

AN ITERATIVE ALGORITHM FOR SOLVING A CLASS OF GENERALIZED COUPLED SYLVESTER-TRANSDPOSE MATRIX EQUATIONS OVER BISYMMETRIC OR SKEW-ANTI-SYMMETRIC MATRICES

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Abstract This paper presents an iterative algorithm to solve a class of generalized coupled Sylvester-transpose matrix equations over bisymmetric or skew-anti-symmetric matrices. When the matrix equations are consistent, the bisymmetric or skew-anti-symmetric solutions can be obtained within finite iteration steps in the absence of round-off errors for any initial bisymmetric or skew-anti-symmetric matrices by the proposed iterative algorithm. In addition, we can obtain the least norm solution by choosing the special initial matrices. Finally, numerical examples are given to demonstrate the iterative algorithm is quite efficient. The merit of our method is that it is easy to implement.

Keywords Generalized coupled Sylvester-transpose matrix equations, Bisymmetric matrix, Skew-anti-symmetric matrix, Iterative algorithm.

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1. Introduction

Matrix equations are widely applied in many fields of science and engineering computing, such as control theory, stability theory, system theory [7, 11, 12, 19]. For instance, in [56], the second-order linear system

$$M\ddot{x} + D\dot{x} + Kx = Bu$$

where M , D , K and B are known matrices with appropriate dimensions, is closely related with a matrix equation of the form

$$MVF^2 + DVF + KV = BW + R$$

where F and R are given matrices, and (V, W) is a pair of matrices to be determined. In recent years, there have been many studies on solving matrix equations [1–6, 8–10, 13–18, 20–61]. The skew-symmetric solution of matrix equation

$$AXB = C \tag{1.1}$$

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was obtained by constructing the orthogonal residual matrices [27]. In [18], Dehghan and Shirilord proposed a generalized modified Hermitian and skew-Hermitian splitting (GMHSS) method for solving complex Sylvester matrix equation

$$AX + XB = C \tag{1.2}$$

where $A \in \mathbb{C}^{m \times m}$, $B \in \mathbb{C}^{n \times n}$ and $C \in \mathbb{C}^{m \times n}$ are given complex matrices. Also A , B and C are large and sparse matrices. Zhang, Wei, Li and Zhao proposed an efficient method for the minimal norm least squares Hermitian solution of the complex matrix equation

$$(AXB, CXD) = (E, F) \tag{1.3}$$

by applying the real representation matrices of complex matrices, the particular structure of real representation matrices, the Kronecker product of matrices and the Moore-Penrose generalized inverse, and then the expression of the minimal norm least squares solution was obtained [59]. where A , B , C , D , E and F are given matrices of suitable sizes, and X is an unknown matrix of suitable size over real number field, complex number field or quaternion field. Very recently, some researchers studied coupled Sylvester matrix equations or generalized coupled Sylvester matrix equations [9, 14, 15, 22, 23, 28, 32, 35, 39, 43, 51, 57, 61]. The form of the equation is

$$\begin{cases} \sum_{i=1}^l A_i X B_i + \sum_{i=1}^l C_i Y D_i = M, \\ \sum_{i=1}^l E_i X F_i + \sum_{i=1}^l G_i Y H_i = N \end{cases} \tag{1.4}$$

and others. With the progress of scientific research the coupled Sylvester-transpose matrix equations and generalized coupled Sylvester-transpose matrix equations are investigated [20, 36, 40, 44, 50]. For instance [50], the authors considered the generalized coupled Sylvester-transpose matrix equation as follows

$$\begin{cases} AXB + CY^T D = S_1, \\ EX^T F + GYH = S_2. \end{cases} \tag{1.5}$$

Also they gave an iterative algorithm to solve the system over reflexive (or anti-reflexive) matrix. In [24], M. Hajarian extended the conjugate direction (CD) method to obtain an efficient method for solving the general coupled Sylvester discrete-time periodic (GCSDTP) matrix equations

$$\begin{cases} \sum_{j=1}^m (A_{ij} X_i B_{ij} + C_{ij} X_{i+1} D_{ij} + E_{ij} Y_i F_{ij} + G_{ij} Y_{i+1} H_{ij}) = M_i, \quad i = 1, 2, \dots, \\ \sum_{j=1}^m (\hat{A}_{ij} X_i \hat{B}_{ij} + \hat{C}_{ij} X_{i+1} \hat{D}_{ij} + \hat{E}_{ij} Y_i \hat{F}_{ij} + \hat{G}_{ij} Y_{i+1} \hat{H}_{ij}) = M_i, \quad i = 1, 2, \dots. \end{cases}$$

In [25], first M. Hajarian developed the BiCOR and CORS methods to solve the coupled Sylvester-transpose matrix equations

$$\begin{cases} \sum_{k=1}^l (A_{1,k} X B_{1,k} + C_{1,k} X^T D_{1,k} + E_{1,k} Y F_{1,k}) = M_1, \\ \sum_{k=1}^l (A_{2,k} X B_{2,k} + C_{2,k} X^T D_{2,k} + E_{2,k} Y F_{2,k}) = M_2. \end{cases}$$

Second based on the developed methods, M. Hajarian proposed matrix methods to solve the coupled periodic Sylvester matrix equations

$$\begin{aligned} \mathcal{A}_{1,j}\mathcal{X}_j\mathcal{B}_{1,j} + \mathcal{C}_{1,j}\mathcal{X}_{j+1}\mathcal{D}_{1,j} + \mathcal{E}_{1,k}\mathcal{Y}_j\mathcal{F}_{1,j} &= \mathcal{M}_{1,j}, \quad j = 1, 2, \dots, \\ \mathcal{A}_{2,j}\mathcal{X}_j\mathcal{B}_{2,j} + \mathcal{C}_{2,j}\mathcal{X}_{j+1}\mathcal{D}_{2,j} + \mathcal{E}_{2,k}\mathcal{Y}_j\mathcal{F}_{2,j} &= \mathcal{M}_{2,j}, \quad j = 1, 2, \dots. \end{aligned}$$

In [26], first M. Hajarian considered the problem of solving the Sylvester-transpose matrix equation:

$$\sum_{i=1}^k (A_i X B_i + C_i X^T D_i) = E.$$

Second he considered the periodic Sylvester matrix equation:

$$\hat{A}_j \hat{X}_j \hat{B}_j + \hat{C}_j \hat{X}_{j+1} \hat{D}_j = \hat{E}_j.$$

In this paper, we derive and analyze an efficient algorithm to solve the generalized coupled Sylvester-transpose matrix equations of the form

$$\begin{cases} A_1 X^T B_1 + C_1 Y D_1 + E_1 Z F_1 = G_1 \\ A_2 X B_2 + C_2 Y^T D_2 + E_2 Z F_2 = G_2 \\ A_3 X B_3 + C_3 Y D_3 + E_3 Z^T F_3 = G_3 \end{cases} \quad (1.6)$$

over bisymmetric or skew-anti-symmetric matrices. where $A_1, A_2, A_3, C_1, C_2, C_3, E_1, E_2, E_3 \in \mathbb{R}^{m \times n}$, $B_1, B_2, B_3, D_1, D_2, D_3, F_1, F_2, F_3 \in \mathbb{R}^{n \times m}$, $G_1, G_2, G_3 \in \mathbb{R}^{m \times m}$, are known constant matrices, $X, Y, Z \in \mathbb{R}^{n \times n}$ are unknown matrices to be solved.

For the convenience of description, the notations and definitions used in this paper are summarized as follows. Let \mathbb{R}^n denotes the set of n -dimensional vector space, $\mathbb{R}^{m \times n}$ denotes the set of $m \times n$ real matrices, For $A \in \mathbb{R}^{m \times n}$, the symbols $A^T, tr(A), \|\cdot\|$ and $\mathcal{R}(A)$ denote the transpose, the trace, the Frobenius norm and the column space of real matrix A , respectively. For arbitrary $A = (a_{ij}), B = (b_{ij})$, the Kroncker product is $A \otimes B = (a_{ij} B)$. For any $A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{m \times n}$, the inner product is defined as $\langle A, B \rangle = tr(B^T A)$. Let a matrix $A \in \mathbb{R}^{m \times n}$, $vec(A)$ denotes the vec operator defined by $vec(A) = (a_1^T, a_2^T, \dots, a_n^T)^T$, where a_i is the i -th column of A . For $S = [e_m, e_{m-1}, \dots, e_1]$, where e_i is an m -order unit vector with i th component 1, the matrix $S \in \mathbb{R}^{m \times m}$ is said to be an m -order quasi-identity matrix.

