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A NEW THIRD ORDER EXPONENTIALLY FITTED DISCRETIZATION FOR THE SOLUTION OF NON-LINEAR TWO POINT BOUNDARY VALUE PROBLEMS ON A GRADED MESH

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Abstract This paper puts forward a novel graded mesh implicit scheme resting upon full step discretization of order three for computation of non-linear two point boundary value problems. The suggested method is compact and employs three nodal points for the unknown function $u(x)$ in spatial axis. We have also performed error analysis of the cited method. The given method was tried (implemented) upon multiple problems in Cartesian and Polar coordinates with extremely favorable outcomes. This method, though meant for scalar equations, was further extended to compute the vector equations of two point nonlinear boundary value problems. To check the validity of the proposed scheme, we applied it to multiple problems and obtained supporting numerical computations.

Keywords Exponentially fitted discretization on a graded mesh, two point non-linear boundary value problems, modified Emden equation, Vanderpol's equation, Hopf bifurcation for the coupled oscillator.

MSC(2010) 65L10, 65L12, 65L20.

1. Introduction

All the physical processes observed in various streams of sciences and technology can be categorized into two: Time Dependent processes and Stationary processes.

Boundary Value Problems (BVPs) are used to depict stationary processes. All physical problems can be expressed in mathematical language - ranging from simple differential equations to extremely complex formulations. To find a perfect solution to these mathematical expressions, scientists reach out to mathematicians for either an analytical or numerical solution. The simpler problems do have analytical solutions but incidentally many real world models have intricate behaviour and hence

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do not possess any readily available analytical solution. Hence scientists connect with numerical analysts for the best possible solution.

We have solved thirteen prominent BVPs, including Modified Emden Equation (MEE),Vander Pol's Equation and Hopf bifurcation for the coupled oscillator using the proposed method.

MEE is one of the models which characterizes nonlinear dynamics. Emden type equation (MEE), also know as the modified Painlev'e-Ince equation, has been widely used by scientist for over a century. This equation finds place in a host of mathematical problems such as single-valued functions specified by differential equations of order two [12], including the Riccati equation [6]. Furthermore, physicists have revealed that this equation appears in various states: such as the study of composition of equilibrium points of a spherical gas cloud influenced by conjoint magnetism of its atoms, and under the laws of energy.

Van der Pol oscillation equation forms the basis of plethora of processes involving oscillations in a number of streams including electronics, physics, biology and mechanics. Van der Pol formulated numerous electronic circuits mimicking the heart of a human being, to figure out the stabilizing aspects of a heart's irregular beating or "arrhythmias". Ever since this equation is widely employed by researchers in various natural processes including, but not limited to, modelling of group of neurons in the digestive grind track of the digestive cyst([13, 29]) (biological implementation) and modelling the interface of two plates in an environmental culpability [2] (seismological implementation).

In the past several schemes employing constant mesh have been established to solve two point BVPs ([3, 4, 10]). We mention some of the work done in the field of BVPs. A group of schemes based on geometric discretization of order three were developed by Jain *et al.* [15] for solving BVPs. The existence results for solution to non-linear boundary value problems of order four were presented by Regan [25]. Usmani developed and analysed finite difference methods of even order BVPs associated with a fourth order linear ODE [32]. Singular two-point BVP were dealt by Tariq *et al.* by developing a three point finite difference method [1]. Chawla presented a new 4th order finite difference method based on constant mesh for the two-point BVP which are singular in nature [5]. Mohanty *et al.* [18,20] proposed the application of two parameter alternating group explicit (TAGE) iterative method to an efficient third order numerical method for two point non-linear boundary value problems on a variable mesh as well as dealt with the singular problems. Dehghan *et al.* have proposed a compact 4th order scheme, for computing the 1-D non-linear Klein Gordon equation [7]. Dehghan and Tatari have presented a numerical scheme which solves BVPs of order three, prescribed with boundary conditions that are non-linear in nature [8]. Selim *et al.* studied numerical behavior of the singular two point BVPs through various numerical techniques [28]. Ghomanjani *et al.* discussed numerical solution of non-linear two point BVPs for ordinary differential equations by Bezier curve method and orthonormal Bernstein polynomials using Gram Schmidt technique [11]. Zhu *et al.* discussed a new approach for solving non-linear singular BVP based on Quasi Newton's method and the simplified reproducig Kernel method [36]. Niu *et al.* proposed an efficent method for solving non-linear singular BVPs using reproducing kernel method [35]. Roul *et al.* developed a scheme for solving singular two point BVP and studied its convergence properties [26]. Mohanty *et al.* [23] have developed fourth order compact approximations in exponential form for the solution of 2D quasilinear elliptic BVPs on a constant mesh. Also, recently Mohanty *et al.* [24] have proposed a third order accurate exponentially fitted compact off-step discretization for the solution of non-linear two point boundary value problems on a variable mesh. However, their method involves more algebra for computation. In this paper, using only three grid points and less algebra, we propose a new variable mesh method in exponential form for the solution of non-linear two point BVP.

Motivated by the above literature survey, we have developed a new robust exponentially fitted finite difference scheme for solving BVPs, which is evident from the computational results. The worthiness of our work lies in the fact that this method can be applied to solve the problems both in Cartesian as well as Polar coordinates. The solution progressively worsens in the proximity of the singularity $r = 0$. To resolve this concern, we have reformed the scheme in such a manner so as to accommodate singular equations as well.

We consider the 2ndorder non-linear boundary value problem of the type

$$u'' = f(x, u, u'), x \in (a, b), \tag{1.1}$$

associated with

$$u(0) = r_1, u(1) = r_2, \tag{1.2}$$

r_1, r_2 being constants.

We have assumed that for $x \in (0, 1)$ $u, u' \in (-\infty, \infty)$:

(i) $f(x, u, u')$ is continuous,

(ii) $\frac{\partial f}{\partial u}$, $\frac{\partial f}{\partial u'}$ exist and are continuous,

(iii) $\frac{\partial f}{\partial u} > 0$ and $\left|\frac{\partial f}{\partial u'}\right| \leq K$, where K is a positive constant.

The above conditions ensure the existence and uniqueness of the boundary value problem (1.1)-(1.2) [16]. We pre-suppose that these conditions are obeyed in the problem which we would solve numerically.

The paper is arranged as follows: Section 2 describes the numerical scheme to solve two point BVPs (1.1)-(1.2). Section 3 gives detailed explanation of the method based on exponential approximations. Section 4 discusses about the convergence analysis. In section 5, we apply our scheme to singular boundary value problems. Section 6 extends our method in order to deal with the vector form of non-linear BVPs of second order. Section 7 is devoted to application of our method to fourth and sixth order BVPs. Further in Section 8, we have solved some standard problems and compared results with other prevailing schemes. Finally, Section 9, concludes the paper.

2. Conception of the scheme

To compute the proposed problem (1.1)-(1.2) numerically, discretize the domain of solution $[0, 1]$ into $(N + 2)$ mesh points $x_k, k = 0(1)N + 1$ such that $0 = x_0 < x_1 < ... < x_{N+1} = 1, N \in \mathbb{Z}^+$, with variable mesh spacing $h_k = x_k - x_{k-1}, k = 1(1)(N+1)$ and the mesh ratio $\eta = (\frac{h_{k+1}}{h_k}) > 0, k = 1(1)N$. For $\eta > 1$ mesh sizes are increasing in order and for $\eta < 1$ the mesh sizes are decreasing in order.

Suppose $U_k = u(x_k)$ be the exact value and u_k be numerical value at the point x_k.

Further, at the nodal point x_k , notate

$$P = \eta^2 + \eta - 1,$$
$$Q = (1 + \eta)(1 + 3\eta + \eta^2),$$
$$R = \eta(1 + \eta - \eta^2).$$

Further, for the third order discretization of the equation (1.1) we employ the below estimations:

$$\overline{U}'_k = \frac{U_{k+1} - (1 - \eta^2)\, U_k - \eta^2\, U_{k-1}}{\eta(1 + \eta)h_k}, \tag{2.1}$$

$$\overline{U}'_{k+1} = \frac{(1 + 2\eta)\, U_{k+1} - (1 + \eta)^2\, U_k + \eta^2\, U_{k-1}}{\eta(1 + \eta)h_k}, \tag{2.2}$$

$$\overline{U}'_{k-1} = \frac{-\, U_{k+1} + (1 + \eta)^2\, U_k - \eta(2 + \eta)\, U_{k-1}}{\eta(1 + \eta)h_k}. \tag{2.3}$$

Define

$$\overline{F}_{k\pm1} = f(x_{k\pm1}, U_{k\pm1}, \overline{U}'_{k\pm1}). \tag{2.4}$$

Let us define the following approximations

$$\hat{U'}_k = \overline{U}'_k + ah_k(\overline{F}_{k+1} - \overline{F}_{k-1}) \tag{2.5}$$

where

$$a = \frac{-\eta}{6(1 + \eta)}.$$

$$\hat{F}_k = f(x_k, U_k, \hat{U'}_k), \tag{2.6}$$

$$\overline{\overline{U}}'_k = \overline{U}'_k + bh_k(\overline{F}_{k+1} - \overline{F}_{k-1}), \tag{2.7}$$

where

$$b = \frac{-\eta(1 - 5\eta + \eta^2)}{6(1 + \eta)(1 - 3\eta + \eta^2)}.$$

$$\overline{\overline{F}}_k = f(x_k, U_k, \overline{\overline{U}}'_k). \tag{2.8}$$

After discretization(1.1) at internal nodal point x_k , we get:

$$U_{k+1} - (1+\eta)U_k + \eta U_{k-1} = \eta(1+\eta)\frac{h_k^2}{2}\hat{F}_k \exp\left(\frac{P\overline{F}_{k+1} + R\overline{F}_{k-1} - (P+R)\overline{\overline{F}}_k}{6\eta(1+\eta)\hat{F}_k}\right) + T_k,$$
$$k = 1(1)N, \quad \text{where} \quad T_k = O(h_k^5). \tag{2.9}$$

3. Derivation of the numerical algorithm

To derive the scheme (2.9), at the nodal point (x_k), use the following notations:

$$\alpha_k = \frac{\partial f}{\partial u_k}, \beta_k = \frac{\partial f}{\partial u'_k}.$$

Using Taylor's expansion the estimations (2.1)-(2.4) reduce to:

$$\overline{U}'_k = U'_k + \frac{\eta {h_k}^2}{6} U'''_k + O(h_k^3), \tag{3.1}$$

$$\overline{U}'_{k+1} = U'_{k+1} - \frac{\eta(1+\eta){h_k}^2}{6} U'''_k + O(h_k^3), \tag{3.2}$$

$$\overline{U}'_{k-1} = U'_{k-1} - \frac{(1+\eta){h_k}^2}{6} U'''_k + O(h_k^3), \tag{3.3}$$

$$\overline{F}_{k+1} = F_{k+1} - \frac{\eta(1+\eta){h_k}^2}{6} U'''_k \beta_k + O(h_k^3), \tag{3.4}$$

$$\overline{F}_{k-1} = F_{k-1} - \frac{(1+\eta){h_k}^2}{6} U'''_k \beta_k + O(h_k^3). \tag{3.5}$$

