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UNIQUENESS AND EXISTENCE OF SOLUTIONS FOR A SINGULAR SYSTEM WITH NONLOCAL OPERATOR VIA PERTURBATION METHOD

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Abstract In this work, we investigate the existence and the uniqueness of solutions for the nonlocal elliptic system involving a singular nonlinearity as follows:

$$\begin{cases} (-\Delta_p)^s u = a(x)|u|^{q-2}u + \frac{1-\alpha}{2-\alpha-\beta}c(x)|u|^{-\alpha}|v|^{1-\beta}, & \text{in } \Omega, \\ (-\Delta_p)^s v = b(x)|v|^{q-2}v + \frac{1-\beta}{2-\alpha-\beta}c(x)|u|^{1-\alpha}|v|^{-\beta}, & \text{in } \Omega, \\ u = v = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary, $0 < \alpha < 1$, $0 < \beta < 1$, $2 - \alpha - \beta < p < q \leq p_s^* = \frac{Np}{N-sp}$, a, b, $c \in C(\overline{\Omega})$ are non-negative weight functions with compact support in Ω, and $(-\Delta)_p^s$ is the fractional p-laplacian operator. We use a perturbation method combine with some variationals methods in order to show the existence of a solution to the above system. We also prove the uniqueness of the solution to the system for some additional condition.

Keywords Singular nonlocal elliptic system, approximated methods, variationals methods, existence of solutions, uniqueness of solutions.

MSC(2010) 34B15, 37C25, 35R20.

1. Introduction

The purpose of this paper is to study the following elliptic system involving nonlocal operator and singular nonlinearity:

$$\begin{cases} (-\Delta_p)^s u = a(x)|u|^{q-2}u + \frac{1-\alpha}{2-\alpha-\beta}c(x)|u|^{-\alpha}|v|^{1-\beta}, & \text{in } \Omega, \\ (-\Delta_p)^s v = b(x)|v|^{q-2}v + \frac{1-\beta}{2-\alpha-\beta}c(x)|u|^{1-\alpha}|v|^{-\beta}, & \text{in } \Omega, \\ u = v = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.1}$$

where Ω is an open bounded domain of $\mathbb{R}^N$ with smooth boundary, $N > ps$, $s \in (0,1)$, $1 < p < \infty$ and $(-\Delta_p)^s$ is the fractional p-laplacian operator which is defined

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as

$$(-\Delta_p)^s u(x) = C_{N,s} P.V. \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}} dy \ , \forall \ p \in [1,\infty)$$

with $C_{N,s}$, being the normalizing constant.

During the past few decades, the study of nonlocal elliptic PDEs involving singularity with Dirichlet boundary condition has drawn interest by many researchers, both from a pure mathematical point of view and for concrete applications, since the problems of this type are important in many fields of sciences, notably the fields of physics, probability, finance, electromagnetism, astronomy, and fluid dynamics, it also they can be used to accurately describe the jump Lévy processes in probability theory and fluid potentials for more details see [1, 6] and references therein.

Set $\alpha=\beta$, $\alpha+\beta=\gamma$ and $u=v$. Then the problem (1.1) becomes the following nonlocal singular problem

$$\begin{aligned} &(-\Delta_p)^s u = \frac{\lambda c(x)}{u^\gamma} + M|u|^{q-2}u \text{ in } \Omega,\\ &u = 0 \text{ in } \mathbb{R}^N \setminus \Omega, \\ &u > 0 \text{ in } \Omega, \end{aligned} \tag{1.2}$$

where $N > ps$, $M \geq 0$, $c : \Omega \to \mathbb{R}$ is a nonnegative bounded function. When $M = 0$ and $p = 2$ (the purely singular problem), Fang [8] had shown that the problem (1.2) has a unique solution $u \in C^{2,\alpha}(\Omega)$ for $0 < \alpha < 1$. In [12], the multiplicity result for the problem (1.2) is proved by converting the nonlocal problem to a local problem. For $1 < p < \infty$, $M = 0$ and $\lambda = 1$ the problem (1.2) was studied by Canino et al. [5]. In [9, 10], the authors established the existence and the multiplicity of weak solutions to the problem (1.2) by using the Nehari manifold method. Recently, Saoudi et al. in [14] has guaranteed the existence of at least two solutions by using min-max method with the help of modified Mountain Pass theorem. In particular, in [13], the author considered the following nonlocal problem

$$\begin{cases} (-\Delta)^s_p u = \lambda a(x)|u|^{q-2}u + \frac{1-\alpha}{2-\alpha-\beta} c(x)|u|^{-\alpha}|v|^{1-\beta}, & \text{in } \Omega, \\ (-\Delta)^s_p v = \mu b(x)|v|^{q-2}v + \frac{1-\beta}{2-\alpha-\beta} c(x)|u|^{1-\alpha}|v|^{-\beta}, & \text{in } \Omega, \\ u = v = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.3}$$

where $0 < \alpha < 1,\ 0 < \beta < 1,\ 2-\alpha-\beta < p < q < p_s^*,\ p_s^* = \frac{N}{N-ps}$ is the fractional Sobolev exponent, λ, μ are two parameters, $a,\ b,\ c \in C(\overline{\Omega})$ are non-negative weight functions with compact support in Ω. With the help of the Nehari manifold and the fibering maps (appropriately modified), the authors proved the existence of at least two non-negatives solutions of problem (1.3).

Motivated by above results, in the present work, we are interested in the existence and the uniqueness of solutions for nonlocal system (1.1) via perturbed method's.

In order to state our result, let us introduce some notations. We define

$$W^{s,p}(\mathbb{R}^N) := \left\{ u \in L^p(\mathbb{R}^N) : u \text{ measurable } , |u|_{s,p} < \infty \right\},$$

the usual fractional Sobolev space with the Gagliardo norm

$$||u||_{s,p} := \left(||u||_p^p + |u|_{s,p}^p \right)^{\frac{1}{p}}.$$

For a detailed account on the properties of $W^{s,p}(\mathbb{R}^N)$ we refer the reader to [11]. Denote

$$\mathcal{Q} = \mathbb{R}^{2N} \setminus \big((\mathbb{R}^N \setminus \Omega) \times (\mathbb{R}^N \setminus \Omega)\big)$$

and we define the space

$$X = \left\{ u : \mathbb{R}^N \to \mathbb{R} \text{ Lebesgue measurable} : \ u\backslash_\Omega \in L^p(\Omega) \text{ and } \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \in L^p(\mathcal{Q}) \right\}$$

with the norm

$$\|u\|_X = \|u\|_{L^p(\Omega)} + \Big(\int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy \Big)^{1/p}.$$

Through this paper we consider the space

$$X_0 = \{ u \in X : u = 0 \ a.e. \text{ in } \mathbb{R}^n \setminus \Omega \},$$

with the norm

$$\|u\|_{X_0} = \Big(\int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy \Big)^{1/p}.$$

It is readily seen that $(X_0, ||.||)$ is a uniformly convex Banach space and that the embedding $X_0 \hookrightarrow L^q(\Omega)$ is continuous for all $1 \le q \le p_s^*$, and compact for all $1 \le q < p_s^*$. The dual space of $(X_0, ||.||)$ is denoted by $(X^*, ||.||_*)$, and $\langle ., . \rangle$ denotes the usual duality between X_0 and X^*.