Definition 1.1. Let S be an m -order quasi-identity matrix, the matrix $X \in \mathbb{R}^{m \times m}$ is said to be a bisymmetric matrix if $X^T = X = SXS$ and $\mathbb{BSR}^{m \times m}$ denotes the set of m -order bisymmetric matrices.

Definition 1.2. Let S be an m -order quasi-identity matrix, the matrix $X \in \mathbb{R}^{m \times m}$ is said to be a skew-anti-symmetric matrix if $X^T = X = -SXS$ and $\mathbb{SASR}^{m \times m}$ denotes the set of m -order skew-anti-symmetric matrices.

The remainder of this paper is structured as follows. In Section 2, after several lemmas, we introduce a MCG (modified conjugate gradient) method for solving the matrix equations (1.6) over bisymmetric or skew-anti-symmetric matrix and prove that a solution (X^*, Y^*, Z^*) of Eq.(1.6) can be obtained by the MCG method within

finite iterative steps in the absence round-off errors for arbitrary initial value. In Section 3, we show that the least norm solution can be obtained by giving a special initial matrix. In Section 4, numerical examples are given to demonstrate that the introduced iterative algorithm is efficient. Finally, we conclude the paper in Section 5.

2. The iterative algorithm for solving the matrix Eq.(1.6)

In this section, we propose a MCG(modified conjugate gradient) iterative method for solving the generalized coupled Sylvester-transpose matrix equations (1.6) over bisymmetric or skew-anti-symmetric matrix. Firstly, on the basis of the properties of the inner product and theorem 4.3.8 and corollary in [30], we give a few useful lemmas.

Lemma 2.1. *Let $X \in \mathbb{R}^{m \times n}$, then*

$$\begin{aligned} \text{vec}(X^T) &= P_{mn}\text{vec}(X), \\ P_{mn}^T &= P_{mn}^{-1} = P_{mn}, \end{aligned} \tag{2.1}$$

where P_{mn} is a permutation matrix, that is

$$P_{mn} = \sum_{i=1}^m \sum_{j=1}^n E_{ij} \otimes E_{ij}^T, \quad E_{ij} = e_i e_j^T, \quad e_i \in \mathbb{R}^m, \quad e_j \in \mathbb{R}^n.$$

Remark 2.1. From the lemma 2.1, we are very easy to obtain $P_{mn}^2 = P_{mn}P_{nm} = I_{mn}$, and $P_{mn}\text{vec}(X^T) = \text{vec}(X)$.

Lemma 2.2. *Let $A \in \mathbb{R}^{p \times m}$, $B \in \mathbb{R}^{n \times q}$, then*

$$\begin{aligned} P_{pn}^T(A \otimes B)P_{mq} &= B \otimes A, \\ (B \otimes A)P_{mq} &= P_{pn}^T(A \otimes B). \end{aligned} \tag{2.2}$$

According to the Kronecker product, the vec operator, and the above lemmas 2.1–2.2, we can transform generalized coupled Sylvester-transpose matrix equations (1.6) into the following system of linear equations:

$$\underbrace{\begin{pmatrix} (B_1^T \otimes A_1)P_{nn} & D_1^T \otimes C_1 & F_1^T \otimes E_1 \\ B_2^T \otimes A_2 & (D_2^T \otimes C_2)P_{nn} & F_2^T \otimes E_2 \\ B_3^T \otimes A_3 & D_3^T \otimes C_3 & (F_3^T \otimes E_3)P_{nn} \end{pmatrix}}_{\mathcal{A}} \underbrace{\begin{pmatrix} \text{vec}(X) \\ \text{vec}(Y) \\ \text{vec}(Z) \end{pmatrix}}_x = \underbrace{\begin{pmatrix} \text{vec}(G_1) \\ \text{vec}(G_2) \\ \text{vec}(G_3) \end{pmatrix}}_b \tag{2.3}$$

Lemma 2.3 ([50]). *The system of linear equation (2.3) is consistent if and only if*

$$\text{rank}(\mathcal{A}) = \text{rank}(\mathcal{A}, b). \tag{2.4}$$

Moreover, if

$$\text{rank}(A) = \text{rank}(A, b) = 3m^2, \quad (2.5)$$

then system (2.3) has a unique solution.

We can use the classical conjugate gradient method to solve Eq.(2.3), but it has no practical significance for this equation. Because the size of Eq.(2.3) is usually very large, it will take a lot of time and occupy a lot of computer storage space in the calculation process. Therefore, we use MCG method in matrix form to solve the Eq.(2.3). Next, we give the MCG algorithms for solving Eq.(2.3) over bisymmetric or skew-anti-symmetric matrices.

Algorithm 2.1. (MCG method over bisymmetric matrix)

Step 1 Input matrices $A_1, A_2, A_3, C_1, C_2, C_3, E_1, E_2, E_3 \in \mathbb{R}^{m \times n}$,
 $B_1, B_2, B_3, D_1, D_2, D_3, F_1, F_2, F_3 \in \mathbb{R}^{n \times m}$, $G_1, G_2, G_3 \in \mathbb{R}^{m \times m}$,
quasi-identity matrix $S \in \mathbb{R}^{n \times n}$, bisymmetric initial matrices $X_1, Y_1, Z_1 \in \mathbb{BSR}^{n \times n}$,
and compute

$$R_1 = \begin{pmatrix} R_1^{(1)} \\ R_1^{(2)} \\ R_1^{(3)} \end{pmatrix} = \begin{pmatrix} G_1 - A_1 X_1^T B_1 - C_1 Y_1 D_1 - E_1 Z_1 F_1 \\ G_2 - A_2 X_1 B_2 - C_2 Y_1^T D_2 - E_2 Z_1 F_2 \\ G_3 - A_3 X_1 B_3 - C_3 Y_1 D_3 - E_3 Z_1^T F_3 \end{pmatrix}, \quad (2.6)$$

$$\tilde{R}_1 = \begin{pmatrix} \tilde{R}_1^{(1)} \\ \tilde{R}_1^{(2)} \\ \tilde{R}_1^{(3)} \end{pmatrix} = \begin{pmatrix} B_1 (R_1^{(1)})^T A_1 + A_2^T R_1^{(2)} B_2^T + A_3^T R_1^{(3)} B_3^T \\ C_1^T R_1^{(1)} D_1^T + D_2 (R_1^{(2)})^T C_2 + C_3^T R_1^{(3)} D_3^T \\ E_1^T R_1^{(1)} F_1^T + E_2^T R_1^{(2)} F_2^T + F_3 (R_1^{(3)})^T E_3 \end{pmatrix}, \quad (2.7)$$

$$P_1 = \frac{1}{4} \begin{pmatrix} (\tilde{R}_1^{(1)} + (\tilde{R}_1^{(1)})^T) + S(\tilde{R}_1^{(1)} + (\tilde{R}_1^{(1)})^T)S \\ (\tilde{R}_1^{(2)} + (\tilde{R}_1^{(2)})^T) + S(\tilde{R}_1^{(2)} + (\tilde{R}_1^{(2)})^T)S \\ (\tilde{R}_1^{(3)} + (\tilde{R}_1^{(3)})^T) + S(\tilde{R}_1^{(3)} + (\tilde{R}_1^{(3)})^T)S \end{pmatrix} := \begin{pmatrix} P_1^{(1)} \\ P_1^{(2)} \\ P_1^{(3)} \end{pmatrix}. \quad (2.8)$$

set $k := 1$.