Using the approximations (3.1),(3.4) and (3.5) in (2.5) we get

$$\hat{U}'_k = U'_k + \frac{h_k^2}{6}\Big(\eta + 6a(1+\eta)\Big) U'''_k + O(h_k^3). \tag{3.6}$$

Note that,

$$\hat{U}'_k = U'_k + O(h_k^3) \quad \text{if} \quad a = \frac{-\eta}{6(1+\eta)}.$$

It can be verified easily that

$$\hat{F}_k = F_k + O(h_k^3). \tag{3.7}$$

Using the approximations (3.1), (3.4) and (3.5) in (2.7), we get

$$\overline{\overline{U}}'_k = U'_k + \frac{h_k^2}{6}\Big(\eta + 6b(1+\eta)\Big) U'''_k + O(h_k^3). \tag{3.8}$$

Now using (3.8) in (2.8), we get

$$\overline{\overline{F}}_k = F_k + \frac{h_k^2}{6}\Big(\eta + 6b(1+\eta)\Big) U'''_k \beta_k + O(h_k^3). \tag{3.9}$$

Further, substituting (3.4), (3.5), (3.7) and (3.9) in (2.9) and using

$$\begin{aligned} U_{k+1} - (1+\eta)U_k + \eta U_{k-1} =& \eta(1+\eta)\frac{h_k^2}{2} F_k \exp\left(\frac{PF_{k+1} + RF_{k-1} - (P+R)F_k}{6\eta(1+\eta)F_k}\right) \\ &+ O(h_k^5), k = 1(1)N, \end{aligned} \tag{3.10}$$

we get the local truncation error as

$$T_k = -\frac{h_k^4}{72}(1+\eta)\big[(1-3\eta+\eta^2)\big(\eta + 6b(1+\eta)\big) - 2\eta^2\big] U''''_k \beta_k + O(h_k^5). \tag{3.11}$$

Hence the parameter b used in the estimate (2.7) is computed as

$$b = \frac{-\eta(1-5\eta+\eta^2)}{6(1+\eta)(1-3\eta+\eta^2)}$$

and the local truncation error as $T_k = O(h_k^5)$.

4. Convergence Analysis

Consider the non-linear problem

$$u'' = f(x, u, u'), 0 < x < 1, \tag{4.1}$$

with the boundary values

$$u(0) = A, u(1) = B.$$

Then the difference method (2.9) for the equation(4.1) becomes

$$U_{k+1} - (1+\eta)U_k + \eta U_{k-1} = \frac{h_k^2}{12}[(P+Q+R)\hat{F}_k + P\overline{F}_{k+1} + R\overline{F}_{k-1} - (P+R)\overline{\overline{F}}_k] + \overline{T}_k,$$
$1 \le k \le N+1,$

where $\overline{T}_k = O(h_k^5)$, $\eta \neq 1$, $\overline{T}_k = O(h_k^6)$, $\eta = 1$. (4.2)

Let

$$m_k = \frac{h_k^2}{12}[(P+Q+R)\hat{F}_k + P\overline{F}_{k+1} + R\overline{F}_{k-1} - (P+R)\overline{\overline{F}}_k].$$

Then the method (4.2) in the form of matrix may be expressed as

$$\boldsymbol{DU} + \boldsymbol{MU} + \boldsymbol{T(h_k)} = 0, \tag{4.3}$$

where

$$\boldsymbol{D = [-\eta, (1+\eta), -1]_{N\times N}}, \quad \boldsymbol{U = [U_1, U_2, ..., U_N]^T},$$
$$\boldsymbol{M(U)} = [m_1 - \eta A, m_2, ..., m_{N-1}, m_n - B]^T,$$
$$\boldsymbol{T(h_k)} = [T_1(h_k), T_2(h_k), ..., T_N(h_k)]^T \quad \text{and}$$
$$\mathbf{0} = [0, 0, ..., 0]^T.$$

Suppose $\boldsymbol{u} = [u_1, u_2, ..., u_n]^T \approx \boldsymbol{U}$.

Then at each x_k we need to solve the difference equation of the form

$$\boldsymbol{Du} + \boldsymbol{M(u)} = 0. \tag{4.4}$$

Consider the error at k^{th} nodal point $\varepsilon_k = u_k - U_k$, k=1(1)N so that

$$\boldsymbol{u - U = E} = [\epsilon_1, \epsilon_2, ..., \epsilon_N]^T. \tag{4.5}$$

Let

$$\overline{f}_{k+1} = f(x_{k+1}, u_{k+1}, \overline{u}'_{k+1}) \approx \overline{F}_{k+1}, \tag{4.6a}$$
$$\overline{f}_{k-1} = f(x_{k-1}, u_{k-1}, \overline{u}'_{k-1}) \approx \overline{F}_{k-1}, \tag{4.6b}$$
$$\hat{f}_k = f(x_k, u_k, \hat{u}'_k) \approx \hat{F}_k, \tag{4.6c}$$
$$\overline{\overline{f}}_k = f(x_k, u_k, \overline{\overline{u}}'_k) \approx \overline{\overline{F}}_k. \tag{4.6d}$$

We may write

$$\overline{f}_{k+1} - \overline{F}_{k+1} = (u_{k+1} - U_{k+1})G_{k+1} + (\overline{u}'_{k+1} - \overline{U}'_{k+1})H_{k+1}, \tag{4.7a}$$

$$\overline{f}_{k-1} - \overline{F}_{k-1} = (u_{k-1} - U_{k-1})G_{k-1} + (\overline{u}'_{k-1} - \overline{U}'_{k-1})H_{k-1}, \tag{4.7b}$$

$$\hat{f}_k - \hat{F}_k = (u_k - U_k)G_k^{(1)} + (\hat{u}'_k - \hat{U}'_k)H_k^{(1)}, \tag{4.7c}$$

$$\overline{\overline{f}}_k - \overline{\overline{F}}_k = (u_k - U_k)G_k^{(2)} + (\overline{\overline{u}}'_k - \overline{\overline{U}}'_k)H_k^{(2)}, \tag{4.7d}$$

for suitable $G's$ and $H's$.

Note that,

$$H_{k\pm1} = H_k \pm h_k H'_k + O(h_k^2), \tag{4.8a}$$

$$G_{k\pm1} = G_k \pm O(h_k). \tag{4.8b}$$

Using equations (4.7a), (4.7b), (4.7c), (4.7d) and (4.8a), (4.8b) we get

$$\boldsymbol{M(u) - M(U) = PE} \tag{4.9}$$

where $\boldsymbol{P} = (P_{i,j})$, $((i = 1(1)N), j = 1(1)N)$ is tri-diagonal matrix with

$$\begin{aligned}
P_{k,k} &= \text{coefficent of} \quad \epsilon_k \\
&= \frac{h_k^2}{12}[6\eta(1+\eta)G_k^{(1)} - (1+\eta)(3\eta - 1 - \eta^2)G_k^{(2)} - 2\eta(1+\eta)H'_k \\
&\quad +2\eta(1+\eta)H_k^{(1)}H_k + \frac{(1-5\eta+\eta^2)(1+\eta)}{3}H_kH_k^{(2)}] \\
&\quad -\frac{h_k}{12}[\frac{(1+\eta)(\eta^3-1)}{\eta}H_k + 6(1-\eta^2)H_k^{(1)} \\
&\quad -\frac{(3\eta-1-\eta^2)(1-\eta^2)}{\eta}H_k^{(2)}] + O(h_k^3), \quad (k = 1, N), \\
P_{k,k+1} &= \text{coefficent of} \quad \epsilon_{k+1} \\
&= \frac{h_k}{12}[6H_k^{(1)} - \frac{(3\eta-1-\eta^2)}{\eta}H_k^{(2)} + \frac{(3\eta^3+2\eta^2-2\eta-1)}{\eta(1+\eta)}H_k] \\
&\quad +\frac{h_k^2}{12}[(\eta^2+\eta-1)G_k - 2\eta H_k^{(1)}H_k \\
&\quad -\frac{(1-5\eta+\eta^2)}{3}H_kH_k^{(2)} + 2\eta^2H'_k] + O(h_k^3), \quad (k = 1, N-1), \\
P_{k,k-1} &= \text{coefficent of} \quad \epsilon_{k-1} \\
&= \frac{h_k}{12}[-6\eta^2H_k^{(1)} + \eta^3(3\eta-1-\eta^2)H_k^{(2)} + \frac{\eta(\eta^3+2\eta^2-2\eta-3)}{(1+\eta)}H_k] \\
&\quad +\frac{\eta h_k^2}{12}[(1+\eta-\eta^2)G_k - 2\eta H_k^{(1)}H_k \\
&\quad -\frac{(1-5\eta+\eta^2)}{3}H_kH_k^{(2)} + 2H'_k] + O(h_k^3), \quad (k = 1, N-1).
\end{aligned}$$

With the help of (4.9) from (4.3) and (4.4), on ignoring round-off errors, we get following error equation:

$$\boldsymbol{(D{+}P)E{=}T} \tag{4.10}$$

where

$$\boldsymbol{D+P}=\begin{bmatrix} (1+\eta)+P_{11} & -1+P_{12} & 0 & \cdots & 0 \\ -\eta+P_{21} & (1+\eta)+P_{22} & -1+P_{23} & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & -\eta+P_{N-1N-2} & (1+\eta)+P_{N-1N-1} & -1+P_{N-1N} \\ 0 & \cdots & 0 & -\eta+P_{NN-1} & (1+\eta)+P_{NN} \end{bmatrix}.$$

Let

$$G_* = \min_{0<x<1} \frac{\partial f}{\partial u} \text{ and } G^* = \max_{0<x<1} \frac{\partial f}{\partial u}$$

then,

$$0 < G_* \leq G_{k+i}, G_k^{(1)}, G_k^{(2)} \leq G^*, i = \pm 1$$

and let

$$|H_{k+i}| \leq H \text{ for some positive constant } \; H, \quad (i = 0, \pm 1),$$

then

$$|H_k^{(1)}| \leq H, |H_k^{(2)}| \leq H \text{ and } |H_k'| \leq H^* \text{ for some positive constant } H^*.$$

Now it can be certainly verified that for howsoever small h_k,

$$-1 < P_{k,k\pm1} < 1.$$

Hence, ***(D+P)*** is an irreducible matrix. [14, 34]