Let $\mathcal{X}_0 = X_0 \times X_0$ be the Cartesian product of two Hilbert spaces, which is (In [6] it is claimed that X_0 is a Hilbert space) a reflexive Banach space endowed with the norm

$$\|(u,v)\| = \Big(\int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy + \int_Q \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}} dxdy \Big)^{1/p}. \tag{1.4}$$

Before stating our main results, we make the following assumptions throughout this paper:

(H1) $0 < \alpha < 1$, $0 < \beta < 1$, $2 - \alpha - \beta < p < q \le p_s^* = \frac{Np}{N-sp}$.

(H2) a, b, $c \in C(\overline{\Omega})$ are non-negative weight functions with compact support in Ω.

(H3) $a = b = 0$, $c \in C(\overline{\Omega})$ is a non-negative weight functions with compact support in Ω, $\alpha = \beta$ and $\frac{1}{2} < \alpha < 1$.

We now list out the results that we will prove in this work.

Theorem 1.1. *Suppose that* **(H1)**-**(H2)** *are fulfilled. Then the system* (1.1) *has at least one positive solution.*

Theorem 1.2. *Suppose that* **(H1)**-**(H3)** *are fulfilled. Then the system* (1.1) *has a unique positive solution.*

This paper is organized as follows: The Section 2 is devoted to study approximated system. While, existence of solution (Theorem 1.1) and uniqueness of solution (Theorem 1.2) will be presented in Section 3 and in Section 4 respectively.

2. The approximated fractional system

In this section, we introduce the following approximated system, for a fixed $n > 1$

$$\begin{cases} (-\Delta_p)^s u = a(x)\,|u(x)|^{q-2}\,u + \frac{1-\alpha}{2-\alpha-\beta}c(x)\left(|u| + \frac{1}{n}\right)^{-\alpha}\left(|v| + \frac{1}{n}\right)^{1-\beta} \\ (-\Delta_p)^s v = b(x)\,|v(x)|^{q-2}\,v + \frac{1-\beta}{2-\alpha-\beta}c(x)\left(|u| + \frac{1}{n}\right)^{1-\alpha}\left(|v| + \frac{1}{n}\right)^{-\beta}. \end{cases} \tag{2.1}$$

Associated to the approximated problem (2.1), we define the functional $E_n : \mathcal{X}_0 \to \mathbb{R}$ by

$$\begin{aligned} E_n(u,v) = & \frac{1}{p}\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy + \frac{1}{p}\int_\Omega \frac{|v(x)-v(y)|^p}{|x-y|^{N+sp}}dxdy \\ & -\frac{1}{q}\int_\Omega \left(a(x)\,|u|^q + b(x)\,|v|^q\right)dx \\ & -\frac{1}{2-\alpha-\beta}\int_\Omega c(x)(|u|+\frac{1}{n})^{1-\alpha}(|v|+\frac{1}{n})^{1-\beta}dx. \end{aligned}$$

Notice that E_n is a C^1 functional and obviously, any critical point of E_n is a weak solution of the problem (2.1).

Definition 2.1. We say that $(u, v) \in \mathcal{X}_0$ is a weak solution of problem (2.1) if $u, v > 0$ in Ω, one has

$$\begin{aligned} & \int_Q \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+sp}}\,dx\,dy \\ & + \int_Q \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))(\psi(x)-\psi(y))}{|x-y|^{N+sp}}\,dx\,dy \\ = & \int_\Omega \Big(a(x)|u|^{q-2}u\phi + b(x)|v|^{q-2}v\psi\Big)dx \\ & + \frac{1-\alpha}{2-\alpha-\beta}\int_\Omega c(x)\Big(|u|+\frac{1}{n}\Big)^{-\alpha}\Big(|v|+\frac{1}{n}\Big)^{1-\beta}\phi\,dx \\ & + \frac{1-\beta}{2-\alpha-\beta}\int_\Omega c(x)\Big(|u|+\frac{1}{n}\Big)^{1-\alpha}\Big(|v|+\frac{1}{n}\Big)^{-\beta}\psi\,dx \end{aligned}$$

for all $(\phi, \psi) \in C_c^\infty(\Omega) \times C_c^\infty(\Omega)$.

Before, given the first result in this section let us recall the following inequality

$$(d+e)^m \le d^m + e^m \tag{2.2}$$

for all $d,\ e \in \mathbb{R}^+$ and $m \in (0,1)$. On the other hand, using Hölder's inequality and Sobolev inequalities, one has

$$\begin{aligned} \int_\Omega \Big(a(x)|u|^q + b(x)|v|^q\Big)dx & \le |\Omega|^{\frac{p_s^*-q}{p_s^*}}\Big(||a||_\infty ||u||_{p_s^*}^q + ||b||_\infty ||v||_{p_s^*}^q\Big) \\ & \le |\Omega|^{\frac{p_s^*-q}{p_s^*}} S^{-\frac{q}{p}}\Big(||a||_\infty ||u||^q + ||b||_\infty ||v||^q\Big) \\ & \le |\Omega|^{\frac{p_s^*-q}{p_s^*}} S^{-\frac{q}{p}}\Big(||a||_\infty^{\frac{p}{p-q}} + ||b||_\infty)^{\frac{p}{p-q}}\Big)^{\frac{p-q}{p}}\Big(||u||^q + ||v||^q\Big) \end{aligned}$$

$$\leq |\Omega|^{\frac{p_s^*-q}{p_s^*}} S^{-\frac{q}{p}} \left(||a||_\infty^{\frac{p}{p-q}} + ||b||_\infty^{\frac{p}{p-q}}\right)^{\frac{p-q}{p}} ||(u,v)||^q. \tag{2.3}$$

