_1(r_2))ds\Big)^{-1}.
\end{aligned}$$

Thus, for any $y\in\Omega$, we get

$$\begin{aligned}
&|(Ty)(t_1)-(Ty)(t_2)|\\
\leq&\int_0^{\tau}|G^*(t_1,s)-G^*(t_2,s)|\varphi_1(s)(g(0)+h_1(S))ds\\
&+\int_{\tau}^{1}|G^*(t_1,s)-G^*(t_2,s)|\varphi_1(s)(g(\frac{1}{\alpha-1}(s-\tau)^{2n-2})+h_1(r_2))ds\\
<&\epsilon.
\end{aligned}$$

Thus $T(\Omega)$ is equicontinuous.

According to (1), (2) above, and by the Ascoli-Arzelà theorem, $T(\Omega)$ is a sequentially compact set. Thus, T is a compact operator.

Let us sum up all of the proof above, and then T is completely continuous. □

Theorem 3.1. *Let* (H_1), (H_2), (H_3) *and* (H_4) *hold, then the boundary value problem* (1.1) *at least has one positive solution.*

Proof. On the one hand, for $y\in\partial\Omega_2$, as similar as (3.10), for $t\in(0,1)$, we obtain

$$y(t)-t^{2n-1-\alpha}\omega(t)\geq\frac{1}{\alpha-1}t^{2n-2}(r_2-c)\geq\frac{1}{\alpha-1}\frac{r_2}{2}t^{2n-2}. \tag{3.11}$$

Then from (H_1), (H_3), (3.7), (3.11), and Remark 2.1, we get

$$
\begin{aligned}
&(Ty)(t)\\
\le&\frac{(\alpha-1)t^{2n-2}}{\Gamma(\alpha)}\int_0^\tau(1-s)^{\alpha-n+1}\varphi_1(s)(g(\eta(s-\tau))+h_1(\eta(s-\tau)))ds\\
&+\frac{(\alpha-1)t^{2n-2}}{\Gamma(\alpha)}\int_\tau^1(1-s)^{\alpha-n+1}\varphi_1(s)\\
&\times(g(\frac{1}{\alpha-1}\frac{r_2}{2}(s-\tau)^{\alpha-1})+h_1(y(s-\tau)-(s-\tau)^{2n-1-\alpha}\omega(s-\tau)))ds\\
\le&\frac{(\alpha-1)t^{2n-2}}{\Gamma(\alpha)}\int_0^\tau(1-s)^{\alpha-n+1}\varphi_1(s)(g(\eta(s-\tau))+h_1(\eta(s-\tau)))ds\\
&+\frac{(\alpha-1)t^{2n-2}}{\Gamma(\alpha)}\int_\tau^1(1-s)^{\alpha-n+1}\varphi_1(s)(g(\frac{1}{\alpha-1}\frac{r_2}{2}(s-\tau)^{2n-2})+h_1(y(s-\tau)))ds\\
\le&\frac{\alpha-1}{\Gamma(\alpha)}\int_0^\tau(1-s)^{\alpha-n+1}\varphi_1(s)(g(\eta(s-\tau))+h_1(\eta(s-\tau)))ds\\
&+\frac{\alpha-1}{\Gamma(\alpha)}\int_\tau^1(1-s)^{\alpha-n+1}\varphi_1(s)(g(\frac{1}{\alpha-1}(s-\tau)^{2n-2})+h_1(r_2))ds\\
\le&\frac{\alpha-1}{\Gamma(\alpha)}\int_0^\tau(1-s)^{\alpha-n+1}\varphi_1(s)(g(0)+h_1(S))ds\\
&+\frac{\alpha-1}{\Gamma(\alpha)}\int_\tau^1(1-s)^{\alpha-n+1}\varphi_1(s)(g(\frac{1}{\alpha-1}(s-\tau)^{2n-2})+er_2)ds\\
<&r_2.
\end{aligned}
$$

Therefore, for $y\in K_1\cap\partial\Omega_2$, we have $\|Ty\|<\|y\|$, then $i(T,\Omega_2,K)=1$.