Let S_k be the sum of the elements of the kth row of $(\boldsymbol{D}+\boldsymbol{P})$, then for $k = 1$

$$\begin{aligned} S_k = \eta + \frac{h_k}{12}[\frac{(-3\eta^4 + 2\eta^2 + 3\eta)}{(1+\eta)} H_k + 6\eta^2 H_k^{(1)} - \eta(3\eta - 1 - \eta^2) H_k^{(2)}] \\ + \frac{h_k^2}{12}[(PG_k + (P+Q+R)G_k^{(1)} - (1+\eta)(3\eta - 1 - \eta^2)G_k^{(2)} - 2\eta H_k') \\ + 2\eta^2 H_k H_k^{(1)} + \frac{\eta(1 - 5\eta + \eta^2)}{3} H_k H_k^{(2)}] + O(h_k^3), \end{aligned} \tag{4.11a}$$

for k= N

$$\begin{aligned} S_k = 1 + \frac{h_k}{12}[\frac{(-3\eta^3 - 2\eta^2 + 2\eta + 1)}{\eta(1+\eta)} H_k - 6H_k^{(1)} + \frac{(3\eta - 1 - \eta^2)(\eta^4 + 1 - \eta^2)}{\eta} H_k^{(2)}] \\ + \frac{h_k^2}{12}[(RG_k + (P+Q+R)G_k^{(1)} - (1+\eta)(3\eta - 1 - \eta^2)G_k^{(2)} - 2\eta^2 H_k') \\ + 2\eta H_k H_k^{(1)} + \frac{(1 - 5\eta + \eta^2)}{3} H_k H_k^{(2)}] + O(h_k^3), \end{aligned} \tag{4.11b}$$

and for $k = 2(1)N-1$

$$\begin{aligned} S_k = \frac{h_k^2}{12}[(P+Q+R)G_k^{(1)} - (1+\eta)(3\eta - 1 - \eta^2)G_k^{(2)} + PG_k + \eta(1 + \eta - \eta^2)G_k] \\ + \frac{h_k}{12}[(\eta^2 - 1)(3\eta^2 - \eta - \eta^3)H_k^{(2)}] + O(h_k^4). \end{aligned} \tag{4.11c}$$

Therefore, for h_k howsoever small

$$S_1 > \frac{(P+P+Q+R)h_k^2}{12}G_*; \quad k=1, \tag{4.12a}$$

$$S_N > \frac{(R+P+Q+R)h_k^2}{12}G_*; \quad k=N, \tag{4.12b}$$

$$S_k \geq \frac{h_k^2}{2}\eta(1+\eta)G_*; \quad k=2(1)N-1. \tag{4.12c}$$

Thus, for h_k howsoever small, $\boldsymbol{(D{+}P)}$ is monotone. Hence $\boldsymbol{(D{+}P)}^{-1}$ exists and $\boldsymbol{(D{+}P)}^{-1} > \mathbf{0}$.

Let $(i,k)^{th}$ element of $\boldsymbol{(D+P)^{-1}}$ be $(D+P)^{-1}_{i,k}$ with $(D+P)^{-1}_{i,k} \geq 0$. Since,

$$\sum_{k=1}^{N}(D+P)^{-1}_{i,k}S_k = 1, \quad i=1(1)N. \tag{4.13a}$$

Hence

$$(D+P)^{-1}_{i,k}S_k \leq 1, \quad i=1(1)N, k=1 \text{ and } N. \tag{4.13b}$$

By the help of (4.12a) and (4.12b),we have

$$(D+P)^{-1}_{i,1} \leq \frac{1}{S_1} \leq \frac{12}{(2P+Q+R)h_kG_*}, \quad i=1(1)N,\ k=1,$$

$$(D+P)^{-1}_{i,N} \leq \frac{1}{S_N} \leq \frac{12}{(2R+P+Q)h_kG_*}, \quad i=1(1)N,\ k=1.$$

Further,

$$\sum_{k=2}^{N-1}(D+P)^{-1}_{i,k}\min_{2\leq k\leq N-1}S_k \leq 1, \quad i=1(1)N,$$

$$\sum_{k=2}^{N-1}(D+P)^{-1}_{i,k} \leq \frac{1}{\min_{2\leq k\leq N-1}S_k}, \quad i=1(1)N.$$

Thus, by the help of (4.12c), we have

$$\sum_{k=2}^{N-1}(D+P)^{-1}_{i,k} \leq \frac{1}{\min_{2\leq k\leq N-1}S_k} \leq \frac{2}{h_k^2\eta(1+\eta)G_*}, \quad i=1(1)N. \tag{4.13c}$$

As $\boldsymbol{(D+P)^{-1}}$ exists, thereby we may write the equation of error as

$$\boldsymbol{E} = \boldsymbol{(D+P)^{-1}(h_k)}. \tag{4.14}$$

Taking norm on both sides, we get

$$\|\boldsymbol{E}\| = \|\boldsymbol{(D+P)^{-1}}\|\|\boldsymbol{T(h_k)}\|. \tag{4.15}$$

Now,

$$\|\boldsymbol{(D+P)^{-1}}\| = \max_{1\leq i\leq N}\sum_{k=1}^{N}|(D+P)^{-1}_{i,k}| = \max_i(|(D+P)^{-1}_{i,1}|$$

$$+\,|\sum_{k=2}^{N-1}|(D+P)^{-1}_{i,k}|+|(D+P)^{-1}_{i,N}|). \tag{4.16}$$

Using inequalities (4.12a),(4.12b),(4.12c),(4.13a),(4.13b) and (4.13c)in (4.16), we get

$$\|\boldsymbol{(D{+}P)}^{-1}\| \le (\frac{1}{2P+Q+R}+\frac{1}{2R+P+Q}+\frac{1}{P+Q+R})\frac{12}{h_k^2 G_*}. \tag{4.17}$$

Finally, for appropriately small h_k, from (4.15)and (4.17)we get

$$\|\boldsymbol{E}\| \le O(h_k^3). \tag{4.18}$$

This builds the convergence of the method of order three for the elliptic equation (4.1) on non-constant mesh.

5. Application to singular boundary value problem

Let us study linear singular two-point boundary value problem

$$u''+\frac{\beta}{r}u'-\frac{\beta}{r^2}u=g(r), r\in(0,1). \tag{5.1}$$

with associated boundary conditions as stated in (1.2).

Here $\alpha\in(0,1)$or it can take values 1 or 2. When $\alpha=1$ or 2, equation (5.1) appears as cylindrical or spherical problem respectively.

With the help of the numerical scheme (2.9), the difference scheme for (5.1) after simplification takes the form:

$$\begin{aligned}U_{k+1}-(1+\eta)U_k+\eta U_{k-1}=\frac{h_k^2}{12}\Big(&-\alpha D_0(I+M_k)\overline{U}'_k-\frac{\alpha}{r_{k+1}}L_k\overline{U}'_{k+1}-\frac{\alpha}{r_{k-1}}N_k\overline{U}'_{k-1}\\&+\alpha E_0(I+M_k)U_k+\frac{\alpha}{r_{k+1}^2}L_kU_{k+1}+\frac{\alpha}{r_{k-1}^2}N_kU_{k-1}\\&+(I+M_k)G_0+L_kg_{k+1}+N_kg_{k-1}\Big)+O(h_k^5),\end{aligned} \tag{5.2}$$

where

$$\begin{aligned}&D_0=\frac{1}{r_k}, E_0=\frac{1}{r_k^2}, I=\frac{\eta(1+\eta)}{2},\\&P=\eta^2+\eta-1, Q=(1+\eta)(1+3\eta+\eta^2), R=\eta(1+\eta-\eta^2),\\&J_k=\frac{h_k^2}{12}P, K_k=\frac{h_k^2}{12}R, M_k=-\frac{h_k^2}{12}(P+R),\\&G_0=g_k=g(r_k), g_{k\pm1}=g(r_{k\pm1}),\\&L_k=J_k-\frac{\alpha}{r_k}(aI+bM_k)h_k,\\&N_k=K_k+\frac{\alpha}{r_k}(aI+bM_k)h_k.\end{aligned}$$

The scheme (5.2) is of $O(h_k^3)$for the solution of (5.1). Although, the scheme does not work when we try to find the solution at k =1. To circumvent this problem, we adopt the below estimations:

$$\frac{1}{r_{k+1}} = \frac{1}{r_k} - \frac{\eta h_k}{r_k^2} + \frac{\eta^2 h_k^2}{r_k^3} + O(h_k^3) \equiv D_1, \tag{5.3}$$

$$\frac{1}{r_{k-1}} = \frac{1}{r_k} + \frac{h_k}{r_k^2} + \frac{h_k^2}{r_k^3} + O(h_k^3) \equiv D_2, \tag{5.4}$$

$$\frac{1}{r_{k+1}^2} = \frac{1}{r_k^2} - \frac{2\eta h_k}{r_k^3} + \frac{3\eta^2 h_k^2}{r_k^4} + O(h_k^3) \equiv E_1, \tag{5.5}$$

$$\frac{1}{r_{k-1}^2} = \frac{1}{r_k^2} + \frac{2h_k}{r_k^3} + \frac{3h_k^2}{r_k^4} + O(h_k^3) \equiv E_2, \tag{5.6}$$

$$g_{k+1} = g_k + \eta h_k g_k' + \frac{\eta^2 h_k^2}{2} g_k'' + O(h_k^3) \equiv G_1, \tag{5.7}$$

$$g_{k-1} = g_k - h_k g_k' + \frac{h_k^2}{2} g_k'' + O(h_k^3) \equiv G_2, \tag{5.8}$$

where $g_k' = \frac{dg_k}{dr}$,...etc. Now substituting approximations (5.3)-(5.8)in (5.2)and combining higher order terms with the local truncation error, we deduce

$$\begin{aligned} U_{k+1} - (1+\eta)U_k + \eta U_{k-1} = \frac{h_k^2}{12}\Big(& -\alpha D_0 (I + M_k)\overline{U}_k' - \alpha D_1 L_k \overline{U}_{k+1}' - \alpha D_2 N_k \overline{U}_{k-1}' \\ & + \alpha E_0 (I + M_k) U_k + \alpha E_1 L_k U_{k+1} + \alpha E_2 N_k U_{k-1} \\ & + (I + M_k) G_0 + L_k g_{k+1} + N_k g_{k-1}\Big) + O(h_k^5). \end{aligned} \tag{5.9}$$

The numerical scheme has local truncation error $O(h_k^3)$ and does not have expressions $\frac{1}{k\pm1}$, so can be readily calculated for $k = 1(1)N$ in the region $(0, 1)$.