Lemma 2.1. *The functional E_n is coercive and bounded below in $\mathcal{X}_0$.*

Proof. From the Hölder's and Sobolev inequalities combine with Eq. (2.2) and Eq. (2.3), we obtain

$$\begin{aligned}
E_n(u,v) \geq& \frac{1}{p}\|u\|^p + \frac{1}{p}\|v\|^p - \frac{1}{q}\int_\Omega \Big(a(x)\,|u|^q + b(x)\,|v|^q\Big)dx \\
&- \frac{1}{2-\alpha-\beta}\int_\Omega c(x)\Big(|u|+1\Big)^{1-\alpha}\Big(|v|+1\Big)^{1-\beta}dx \\
\geq& \frac{1}{p}\|(u,v)\|^p - c\,|\Omega|^{\frac{p_s^*-p}{p_s^*}} S^{-q/p}\Big(\|a\|_\infty^{p/p-q} + \|b\|_\infty^{p/p-q}\Big)^{\frac{p-q}{p}}\|(u,v)\|^q \\
&- \frac{1}{2-\alpha-\beta}\Big(\|c\|_{r_1}\|u\|_{p_s^*}^{1-\alpha}\|v\|_{p_s^*}^{1-\beta} - \|c\|_{r_2}\|u\|_{p_s^*}^{1-\alpha} - \|c\|_{r_3}\|v\|_{p_s^*}^{1-\beta} - \|c\|_1\Big) \\
\geq& \frac{1}{p}\|(u,v)\|^p - c\,|\Omega|^{\frac{p_s^*-p}{p_s^*}} S^{-q/p}\Big(\|a\|_\infty^{p/p-q} + \|b\|_\infty^{p/p-q}\Big)^{\frac{p-q}{p}}\|(u,v)\|^q \\
&- \frac{1}{2-\alpha-\beta}\Big(S^{2-\alpha-\beta}\|c\|_{r_1}\|u\|_p^{1-\alpha}\|v\|_p^{1-\beta} - S^{1-\alpha}\|c\|_{r_2}\|u\|_p^{1-\alpha} \\
&- S^{1-\beta}\|c\|_{r_3}\|v\|_p^{1-\beta} - \|c\|_1\Big)
\end{aligned}$$

with
$r_1 = \frac{Np}{(\alpha+\beta)(N-sp)+Np+2sp-2N}$, $r_2 = \frac{Np}{(\alpha-1)(N-sp)+Np}$, $r_3 = \frac{Np}{(\beta-1)(N-sp)+Np}$. On the other hand, it is very simple to see that $1 \leq \max(r_2, r_3) \leq r_1 \leq \frac{N}{sp}$. Hence, E_n is coercive and bounded below on $\mathcal{X}_0$. □

Consider the following minimization problem

$$c = \inf\left\{E_n(u,v), (u,v) \in \mathcal{X}_0\right\}. \tag{2.4}$$

Hence, from Lemma 2.1 combine with the Ekeland's variational principle, we obtain the existence of the sequences $(u_k, v_k) \in \mathcal{X}_0$ such that $E_n(u_k, v_k) \to c$ and $E'_n(u_k, v_k) \to 0$ in $\mathcal{X}_0^*$ as $k \to \infty$.

Now, we prove the following crucial result.

Lemma 2.2. *the approximated system* (2.1) *has a nonnegative solution.*

Proof. At first, it is simple to see from the coerviness of the functional energy E_n, the boundedness of the sequence $\{u_k, v_k\}$ in $\mathcal{X}_0$. So, there exists $\{U_n, V_n\}$ in $\mathcal{X}_0$ such that

$$\begin{aligned}
&(u_k, v_k) \to (U_n, V_n) \ \text{ weakly in } \ \mathcal{X}_0 \\
&(u_k, v_k) \to (U_n, V_n) \ \text{ strongly in } \ L^k(\Omega) \text{ for } \ 1 \leq k < p_s^* \\
&(u_k, v_k) \to (U_n, V_n) \ \text{ pointwise a.e. in } \ \Omega.
\end{aligned}$$

By dominated convergence theorem, we claim that

$$\int_\Omega |u_k|^{q-2}u_k\Phi dx \to \int_\Omega |U_n|^{q-2}U_n\Phi dx \quad \forall\, \Phi \in C_c^\infty(\Omega). \tag{2.5}$$

Indeed, from [3], there exists $l \in L^r(\mathbb{R}^N)$ such that

$$|u_k(x)| \le l(x), \quad |v_k(x)| \le l(x), \ \text{ as } k \to \infty$$

for any $1 \le r < p_s^*$. The fact that, $(u_k, v_k) \to (U_n, V_n)$ pointwise a.e. in Ω. Therefore, by Dominated convergence theorem, our claim is true. Similarly,

$$\int_\Omega |v_k|^{q-2} v_k \Psi dx \to \int_\Omega |V_n|^{q-2} V_n \Psi dx \quad \forall\ \Psi \in C_c^\infty(\Omega). \tag{2.6}$$

On the other hand, note that $\{u_k\}$ is bounded, by the Sobolev embedding theorem, so there exists a constant $C > 0$ such that $|u_k|_{p_s^*} \le C < \infty$. Moreover, by Hölder inequalities we have

$$\int_\Omega c(x) u_k^{1-\alpha} dx \le \|c\|_\infty \int_\Omega |u_k|^{1-\alpha} dx \le \|c\|_\infty |\Omega|^{\frac{p_s^*}{p_s^* + \alpha - 1}} |u_k|_{p_s^*}^{1-\alpha}. \tag{2.7}$$

Thus, using Eq. (2.7), there exists $l_1 \in L^{(1-\alpha)}(\Omega)$, and up to a subsequence,

$$c(x)\left(|u_k| + \frac{1}{n}\right)^{1-\alpha} \left(|v_k| + \frac{1}{n}\right)^{-\beta} \Psi \to c(x)\left(|U_n| + \frac{1}{n}\right)^{1-\alpha} \left(|V_n| + \frac{1}{n}\right)^{-\beta} \Psi \text{a.e.}$$

and

$$\left| c(x)\left(|u_k| + \frac{1}{n}\right)^{1-\alpha} \left(|v_k| + \frac{1}{n}\right)^{-\beta} \Psi \right| \le n^\beta |c| (|l_1|^{1-\alpha} + 1)\Psi \in L^1(\Omega).$$