Step 2 If $R_k = 0$, or $R_k \neq 0$ and $P_k = 0$, stop; otherwise, Compute

$$\alpha_k = \frac{\|R_k\|^2}{\|P_k\|^2} = \frac{\|R_k^{(1)}\|^2 + \|R_k^{(2)}\|^2 + \|R_k^{(3)}\|^2}{\|P_k^{(1)}\|^2 + \|P_k^{(2)}\|^2 + \|P_k^{(3)}\|^2}, \quad (2.9)$$

$$\begin{pmatrix} X_{k+1} \\ Y_{k+1} \\ Z_{k+1} \end{pmatrix} = \begin{pmatrix} X_k \\ Y_k \\ Z_k \end{pmatrix} + \alpha_k \begin{pmatrix} P_k^{(1)} \\ P_k^{(2)} \\ P_k^{(3)} \end{pmatrix}. \quad (2.10)$$

Step 3 Calculate

$$R_{k+1} = \begin{pmatrix} R_{k+1}^{(1)} \\ R_{k+1}^{(2)} \\ R_{k+1}^{(3)} \end{pmatrix} = \begin{pmatrix} G_1 - A_1 X_{k+1}^T B_1 - C_1 Y_{k+1} D_1 - E_1 Z_{k+1} F_1 \\ G_2 - A_2 X_{k+1} B_2 - C_2 Y_{k+1}^T D_2 - E_2 Z_{k+1} F_2 \\ G_3 - A_3 X_{k+1} B_3 - C_3 Y_{k+1} D_3 - E_3 Z_{k+1}^T F_3 \end{pmatrix}, \quad (2.11)$$

$$\tilde{R}_{k+1} = \begin{pmatrix} B_1(R_{k+1}^{(1)})^T A_1 + A_2^T R_{k+1}^{(2)} B_2^T + A_3^T R_{k+1}^{(3)} B_3^T \\ C_1^T R_{k+1}^{(1)} D_1^T + D_2(R_{k+1}^{(2)})^T C_2 + C_3^T R_{k+1}^{(3)} D_3^T \\ E_1^T R_{k+1}^{(1)} F_1^T + E_2^T R_{k+1}^{(2)} F_2^T + F_3(R_{k+1}^{(3)})^T E_3 \end{pmatrix} := \begin{pmatrix} \tilde{R}_{k+1}^{(1)} \\ \tilde{R}_{k+1}^{(2)} \\ \tilde{R}_{k+1}^{(3)} \end{pmatrix}, \quad (2.12)$$

$$\beta_{k+1} = \frac{\|R_{k+1}\|^2}{\|R_k\|^2} = \frac{\|R_{k+1}^{(1)}\|^2 + \|R_{k+1}^{(2)}\|^2 + \|R_{k+1}^{(3)}\|^2}{\|R_k^{(1)}\|^2 + \|R_k^{(2)}\|^2 + \|R_k^{(3)}\|^2}, \quad (2.13)$$

$$P_{k+1} = \frac{1}{4} \begin{pmatrix} (\tilde{R}_{k+1}^{(1)} + (\tilde{R}_{k+1}^{(1)})^T) + S(\tilde{R}_{k+1}^{(1)} + (\tilde{R}_{k+1}^{(1)})^T)S \\ (\tilde{R}_{k+1}^{(2)} + (\tilde{R}_{k+1}^{(2)})^T) + S(\tilde{R}_{k+1}^{(2)} + (\tilde{R}_{k+1}^{(2)})^T)S \\ (\tilde{R}_{k+1}^{(3)} + (\tilde{R}_{k+1}^{(3)})^T) + S(\tilde{R}_{k+1}^{(3)} + (\tilde{R}_{k+1}^{(3)})^T)S \end{pmatrix} + \beta_{k+1} P_k = \begin{pmatrix} P_{k+1}^{(1)} \\ P_{k+1}^{(2)} \\ P_{k+1}^{(3)} \end{pmatrix}. \quad (2.14)$$

Step 4 Set $k := k + 1$, return to Step 2.

Algorithm 2.2. (MCG method over skew-anti-symmetric matrix).

Step 1 Input matrices $A_1, A_2, A_3, C_1, C_2, C_3, E_1, E_2, E_3 \in \mathbb{R}^{m \times n}$, $B_1, B_2, B_3, D_1, D_2, D_3, F_1, F_2, F_3 \in \mathbb{R}^{n \times m}$, $G_1, G_2, G_3 \in \mathbb{R}^{m \times m}$, quasi-identity matrices $S \in \mathbb{R}^{n \times n}$, skew-anti-symmetric initial matrices $X_1, Y_1, Z_1 \in \mathbb{B}\mathbb{S}\mathbb{R}^{n \times n}$, and compute

$$R_1 = \begin{pmatrix} R_1^{(1)} \\ R_1^{(2)} \\ R_1^{(3)} \end{pmatrix} = \begin{pmatrix} G_1 - A_1 X_1^T B_1 - C_1 Y_1 D_1 - E_1 Z_1 F_1 \\ G_2 - A_2 X_1 B_2 - C_2 Y_1^T D_2 - E_2 Z_1 F_2 \\ G_3 - A_3 X_1 B_3 - C_3 Y_1 D_3 - E_3 Z_1^T F_3 \end{pmatrix}, \quad (2.15)$$

$$\tilde{R}_1 = \begin{pmatrix} \tilde{R}_1^{(1)} \\ \tilde{R}_1^{(2)} \\ \tilde{R}_1^{(3)} \end{pmatrix} = \begin{pmatrix} B_1(R_1^{(1)})^T A_1 + A_2^T R_1^{(2)} B_2^T + A_3^T R_1^{(3)} B_3^T \\ C_1^T R_1^{(1)} D_1^T + D_2(R_1^{(2)})^T C_2 + C_3^T R_1^{(3)} D_3^T \\ E_1^T R_1^{(1)} F_1^T + E_2^T R_1^{(2)} F_2^T + F_3(R_1^{(3)})^T E_3 \end{pmatrix}, \quad (2.16)$$

$$P_1 = \frac{1}{4} \begin{pmatrix} (\tilde{R}_1^{(1)} + (\tilde{R}_1^{(1)})^T) - S(\tilde{R}_1^{(1)} + (\tilde{R}_1^{(1)})^T)S \\ (\tilde{R}_1^{(2)} + (\tilde{R}_1^{(2)})^T) - S(\tilde{R}_1^{(2)} + (\tilde{R}_1^{(2)})^T)S \\ (\tilde{R}_1^{(3)} + (\tilde{R}_1^{(3)})^T) - S(\tilde{R}_1^{(3)} + (\tilde{R}_1^{(3)})^T)S \end{pmatrix} := \begin{pmatrix} P_1^{(1)} \\ P_1^{(2)} \\ P_1^{(3)} \end{pmatrix}. \quad (2.17)$$

set $k := 1$.

Step 2 If $R_k = 0$, or $R_k \neq 0$ and $P_k = 0$, stop; otherwise, Compute

$$\alpha_k = \frac{\|R_k\|^2}{\|P_k\|^2} = \frac{\|R_k^{(1)}\|^2 + \|R_k^{(2)}\|^2 + \|R_k^{(3)}\|^2}{\|P_k^{(1)}\|^2 + \|P_k^{(2)}\|^2 + \|P_k^{(3)}\|^2}, \quad (2.18)$$

$$\begin{pmatrix} X_{k+1} \\ Y_{k+1} \\ Z_{k+1} \end{pmatrix} = \begin{pmatrix} X_k \\ Y_k \\ Z_k \end{pmatrix} + \alpha_k \begin{pmatrix} P_k^{(1)} \\ P_k^{(2)} \\ P_k^{(3)} \end{pmatrix}. \quad (2.19)$$