6. Method extended to vector form

Here, we consider the vector form of nonlinear ordinary differential equations

$$\frac{d^2u^{(i)}}{dx^2} = f^{(i)}\Big(x, u^{(1)}, u^{(2)}, ..., u^{(M)}, \frac{du^{(1)}}{dx}, \frac{du^{(2)}}{dx}, ..., \frac{du^{(M)}}{dx}\Big), a < x < b, \quad i = 1(1)M, \tag{6.1}$$

with associated boundary conditions of the type

$$u^i(0) = r_1^{(i)}, u^i(1) = r_2^{(i)}, \quad i = 1(1)M; \qquad r_1^{(i)} \text{and} r_2^{(i)} \text{being the constants.} \tag{6.2}$$

We suppose , that for $a < x < b, -\infty < u^{(i)}, u'^{(i)} < \infty$

(i) $f^{(i)}(x, u^{(1)}, u^{(2)}, ..., u^{(M)}, u'^{(1)}, u'^{(2)}, ..., u'^{(M)})$ are continuous in (a, b) ,

(ii) $\frac{\partial f^{(i)}}{\partial u^{(j)}}, j = 1(1)M$, exist and are continuous,

(iii) $\frac{\partial f^{(i)}}{\partial u^{(j)}} > 0$ and $\left|\frac{\partial f^{(i)}}{\partial u'^{(j)}}\right| \leq K_{(j)}^{(i)}$ for some positive constants$K_{(j)}^{(i)}$ $j = 1(1)M$.

The above conditions guarantee the uniqueness of solution of (6.1)-(6.2) [16].

For $i = 1(1)M$, let $U_k^{(i)} = u^{(i)}(x_k)$ be the exact value of $u^{(i)}$ at the mesh point x_k and $u_k^{(i)}$ be the approximate value.

Further, for $i = 1(1)M$ define the following estimates:

$$\overline{U}'^{(i)}_{k} = \frac{U^{(i)}_{k+1} - (1-\eta^2)U^{(i)}_{k} - \eta^2 U^{(i)}_{k-1}}{\eta(1+\eta)h_k}, \tag{6.3}$$

$$\overline{U}'^{(i)}_{k+1} = \frac{(1+2\eta)U^{(i)}_{k+1} - (1+\eta)^2 U^{(i)}_{k} + \eta^2 U^{(i)}_{k-1}}{\eta(1+\eta)h_k}, \tag{6.4}$$

$$\overline{U}'^{(i)}_{k-1} = \frac{-U^{(i)}_{k+1} + (1+\eta)^2 U^{(i)}_{k} - \eta(2+\eta) U^{(i)}_{k-1}}{\eta(1+\eta)h_k}. \tag{6.5}$$

Then we define

$$\overline{F}^{(i)}_{k\pm1} = f^{(i)}(x_{k\pm1}, U^{(1)}_{k\pm1}, U^{(2)}_{k\pm1}, ..., U^{(M)}_{k\pm1}, \overline{U}'^{(1)}_{k\pm1}, \overline{U}'^{(2)}_{k\pm1}, ..., \overline{U}'^{(M)}_{k\pm1}). \tag{6.6}$$

Let

$$\hat{U}'^{(i)}_{k} = \overline{U}'^{(i)}_{k} + ah_k(\overline{F}^{(i)}_{k+1} - \overline{F}^{(i)}_{k-1}), \tag{6.7}$$

$$\hat{F}^{(i)}_{k} = f^{(i)}(x_k, U^{(1)}_{k}, U^{(2)}_{k}, ...,, U^{(M)}_{k}, \hat{U}'^{(1)}_{k}, \hat{U}'^{(2)}_{k}, ...,, \hat{U}'^{(M)}_{k}), \tag{6.8}$$

$$\overline{\overline{U}}'^{(i)}_{k} = \overline{U}'^{(i)}_{k} + bh_k(\overline{F}^{(i)}_{k+1} - \overline{F}^{(i)}_{k-1}), \tag{6.9}$$

$$\overline{\overline{F}}^{(i)}_{k} = f^{(i)}(x_k, U^{(1)}_{k}, U^{(2)}_{k}, ..., U^{(M)}_{k}, \overline{\overline{U}}'^{(1)}_{k}, \overline{\overline{U}}'^{(2)}_{k}, ..., \overline{\overline{U}}'^{(M)}_{k}), \tag{6.10}$$

where the values of 'a' and 'b' are defined in section 2. Then at each internal grid point x_k, vector form of equations (6.1) and (6.2) are discretized by:

$$U^{(i)}_{k+1} - (1+\eta)U^{(i)}_{k} + \eta U^{(i)}_{k-1} = \eta(1+\eta)\frac{h_k^2}{2}\hat{F}^{(i)}_{k} \exp\left(\frac{P\overline{F}^{(i)}_{k+1} + R\overline{F}^{(i)}_{k-1} - (P+R)\overline{\overline{F}}^{(i)}_{k}}{6\eta(1+\eta)\hat{F}^{(i)}_{k}}\right) + T^{(i)}_{k},$$

$$k = 1(1)N, \quad \text{where} \quad T_k = O(h_k^5). \tag{6.11}$$

7. Application to fourth and sixth order boundary value problems

7.1. Fourth order boundary value problems

We study the following 4^{th} order non-linear differential equation

$$\frac{d^4u}{dx^4} = f(x, u, u', u'', u'''), \quad x \in (a, b), \tag{7.1.1}$$

associated with the following conditions

$$u(a) = A_0, u''(a) = A_1, u(b) = B_0, u''(b) = B_1, \tag{7.1.2}$$

where $A_i, B_i, i = 0, 1$ are constants.

Now, (7.1.1) can be written in equivalent form as:

$$u'' = v, \quad a < x < b, \tag{7.1.3}$$
$$v'' = f(x, u, v, u', v'), \quad a < x < b. \tag{7.1.4}$$

The boundary conditions (7.1.2) reduce to

$$u(a) = A_0, v(a) = A_1, u(b) = B_0, v(b) = B_1. \tag{7.1.5}$$

Using the method (6.11) in the system of differential equations (7.1.3)-(7.1.4), we get

$$U_{k+1} - (1+\eta)U_k + \eta U_{k-1} = \eta(1+\eta)\frac{h_k^2}{2}\hat{F}_k^{(1)} \exp\left(\frac{P\overline{F}_{k+1}^{(1)} + R\overline{F}_{k-1}^{(1)} - (P+R)\overline{\overline{F}}_k^{(1)}}{6\eta(1+\eta)\hat{F}_k^{(1)}}\right) + O(h_k^5), \tag{7.1.6}$$

$$V_{k+1} - (1+\eta)V_k + \eta V_{k-1} = \eta(1+\eta)\frac{h_k^2}{2}\hat{F}_k^{(2)} \exp\left(\frac{P\overline{F}_{k+1}^{(2)} + R\overline{F}_{k-1}^{(2)} - (P+R)\overline{\overline{F}}_k^{(2)}}{6\eta(1+\eta)\hat{F}_k^{(2)}}\right) + O(h_k^5), \tag{7.1.7}$$

where

$$\overline{F}_{k\pm1}^{(1)} = V_{k\pm1}, \overline{\overline{F}}_k^{(1)} = \hat{F}_k^{(1)} = V_k,$$
$$\overline{F}_{k\pm1}^{(2)} = f(x_{k\pm1}, U_{k\pm1}, V_{k\pm1}, \overline{U}'_{k\pm1}, \overline{V}'_{k\pm1}),$$
$$\overline{\overline{F}}_k^{(2)} = f(x_k, U_k, V_k, \overline{\overline{U}}'_k, \overline{\overline{V}}'_k),$$
$$\hat{F}_k^{(2)} = f(x_k, U_k, V_k, \hat{U}'_k, \hat{V}'_k)$$

and the approximations for first order derivatives are already defined in Section 2.

7.2. Sixth order boundary value problems

We now study the sixth order non-linear differential equation

$$\frac{d^6u}{dx^6} = f(x, u, u', u'', u''', u^{iv}, u^{v}), \quad x \in (a, b), \tag{7.2.1}$$

associated with the following conditions

$$u(a) = A_0, u''(a) = A_1, u^{iv}(a) = A_2, u(b) = B_0, u''(b) = B_1, u^{iv}(b) = B_2, \tag{7.2.2}$$

where $A_i, B_i, i = 0, 1, 2$ are constants.

Equation (7.2.1) can be written in equivalent form as:

$$u'' = v, \quad a < x < b, \tag{7.2.3}$$
$$v'' = w, \quad a < x < b, \tag{7.2.4}$$
$$w'' = f(x, u, v, w, u', v', w'), \quad a < x < b. \tag{7.2.5}$$

The boundary conditions (7.2.2) reduce to

$$u(a) = A_0, v(a) = A_1, w(a) = A_2, u(b) = B_0, v(b) = B_1, w(b) = B_2. \quad (7.2.6)$$

Applying the method (6.11) to the system of differential equations (7.2.3)-(7.2.5), we get

$$U_{k+1} - (1+\eta)U_k + \eta U_{k-1} = \eta(1+\eta)\frac{h_k^2}{2}\hat{F}_k^{(1)} \exp\left(\frac{P\overline{F}_{k+1}^{(1)} + R\overline{F}_{k-1}^{(1)} - (P+R)\overline{\overline{F}}_k^{(1)}}{6\eta(1+\eta)\hat{F}_k^{(1)}}\right) + O(h_k^5), \quad (7.2.7)$$

$$V_{k+1} - (1+\eta)V_k + \eta V_{k-1} = \eta(1+\eta)\frac{h_k^2}{2}\hat{F}_k^{(2)} \exp\left(\frac{P\overline{F}_{k+1}^{(2)} + R\overline{F}_{k-1}^{(2)} - (P+R)\overline{\overline{F}}_k^{(2)}}{6\eta(1+\eta)\hat{F}_k^{(2)}}\right) + O(h_k^5), \quad (7.2.8)$$

$$W_{k+1} - (1+\eta)W_k + \eta W_{k-1} = \eta(1+\eta)\frac{h_k^2}{2}\hat{F}_k^{(3)} \exp\left(\frac{P\overline{F}_{k+1}^{(3)} + R\overline{F}_{k-1}^{(3)} - (P+R)\overline{\overline{F}}_k^{(3)}}{6\eta(1+\eta)\hat{F}_k^{(3)}}\right) + O(h_k^5), \quad (7.2.9)$$

where

$$\begin{aligned}
\overline{F}_{k\pm1}^{(1)} &= V_{k\pm1}, \overline{\overline{F}}_k^{(1)} = \hat{F}_k^{(1)} = V_k, \\
\overline{F}_{k\pm1}^{(2)} &= W_{k\pm1}, \overline{\overline{F}}_k^{(2)} = \hat{F}_k^{(2)} = W_k, \\
\overline{F}_{k\pm1}^{(3)} &= f(x_{k\pm1}, U_{k\pm1}, V_{k\pm1}, W_{k\pm1}, \overline{U}'_{k\pm1}, \overline{V}'_{k\pm1}, \overline{W}'_{k\pm1}), \\
\overline{\overline{F}}_k^{(3)} &= f(x_k, U_k, V_k, W_k, \overline{\overline{U}}'_k, \overline{\overline{V}}'_k, \overline{\overline{W}}'_k), \\
\hat{F}_k^{(3)} &= f(x_k, U_k, V_k, W_k, \hat{U}'_k, \hat{V}'_k, \hat{W}'_k)
\end{aligned}$$

and the approximations for first order derivatives are already defined in Section 2.