On the other hand, for $y\in\partial\Omega_1$, from (H_1), (H_4), (3.10), and Remark 2.1, we have

$$
\begin{aligned}
\|Ty\|&\ge\int_a^b\min_{t\in[a,b]}G^*(t,s)(f(s,(s-\tau)^{\alpha-2n+1}y(s-\tau)-\omega(s-\tau))+\rho(s))ds\\
&\ge\int_a^b\min_{t\in[a,b]}G^*(t,s)\varphi_2(s)h_2(\frac{1}{\alpha-1}\frac{r_1}{2}(s-\tau)^{2n-2})ds\\
&\ge\frac{\zeta_2}{\Gamma(\alpha)}h_2(\frac{r_1\zeta_1}{2})\int_a^b s(1-s)^{\alpha-n+1}\varphi_2(s)ds\\
&=\frac{a^{2n-2}}{\Gamma(\alpha)}h_2(\frac{r_1(a-\tau)^{2n-2}}{2})\int_a^b s(1-s)^{\alpha-n+1}\varphi_2(s)ds\\
&=\xi_1\\
&>r_1.
\end{aligned}
$$

Therefore, for $y\in K_1\cap\partial\Omega_1$, we have $\|Ty\|>\|y\|$, then, $i(T,\Omega_1,K)=0$.

Thus, $i(T,\Omega_2\backslash\overline{\Omega_1},K)=i(T,\Omega_2,K)-i(T,\Omega_1,K)=1$. Then, T defined by (3.2) has a fixed point $\widetilde{y}\in\Omega_2\backslash\overline{\Omega_1}$.

In view of (3.10), we have

$$
\begin{aligned}
\widetilde{x}(t)-\omega(t)&=t^{\alpha-2n+1}\widetilde{y}(t)-\omega(t)\\
&=t^{\alpha-2n+1}(\widetilde{y}(t)-t^{2n-1-\alpha}\omega(t))
\end{aligned}
$$

$$
\begin{aligned}
&> t^{\alpha-2n+1}\frac{1}{\alpha-1}\frac{r_1}{2}t^{2n-2}\\
&= \frac{1}{\alpha-1}\frac{r_1}{2}t^{\alpha-1}\\
&> 0.
\end{aligned}
$$

It is easy to know that (3.4) is satisfied. Therefore, $x(t)=\widetilde{x}(t)-\omega(t)$ is a positive solution of the boundary value problem (1.1). The proof is completed. □

We choose a number $n_0\in\{1,2,\cdots\}$, let $N_0=\{n_0,n_0+1,\cdots\}$. Fixing $n\in N_0$ and considering the family of integral equation

$$
y(t)=\begin{cases}\kappa\displaystyle\int_0^1 G^*(t,s)(f_n(s,y^*(s-\tau))+\rho(s))ds+\frac{1}{n},\ t\in(0,1),\\ \dfrac{1}{n},\quad t\in[-\tau,0],\end{cases} \tag{3.12}
$$

where $\kappa\in(0,1)$,

$$
f_n(t,y^*(t-\tau))+\rho(t)=\begin{cases}f(t,y^*(t-\tau))+\rho(t),\ y^*(t-\tau)\geq\dfrac{1}{n},\\ f(t,\dfrac{1}{n})+\rho(t),\quad y^*(t-\tau)<\dfrac{1}{n}.\end{cases}
$$

Next, we give another condition:

(H_5) Let

$$
\lim_{z\to+\infty}\sup\frac{h_1(z)}{z}<\frac{\Gamma(\alpha)}{(\alpha-1)\int_\tau^1(1-s)^{\alpha-n+1}\varphi_1(s)ds}.
$$

Theorem 3.2. *Let (H_1), (H_2), (H_5) hold. Then boundary value problem (1.1) at least has one positive solution.*

Proof. In view of (H_5), there exists a positive r satisfies

$$
\begin{aligned}
r>\frac{\alpha-1}{\Gamma(\alpha)}\Big(&\int_0^\tau(1-s)^{\alpha-n+1}\varphi_1(s)(g(0)+h_1(S))ds\\
&+\int_\tau^1(1-s)^{\alpha-n+1}\varphi_1(s)(g(\frac{1}{\alpha-1}(s-\tau)^{2n-2})+h_1(r))ds\Big),
\end{aligned}
$$

and meanwhile, we let $r>\max\{2,2c\}$, c defined as (3.8).