Step 3 Calculate

$$R_{k+1} = \begin{pmatrix} R_{k+1}^{(1)} \\ R_{k+1}^{(2)} \\ R_{k+1}^{(3)} \end{pmatrix} = \begin{pmatrix} G_1 - A_1 X_{k+1}^T B_1 - C_1 Y_{k+1} D_1 - E_1 Z_{k+1} F_1 \\ G_2 - A_2 X_{k+1} B_2 - C_2 Y_{k+1}^T D_2 - E_2 Z_{k+1} F_2 \\ G_3 - A_3 X_{k+1} B_3 - C_3 Y_{k+1} D_3 - E_3 Z_{k+1}^T F_3 \end{pmatrix}, \quad (2.20)$$

$$\tilde{R}_{k+1} = \begin{pmatrix} B_1 (R_{k+1}^{(1)})^T A_1 + A_2^T R_{k+1}^{(2)} B_2^T + A_3^T R_{k+1}^{(3)} B_3^T \\ C_1^T R_{k+1}^{(1)} D_1^T + D_2 (R_{k+1}^{(2)})^T C_2 + C_3^T R_{k+1}^{(3)} D_3^T \\ E_1^T R_{k+1}^{(1)} F_1^T + E_2^T R_{k+1}^{(2)} F_2^T + F_3 (R_{k+1}^{(3)})^T E_3 \end{pmatrix} := \begin{pmatrix} \tilde{R}_{k+1}^{(1)} \\ \tilde{R}_{k+1}^{(2)} \\ \tilde{R}_{k+1}^{(3)} \end{pmatrix}, \quad (2.21)$$

$$\beta_{k+1} = \frac{\|R_{k+1}\|^2}{\|R_k\|^2} = \frac{\|R_{k+1}^{(1)}\|^2 + \|R_{k+1}^{(2)}\|^2 + \|R_{k+1}^{(3)}\|^2}{\|R_k^{(1)}\|^2 + \|R_k^{(2)}\|^2 + \|R_k^{(3)}\|^2}, \quad (2.22)$$

$$P_{k+1} = \frac{1}{4} \begin{pmatrix} (\tilde{R}_{k+1}^{(1)} + (\tilde{R}_{k+1}^{(1)})^T) - S(\tilde{R}_{k+1}^{(1)} + (\tilde{R}_{k+1}^{(1)})^T)S \\ (\tilde{R}_{k+1}^{(2)} + (\tilde{R}_{k+1}^{(2)})^T) - S(\tilde{R}_{k+1}^{(2)} + (\tilde{R}_{k+1}^{(2)})^T)S \\ (\tilde{R}_{k+1}^{(3)} + (\tilde{R}_{k+1}^{(3)})^T) - S(\tilde{R}_{k+1}^{(3)} + (\tilde{R}_{k+1}^{(3)})^T)S \end{pmatrix} + \beta_{k+1} P_k = \begin{pmatrix} P_{k+1}^{(1)} \\ P_{k+1}^{(2)} \\ P_{k+1}^{(3)} \end{pmatrix}. \quad (2.23)$$

Step 4 Set $k := k + 1$, return to Step 2.

Remark 2.2. The definition of bisymmetric matrix shows that the sequences $\{X_k\}$, $\{Y_k\}$ and $\{Z_k\}$ generated by algorithm 2.1 are bisymmetric matrices. Analogous, the definition of skew-anti-symmetric matrix shows that the sequences $\{X_k\}$, $\{Y_k\}$ and $\{Z_k\}$ generated by algorithm 2.2 are skew-anti-symmetric matrices.

In order to prove that the sequences $\{X_k, Y_k, Z_k\}$ generated by algorithm 2.1 converges to the solution (X^*, Y^*, Z^*) , We can list the following results see [50] and lemmas. Let $A, B \in \mathbb{R}^{m \times n}$, $C \in \mathbb{R}^{n \times m}$, $X \in \mathbb{R}^{n \times n}$, then the following equalities hold.

$$\langle A, B \rangle = \text{tr}(B^T A) = \text{tr}(AB^T) = \text{tr}(BA^T) = \text{tr}(A^T B) = \langle B, A \rangle = \langle A^T, B^T \rangle, \quad (2.24)$$

$$\langle A, BXC \rangle = \text{tr}(C^T X^T B^T A) = \text{tr}(X^T B^T AC^T) = \langle B^T AC^T, X \rangle. \quad (2.25)$$

Lemma 2.4. If $X \in \mathbb{B}\mathbb{S}\mathbb{R}^{n \times n}$, $R \in \mathbb{R}^{n \times n}$ and S be an n -order quasi-identity matrix, then

$$\langle X, \frac{(R + R^T) + S(R + R^T)S}{4} \rangle = \langle X, R \rangle. \quad (2.26)$$

Proof. By the quasi-identity matrix and definition 1.1, we can obtain $S^T = S$ and $SXS = X = X^T$, then

$$\begin{aligned} & \langle X, \frac{(R + R^T) + S(R + R^T)S}{4} \rangle \\ &= \frac{1}{4} (\langle X, R \rangle + \langle X, R^T \rangle + \langle X, SRS \rangle + \langle X, SR^T S \rangle) \\ &= \frac{1}{4} (\langle X, R \rangle + \langle X, R^T \rangle + \langle SXS, R \rangle + \langle SXS, R^T \rangle) \\ &= \langle X, R \rangle. \end{aligned} \quad \square$$

Lemma 2.5. For the sequences $\{R_k\}, \{\tilde{R}_k\}, \{P_k\}$, $\{\alpha_k\}$ and $\{\beta_k\}$ generated by algorithm 2.1, there is

$$\langle R_{k+1}, R_j \rangle = \langle R_k, R_j \rangle - \alpha_k \langle P_k, \tilde{R}_k \rangle, \quad k, j = 1, 2, \dots \quad (2.27)$$