8. Numerical Illustrations

The given interval $[0,1]$ is divided into $(N+1)$ parts with $0 = x_0 < x_1 < ... < x_{N+1} = 1$, $N \in \mathbb{Z}^+$ with variable mesh spacing $h_k = x_k - x_{k-1}$, $k = 1(1)(N+1)$ and the mesh ratio $\eta = (\frac{h_{k+1}}{h_k}) > 0, k = 1(1)N$.

Now, consider

$$\begin{aligned}
1 = x_{N+1} - x_0 &= (x_{N+1} - x_N) + (x_N - x_{N-1}) + ... + (x_1 - x_0) \\
&= h_{N+1} + h_N + ... + h_1 = (1 + \eta + \eta^2 + ... + \eta^N)h_1.
\end{aligned}$$

Thus,

$$h_1 = 1/(1 + \eta + \eta^2 + ... + \eta^N). \quad (8.1)$$

From the above calculation, we have determined the starting value of the first step length and the subsequent step lengths as $h_2 = \eta h_1, h_3 = \eta h_2, ...$and so on. Thus by prescribing the total number of $(N+2)$ mesh points, we can compute the value of h_1 from (8.1). This is the first mesh spacing on the left and the remaining mesh is determined by $h_{k+1} = \eta h_k, k = 1(1)N$.

We have implemented the purported method on thirteen problems of two-point BVPs. The analytical solution is stated in each problem. The imposing function and boundary conditions are computed from the analytical solution in every problem. We have computed the vector form of linear difference equations and that of non - linear difference equations by employing the block Gauss – Seidel and Newton–Raphson iteration method respectively [14]. The iterations are put to an end once tolerance $\leq 10^{-12}$ in the maximum absolute error is attained, and the initial guess made in each problem is $\boldsymbol{u} = 0$. All calculations were carried out using MATLAB coding.

Problem 8.1 (Convection-Diffusion equation, [19])**.**

$$u'' = \alpha u', \quad 0 < x < 1. \tag{8.2}$$

The analytical solution is given by

$$u(x) = \frac{1 - exp^{-\alpha(1-x)}}{1 - exp^{-\alpha}}.$$

The maximum absolute errors (MAEs) in u are listed in Tables 8.1a and 8.1b for $\eta = 1$ and $\eta = 0.8$ respectively. Figures 8.1a and 8.1b depict the analytical and approximate solutions for $N = 40$ and $\alpha = 100$ with $\eta = 0.8$.

Table 8.1a. Problem 8.1: The maximum absolute errors for problem 8.1 with $\eta = 1$

	Proposed method (2.9)		Method discussed in [17]	
N	$\alpha = 10$	$\alpha = 100$	$\alpha = 10$	$\alpha = 100$
20 CPU time in secs	2.6577(-05) (0.0063)	8.7966(-02) (0.0066)	8.4151(-04) (0.0078)	1.4908(-01) (0.0072)
40 CPU time in secs	1.8127(-06) (0.0122)	4.4485(-03) (0.0143)	5.2924(-05) (0.0142)	9.5266(-03) (0.0218)
80 CPU time in secs	1.1855(-07) (0.0320)	2.8657(-04) (0.0453)	3.3208(-06) (0.0361)	5.9841(-04) (0.0512)

Table 8.1b. Problem 8.1: The maximum absolute errors for Problem 8.1 with $\eta = 0.8$

	Proposed method (2.9)		Method discussed in [17]	
N	$\alpha = 10$	$\alpha = 100$	$\alpha = 10$	$\alpha = 100$
20 CPU time in secs	1.8648(-05) (0.0091)	4.0872(-05) (0.0051)	1.1302(-04) (0.0112)	2.0263(-04) (0.0106)
40 CPU time in secs	1.7114(-05) (0.0155)	1.8523(-05) (0.0131)	7.6064(-05) (0.0187)	8.2235(-05) (0.0214)
80 CPU time in secs	1.7097(-05) (0.0281)	1.8352(-05) (0.0254)	5.3767(-05) (0.0345)	5.7857(-05) (0.0336)

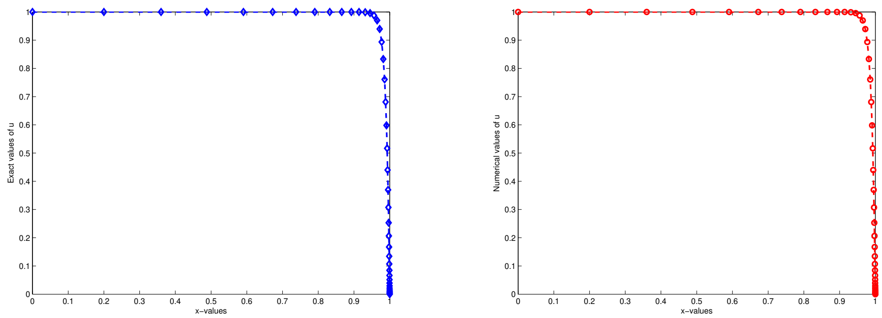

Figure 8.1a. Analytical(Exact) solution of Problem 8.1

Figure 8.1b. Approximate (Numerical) solution of Problem 8.1

Problem 8.2 (Singular equation, [19]).

$$u'' + \frac{\beta}{x}u' = g(x), \quad 0 < x < 1. \tag{8.3}$$

The analytical solution is given by

$$u(x) = \exp^{x^4}.$$

The maximum absolute errors (MAEs) in u are listed in Tables 8.2a and 8.2b for $\eta = 1$ and $\eta = 0.8$ respectively. Figures 8.2a and 8.2b depict the analytical and approximate solutions for $N = 40$ and $\beta = 2$ with $\eta = 0.8$.

Table 8.2a. Problem 8.2: The maximum absolute errors for problem 8.2 with $\eta = 1$

	Proposed method (2.9)		Method discussed in [17]	
N	$\beta = 1$	$\beta = 2$	$\beta = 1$	$\beta = 2$
20 CPU time in secs	9.7432(-05) (0.0218)	1.4856(-04) (0.0181)	1.4911(-04) (0.0277)	3.6677(-04) (0.0212)
40 CPU time in secs	6.9942(-06) (0.0718)	1.1130(-05) (0.0581)	9.5378(-06) (0.0821)	2.3247(-05) (0.0775)
80 CPU time in secs	4.7419(-07) (0.2744)	7.8804(-07) (0.2269)	5.9833(-07) (0.3124)	1.4694(-06) (0.2884)

Table 8.2b. Problem 8.2: The maximum absolute errors for Problem 8.2 with $\eta = 0.8$

	Proposed method (2.9)		Method discussed in [17]	
N	$\beta = 1$	$\beta = 2$	$\beta = 1$	$\beta = 2$
20 CPU time in secs	6.8618(-03) (0.0161)	7.6978(-03) (0.0162)	1.1621(-02) (0.0246)	8.8142(-03) (0.0248)
40 CPU time in secs	6.6183(-03) (0.0309)	7.4182(-03) (0.0304)	8.9971(-03) (0.0443)	8.7818(-03) (0.0446)
80 CPU time in secs	6.6155(-03) (0.0548)	7.4150(-03) (0.0546)	8.8952(-03) (0.0682)	8.7510(-03) (0.0687)

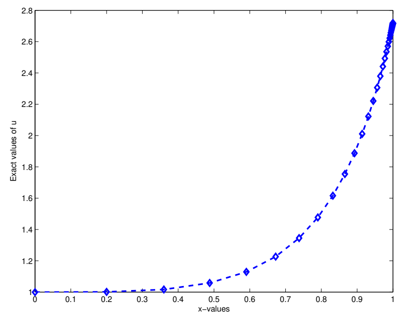

Figure 8.2a. Analytical(Exact) solution of Problem 8.2

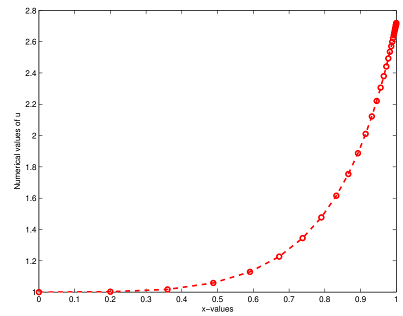

Figure 8.2b. Approximate (Numerical) solution of Problem 8.2

Problem 8.3 (Burger's equation in cartesian coordinates)**.**

$$\nu u'' = (u - \alpha)u' + g(x), \quad 0 < x < 1. \tag{8.4}$$

The analytical solution is given by

$$u(x) = \alpha[1 - \tanh(\frac{\alpha x}{2\nu})] \text{ where } R_e = \nu^{-1} > 0.$$

The maximum absolute errors (MAEs) in u are listed in Tables 8.3a and 8.3b for $\eta = 1$ and $\eta = 1.2$ respectively. Figures 8.3a and 8.3b depict the analytical and approximate solutions for $N = 40$ and $\alpha = \frac{1}{2}$ with $\eta = 1.2$.