Now, apply the Lebesque's dominated convergence theorem, we conclude that, $\forall\ \Psi \in C_c^\infty(\Omega)$

$$\begin{aligned} &\int_\Omega c(x)\left(|u_k| + \frac{1}{n}\right)^{1-\alpha} \left(|v_k| + \frac{1}{n}\right)^{-\beta} \Psi dx \to \\ &\int_\Omega c(x)\left(|U_n| + \frac{1}{n}\right)^{1-\alpha} |U_n| \left(|V_n| + \frac{1}{n}\right)^{-\beta} \Psi dx. \end{aligned} \tag{2.8}$$

Similarly, we deduce that, $\forall\ \Phi \in C_c^\infty(\Omega)$,

$$\begin{aligned} &\int_\Omega c(x)\left(|u_k| + \frac{1}{n}\right)^{-\alpha} \left(|v_k| + \frac{1}{n}\right)^{1-\beta} \Phi dx \to \\ &\int_\Omega c(x)\left(|U_n| + \frac{1}{n}\right)^{-\alpha} \left(|V_n| + \frac{1}{n}\right)^{1-\beta} \Phi dx. \end{aligned} \tag{2.9}$$

Moreover, let p' the Hölder conjugate of p given by $p' = \frac{p}{p-1}$, then $\{|u_k(x) - u_k(y)|^{p-2}(u_k(x) - u_k(y))/|x-y|^{(N+sp)/p'}\}$ is bounded in $L^{p'}(\mathbb{R}^{2N})$ and by pointwise convergence $u_k \to U_n$ a.e. in Ω

$$\frac{|u_k(x) - u_k(y)|^{p-2}(u_k(x) - u_k(y))}{|x-y|^{(N+sp)/p'}} \to \frac{|U_n(x) - U_n(y)|^{p-2}(U_n(x) - U_n(y))}{|x-y|^{(N+sp)/p'}} \text{ a.e. in } \mathbb{R}^{2N}.$$

Furthermore, since $(\Phi(x) - \Phi(y))/|x-y|^{(N+sp)/p} \in L^p(\mathbb{R}^{2N})$, we deduce that

$$\int_Q \frac{|u_k(x) - u_k(y)|^{p-2}(u_k(x) - u_k(y))(\Phi(x) - \Phi(y))}{|x-y|^{N+sp}} \, dx \, dy$$

$$\to \int_Q \frac{|U_n(x)-U_n(y)|^{p-2}(U_n(x)-U_n(y))(\Phi(x)-\Phi(y))}{|x-y|^{N+sp}}\,dx\,dy \quad \text{for any } \Phi \in C_c^\infty(\Omega). \tag{2.10}$$

Using the same argument, as above, we obtain

$$\int_Q \frac{|v_k(x)-v_k(y)|^{p-2}(v_k(x)-v_k(y))(\Psi(x)-\Psi(y))}{|x-y|^{N+sp}}\,dx\,dy$$
$$\to \int_Q \frac{|V_n(x)-V_n(y)|^{p-2}(V_n(x)-V_n(y))(\Psi(x)-\Psi(y))}{|x-y|^{N+sp}}\,dx\,dy \quad \text{for any } \Psi \in C_c^\infty(\Omega). \tag{2.11}$$

Therefore, from Eqs. (2.5), (2.6), (2.8), (2.9), (2.10) and (2.11), we deduce that

$$\langle I_n'(u_k,v_k)(\Phi,\Psi)\rangle \longrightarrow \langle I_n'(U_n,V_n)(\Phi,\Psi)\rangle = 0 \;\forall\,(\Phi,\Psi)\in C_c^\infty(\Omega)\times C_c^\infty(\Omega).$$

Now, we will prove that the sequence (u_k, v_k) converge strongly to (U_n, V_n) in $\mathcal{X}_0$. Indeed, since

$$\langle I_n'(u_k,v_k) - I_n'(U_n,V_n), (u_k-U_n, v_k-V_n)\rangle \to 0,$$

as $k\to\infty$, we conclude that,

$$\begin{aligned}
o_k(1) &= \langle I_n'(u_k,v_k) - I_n'(U_n,V_n), (u_k-U_n, v_k-V_n)\rangle\\
&= \int_Q \frac{|u_k(x)-u_k(y)|^{p-2}(u_k(x)-u_k(y))((u_k-U_n)(x)-(u_k-U_n)(y))dxdy}{|x-y|^{N+sp}}\\
&\quad - \int_Q \frac{|U_n(x)-U_n(y)|^{p-2}(U_n(x)-U_n(y))((u_k-U_n)(x)-(u_k-U_n)(y))}{|x-y|^{N+sp}}dxdy\\
&\quad + \int_Q \frac{|v_k(x)-v_k(y)|^{p-2}(v_k(x)-v_k(y))((v_k-V_n)(x)-(v_k-V_n)(y))}{|x-y|^{N+sp}}dxdy\\
&\quad - \int_Q \frac{|V_n(x)-V_n(y)|^{p-2}(V_n(x)-V_n(y))((v_k-V_n)(x)-(v_k-V_n)(y))}{|x-y|^{N+sp}}dxdy\\
&\quad - \int_\Omega a(x)\left(|u_k|^{q-2}u_k - |V_n|^{q-2}V_n\right)(u_k-U_n)\,dx\\
&\quad - \int_\Omega b(x)\left(|v_k|^{q-2}v_k - |V_n|^{q-2}V_n\right)(v_k-V_n)\,dx\\
&\quad - \frac{1-\alpha}{2-\alpha-\beta}\int c(x)\left[\left(|u_k|+\frac{1}{n}\right)^{-\alpha}(u_k-U_n)\left(|v_k|+\frac{1}{n}\right)^{1-\beta}\right.\\
&\quad \left. - \left(|U_n|+\frac{1}{n}\right)^{-\alpha}(u_k-U_n)\left(|V_n|+\frac{1}{n}\right)^{1-\beta}\right]dx\\
&\quad - \frac{1-\beta}{2-\alpha-\beta}\int c(x)\left[\left(|u_k|+\frac{1}{n}\right)^{1-\alpha}(v_k-V_n)\left(|v_k|+\frac{1}{n}\right)^{-\beta}\right.\\
&\quad \left. - \left(|U_n|+\frac{1}{n}\right)^{1-\alpha}(v_k-V_n)\left(|V_n|+\frac{1}{n}\right)^{-\beta}\right]dx.
\end{aligned}$$