So, we can use $n_0\in\{1,2,\cdots\}$ such that

$$
\begin{aligned}
r>\frac{\alpha-1}{\Gamma(\alpha)}\Big(&\int_0^\tau(1-s)^{\alpha-n+1}\varphi_1(s)(g(0)+h_1(S))ds\\
&+\int_\tau^1(1-s)^{\alpha-n+1}\varphi_1(s)(g(\frac{1}{\alpha-1}(s-\tau)^{2n-2})+h_1(r))ds\Big)+\frac{1}{n_0}.
\end{aligned}
$$

We claim that any solution y of (3.12) for any $\kappa\in(0,1)$ must satisfy $\|y\|\neq r$. Otherwise, supposing that y is a solution of (3.12) for some $\kappa\in(0,1)$ such that $\|y\|=r$. We just prove the situation that $y^*(t-\tau)\geq\frac{1}{n}$ for $t\in(0,1)$. In view of Lemma 2.2 (1), we can get

$$
y(t)\leq\frac{\kappa(\alpha-1)t^{2n-1-\alpha}}{\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-n+1}(f_n(s,y^*(s-\tau))+\rho(s))ds+\frac{1}{n},
$$

therefore,

$$\|y\| \le \frac{\kappa(\alpha-1)}{\Gamma(\alpha)} \int_0^1 s(1-s)^{\alpha-n+1}(f_n(s, y^*(s-\tau)) + \rho(s))ds + \frac{1}{n}. \tag{3.13}$$

Thus, by Remark 2.1, for $t \in (0,1)$, we have

$$\begin{aligned} y(t) &\ge \frac{1}{n} + \frac{\kappa t^{2n-2}}{\Gamma(\alpha)} \int_0^1 s(1-s)^{\alpha-n+1}(f_n(s, y^*(s-\tau)) + \rho(s))ds \\ &\ge \frac{1}{n} + \frac{1}{\alpha-1} t^{2n-2}(\|y\| - \frac{1}{n}) \\ &= \frac{1}{n} + \frac{1}{\alpha-1} t^{2n-2}\|y\| - \frac{1}{\alpha-1} t^{2n-2} \cdot \frac{1}{n} \\ &= (1 - \frac{1}{\alpha-1} t^{2n-2}) \cdot \frac{1}{n} + \frac{1}{\alpha-1} t^{2n-2}\|y\| \\ &\ge \frac{1}{\alpha-1} t^{2n-2}\|y\| \\ &= \frac{1}{\alpha-1} t^{2n-2} r. \end{aligned}$$

Then like for (3.10), for $t \in (0,1)$, we can get

$$y(t) - t^{2n-1-\alpha}\omega(t) \ge \frac{1}{\alpha-1} t^{2n-2}(r-c) \ge \frac{1}{\alpha-1} \frac{r}{2} t^{2n-2}. \tag{3.14}$$

Then from (H_1), (3.14), for $t \in [0,1]$, $\kappa \in (0,1)$, we have

$$\begin{aligned} y(t) &= \frac{1}{n} + \kappa \int_0^1 G^*(t,s)(f_n(s, y^*(s-\tau)) + \rho(s))ds \\ &= \frac{1}{n} + \kappa \int_0^1 G^*(t,s)(f(s, y^*(s-\tau)) + \rho(s))ds \\ &\le \frac{1}{n} + \frac{(\alpha-1)t^{2n-2}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-n+1}(f(s, y^*(s-\tau)) + \rho(s))ds \\ &\le \frac{1}{n_0} + \frac{\alpha-1}{\Gamma(\alpha)} \int_0^\tau (1-s)^{\alpha-n+1}\varphi_1(s)(g(\eta(s-\tau)) + h_1(\eta(s-\tau)))ds \\ &\quad + \frac{\alpha-1}{\Gamma(\alpha)} \int_\tau^1 (1-s)^{\alpha-n+1}\varphi_1(s) \\ &\quad \times (g(\frac{1}{\alpha-1}\frac{r}{2}(s-\tau)^{2n-2}) + h_1(y(s-\tau) - (s-\tau)^{2n-1-\alpha}\omega(s-\tau)))ds. \end{aligned}$$