Table 8.3a. Problem 8.3: The maximum absolute errors for problem 8.3 with $\eta = 1$

	Proposed method (2.9)			Method discussed in [17]		
N	$R_e = 10$	$R_e = 100$	$R_e = 1000$	$R_e = 10$	$R_e = 100$	$R_e = 1000$
20	2.1820(-06)	2.4587(-02)	8.8050(-02)	3.6678(-06)	3.6150(-02)	1.1121(-01)
CPU time in secs	(0.0158)	(0.0168)	(0.0175)	(0.0187)	(0.0191)	(0.0194)
40	1.5004(-07)	1.6215(-03)	2.4022(-02)	2.3248(-07)	2.4607(-03)	6.6018(-02)
CPU time in secs	(0.1221)	(0.1455)	(0.1558)	(0.1483)	(0.1662)	(0.1773)
80	9.3143(-09)	1.0251(-04)	1.5483(-03)	1.4695(-08)	1.5669(-04)	4.1388(-03)
CPU time in secs	(0.3463)	(0.3693)	(0.3852)	(0.3847)	(0.4142)	0.5252

Table 8.3b. Problem 8.3: The maximum absolute errors for Problem 8.3 with $\eta = 1.2$

	Proposed method (2.9)			Method discussed in [17]		
N	$R_e = 10$	$R_e = 100$	$R_e = 1000$	$R_e = 10$	$R_e = 100$	$R_e = 1000$
20	4.0988(-06)	8.5989(-06)	3.3782(-03)	2.3272(-05)	3.9915(-05)	7.5577(-03)
CPU time in secs	(0.0111)	(0.0159)	(0.0193)	(0.0162)	(0.0177)	(0.0187)
40	3.5048(-06)	3.4117(-06)	3.8317(-06)	1.4671(-05)	1.7468(-05)	2.1785(-05)
CPU time in secs	(0.1206)	(0.1253)	(0.1279)	(0.1324)	(0.1363)	(0.1372)
80	3.4900(-06)	3.3679(-06)	3.3061(-06)	1.0344(-05)	1.5632(-04)	1.8636(-05)
CPU time in secs	(0.3352)	(0.3591)	(0.3811)	(0.4472)	(0.4678)	(0.4892)

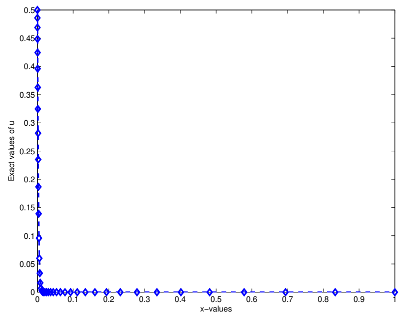

Figure 8.3a. Analytical(Exact) solution of Problem 8.3

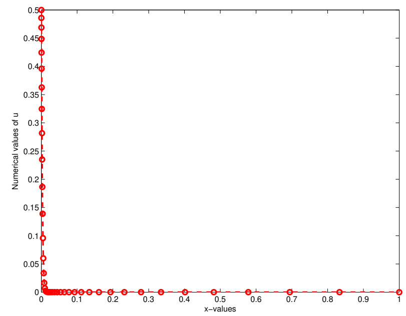

Figure 8.3b. Approximate(Numerical) solution of Problem 8.3

Problem 8.4 (Linear Singular equation)**.**

$$u'' + \frac{\alpha}{x}u' - \frac{\alpha}{x^2}u = f(x), \quad 0 < x < 1. \tag{8.5}$$

The analytical solution is given by

$$u(x) = \exp^{x^4}.$$

The maximum absolute errors (MAEs) in u are listed in Tables 8.4a and 8.4b for $\eta = 1$ and $\eta = 0.7$ respectively. Figures 8.4a and 8.4b depict the analytical and approximate solutions for $N = 40$ and $\alpha = 2$ with $\eta = 0.7$.

Table 8.4a. Problem 8.4: The maximum absolute errors for problem 8.4 with $\eta = 1$

	Proposed method (2.9)		Method discussed in [17]	
N	$\alpha = 1$	$\alpha = 2$	$\alpha = 1$	$\alpha = 2$
20 CPU time in secs	6.5727(-05) (0.0197)	7.7562(-05) (0.0160)	1.0034(-04) (0.0212)	2.3858(-04) (0.0204)
40 CPU time in secs	4.6080(-06) (0.0569)	5.4277(-06) (0.0447)	6.6873(-06) (0.0662)	1.5502(-05) (0.0514)
80 CPU time in secs	3.0362(-07) (0.2180)	3.5787(-07) (0.1765)	4.8435(-07) (0.3118)	9.7191(-05) (0.2566)

Table 8.4b. Problem 8.4: The maximum absolute errors for Problem 8.4 with $\eta = 0.7$

	Proposed method (2.9)		Method discussed in [17]	
N	$\alpha = 1$	$\alpha = 2$	$\alpha = 1$	$\alpha = 2$
20 CPU time in secs	6.8618(-04) (0.0159)	7.6978(-05) (0.0133)	0.1276(-02) (0.0218)	0.7489(-03) (0.0204)
40 CPU time in secs	6.6183(-04) (0.0283)	7.4182(-05) (0.0233)	0.9002(-03) (0.0362)	0.5279(-03) (0.0294)
80 CPU time in secs	6.6155(-04) (0.0404)	7.4150(-05) (0.0379)	0.6365(-03) (0.0524)	0.3732(-03) (0.0488)

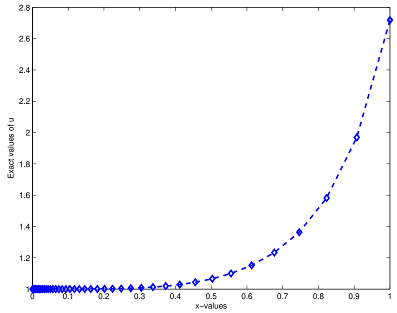

Figure 8.4a. Analytical(Exact) solution of Problem 8.4

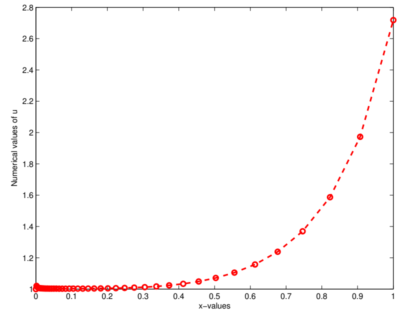

Figure 8.4b. Approximate(Numerical) solution of Problem 8.4

Problem 8.5 (Modified Emden's equation)**.**

$$u'' = g(x) - 3uu' - u^3, \quad 0 < x < 1. \tag{8.6}$$

The analytical solution is given by

$$u(x) = \sinh x.$$

The maximum absolute errors (MAEs) in u are listed in Tables 8.5a and 8.5b for $\eta = 1$ and $\eta = 0.7$ respectively. Figures 8.5a and 8.5b depict the analytical and approximate solutions for $N = 40$ and $\eta = 0.7$.

Table 8.5a. Problem 8.5: The maximum absolute errors for problem 8.5 with $\eta = 1$

N	Proposed method (2.9)	Method discussed in [17]
20	5.2105(-06)	1.4951(-05)
CPU time in secs	(0.0212)	(0.0322)
40	3.5978(-07)	8.6728(-07)
CPU time in secs	(0.0603)	(0.0814)
80	2.2396(-08)	5.1945(-08)
CPU time in secs	(0.1325)	(0.1556)

Table 8.5b. Problem 8.5: The maximum absolute errors for Problem 8.5 with $\eta = 0.7$

N	Proposed method (2.9)	Method discussed in [17]
20	6.6023(-04)	2.6590(-03)
CPU time in secs	(0.0123)	(0.0168)
40	6.5844(-04)	2.6413(-03)
CPU time in secs	(0.0241)	(0.0288)
80	6.5844(-04)	2.6413(-03)
CPU time in secs	(0.0497)	(0.0588)

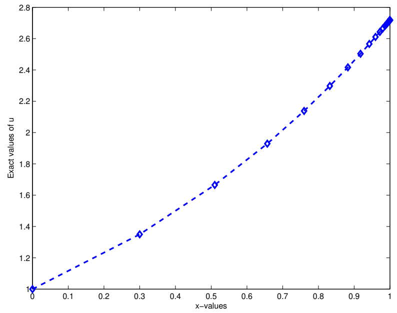

Figure 8.5a. Analytical(Exact)solution of Problem 8.5

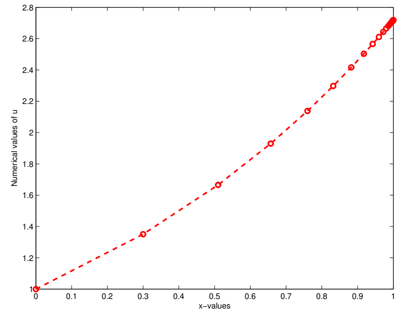

Figure 8.5b. Approximate(Numerical)solution of Problem 8.5

Problem 8.6 (Vanderpol's equation)**.**

$$\nu u'' = \mu(1-u^2)u' - u + g(x), \quad 0 < x < 1. \tag{8.7}$$

The analytical solution is given by

$$u(x) = \exp x.$$

The maximum absolute errors (MAEs) in u are listed in Tables 8.6a and 8.6b for $\eta = 1$ and $\eta = 0.8$ respectively. Figures 8.6a and 8.6b depict the analytical and approximate solutions for $N = 40$ and $\eta = 0.8$.

Table 8.6a. Problem 8.6: The maximum absolute errors for problem 8.6 with $\eta = 1$

N	Proposed method (2.9)	Method discussed in [17]
20	8.3525(-07)	1.9650(-06)
CPU time in secs	(0.0085)	(0.0112)
40	5.7432(-08)	1.1851(-07)
CPU time in secs	(0.0162)	(0.0231)
80	3.5450(-09)	7.2725(-09)
CPU time in secs	(0.0231)	(0.0284)

Table 8.6b. Problem 8.6: The maximum absolute errors for Problem 8.6 with $\eta = 0.8$

N	Proposed method (2.9)	Method discussed in [17]
20	5.7643(-05)	1.4328(-04)
CPU time in secs	(0.0087)	(0.0115)
40	5.5049(-05)	1.3979(-04)
CPU time in secs	(0.0179)	(0.0223)
80	5.5019(-05)	1.3975(-04)
CPU time in secs	(0.0293)	(0.0344)

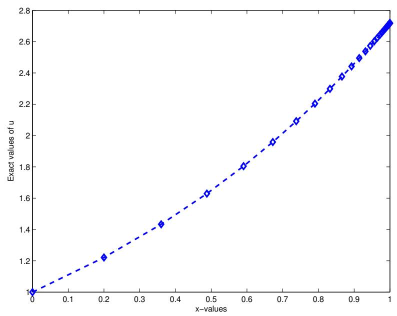

Figure 8.6a. Analytical(Exact) solution of Problem 8.6

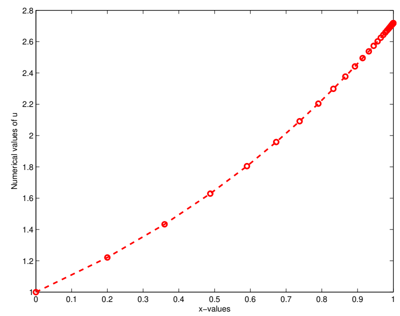

Figure 8.6b. Approximate (Numerical) solution of Problem 8.6

Problem 8.7 (Non-linear fourth order 1-D Elliptic Equation, [9]). *Consider the BVP originating from time independent Navier-Stokes equation for axis symmetric flow of an incompressible fluid contained between infinite discs*

$$u^{iv}(x) = \tau u(x)u''(x) + g(x), \quad 0 < x < 1 \tag{8.8}$$

where $\tau > 0$ is a parameter.

The analytical solution is given by

$$u(x) = (1 - x^2)\exp(x).$$

The maximum absolute errors (MAEs) in u are listed in Tables 8.7a and 8.7b for $\eta = 1$ and $\eta = 0.985$ respectively. Figures 8.7a and 8.7b depict the analytical and approximate solutions for $N = 40$ and $\tau = 10^3$ with $\eta = 0.985$.