By Hölder inequality, it follows that

$$\int_\Omega a(x)\,|u_k|^{q-2}\,u_k\,(u_k-U_n)\,dx \le \|a\|_{L^{\frac{r}{r-q}}(\Omega)}\,\|u_k\|_{L^r}^{q-1}\,\|(u_k-U_n)\|_{L^r(\Omega)} \to 0 \quad \text{as } \ k\to\infty. \tag{2.12}$$

Similarly,

$$\int_\Omega b(x)\,|v_k|^{q-2}\,v_k\,(v_k-V_n)\,dx \le \|b\|_{L^{\frac{r}{r-q}}(\Omega)}\,\|v_k\|_{L^r}^{q-1}\,\|(v_k-V_n)\|_{L^r(\Omega)} \to 0 \quad \text{as } \ k\to\infty. \tag{2.13}$$

On the other hand, let $\epsilon>0$, there exists $R_\epsilon>0$ such that

$$\begin{aligned} 0 &\le \int_\Omega c(x)\,|u_k-U_n|^{2(1-\alpha)}\,dx \\ &\le \|c\|_{\frac{m}{m+2\alpha-2}}\left(\int_{B_{R_\epsilon}}|u_k-U_n|^m\,dx\right)^{2\frac{(1-\alpha)}{m}} + C_1\epsilon \end{aligned}$$

which gives that

$$\int_\Omega c(x)\,|u_k-U_n|^{2(1-\alpha)}\,dx \to 0$$

and

$$c(x)^{\frac{1}{2(1-\alpha)}}U_k \to c(x)^{\frac{1}{2(1-\alpha)}}U_n \text{ in } L^{2(1-\alpha)}(\Omega). \tag{2.14}$$

Similarly, we obtain

$$c(x)^{\frac{1}{2(1-\beta)}}V_k \to c(x)^{\frac{1}{2(1-\beta)}}V_n \text{ in } L^{2(1-\beta)}(\Omega). \tag{2.15}$$

Therefore, from Eqs. (2.14) and (2.15), there exists $L_1 \in L^{2(1-\alpha)}(\Omega)$, $L_2 \in L^{2(1-\beta)}(\Omega)$ and up to a subsequence

$$\begin{aligned} & c(x)\left(|u_k|+\frac{1}{n}\right)^{-\alpha}(u_k-U_n)\left(|v_k|+\frac{1}{n}\right)^{1-\beta} \to \\ & c(x)\left(|U_n|+\frac{1}{n}\right)^{-\alpha}(u_k-U_n)\left(|V_n|+\frac{1}{n}\right)^{1-\beta} \quad \text{a.e. in} \quad \Omega, \quad \text{and} \\ & \left|c(x)\left(|u_k|+\frac{1}{n}\right)^{-\alpha}(u_k-U_n)\left(|v_k|+\frac{1}{n}\right)^{1-\beta}\right| \\ \le\ & |c|\,(|u_k|)^{-\alpha}\,(|u_k-U_n|)\left(|v_k|^{1-\beta}+1\right) \\ \le\ & |c|\,(|u_k|)^{-\alpha}\,(|u_k-U_n|)\left(|v_k|^{1-\beta}+1\right) \\ \le\ & |c|\,(|u_k|)^{1-\alpha}\left(|v_k|^{1-\beta}+1\right) \le |c|\left(|L_1|^{1-\alpha}\,|L_2|^{1-\beta}+|L_1|^{1-\alpha}\right) \in L^1(\Omega). \end{aligned}$$

Therefore, by Dominated convergence theorem, we conclude that

$$\int_\Omega c(x)\left(|u_k|+\frac{1}{n}\right)^{-\alpha}(u_k-U_n)\left(|v_k|+\frac{1}{n}\right)^{1-\beta}dx \to 0 \quad \text{as } \ k\to\infty. \tag{2.16}$$

Using the same argument, we conclude that

$$\int_{\Omega} C(x) \left(|v_k| + \frac{1}{n}\right)^{1-\alpha} \left(|v_k| + \frac{1}{n}\right)^{-\beta} (v_k - V_n)\, dx \to 0 \quad \text{as} \quad k \to \infty. \tag{2.17}$$

Now, from the inequality $|a-b|^l \leq 2^{l-2}\left(|a|^{l-2}a - |b|^{l-2}b\right)(a-b)$ for all $a, b \in \mathbb{R}^N$ and $l \geq 2$, we obtain:

$$o_k(1) = \langle I_n'(u_k, v_k) - I_n'(U_n, V_n), (u_k - U_n, v_k - V_n)\rangle \geq \|(u_k - U_n)\|^p + \|v_k - V_n\|^p.$$

Now, combine Eqs. (2.12), (2.13), (2.16) and (2.17), we deduce that

$$\|u_k - U_n\| \to 0 \quad \text{and} \quad \|v_k - V_n\| \to 0 \quad \text{as} \quad k \to \infty.$$

Hence, there exists $(U_n, V_n) \in \mathcal{X}_0$ such that

$$E_n(U_n, V_n) = \inf_{(u,v)\in\mathcal{X}_0} E_n(u, v).$$

Moreover, since $E_n(U_n, V_n) = E_n(|U_n|, |V_n|)$ and $(|U_n|, |V_n|) \in \mathcal{X}_0$, so we may assume $(U_n, V_n) \geq 0$. Hence, (U_n, V_n) is a non-trivial non-negative solution of the problem (2.1). □

3. Proof of Theorem 1.1

This section is devoted to prove the existence of a solution to the problem (1.1). The proof is done in servals Steps.

Step 1: The solutions (U_n) and (V_n) of the problem (2.1) are uniformly bounded.