Hence we obtain

$$\begin{aligned} r &= \|y\| \\ &= \max_{t\in[0,1]} y(t) \\ &\le \frac{1}{n_0} + \frac{\alpha-1}{\Gamma(\alpha)} \int_0^\tau (1-s)^{\alpha-n+1}(\varphi_1(s)(g(\eta(s-\tau)) + h_1(\eta(s-\tau))))ds \\ &\quad + \frac{\alpha-1}{\Gamma(\alpha)} \int_\tau^1 (1-s)^{\alpha-n+1}\varphi_1(s) \end{aligned}$$

$$
\begin{aligned}
&\times (g(\frac{1}{\alpha-1}\frac{r}{2}(s-\tau)^{2n-2}) + h_1(y(s-\tau) - (s-\tau)^{2n-1-\alpha}\omega(s-\tau)))ds \\
\le\ & \frac{1}{n_0} + \frac{\alpha-1}{\Gamma(\alpha)}\int_0^\tau (1-s)^{\alpha-n+1}(\varphi_1(s)(g(\eta(s-\tau)) + h_1(\eta(s-\tau))))ds \\
&+ \frac{\alpha-1}{\Gamma(\alpha)}\int_\tau^1 (1-s)^{\alpha-n+1}(\varphi_1(s)(g(\frac{1}{\alpha-1}\frac{r}{2}(s-\tau)^{2n-2}) + h_1(y(s-\tau))))ds \\
\le\ & \frac{1}{n_0} + \frac{\alpha-1}{\Gamma(\alpha)}\int_0^\tau (1-s)^{\alpha-n+1}(\varphi_1(s)(g(\eta(s-\tau)) + h_1(\eta(s-\tau))))ds \\
&+ \frac{\alpha-1}{\Gamma(\alpha)}\int_\tau^1 (1-s)^{\alpha-n+1}(\varphi_1(s)(g(\frac{1}{\alpha-1}(s-\tau)^{2n-2}) + h_1(r)))ds \\
\le\ & \frac{1}{n_0} + \frac{\alpha-1}{\Gamma(\alpha)}\int_0^\tau (1-s)^{\alpha-n+1}(\varphi_1(s)(g(0) + h_1(S)))ds \\
&+ \frac{\alpha-1}{\Gamma(\alpha)}\int_\tau^1 (1-s)^{\alpha-n+1}(\varphi_1(s)(g(\frac{1}{\alpha-1}(s-\tau)^{2n-2}) + h_1(r)))ds.
\end{aligned}
$$

This is a contradiction and the claim is proved.

We set a cone

$$K_1 = \{x \in X \mid x(t) \ge \frac{1}{\alpha-1}t^{2n-2}\|x\| \ for \ t \in [0,1]\}.$$

Now the Lemma 2.4 guarantees that the equation

$$y(t) = \int_0^1 G^*(t,s)(f_n(s, y^*(s-\tau)) + \rho(s))ds$$

has a solution y_n in $\Omega_3 = \{y \in K : \frac{1}{2}r < \|y\| < r\}$, for $t \in (0,1)$.

And $\forall\ t \in (0,1)$, by $c < \frac{1}{2}r < \|y_n\| < r$, we can get $\{y_n\}_{n\in N_0}$ is a uniformly bounded set on $(0,1)$.