Table 8.7a. Problem 8.7: The maximum absolute errors for problem 8.7 with $\eta = 1$

	Proposed method (2.9)			Method discussed in [17]		
N	$R_e = 10$	$R_e = 100$	$R_e = 1000$	$R_e = 10$	$R_e = 100$	$R_e = 1000$
20	1.1494(-07)	1.0435(-07)	1.0243(-07)	4.2570(-05)	3.8517(-05)	2.2123(-05)
CPU time in secs	(0.0163)	(0.0172)	(0.0173)	(0.0182)	(0.0186)	(0.0187)
40	7.8443(-09)	7.0858(-09)	6.9432(-09)	2.6606(-06)	2.4071(-06)	1.5867(-06)
CPU time in secs	(0.0233)	(0.0235)	(0.0240)	(0.0245)	(0.0248)	(0.0249)
80	1.0958(-10)	9.5734(-11)	7.8566(-11)	1.6596(-07)	1.5013(-07)	1.0516(-07)
CPU time in secs	(0.0486)	(0.0488)	(0.0488)	(0.0512)	(0.0517)	(0.0518)

Table 8.7b. Problem 8.7: The maximum absolute errors for Problem 8.7 with $\eta = 0.985$

	Proposed method (2.9)			Method discussed in [17]		
N	$R_e = 10$	$R_e = 100$	$R_e = 1000$	$R_e = 10$	$R_e = 100$	$R_e = 1000$
20	1.7895(-07)	1.6690(-07)	1.6464(-07)	1.1918(-05)	1.1866(-05)	1.1617(-05)
CPU time in secs	(0.0064)	(0.0076)	(0.0085)	(0.0084)	(0.0088)	(0.0093)
40	3.2915(-08)	3.1067(-08)	3.0837(-08)	1.4245(-06)	1.3955(-06)	1.3653(-06)
CPU time in secs	(0.0160)	(0.0161)	(0.0169)	(0.0184)	(0.0187)	(0.0192)
80	6.2241(-09)	5.9110(-09)	5.8806(-09)	1.6363(-07)	1.6113(-07)	1.5928(-07)
CPU time in secs	(0.0390)	(0.0399)	(0.0402)	(0.0448)	(0.0453)	(0.0457)

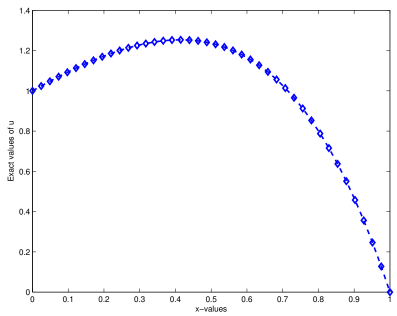

Figure 8.7a. Analytical(Exact) solution of Problem 8.7

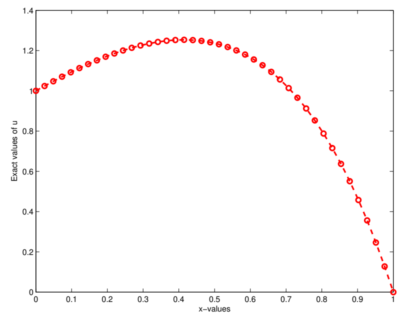

Figure 8.7b. Approximate(Numerical) solution of Problem 8.7

Problem 8.8 (Non-linear sixth order Elliptic Equation, see [21]). *Consider the non-linear problem:*

$$u^{vi}(x) = \gamma[u(x)u''(x) + u''(x)u^{iv}(x) + u(x)u^{iv}(x)] + g(x), \quad 0 < x < 1. \tag{8.9}$$

The analytical solution is given by

$$u(x) = \sinh(x).$$

The maximum absolute errors (MAEs) in u are listed in Tables 8.8a and 8.8b for $\eta = 1$ and $\eta = 0.988$ respectively and $\gamma = 2^{13}$. Figures 8.8a and 8.8b depict the analytical and approximate solutions for $N = 40$ $\gamma = 2^{13}$ with $\eta = 0.988$.

Table 8.8a. Problem 8.8: The maximum absolute errors for problem 8.8 with $\eta = 1$

	Proposed method (2.9)	Method discussed in [17]
N	$\gamma = 2^{13}$	$\gamma = 2^{13}$
20 CPU time in secs	6.2757(-10) (0.0082)	Overflow (0.0066)
40 CPU time in secs	9.6478(-11) (0.0214)	8.1775(-10) (0.0319)
80 CPU time in secs	4.8430(-12) (0.0482)	4.8912(-11) (0.0568)

Table 8.8b. Problem 8.8: The maximum absolute errors for problem 8.8 with $\eta = 0.988$

	Proposed method (2.9)	Method discussed in [17]
N	$\gamma = 2^{13}$	$\gamma = 2^{13}$
20 CPU time in secs	8.5932(-09) (0.0104)	4.4644(-08) (0.0156)
40 CPU time in secs	1.2361(-09) (0.0266)	6.5159(-09) (0.0292)
80 CPU time in secs	6.4639(-12) (0.0462)	9.4428(-10) (0.0486)

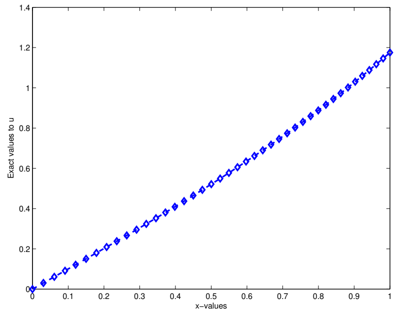

Figure 8.8a. Analytical(Exact)solution of Problem 8.8

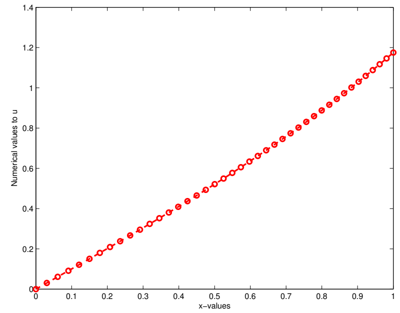

Figure 8.8b. Approximate(Numerical) solution of Problem 8.8

Problem 8.9 (Hopf bifurcation for the coupled oscillator, see [30])**.**

$$\begin{aligned} u'' &= -u + (\mu - u^2 - \alpha v^2)u' + f(x), \quad 0 < x < 1, \\ v'' &= -v + (\mu - v^2 - \alpha u^2)u' + g(x), \quad 0 < x < 1. \end{aligned} \tag{8.10}$$

The analytical solution is given by

$$u(x) = \exp{(x^2)}, \; v(x) = \sin{(\pi x)}.$$

The maximum absolute errors (MAEs) in u are listed in Tables 8.9a and 8.9b for $\eta = 1$ and $\eta = 0.988$ respectively and $\mu = 1$ and $\alpha = 1$. Figures 8.9a, 8.9b, 8.9c and 8.9d depict the analytical and approximate solutions for $N = 40$, $\mu = 1, \alpha = 1$, with $\eta = 1$ and $\eta = 0.988$.

Table 8.9a. Problem 8.9: The maximum absolute errors for problem 8.9 with $\eta = 1$

	Proposed method (2.9)		Method discussed in [17]	
	$\alpha = \mu = 1$		$\alpha = \mu = 1$	
N	u	v	u	v
20 CPU time in secs	4.3692(-05) (0.0281)	3.8135(-06) (0.0281)	1.9477(-04) (0.0381)	1.5959(-05) (0.0381)
40 CPU time in secs	3.0397(-06) (0.1044)	2.6522(-07) (0.1044)	1.1255(-05) (0.1457)	9.9834(-07) (0.1457)
80 CPU time in secs	1.9955(-07) (0.2989)	1.7474(-08) (0.2989)	6.9082(-07) (0.3216)	6.5122(-08) (0.3216)

Table 8.9b. Problem 8.9: The maximum absolute errors for problem 8.9 with $\eta = 0.988$

	Proposed method (2.9)		Method discussed in [17]	
	$\alpha = \mu = 1$		$\alpha = \mu = 1$	
N	u	v	u	v
20 CPU time in secs	2.5054(-05) (0.0268)	1.5070(-06) (0.0268)	3.1267(-04) (0.0364)	2.4612(-05) (0.0364)
40 CPU time in secs	9.0577(-07) (0.0749)	1.2805(-07) (0.0749)	1.6118(-05) (0.0814)	1.8121(-06) (0.0814)
80 CPU time in secs	1.2431(-08) (0.0326)	3.2176(-08) (0.0355)	9.1216(-07) (0.0236)	4.1015(-07) (0.0508)

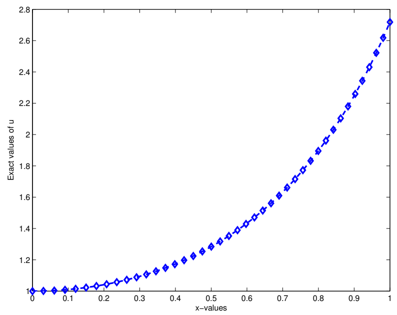

Figure 8.9a. Analytical(Exact) solution of u of Problem 8.9

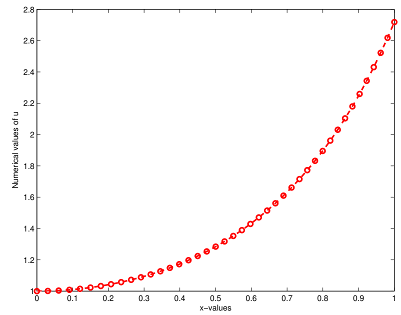

Figure 8.9b. Approximate(Numerical) solution of u of Problem 8.9

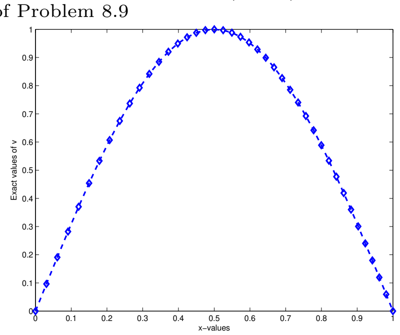

Figure 8.9c. Analytical(Exact) solution of v of Problem 8.9

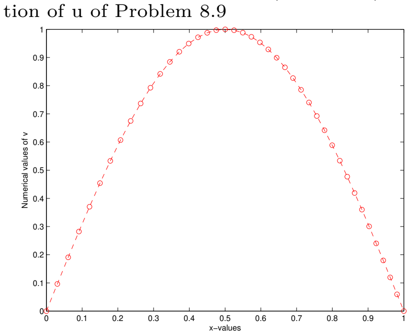

Figure 8.9d. Approximate(Numerical) solution of v of Problem 8.9

Problem 8.10 ([33]). *Consider the fourth order linear boundary value problem of the form:*

$$u^{iv} + xu = -(8 + 7x + x^3)\exp x, \quad 0 \le x \le 1, \qquad (8.11)$$
$$u(0) = u(1) = 0, u''(0) = 0, u''(1) = -4e.$$

The analytical solution is given by

$$u(x) = x(1-x)\exp(x).$$

The maximum absolute errors (MAEs) in u are listed in Tables 8.10 for $\eta = 1$. Figures 8.10a and 8.10b depict the analytical and approximate solutions for $N = 32$ with $\eta = 1$.