Firstly, we have proved in section 2 that (U_n, V_n) is a solution to the problem 2.1. Then, taking $(U_n, 0)$ as test function in the weak formulation of the problem (2.1), we obtain

$$\begin{aligned}
&\int_Q \frac{|U_n(x) - U_n(y)|^p}{|x-y|^{N+sp}}\, dx\, dy = \int_{\Omega} a(x)|U_n|^q \\
&+ \frac{1-\alpha}{2-\alpha-\beta}\int_{\Omega} c(x)(|U_n| + \frac{1}{n})^{-\alpha}(|V_n| + \frac{1}{n})^{1-\beta} U_n\, dx \\
&\leq ||a||_\infty |\Omega|^{\frac{p_s^*-q}{p_s^*}} ||U_n||^q_{p_s^*} + \frac{1-\alpha}{2-\alpha-\beta}\int_{\Omega} c(x)|U_n|^{1-\alpha}(|V_n|^{1-\beta} + 1)\, dx.
\end{aligned} \tag{3.1}$$

In the same way, taking $(0, V_n)$ as test function in the weak formulation of the problem (2.1), we get

$$\begin{aligned}
&\int_Q \frac{|V_n(x) - V_n(y)|^p}{|x-y|^{N+sp}}\, dx\, dy = \int_{\Omega} a(x)|V_n|^q \\
&+ \frac{1-\alpha}{2-\alpha-\beta}\int_{\Omega} c(x)(|U_n| + \frac{1}{n})^{1-\alpha}(|V_n| + \frac{1}{n})^{-\beta} V_n\, dx \\
&\leq ||a||_\infty |\Omega|^{\frac{p_s^*-q}{p_s^*}} ||V_n||^q_{p_s^*} + \frac{1-\alpha}{2-\alpha-\beta}\int_{\Omega} c(x)(|U_n|^{1-\alpha} + 1)|V_n|^{1-\beta}\, dx.
\end{aligned} \tag{3.2}$$

Now, combine Eq. (3.1) with Eq. (3.2) and using Hölder and Sobolev inequalities, we deduce that

$$\begin{aligned}||(U_n, V_n)||^p \leq c_0||(U_n, V_n)||^q + c_1||(U_n, V_n)||^{2-\alpha-\beta} + c_2||(U_n, V_n)||^{1-\alpha}\\ +c_3||(U_n, V_n]||^{1-\beta} + c_4\end{aligned}$$

which yields that (U_n) and (V_n) are uniformly bounded in $\mathcal{X}_0$.

Now, let us prove a priori estimate in $L^\infty(\Omega)$ of the solution U_n and V_n. Define $v_M = (M - U_n)^+$. For $0 < t < 1$ define $\xi(t) = I_n(U_n + tv_M, V_n + tv_M)$. Thus, using $(v_M, 0)$ as test function, we obtain

$$\begin{aligned}\xi'(t) &= \langle I_n(U_n + tv_M, V_n + tv_M), (v_M, 0)\rangle\\ &= \langle(-\Delta_p)^s(U_n + tv_M, V_n + tv_M) - a(x)|U_n + tv_M|^{q-2}(U_n + tv_M)\\ &\quad - \frac{1-\alpha}{2-\alpha-\beta}\Big(|U_n + tv_M| + \frac{1}{n}\Big)^{-\alpha}\Big(|V_n + tv_M| + \frac{1}{n}\Big)^{1-\beta}, v_M\rangle.\end{aligned} \tag{3.3}$$

Similarly,

$$\begin{aligned}\xi'(1) &= \langle I_n(U_n + v_M, V_n + v_M), (v_M, 0)\rangle\\ &= \langle I_n(M, V_n + v_M), (v_M, 0)\rangle\\ &= \langle(-\Delta_p)^s(M) - a(x)|M|^{q-2}M\\ &\quad - \frac{1-\alpha}{2-\alpha-\beta}\Big(|M| + \frac{1}{n}\Big)^{-\alpha}\Big(|V_n + v_M| + \frac{1}{n}\Big)^{1-\beta}, v_M\rangle < 0\end{aligned} \tag{3.4}$$

for sufficiently small $M > 0$. Moreover,

$$\begin{aligned}-\xi'(1) + \xi'(t) =&\langle(-\Delta_p)^s(U_n + tv_M) - (-\Delta_p)^s(U_n + v_M)\\ &+ a(x)|U_n + v_M|^{q-2}(U_n + v_M) - a(x)|U_n + tv_M|^{q-2}(U_n + tv_M)\\ &+ c(x)\frac{1-\alpha}{2-\alpha-\beta}\Big(|U_n + v_M| + \frac{1}{n}\Big)^{-\alpha}\Big(|V_n + v_M| + \frac{1}{n}\Big)^{1-\beta}\\ &- c(x)\frac{1-\alpha}{2-\alpha-\beta}\Big(|U_n + tv_M| + \frac{1}{n}\Big)^{-\alpha}\Big(|V_n + tv_M| + \frac{1}{n}\Big)^{1-\beta}, v_M\rangle.\end{aligned}$$

Since $|s|^{q-2}s + \Big(|s| + \frac{1}{n}\Big)^{-\alpha}\Big(|s| + \frac{1}{n}\Big)^{1-\beta}$ is a uniformly nonincreasing function with respect to $x \in \Omega$ for sufficiently small $s > 0$. Also from the monotonicity of $(-\Delta_p)^s$ we have, for sufficiently small $M > 0$, $0 \leq \xi'(1) - \xi'(t)$. From the Taylor series expansion, we have $\exists\, 0 < \theta < 1$ such that

$$\begin{aligned}0 &\leq I_n(U_n + v_M, V_n + v_M) - I_n(U_n, V_n)\\ &= \langle I_n'(U_n + \theta v_M, V_n + \theta v_M), v_M\rangle\\ &= \xi'(\theta).\end{aligned}$$

Thus for $t = \theta$ we have $\xi'(\theta) \geq 0$ which is a contradiction to $\xi'(\theta) \leq \xi'(1) < 0$. We have verified $v_M \equiv 0$ in Ω, that is $U_n \leq M$. Similarly, by using $(0, (M - V_n)^+)$ as test function, we obtain $V_n \leq M$.

Step 2: The solutions (U_n) and (V_n) of the problem (2.1) are positive almost every where in Ω.

At first, taking into account that (U_n, V_n) is a solution of the problem (2.1) and taking $(U_n^-, 0)$ as test function, we obtain

$$\begin{aligned}&\int_Q \frac{|U_n(x)-U_n(y)|^{p-2}(U_n(x)-U_n(y))(U_n^-(x)-U_n^-(y))}{|x-y|^{N+sp}}\,dx\,dy\\&=\int_\Omega a(x)|U_n|^{q-2}U_nU_n^- - \frac{1-\alpha}{2-\alpha-\beta}\int_\Omega c(x)(|U_n|+\frac{1}{n})^{-\alpha}(|V_n|+\frac{1}{n})^{1-\beta}U_n^-\,dx.\end{aligned} \tag{3.5}$$

Then, by the elementary inequality $(a-b)(a_- - b_-) \leq -(a_- - b_-)^2$ we have

$$\begin{aligned}0 &\leq \int_Q \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(u_-(x)-u_-(y))}{|x-y|^{N+ps}}dxdy\\&\leq -\int_Q \frac{|u(x)-u(y)|^{p-2}(u_-(x)-u_-(y))^2}{|x-y|^{N+ps}}dxdy.\end{aligned} \tag{3.6}$$

Therefore, since the right hand side of the Eq. (3.5) is nonnegative and not equivalently to 0, we can deduce from (3.6) that $||U_n^-|| = 0$. Which implies that U_n is nonnegative and by the strong maximum principle we conclude that U_n is positive almost every where in Ω. In the same manner, we can prove that V_n is positive almost every where in Ω.