Next we claim that $y_n(t)$ has a lower bound. In view of (H_1), we can get that

$$
\begin{aligned}
y_n(t) &= \int_0^1 G^*(t,s)(f_n(s, y_n^*(s-\tau)) + \rho(s))ds \\
&\ge \int_a^b G^*(t,s)(f(s, y_n^*(s-\tau)) + \rho(s))ds \\
&\ge \frac{t^{2n-2}}{\Gamma(\alpha)}\int_a^b s(1-s)^{\alpha-n+1}(f(s, y_n^*(s-\tau)) + \rho(s))ds \\
&\ge \frac{t^{2n-2}}{\Gamma(\alpha)}\int_a^b s(1-s)^{\alpha-n+1}(\varphi_2(s)h_2(\frac{1}{\alpha-1}(\|y_n\| - c)(s-\tau)^{2n-2}))ds \\
&\ge \frac{t^{2n-2}}{\Gamma(\alpha)}h_2((\|y_n\| - c)\zeta_1)\int_a^b s(1-s)^{\alpha-n+1}\varphi_2(s)ds \\
&\ge \frac{t^{2n-2}}{\Gamma(\alpha)}h_2(\frac{(\|y_n\| - c)(a-\tau)^{2n-2}}{\alpha-1})\int_a^b s(1-s)^{\alpha-n+1}\varphi_2(s)ds \\
&> 0.
\end{aligned}
$$

Then we prove that $\{y_n\}_{n\in N_0}$ is an equicontinuous family on $(0,1)$. Since G^* is uniformly continuous for $(t,s) \in [0,1]\times[0,1]$, that is, for any $\epsilon > 0$, there exists

$\zeta_0 > 0$, when $t_1,\ t_2,\ s \in [0,1]$ and $|t_1 - t_2| < \zeta_0$, we can get

$$\begin{aligned}
&|G^*(t_1,s) - G^*(t_2,s)| \\
&= \epsilon(\int_0^\tau \varphi_1(s)(g(\eta(s-\tau)) + h_1(\eta(s-\tau)))ds \\
&\quad + \int_\tau^1 \varphi_1(s)(g(\frac{1}{\alpha-1}(\|y_n\| - c)(s-\tau)^{2n-2}) + h_1(r))ds)^{-1}.
\end{aligned}$$

Thus

$$\begin{aligned}
&|(y_n)(t_1) - (y_n)(t_2)| \\
&\le \int_0^\tau |G^*(t_1,s) - G^*(t_2,s)|\varphi_1(s)(g(\eta(s-\tau)) + h_1(\eta(s-\tau)))ds \\
&\quad + \int_\tau^1 |G^*(t_1,s) - G^*(t_2,s)|\varphi_1(s) \\
&\quad \times (g(\frac{1}{\alpha-1}(\|y_n\| - c)(s-\tau)^{2n-2}) + h_1(r))ds \\
&< \epsilon.
\end{aligned}$$

Therefore, $\{y_n\}_{n\in N_0}$ is an equicontinuous set on $(0,1)$. By the Arzelà-Ascoli theorem, as $\{y_n\}_{n\in N_0}$ is a sequentially compact set, there exist a subsequence N_1 of N_0 and $y \in \Omega_3$ such that $\{y_n\}_{n\in N_1}$ is uniformly convergent to y and y satisfies the relationship that $0 < y(t) < r$ for any $t \in (0,1)$. Because

$$\begin{aligned}
&f(s,(s-\tau)^{\alpha-2n+1}y_n(s-\tau) - \omega(s-\tau)) + \rho(s) \\
&= f(s,(s-\tau)^{\alpha-2n+1}(y_n(s-\tau) - (s-\tau)^{2n-\alpha-1}\omega(s-\tau))) + \rho(s) \\
&\le \varphi_1(s)(g(y_n(s-\tau) - (s-\tau)^{2n-\alpha-1}\omega(s-\tau)) \\
&\quad + h_1(y_n(s-\tau) - (s-\tau)^{2n-\alpha-1}\omega(s-\tau))) \\
&\le \varphi_1(s)(g(\frac{1}{\alpha-1}(\|y_n\| - c)(s-\tau)^{2n-2}) + h_1(r)),
\end{aligned}$$

and similarly,

$$\begin{aligned}
&f(s,(s-\tau)^{\alpha-2n+1}y(s-\tau) - \omega(s-\tau)) + \rho(s) \\
&\le \varphi_1(s)(g(\frac{1}{\alpha-1}(\|y\| - c)(s-\tau)^{2n-2}) + h_1(r)).
\end{aligned}$$