Table 8.10. Problem 8.10: The maximum absolute errors for problem 8.10 with $\eta = 1$

N	Proposed method (2.9)	Method discussed in [33]
8	5.2219(-06)	4.24(-04)
16	3.3066(-07)	1.08(-04)
32	2.0664(-08)	2.70(-05)
64	9.9551(-10)	6.75(-06)

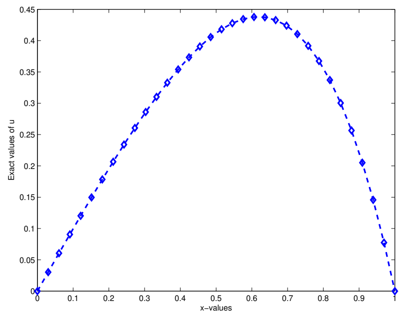

Figure 8.10a. Analytical(Exact) solution of Problem 8.10

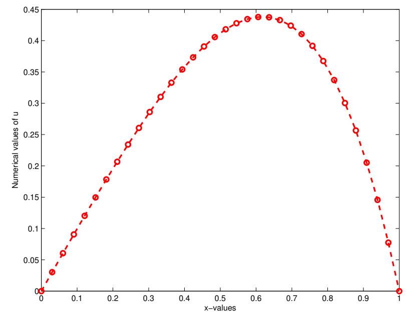

Figure 8.10b. Approximate(Numerical) solution of Problem 8.10

Problem 8.11 ([27]). *Consider the fourth order linear boundary value problem of the form:*

$$
\begin{aligned}
&u^{vi} + u = 6(2x\cos x + 5\sin x), \quad -1 \leq x \leq 1, \qquad (8.12)\\
&u(-1) = u(1) = 0, u''(-1) = -4\cos(-1) + 2\sin(-1),\\
&u''(1) = 4\cos(1) + 2\sin(1),\\
&u^{iv}(-1) = 8\cos(-1) - 12\sin(-1),\\
&u^{iv}(1) = -8\cos(1) - 12\sin(1).
\end{aligned}
$$

The analytical solution is given by

$$u(x) = (x^2 - 1)\sin x.$$

The maximum absolute errors (MAEs) in u are listed in Tables 8.11 for $\eta = 1$. Figures 8.11a and 8.11b depict the analytical and approximate solutions for $N = 32$ with $\eta = 1$.

Table 8.11. Problem 8.11: The maximum absolute errors for problem 8.11 with $\eta = 1$

N	Proposed method (2.9)	Method discussed in [27]
8	3.0101(-06)	8.1514(-05)
16	1.9070(-07)	2.1052(-05)
32	1.1801(-08)	5.3084(-06)

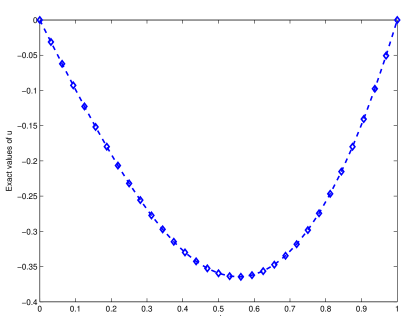

Figure 8.11a. Analytical(Exact) solution of Problem 8.11

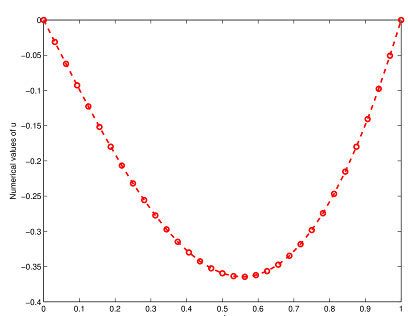

Figure 8.11b. Approximate(Numerical) solution of Problem 8.11

Problem 8.12 ([31]). *Consider the non linear boundary value problem of the form:*

$$u'' = \frac{1}{2}(1+x+u)^3, \quad 0 < x < 1, \tag{8.13}$$
$$u(0) = u(1) = 0.$$

The analytical solution is given by

$$u(x) = \frac{2}{2-x} - x - 1.$$

The maximum absolute errors (MAEs) in u are listed in Tables 8.12 for $\eta = 1$. Figures 8.12a and 8.12b depict the analytical and approximate solutions for $N = 32$ with $\eta = 1$.

Table 8.12. Problem 8.12: The maximum absolute errors for problem 8.12 with $\eta = 1$

N	Proposed method (2.9)	Method discussed in [31]
8	2.0165(-06)	7.7(-05)
16	1.2865(-07)	1.9(-05)
32	7.9875(-09)	4.7(-06)
64	2.3096(-10)	1.2(-06)

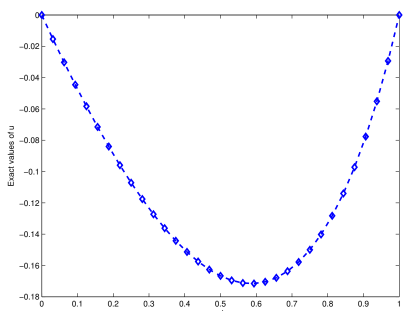

Figure 8.12a. Analytical(Exact) solution of Problem 8.12

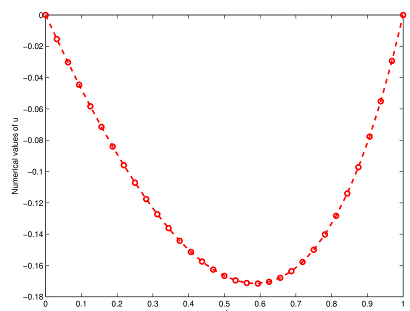

Figure 8.12b. Approximate(Numerical) solution of Problem 8.12

Problem 8.13 ([31]). *Consider the non linear boundary value problem of the form:*

$$u'' = -\exp(-2u), \quad 1 \leq x \leq 2, \tag{8.14}$$
$$u(1) = 0, u(2) = \ln 2.$$

The analytical solution is given by

$$u(x) = \ln(x).$$

The maximum absolute errors (MAEs) in u are listed in Tables 8.10 for $\eta = 1$. Figures 8.13a and 8.13b depict the analytical and approximate solutions for $N = 32$ with $\eta = 1$.

Table 8.10. Problem 8.13: The maximum absolute errors for problem 8.13 with $\eta = 1$

N	Proposed method (2.9)	Method discussed in [31]
8	1.6424(-05)	4.0(-04)
16	1.0481(-06)	9.8(-05)
32	6.5976(-08)	2.4(-05)
64	3.8966(-09)	6.1(-06)

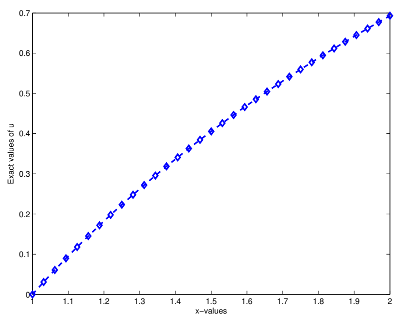

Figure 8.13a. Analytical(Exact) solution of Problem 8.13

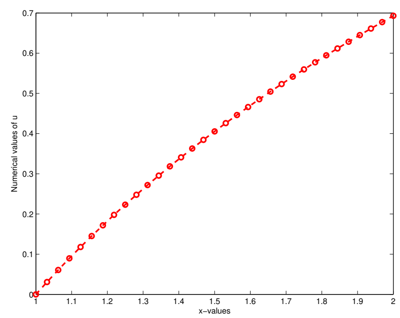

Figure 8.13b. Approximate(Numerical) solution of Problem 8.13

Table 8.14. : The rate of convergence for $\eta = 1$

Problem	Mesh sizes	Parameters	Rate of convergence for the proposed method	Refrence	Rate of convergence in the refrence cited
8.1	$h_1 = \frac{1}{40}, h_2 = \frac{1}{80}$	$\alpha = 10$ $\alpha = 100$	3.9346 3.9564	[17]	3.9943 3.9928
8.2	$h_1 = \frac{1}{40}, h_2 = \frac{1}{80}$	$\beta = 1$ $\beta = 2$	3.8826 3.8200	[17]	3.9946 3.9837
8.3	$h_1 = \frac{1}{40}, h_2 = \frac{1}{80}$	$R_e = 10$ $R_e = 100$ $R_e = 1000$	4.0098 3.9835 3.9556	[17]	3.9837 3.9731 3.9956
8.4	$h_1 = \frac{1}{40}, h_2 = \frac{1}{80}$	$\alpha = 1$ $\alpha = 2$	3.9238 3.9228	[17]	3.7873 2.6484
8.5	$h_1 = \frac{1}{40}, h_2 = \frac{1}{80}$		4.0058	[17]	4.0614
8.6	$h_1 = \frac{1}{40}, h_2 = \frac{1}{80}$		4.0180	[17]	4.0264
8.7	$h_1 = \frac{1}{20}, h_2 = \frac{1}{40}$	$R_e = 10$ $R_e = 100$ $R_e = 1000$	4.0098 3.8804 3.8829	[17]	4.0022 4.0030 3.9154
8.8	$h_1 = \frac{1}{40}, h_2 = \frac{1}{80}$	$\gamma = 2^{13}$	4.3162	[17]	4.0634
8.9	$h_1 = \frac{1}{32}, h_2 = \frac{1}{64}$		4.3174	[17]	1.9951
8.10	$h_1 = \frac{1}{32}, h_2 = \frac{1}{64}$		4.3755	[33]	2.0
8.11	$h_1 = \frac{1}{16}, h_2 = \frac{1}{32}$		4.0143	[27]	1.9876
8.12	$h_1 = \frac{1}{16}, h_2 = \frac{1}{32}$		4.0096	[31]	1.9696
8.13	$h_1 = \frac{1}{32}, h_2 = \frac{1}{64}$		4.0817	[31]	1.9762

9. Conclusions

We have derived a new third order compact numerical method in exponential form for the numerical solution of the system of nonlinear two point boundary value problems on a non-uniform mesh. We have used only three grid points. The proposed method is applicable to solve singular problems. We have solved thirteen benchmark problems and compared the numerical results with the results obtained by using other available methods. The advantage of exponential method is that it gives higher accuracy results as compared to existing methods in literature.

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