Step 3: The solutions (U_n) and (V_n) of the problem (2.1) are bounded from below.

From **step 1**, we know that (U_n) and (V_n) are bounded in $\mathcal{X}_0$. Therefore, by using a standard comparison argument (see Lemma 2.1 in [2]) combine with the strong maximum principle version of the fractional p-laplace operator in [4] we conclude that for all $K \subset\subset \Omega$ there exists C_K such that $U_n(x) \geq C_K > 0$ for a. e. $x \in K$ and for any $n \in \mathbb{N}$. The boundedness from below of V_n follows by the same manner.

Step 4: The problem (1.1) has a positive weak solution.

Firstly, by Step 1, (U_n) and (V_n) are bounded in $\mathcal{X}_0$. Thus, since the space $\mathcal{X}_0$ is reflexive, there exists a subsequence, still denoted by $\{U_n\}$ and $\{V_n\}$, which weakly converges to, say, $U,\ V \in \mathcal{X}_0$ such that

$$\begin{aligned}&U_n \to U \text{ and } V_n \to V \quad \text{weakly in } \mathcal{X}_0\\&U_n \to U \text{ and } V_n \to V \quad \text{strongly in } L^r(\Omega) \text{ for } 1 \leq r < p_s^*\\&U_n \to U \text{ and } V_n \to V \quad \text{pointwise a.e. in } \Omega.\end{aligned} \tag{3.7}$$

Now, since (U_n, V_n) is a positive solution of the problem (2.1), ones has

$$\begin{aligned}&\int_Q \frac{|U_n(x)-U_n(y)|^{p-2}(U_n(x)-U_n(y))(\phi(x)-\phi(y))}{|x-y|^{N+sp}}\,dx\,dy\\&+\int_Q \frac{|V_n(x)-V_n(y)|^{p-2}(V_n(x)-V_n(y))(\psi(x)-\psi(y))}{|x-y|^{N+sp}}\,dx\,dy\\&=\int_\Omega \Big(a(x)|U_n|^{q-2}U_n\phi + b(x)|V_n|^{q-2}V_n\psi\Big)dx\end{aligned}$$

$$+\frac{1-\alpha}{2-\alpha-\beta}\int_\Omega c(x)\Big(|U_n|+\frac{1}{n}\Big)^{-\alpha}\Big(|V_n|+\frac{1}{n}\Big)^{1-\beta}\phi\,dx$$
$$+\frac{1-\beta}{2-\alpha-\beta}\int_\Omega c(x)\Big(|U_n|+\frac{1}{n}\Big)^{1-\alpha}\Big(|V_n|+\frac{1}{n}\Big)^{-\beta}\psi\,dx$$

for all $(\phi,\psi)\in\mathcal{X}_0$. Then, by the weak convergence of U_n and V_n to U and V respectively, we have $|U_n(x)-U_n(y)|^{p-2}(U_n(x)-U_n(y))/|x-y|^{(N+sp)/p'}$ is bounded in $L^{p'}(\mathbb{R}^{2N})$ where p' is the Hölder conjugate of p given by $p'=\frac{p}{p-1}$, and converges to $|U(x)-U(y)|^{p-2}(U(x)-U(y))/|x-y|^{(N+sp)/p'}$ in $\mathbb{R}^{2N}$ and $(U(x)-U(y))/|x-y|^{(N+sp)/p}\in L^p(\mathbb{R}^{2N})$, so

$$\int_Q \frac{|U_n(x)-U_n(y)|^{p-2}(U_n(x)-U_n(y))(\phi(x)-\phi(y))}{|x-y|^{N+sp}}\,dx\,dy$$
$$\to\int_Q \frac{|U(x)-U(y)|^{p-2}(U(x)-U(y))(\phi(x)-\phi(y))}{|x-y|^{N+sp}}\,dx\,dy,$$

for any $\phi\in C_c^\infty(\Omega)$. Similarly,

$$\int_Q \frac{|V_n(x)-V_n(y)|^{p-2}(V_n(x)-V_n(y))(\psi(x)-\psi(y))}{|x-y|^{N+sp}}\,dx\,dy$$
$$\to\int_Q \frac{|V(x)-V(y)|^{p-2}(V(x)-V(y))(\psi(x)-\psi(y))}{|x-y|^{N+sp}}\,dx\,dy,\quad\text{for any }\psi\in X_0.$$

On the other hand, since $U_n\to U$ in $L^r(\text{supp}(\phi))$ and $V_n\to V$ in $L^r(\text{ supp}(\psi))$. We deduce, using the dominated convergence theorem, that

$$\int_\Omega a(x)|U_n|^{q-2}U_n\phi dx\to\int_\Omega a(x)|U|^{q-1}\phi \tag{3.8}$$

and

$$\int_\Omega b(x)|V_n|^{q-2}V_n\psi dx\to\int_\Omega b(x)|V|^{q-1}\psi. \tag{3.9}$$

Moreover, let $\phi\in C_0^\infty(\Omega)$ with $supp(\phi)=K\subset\subset\Omega$. By **Step 3**, there exists C_K such that

$$c(x)\left(|U_n|+\frac{1}{n}\right)^{-\alpha}\left(|V_n|+\frac{1}{n}\right)^{1-\beta}\phi\to c(x)U^{-\alpha}V^{1-\beta}\phi\ \text{ a.e.}$$

and

$$\left|c(x)\left(|U_n|+\frac{1}{n}\right)^{-\alpha}\left(|V_n|+\frac{1}{n}\right)^{1-\beta}\phi\right|\le C_k^{-\alpha}(M+1)^{1-\beta}|c||\phi|\in L^1(\Omega).$$

Applying, Lebsgue's dominated convergence theorem

$$\int_\Omega c(x)(|U_n|+\frac{1}{n})^{-\alpha}(|V_n|+\frac{1}{n})^{1-\beta}\phi\,dx\to\int_\Omega c(x)U^{-\alpha}V^{1-\beta}\phi\,dx. \tag{3.10}$$