Proof. By direct calculation, we can get

$$\begin{aligned} & \langle R_{k+1}, R_j \rangle \\ &= \langle R_{k+1}^{(1)}, R_j^{(1)} \rangle + \langle R_{k+1}^{(2)}, R_j^{(2)} \rangle + \langle R_{k+1}^{(3)}, R_j^{(3)} \rangle \\ &= \langle G_1 - A_1 X_{k+1}^T B_1 - C_1 Y_{k+1} D_1 - E_1 Z_{k+1} F_1, R_j^{(1)} \rangle \\ & \quad + \langle G_2 - A_2 X_{k+1} B_2 - C_2 Y_{k+1}^T D_2 - E_2 Z_{k+1} F_2, R_j^{(2)} \rangle \\ & \quad + \langle G_3 - A_3 X_{k+1} B_3 - C_3 Y_{k+1} D_3 - E_3 Z_{k+1}^T F_3, R_j^{(3)} \rangle \\ &= \langle G_1 - A_1 (X_k + \alpha_k P_k^{(1)})^T B_1 - C_1 (Y_k + \alpha_k P_k^{(2)}) D_1 - E_1 (Z_k + \alpha_k P_k^{(3)}) F_1, R_j^{(1)} \rangle \\ & \quad + \langle G_2 - A_2 (X_k + \alpha_k P_k^{(1)}) B_2 - C_2 (Y_k + \alpha_k P_k^{(2)})^T D_2 - E_2 (Z_k + \alpha_k P_k^{(3)}) F_2, R_j^{(2)} \rangle \\ & \quad + \langle G_3 - A_3 (X_k + \alpha_k P_k^{(1)}) B_3 - C_3 (Y_k + \alpha_k P_k^{(2)}) D_3 - E_3 (Z_k + \alpha_k P_k^{(3)})^T F_3, R_j^{(3)} \rangle \\ &= \langle G_1 - A_1 X_k^T B_1 - \alpha_k A_1 (P_k^{(1)})^T B_1 - C_1 Y_k D_1 - \alpha_k C_1 P_k^{(2)} D_1 - E_1 Z_k F_1 \\ & \quad - \alpha_k E_1 P_k^{(3)} F_1, R_j^{(1)} \rangle + \langle G_2 - A_2 X_k B_2 - \alpha_k A_2 P_k^{(1)} B_2 - C_2 Y_k^T D_2 \\ & \quad - \alpha_k C_2 (P_k^{(2)})^T D_2 - E_2 Z_k F_2 - \alpha_k E_2 P_k^{(3)} F_2, R_j^{(2)} \rangle + \langle G_3 - A_3 X_k B_3 \\ & \quad - \alpha_k A_3 P_k^{(1)} B_3 - C_3 Y_k D_3 - \alpha_k C_3 P_k^{(2)} D_3 - E_3 Z_k^T F_3 - \alpha_k E_3 (P_k^{(3)})^T F_3, R_j^{(3)} \rangle \\ &= \langle G_1 - A_1 X_k^T B_1 - C_1 Y_k D_1 - E_1 Z_k F_1, R_j^{(1)} \rangle \\ & \quad - \alpha_k \langle A_1 (P_k^{(1)})^T B_1 + C_1 P_k^{(2)} D_1 + E_1 P_k^{(3)} F_1, R_j^{(1)} \rangle \\ & \quad + \langle G_2 - A_2 X_k B_2 - C_2 Y_k^T D_2 - E_2 Z_k F_2, R_j^{(2)} \rangle \\ & \quad - \alpha_k \langle A_2 P_k^{(1)} B_2 + C_2 (P_k^{(2)})^T D_2 + E_2 P_k^{(3)} F_2, R_j^{(2)} \rangle \\ & \quad + \langle G_3 - A_3 X_k B_3 - C_3 Y_k D_3 - E_3 Z_k^T F_3, R_j^{(3)} \rangle \\ & \quad - \alpha_k \langle A_3 P_k^{(1)} B_3 + C_3 P_k^{(2)} D_3 + E_3 (P_k^{(3)})^T F_3, R_j^{(3)} \rangle \\ &= \langle R_k^{(1)}, R_j^{(1)} \rangle - \alpha_k (\langle A_1 (P_k^{(1)})^T B_1, R_j^{(1)} \rangle + \langle C_1 P_k^{(2)} D_1, R_j^{(1)} \rangle + \langle E_1 P_k^{(3)} F_1, R_j^{(1)} \rangle) \\ & \quad + \langle R_k^{(2)}, R_j^{(2)} \rangle - \alpha_k (\langle A_2 P_k^{(1)} B_2, R_j^{(2)} \rangle + \langle C_2 (P_k^{(2)})^T D_2, R_j^{(2)} \rangle + \langle E_2 P_k^{(3)} F_2, R_j^{(2)} \rangle) \\ & \quad + \langle R_k^{(3)}, R_j^{(3)} \rangle - \alpha_k (\langle A_3 P_k^{(1)} B_3, R_j^{(3)} \rangle + \langle C_3 P_k^{(2)} D_3, R_j^{(3)} \rangle + \langle E_3 (P_k^{(3)})^T F_3, R_j^{(3)} \rangle) \\ &= \langle R_k, R_j \rangle - \alpha_k (\langle B_1 (R_j^{(1)})^T A_1 + A_2^T R_j^{(2)} B_2^T + A_3^T R_j^{(3)} B_3^T, P_k^{(1)} \rangle) \\ & \quad - \alpha_k (\langle C_1^T R_j^{(1)} D_1^T + D_2 (R_j^{(2)})^T C_2 + C_3^T R_j^{(3)} D_3^T, P_k^{(2)} \rangle) \\ & \quad - \alpha_k (\langle E_1^T R_j^{(1)} F_1^T + E_2^T R_j^{(2)} F_2^T + F_3 (R_j^{(3)})^T E_3, P_k^{(3)} \rangle) \\ &= \langle R_k, R_j \rangle - \alpha_k (\langle \tilde{R}_k^{(1)}, P_k^{(1)} \rangle + \langle \tilde{R}_k^{(2)}, P_k^{(2)} \rangle + \langle \tilde{R}_k^{(3)}, P_k^{(3)} \rangle) \\ &= \langle R_k, R_j \rangle - \alpha_k \langle P_k, \tilde{R}_k \rangle. \quad \square \end{aligned}$$

Lemma 2.6. Let $k \geq 2$, for the sequences $\{R_i\}, \{\tilde{R}_i\}, \{P_i\}$, $\{\alpha_k\}$ and $\{\beta_k\}$ generated by algorithm 2.1 satisfy

$$\langle R_i, R_j \rangle = 0 \quad \text{and} \quad \langle P_i, P_j \rangle = 0, \quad (i, j = 1, 2, \dots, k, \quad 1 \leq j < i \leq k). \quad (2.28)$$

Proof. We use induction to prove Eq.(2.28). For $k = 2$, we have

$$\begin{aligned}
& \langle R_2, R_1 \rangle \\
&= \langle R_1, R_1 \rangle - \alpha_1 (\langle P_1^{(1)}, \tilde{R}_1^{(1)} \rangle + \langle P_1^{(2)}, \tilde{R}_1^{(2)} \rangle + \langle P_1^{(3)}, \tilde{R}_1^{(3)} \rangle) \\
&= \|R_1\|^2 - \alpha_1 (\langle P_1^{(1)}, \frac{1}{4}((\tilde{R}_1^{(1)} + (\tilde{R}_1^{(1)})^T) + S(\tilde{R}_1^{(1)} + (\tilde{R}_1^{(1)})^T)S) \rangle) \\
&\quad + \langle P_1^{(2)}, \frac{1}{4}((\tilde{R}_1^{(2)} + (\tilde{R}_1^{(2)})^T) + S(\tilde{R}_1^{(2)} + (\tilde{R}_1^{(2)})^T)S) \rangle) \\
&\quad + \langle P_1^{(3)}, \frac{1}{4}((\tilde{R}_1^{(3)} + (\tilde{R}_1^{(3)})^T) + S(\tilde{R}_1^{(3)} + (\tilde{R}_1^{(3)})^T)S) \rangle) \\
&= \|R_1\|^2 - \frac{\|R_1\|^2}{\|P_1\|^2} (\langle P_1^{(1)}, P_1^{(1)} \rangle + \langle P_1^{(2)}, P_1^{(2)} \rangle + \langle P_1^{(3)}, P_1^{(3)} \rangle) \\
&= \|R_1\|^2 - \frac{\|R_1\|^2}{\|P_1\|^2} \|P_1\|^2 \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
& \langle P_2, P_1 \rangle \\
&= \langle \frac{1}{4}((\tilde{R}_2^{(1)} + (\tilde{R}_2^{(1)})^T) + S(\tilde{R}_2^{(1)} + (\tilde{R}_2^{(1)})^T)S) + \beta_2 P_1^{(1)}, P_1^{(1)} \rangle \\
&\quad + \langle \frac{1}{4}((\tilde{R}_2^{(2)} + (\tilde{R}_2^{(2)})^T) + S(\tilde{R}_2^{(2)} + (\tilde{R}_2^{(2)})^T)S) + \beta_2 P_1^{(2)}, P_1^{(2)} \rangle \\
&\quad + \langle \frac{1}{4}((\tilde{R}_2^{(3)} + (\tilde{R}_2^{(3)})^T) + S(\tilde{R}_2^{(3)} + (\tilde{R}_2^{(3)})^T)S) + \beta_2 P_1^{(3)}, P_1^{(3)} \rangle \\
&= \langle \frac{1}{4}((\tilde{R}_2^{(1)} + (\tilde{R}_2^{(1)})^T) + S(\tilde{R}_2^{(1)} + (\tilde{R}_2^{(1)})^T)S), P_1^{(1)} \rangle + \langle \beta_2 P_1^{(1)}, P_1^{(1)} \rangle \\
&\quad + \langle \frac{1}{4}((\tilde{R}_2^{(2)} + (\tilde{R}_2^{(2)})^T) + S(\tilde{R}_2^{(2)} + (\tilde{R}_2^{(2)})^T)S), P_1^{(2)} \rangle + \langle \beta_2 P_1^{(2)}, P_1^{(2)} \rangle \\
&\quad + \langle \frac{1}{4}((\tilde{R}_2^{(3)} + (\tilde{R}_2^{(3)})^T) + S(\tilde{R}_2^{(3)} + (\tilde{R}_2^{(3)})^T)S), P_1^{(3)} \rangle + \langle \beta_2 P_1^{(3)}, P_1^{(3)} \rangle \\
&= \langle P_1^{(1)}, \tilde{R}_2^{(1)} \rangle + \beta_2 \langle P_1^{(1)}, P_1^{(1)} \rangle + \langle P_1^{(2)}, \tilde{R}_2^{(2)} \rangle + \beta_2 \langle P_1^{(2)}, P_1^{(2)} \rangle \\
&\quad + \langle P_1^{(3)}, \tilde{R}_2^{(3)} \rangle + \beta_2 \langle P_1^{(3)}, P_1^{(3)} \rangle \\
&= \frac{1}{\alpha_1} (\langle R_1, R_2 \rangle - \langle R_1, R_2 \rangle) + \beta_2 (\langle P_1^{(1)}, P_1^{(1)} \rangle + \langle P_1^{(2)}, P_1^{(2)} \rangle + \langle P_1^{(3)}, P_1^{(3)} \rangle) \\
&= -\frac{\|P_1\|^2}{\|R_1\|^2} \|R_2\|^2 + \frac{\|R_2\|^2}{\|R_1\|^2} \|P_1\|^2 \\
&= 0.
\end{aligned}$$