Because

$$\begin{aligned}
&\varphi_1(s)(g(\frac{1}{\alpha-1}(\|y_n\| - c)(s-\tau)^{2n-2}) + h_1(r)) \in L^1(0,1), \\
&\varphi_1(s)(g(\frac{1}{\alpha-1}(\|y\| - c)(s-\tau)^{2n-2}) + h_1(r)) \in L^1(0,1),
\end{aligned}$$

and

$$\begin{aligned}
&|f(s,(s-\tau)^{\alpha-2n+1}y_n(s-\tau) - \omega(s-\tau)) + \rho(s) \\
&\quad - (f(s,(s-\tau)^{\alpha-2n+1}y(s-\tau) - \omega(s-\tau)) + \rho(s))| \\
&\le \varphi_1(s)(g(\frac{1}{\alpha-1}(\|y_n\| - c)(s-\tau)^{2n-2}) + h_1(r)) \\
&\quad + \varphi_1(s)(g(\frac{1}{\alpha-1}(\|y\| - c)(s-\tau)^{2n-2}) + h_1(r)).
\end{aligned}$$

By the Lebesgue dominated convergence theorem, in view of

$$\begin{aligned} y_n(t) &= \int_0^1 G^*(t,s)(f_n(s, y_n^*(s-\tau)) + \rho(s))ds \\ &= \int_0^1 G^*(t,s)(f(s, y_n^*(s-\tau)) + \rho(s))ds, \end{aligned}$$

and

$$\begin{aligned} |y_n(t) - y(t)| = &| \int_\tau^1 G^*(t,s)[f(s,(s-\tau)^{\alpha-2n+1}y_n(s-\tau) - \omega(s-\tau)) + \rho(s) \\ &- (f(s,(s-\tau)^{\alpha-2n+1}y(s-\tau) - \omega(s-\tau)) + \rho(s))]ds| \\ \le & \int_\tau^1 G^*(t,s)|f(s,(s-\tau)^{\alpha-2n+1}y_n(s-\tau) - \omega(s-\tau)) + \rho(s) \\ &- (f(s,(s-\tau)^{\alpha-2n+1}y(s-\tau) - \omega(s-\tau)) + \rho(s))|ds, \end{aligned}$$

and

$$\begin{aligned} &\lim_{n\to+\infty} | \int_\tau^1 G^*(t,s)[f(s,(s-\tau)^{\alpha-2n+1}y_n(s-\tau) - \omega(s-\tau)) + \rho(s) \\ &- (f(s,(s-\tau)^{\alpha-2n+1}y(s-\tau) - \omega(s-\tau)) + \rho(s))]ds| \\ \le & \int_\tau^1 G^*(t,s) \lim_{m\to+\infty} |f(s,(s-\tau)^{\alpha-2n+1}y_n(s-\tau) - \omega(s-\tau)) + \rho(s) \\ &- (f(s,(s-\tau)^{\alpha-2n+1}y(s-\tau) - \omega(s-\tau)) + \rho(s))|ds = 0. \end{aligned}$$

This implies that

$$\lim_{n\to+\infty} y_n(t) = \int_0^1 G^*(t,s) \lim_{n\to+\infty} (f(s, y_n^*(s-\tau)) + \rho(s))ds.$$

So,

$$y(t) = \int_0^1 G^*(t,s)(f(s, y^*(s-\tau)) + \rho(s))ds.$$

Therefore, T defined by (3.2) has a fixed point $\widetilde{y}$ in Ω_3 with $c < \|\widetilde{y}\| < r$. Similar to (3.14), we have

$$\begin{aligned} \widetilde{x}(t) - \omega(t) &= t^{\alpha-2n+1}\widetilde{y}(t) - \omega(t) = t^{\alpha-2n+1}(\widetilde{y}(t) - t^{2n-1-\alpha}\omega(t)) \\ &\ge t^{\alpha-2n+1}\frac{1}{\alpha-1}(\|\widetilde{y}\| - c)t^{2n-2} = \frac{1}{\alpha-1}(\|\widetilde{y}\| - c)t^{\alpha-1} > 0. \end{aligned}$$

It is easy to know that (3.4) is satisfied. Therefore, $x(t) = \widetilde{x}(t) - \omega(t)$ is a positive solution of the boundary value problem (1.1). The proof is completed. □

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