Similarly, we obtain

$$\int_\Omega c(x)(|U_n|+\frac{1}{n})^{1-\alpha}(|V_n|+\frac{1}{n})^{-\beta}\psi\,dx\to\int_\Omega c(x)U^{1-\alpha}V^{-\beta}\psi\,dx. \tag{3.11}$$

Therefore, using Eq. (3.8), Eq. (3.8), Eq. (3.8), Eq. (3.9), Eq. (3.10) and Eq. (3.11), we conclude that

$$\begin{aligned}&\int_Q \frac{|U(x)-U(y)|^{p-2}(U(x)-U(y))(\phi(x)-\phi(y))}{|x-y|^{N+sp}}\,dx\,dy\\&+\int_Q \frac{|Vx)-V(y)|^{p-2}(V(x)-V(y))(\psi(x)-\psi(y))}{|x-y|^{N+sp}}\,dx\,dy\\=&\int_\Omega \Big(a(x)|U|^{q-2}U\phi+b(x)|V|^{q-2}v\psi\Big)dx+\frac{1-\alpha}{2-\alpha-\beta}\int_\Omega c(x)U^{-\alpha}V^{1-\beta}\phi\,dx\\&+\frac{1-\beta}{2-\alpha-\beta}\int_\Omega c(x)U^{1-\alpha}V^{-\beta}\psi\,dx\end{aligned}$$

for all $(\phi,\psi)\in C_c^\infty(\Omega)\times C_c^\infty(\Omega)$. Yielding, that (U,V) is a positive solution to the problem (1.1) and finished the prove of the Theorem 1.1.

4. Uniqueness of solution

Firstly, let us recall the following elementary inequality needed for the proof of uniqueness of solutions.

Lemma 4.1 (cf. [7, Lemma 2.3])**.** *Let $v_1,\ v_2\in W_0^{s,p}(\Omega)\backslash\{0\}$. There exists a positive constant c_p, depending only on p, such that*

$$\langle(-\Delta_p)^s v_1-(-\Delta_p)^s v_2, v_1-v_2\rangle\geq\begin{cases}\frac{||v_1-v_2||^2}{\left(||v_1||^p+||v_2||^p\right)^{\frac{2-p}{p}}}, & \text{if } 1<p<2\\ ||v_1-v_2||^p, & \text{if } p\geq 2.\end{cases}$$

We begin this section by introducing the following nonlocal pure singular problem:

$$\begin{cases}(-\Delta_p)^s u=\frac{1}{2}c(x)|u|^{1-2\alpha}, & \text{in } \Omega,\\ u=0, \quad \text{in } \mathbb{R}^N\setminus\Omega.\end{cases}\tag{4.1}$$

Now, we state our first result.

Lemma 4.2. *The problem* (4.1) *has at most a positive solution.*

Proof. We proceed by contradiction. Suppose that the problem (4.1) has two positives solutions u_1 and u_2 that is

$$\langle(-\Delta_p)^s u_1,\Phi\rangle=\frac{1}{2}\int_\Omega c(x)u_1^{1-2\alpha}\Phi\,\mathrm{d}x\tag{4.2}$$

and

$$\langle(-\Delta_p)^s u_2,\Phi\rangle=\frac{1}{2}\int_\Omega c(x)u_2^{1-2\alpha}\Phi\,\mathrm{d}x\tag{4.3}$$

for all Φ in $C_c^\infty(\Omega)$. Subtracting (4.2) from (4.3), we have

$$\langle(-\Delta_p)^s u_1-(-\Delta_p)^s u_2,\Phi\rangle=\frac{1}{2}\int_\Omega c(x)\left(u_1^{1-2\alpha}-u_2^{1-2\alpha}\right)\Phi\,\mathrm{d}x.\tag{4.4}$$

Choose $\Phi = (u_1 - u_2)$ as a test function. So, from the equation (4.4) we obtain

$$\begin{aligned} &\langle (-\Delta_p)^s u_1 - (-\Delta_p)^s u_2, (u_1 - u_2)\rangle \\ &= \frac{1}{2}\int_\Omega c(x)\left(u_1^{1-2\alpha} - u_2^{1-2\alpha}\right)(u_1 - u_2)\,\mathrm{d}x \\ &\le 0. \end{aligned} \tag{4.5}$$

Hence, according to Lemma 4.1, we must have $||u_1 - u_2|| = 0$, it follows that $u_1 = u_2$ a.e. in Ω. The proof of the Lemma 4.2. □

Proof of Theorem 1.2. Again as in Lemma 4.2, we proceed by contradiction. We suppose that (u_1, u_2) be the positive solution of the system

$$\begin{cases} (-\Delta_p)^s u = \frac{1}{2}c(x)|u|^{-\alpha}|v|^{1-\alpha}, & \text{in } \Omega, \\ (-\Delta_p)^s v = \frac{1}{2}c(x)|u|^{1-\alpha}|v|^{-\alpha}, & \text{in } \Omega, \\ u = v = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{4.6}$$

That is,

$$\langle (-\Delta_p)^s u_1, \Phi\rangle = \frac{1}{2}\int_\Omega c(x) u_1^{-\alpha} u_2^{1-\alpha}\Phi\,\mathrm{d}x \tag{4.7}$$

and

$$\langle (-\Delta_p)^s u_2, \Phi\rangle = \frac{1}{2}\int_\Omega c(x) u_1^{1-\alpha} u_2^{-\alpha}\Phi\,\mathrm{d}x \tag{4.8}$$

for all Φ in $C_c^\infty(\Omega)$. Now, taking $\Phi = (u_1 - u_2)$ as a test function, we have

$$\begin{aligned} 0 &\le \langle (-\Delta_p)^s u_1 - (-\Delta_p)^s u_2, (u_1 - u_2)\rangle \\ &= \frac{1}{2}\int_\Omega c(x)\left(u_1^{-\alpha} - u_2^{-\alpha}\right)(u_1 - u_2)(u_1 - u_2)\,\mathrm{d}x \\ &\le 0. \end{aligned} \tag{4.9}$$

So, from equation (4.9) and according to Lemma 4.1, we must have $||u_1 - u_2|| = 0$. Therefore, we get $u_1 = u_2$ almost everywhere in Ω. Hence, (u_1, u_1) is the unique solution to the system (4.6). This completes the proof of the Theorem 1.2. □

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