Assuming that Eq.(2.28) holds for $k = s$, then for $k = s + 1$, we can obtain

$$\begin{aligned}
& \langle R_s, R_{s+1} \rangle \\
&= \langle R_s, R_s \rangle - \alpha_s (\langle P_s^{(1)}, \tilde{R}_s^{(1)} \rangle + \langle P_s^{(2)}, \tilde{R}_s^{(2)} \rangle + \langle P_s^{(3)}, \tilde{R}_s^{(3)} \rangle) \\
&= \|R_s\|^2 - \alpha_s (\langle P_s^{(1)}, \frac{1}{4}((\tilde{R}_s^{(1)} + (\tilde{R}_s^{(1)})^T) + S(\tilde{R}_s^{(1)} + (\tilde{R}_s^{(1)})^T)S) \rangle) \\
&\quad + \langle P_s^{(2)}, \frac{1}{4}((\tilde{R}_s^{(2)} + (\tilde{R}_s^{(2)})^T) + S(\tilde{R}_s^{(2)} + (\tilde{R}_s^{(2)})^T)S) \rangle) \\
&\quad + \langle P_s^{(3)}, \frac{1}{4}((\tilde{R}_s^{(3)} + (\tilde{R}_s^{(3)})^T) + S(\tilde{R}_s^{(3)} + (\tilde{R}_s^{(3)})^T)S) \rangle)
\end{aligned}$$

$$\begin{aligned}
 &= \|R_s\|^2 - \alpha_s(\langle P_s^{(1)}, P_s^{(1)} - \beta_s P_{s-1}^{(1)} \rangle + \langle P_s^{(2)}, P_s^{(2)} - \beta_s P_{s-1}^{(2)} \rangle \\
 &\quad + \langle P_s^{(3)}, P_s^{(3)} - \beta_s P_{s-1}^{(3)} \rangle) \\
 &= \|R_s\|^2 - \alpha_s(\langle P_s^{(1)}, P_s^{(1)} \rangle + \langle P_s^{(2)}, P_s^{(2)} \rangle + \langle P_s^{(3)}, P_s^{(3)} \rangle) \\
 &\quad - \beta_s(\langle P_s^{(1)}, P_{s-1}^{(1)} \rangle + \langle P_s^{(2)}, P_{s-1}^{(2)} \rangle + \langle P_s^{(3)}, P_{s-1}^{(3)} \rangle) \\
 &= \|R_s\|^2 - \frac{\|R_s\|^2}{\|P_s\|^2} (\|P_s\|^2 - \frac{\|R_{s+1}\|^2}{\|R_s\|^2} 0) \\
 &= 0
 \end{aligned}$$

and

$$\begin{aligned}
 &\langle P_{s+1}, P_s \rangle \\
 &= \langle P_{s+1}^{(1)}, P_s^{(1)} \rangle + \langle P_{s+1}^{(2)}, P_s^{(2)} \rangle + \langle P_{s+1}^{(3)}, P_s^{(3)} \rangle \\
 &= \langle \frac{1}{4}((\tilde{R}_{s+1}^{(1)} + (\tilde{R}_{s+1}^{(1)})^T) + S(\tilde{R}_{s+1}^{(1)} + (\tilde{R}_{s+1}^{(1)})^T)S) + \beta_{s+1}P_s^{(1)}, P_s^{(1)} \rangle \\
 &\quad + \langle \frac{1}{4}((\tilde{R}_{s+1}^{(2)} + (\tilde{R}_{s+1}^{(2)})^T) + S(\tilde{R}_{s+1}^{(2)} + (\tilde{R}_{s+1}^{(2)})^T)S) + \beta_{s+1}P_s^{(2)}, P_s^{(2)} \rangle \\
 &\quad + \langle \frac{1}{4}((\tilde{R}_{s+1}^{(3)} + (\tilde{R}_{s+1}^{(3)})^T) + S(\tilde{R}_{s+1}^{(3)} + (\tilde{R}_{s+1}^{(3)})^T)S) + \beta_{s+1}P_s^{(3)}, P_s^{(3)} \rangle \\
 &= \langle \frac{1}{4}((\tilde{R}_{s+1}^{(1)} + (\tilde{R}_{s+1}^{(1)})^T) + S(\tilde{R}_{s+1}^{(1)} + (\tilde{R}_{s+1}^{(1)})^T)S), P_s^{(1)} \rangle + \langle \beta_{s+1}P_s^{(1)}, P_s^{(1)} \rangle \\
 &\quad + \langle \frac{1}{4}((\tilde{R}_{s+1}^{(2)} + (\tilde{R}_{s+1}^{(2)})^T) + S(\tilde{R}_{s+1}^{(2)} + (\tilde{R}_{s+1}^{(2)})^T)S), P_s^{(2)} \rangle + \langle \beta_{s+1}P_s^{(2)}, P_s^{(2)} \rangle \\
 &\quad + \langle \frac{1}{4}((\tilde{R}_{s+1}^{(3)} + (\tilde{R}_{s+1}^{(3)})^T) + S(\tilde{R}_{s+1}^{(3)} + (\tilde{R}_{s+1}^{(3)})^T)S), P_s^{(3)} \rangle + \langle \beta_{s+1}P_s^{(3)}, P_s^{(3)} \rangle \\
 &= \langle \tilde{R}_{s+1}^{(1)}, P_s^{(1)} \rangle + \langle \tilde{R}_{s+1}^{(2)}, P_s^{(2)} \rangle + \langle \tilde{R}_{s+1}^{(3)}, P_s^{(3)} \rangle + \beta_{s+1}(\langle P_s^{(1)}, P_s^{(1)} \rangle \\
 &\quad + \langle P_s^{(2)}, P_s^{(2)} \rangle + \langle P_s^{(3)}, P_s^{(3)} \rangle) \\
 &= \frac{1}{\alpha_s}(\langle R_s, R_{s+1} \rangle - \langle R_{s+1}, R_{s+1} \rangle) + \beta_{s+1}(\langle P_s^{(1)}, P_s^{(1)} \rangle + \langle P_s^{(2)}, P_s^{(2)} \rangle + \langle P_s^{(3)}, P_s^{(3)} \rangle) \\
 &= -\frac{\|P_s\|^2}{\|R_s\|^2} \|R_{s+1}\|^2 + \frac{\|R_{s+1}\|^2}{\|R_s\|^2} \|P_s\|^2 \\
 &= 0.
 \end{aligned}$$

For $j = 1$, we have

$$\begin{aligned}
 \langle R_{s+1}, R_1 \rangle &= \langle R_s, R_1 \rangle - \alpha_s(\langle P_s^{(1)}, \tilde{R}_1^{(1)} \rangle + \langle P_s^{(2)}, \tilde{R}_1^{(2)} \rangle + \langle P_s^{(3)}, \tilde{R}_1^{(3)} \rangle) \\
 &= -\alpha_s(\langle P_s^{(1)}, \frac{1}{4}((\tilde{R}_1^{(1)} + (\tilde{R}_1^{(1)})^T) + S(\tilde{R}_1^{(1)} + (\tilde{R}_1^{(1)})^T)S) \rangle \\
 &\quad + \langle P_s^{(2)}, \frac{1}{4}((\tilde{R}_1^{(2)} + (\tilde{R}_1^{(2)})^T) + S(\tilde{R}_1^{(2)} + (\tilde{R}_1^{(2)})^T)S) \rangle \\
 &\quad + \langle P_s^{(3)}, \frac{1}{4}((\tilde{R}_1^{(3)} + (\tilde{R}_1^{(3)})^T) + S(\tilde{R}_1^{(3)} + (\tilde{R}_1^{(3)})^T)S) \rangle) \\
 &= -\alpha_s(\langle P_s^{(1)}, P_1^{(1)} \rangle + \langle P_s^{(2)}, P_1^{(2)} \rangle + \langle P_s^{(3)}, P_1^{(3)} \rangle) \\
 &= 0.
 \end{aligned}$$

Similarly, for $j = 2, 3, \dots, s - 1$, we can obtain

$$\langle R_{s+1}, R_j \rangle$$

$$\begin{aligned}
 &= \langle R_s, R_j \rangle - \alpha_s (\langle P_s^{(1)}, \tilde{R}_j^{(1)} \rangle + \langle P_s^{(2)}, \tilde{R}_j^{(2)} \rangle + \langle P_s^{(3)}, \tilde{R}_j^{(3)} \rangle) \\
 &= -\alpha_s (\langle P_s^{(1)}, \frac{1}{4} ((\tilde{R}_j^{(1)} + (\tilde{R}_j^{(1)})^T) + S(\tilde{R}_j^{(1)} + (\tilde{R}_j^{(1)})^T)S) \rangle \\
 &\quad + \langle P_s^{(2)}, \frac{1}{4} ((\tilde{R}_j^{(2)} + (\tilde{R}_j^{(2)})^T) + S(\tilde{R}_j^{(2)} + (\tilde{R}_j^{(2)})^T)S) \rangle \\
 &\quad + \langle P_s^{(3)}, \frac{1}{4} ((\tilde{R}_j^{(3)} + (\tilde{R}_j^{(3)})^T) + S(\tilde{R}_j^{(3)} + (\tilde{R}_j^{(3)})^T)S) \rangle) \\
 &= -\alpha_s (\langle P_s^{(1)}, P_j^{(1)} - \beta_j P_{j-1}^{(1)} \rangle + \langle P_s^{(2)}, P_j^{(2)} - \beta_j P_{j-1}^{(2)} \rangle + \langle P_s^{(3)}, P_j^{(3)} - \beta_j P_{j-1}^{(3)} \rangle) \\
 &= -\alpha_s (\langle P_s^{(1)}, P_j^{(1)} \rangle + \langle P_s^{(2)}, P_j^{(2)} \rangle + \langle P_s^{(3)}, P_j^{(3)} \rangle) \\
 &\quad - \beta_j (\langle P_s^{(1)}, P_{j-1}^{(1)} \rangle + \langle P_s^{(2)}, P_{j-1}^{(2)} \rangle + \langle P_s^{(3)}, P_{j-1}^{(3)} \rangle) \\
 &= 0.
 \end{aligned}$$

Besides, while $j = 2, 3, \dots, s - 1$, We can also obtain

$$\begin{aligned}
 &\langle P_{s+1}, P_j \rangle \\
 &= \langle P_{s+1}^{(1)}, P_j^{(1)} \rangle + \langle P_{s+1}^{(2)}, P_j^{(2)} \rangle + \langle P_{s+1}^{(3)}, P_j^{(3)} \rangle \\
 &= \langle \frac{1}{4} ((\tilde{R}_{s+1}^{(1)} + (\tilde{R}_{s+1}^{(1)})^T) + S(\tilde{R}_{s+1}^{(1)} + (\tilde{R}_{s+1}^{(1)})^T)S) + \beta_{s+1} P_s^{(1)}, P_j^{(1)} \rangle \\
 &\quad + \langle \frac{1}{4} ((\tilde{R}_{s+1}^{(2)} + (\tilde{R}_{s+1}^{(2)})^T) + S(\tilde{R}_{s+1}^{(2)} + (\tilde{R}_{s+1}^{(2)})^T)S) + \beta_{s+1} P_s^{(2)}, P_j^{(2)} \rangle \\
 &\quad + \langle \frac{1}{4} ((\tilde{R}_{s+1}^{(3)} + (\tilde{R}_{s+1}^{(3)})^T) + S(\tilde{R}_{s+1}^{(3)} + (\tilde{R}_{s+1}^{(3)})^T)S) + \beta_{s+1} P_s^{(3)}, P_j^{(3)} \rangle \\
 &= \langle \tilde{R}_{s+1}^{(1)}, P_j^{(1)} \rangle + \langle \tilde{R}_{s+1}^{(2)}, P_j^{(2)} \rangle + \langle \tilde{R}_{s+1}^{(3)}, P_j^{(3)} \rangle \\
 &\quad + \beta_{s+1} (\langle P_s^{(1)}, P_j^{(1)} \rangle + \langle P_s^{(2)}, P_j^{(2)} \rangle + \langle P_s^{(3)}, P_j^{(3)} \rangle) \\
 &= -\frac{1}{\alpha_j} (\langle R_j, R_{s+1} \rangle - \langle R_{j+1}, R_{s+1} \rangle) \\
 &= 0.
 \end{aligned}$$

By the properties of the inner product, we have

$$\begin{aligned}
 &\langle R_j, R_{s+1} \rangle = \langle R_{s+1}, R_j \rangle = 0, \quad j = 1, 2, \dots, s. \\
 &\langle P_j^{(1)}, P_{s+1}^{(1)} \rangle + \langle P_j^{(2)}, P_{s+1}^{(2)} \rangle + \langle P_j^{(3)}, P_{s+1}^{(3)} \rangle \\
 &= \langle P_{s+1}^{(1)}, P_j^{(1)} \rangle + \langle P_{s+1}^{(2)}, P_j^{(2)} \rangle + \langle P_{s+1}^{(3)}, P_j^{(3)} \rangle = 0.
 \end{aligned}$$

Thus, for $k = s + 1$, holds Eq.(2.28). According to the principle of mathematical induction, the lemma 2.6 holds. □

Lemma 2.7. *Let the matrix equations (1.6) be consistent over bisymmetric matrices, and (X^*, Y^*, Z^*) be an arbitrary bisymmetric solution of Eq.(1.6). Then, for any initial matrix (X_1, Y_1, Z_1) with $X_1 \in \mathbb{B}\mathbb{S}\mathbb{R}^{n \times n}, Y_1 \in \mathbb{B}\mathbb{S}\mathbb{R}^{n \times n}, Z_1 \in \mathbb{B}\mathbb{S}\mathbb{R}^{n \times n}$, the sequences $\{X_k\}, \{Y_k\}, \{Z_k\}, \{R_k\}, \{P_k^{(1)}\}, \{P_k^{(2)}\}$ and $\{P_k^{(3)}\}$ generated by Algorithm 2.1 satisfy*

$$\langle X^* - X_k, P_k^{(1)} \rangle + \langle Y^* - Y_k, P_k^{(2)} \rangle + \langle Z^* - Z_k, P_k^{(3)} \rangle = \|R_k\|^2, \quad (k = 1, 2, \dots). \tag{2.29}$$

Proof. By direct calculation, for $k = 1$, we have

$$\langle X^* - X_1, P_1^{(1)} \rangle + \langle Y^* - Y_1, P_1^{(2)} \rangle + \langle Z^* - Z_1, P_1^{(3)} \rangle$$

$$\begin{aligned}
&= \langle X^* - X_1, \frac{1}{4}((\tilde{R}_1^{(1)} + (\tilde{R}_1^{(1)})^T) + S(\tilde{R}_1^{(1)} + (\tilde{R}_1^{(1)})^T)S) \rangle \\
&\quad + \langle Y^* - Y_1, \frac{1}{4}((\tilde{R}_1^{(2)} + (\tilde{R}_1^{(2)})^T) + S(\tilde{R}_1^{(2)} + (\tilde{R}_1^{(2)})^T)S) \rangle \\
&\quad + \langle Z^* - Z_1, \frac{1}{4}((\tilde{R}_1^{(3)} + (\tilde{R}_1^{(3)})^T) + S(\tilde{R}_1^{(3)} + (\tilde{R}_1^{(3)})^T)S) \rangle \\
&= \langle X^* - X_1, \tilde{R}_1^{(1)} \rangle + \langle Y^* - Y_1, \tilde{R}_1^{(2)} \rangle + \langle Z^* - Z_1, \tilde{R}_1^{(3)} \rangle \\
&= \langle X^* - X_1, B_1(R_1^{(1)})^T A_1 + A_2^T R_1^{(2)} B_2^T + A_3^T R_1^{(3)} B_3^T \rangle \\
&\quad + \langle Y^* - Y_1, C_1^T R_1^{(1)} D_1^T + D_2(R_1^{(2)})^T C_2 + C_3^T R_1^{(3)} D_3^T \rangle \\
&\quad + \langle Z^* - Z_1, E_1^T R_1^{(1)} F_1^T + E_2^T R_1^{(2)} F_2^T + F_3(R_1^{(3)})^T E_3 \rangle \\
&= \langle X^* - X_1, B_1(R_1^{(1)})^T A_1 \rangle + \langle X^* - X_1, A_2^T R_1^{(2)} B_2^T \rangle + \langle X^* - X_1, A_3^T R_1^{(3)} B_3^T \rangle \\
&\quad + \langle Y^* - Y_1, C_1^T R_1^{(1)} D_1^T \rangle + \langle Y^* - Y_1, D_2(R_1^{(2)})^T C_2 \rangle + \langle Y^* - Y_1, C_3^T R_1^{(3)} D_3^T \rangle \\
&\quad + \langle Z^* - Z_1, E_1^T R_1^{(1)} F_1^T \rangle + \langle Z^* - Z_1, E_2^T R_1^{(2)} F_2^T \rangle + \langle Z^* - Z_1, F_3(R_1^{(3)})^T E_3 \rangle \\
&= \langle A_1(X^* - X_1)^T B_1, R_1^{(1)} \rangle + \langle C_1(Y^* - Y_1) D_1, R_1^{(1)} \rangle + \langle E_1(Z^* - Z_1) F_1, R_1^{(1)} \rangle \\
&\quad + \langle A_2(X^* - X_1) B_2, R_1^{(2)} \rangle + \langle C_2(Y^* - Y_1)^T D_2, R_1^{(2)} \rangle + \langle E_2(Z^* - Z_1) F_2, R_1^{(2)} \rangle \\
&\quad + \langle A_3(X^* - X_1) B_3, R_1^{(3)} \rangle + \langle C_3(Y^* - Y_1) D_3, R_1^{(3)} \rangle + \langle E_3(Z^* - Z_1)^T F_3, R_1^{(3)} \rangle \\
&= \langle A_1(X^* - X_1)^T B_1 + C_1(Y^* - Y_1) D_1 + E_1(Z^* - Z_1) F_1, R_1^{(1)} \rangle \\
&\quad + \langle A_2(X^* - X_1) B_2 + C_2(Y^* - Y_1)^T D_2 + E_2(Z^* - Z_1) F_2, R_1^{(2)} \rangle \\
&\quad + \langle A_3(X^* - X_1) B_3 + C_3(Y^* - Y_1) D_3 + E_3(Z^* - Z_1)^T F_3, R_1^{(3)} \rangle \\
&= \langle R_1^{(1)}, R_1^{(1)} \rangle + \langle R_1^{(2)}, R_1^{(2)} \rangle + \langle R_1^{(3)}, R_1^{(3)} \rangle \\
&= \|R_1\|^2.
\end{aligned}$$

Assume that (2.12) holds for $k = s (s \geq 2)$. Then for $k = s + 1$, we obtain

$$\begin{aligned}
&\langle X^* - X_{s+1}, P_s^{(1)} \rangle + \langle Y^* - Y_{s+1}, P_s^{(2)} \rangle + \langle Z^* - Z_{s+1}, P_s^{(3)} \rangle \\
&= \langle X^* - X_s - \alpha_s P_s^{(1)}, P_s^{(1)} \rangle + \langle Y^* - Y_s - \alpha_s P_s^{(2)}, P_s^{(2)} \rangle + \langle Z^* - Z_s - \alpha_s P_s^{(3)}, P_s^{(3)} \rangle \\
&= \langle X^* - X_s, P_s^{(1)} \rangle + \langle Y^* - Y_s, P_s^{(2)} \rangle + \langle Z^* - Z_s, P_s^{(3)} \rangle \\
&\quad - \alpha_s (\langle P_s^{(1)}, P_s^{(1)} \rangle + \langle P_s^{(2)}, P_s^{(2)} \rangle + \langle P_s^{(3)}, P_s^{(3)} \rangle) \\
&= \|R_s\|^2 - \frac{\|R_s\|^2}{\|P_s\|^2} \|P_s\|^2 \\
&= 0
\end{aligned}$$

so

$$\begin{aligned}
&\langle X^* - X_{s+1}, P_{s+1}^{(1)} \rangle + \langle Y^* - Y_{s+1}, P_{s+1}^{(2)} \rangle + \langle Z^* - Z_{s+1}, P_{s+1}^{(3)} \rangle \\
&= \langle X^* - X_{s+1}, \frac{1}{4}((\tilde{R}_{s+1}^{(1)} + (\tilde{R}_{s+1}^{(1)})^T) + S(\tilde{R}_{s+1}^{(1)} + (\tilde{R}_{s+1}^{(1)})^T)S) + \beta_{s+1} P_s^{(1)} \rangle \\
&\quad + \langle Y^* - Y_{s+1}, \frac{1}{4}((\tilde{R}_{s+1}^{(2)} + (\tilde{R}_{s+1}^{(2)})^T) + S(\tilde{R}_{s+1}^{(2)} + (\tilde{R}_{s+1}^{(2)})^T)S) + \beta_{s+1} P_s^{(2)} \rangle \\
&\quad + \langle Z^* - Z_{s+1}, \frac{1}{4}((\tilde{R}_{s+1}^{(3)} + (\tilde{R}_{s+1}^{(3)})^T) + S(\tilde{R}_{s+1}^{(3)} + (\tilde{R}_{s+1}^{(3)})^T)S) + \beta_{s+1} P_s^{(3)} \rangle \\
&= \langle X^* - X_{s+1}, \frac{1}{4}((\tilde{R}_{s+1}^{(1)} + (\tilde{R}_{s+1}^{(1)})^T) + S(\tilde{R}_{s+1}^{(1)} + (\tilde{R}_{s+1}^{(1)})^T)S) \rangle + \langle X^* - X_{s+1}, \beta_{s+1} P_s^{(1)} \rangle
\end{aligned}$$

$$\begin{aligned}
& + \langle Y^* - Y_{s+1}, \frac{1}{4}((\tilde{R}_{s+1}^{(2)} + (\tilde{R}_{s+1}^{(2)})^T) + S(\tilde{R}_{s+1}^{(2)} + (\tilde{R}_{s+1}^{(2)})^T)S) \rangle + \langle Y^* - Y_{s+1}, \beta_{s+1} P_s^{(2)} \rangle \\
& + \langle Z^* - Z_{s+1}, \frac{1}{4}((\tilde{R}_{s+1}^{(3)} + (\tilde{R}_{s+1}^{(3)})^T) + S(\tilde{R}_{s+1}^{(3)} + (\tilde{R}_{s+1}^{(3)})^T)S) \rangle + \langle Z^* - Z_{s+1}, \beta_{s+1} P_s^{(3)} \rangle \\
= & \langle X^* - X_{s+1}, \tilde{R}_{s+1}^{(1)} \rangle + \langle Y^* - Y_{s+1}, \tilde{R}_{s+1}^{(2)} \rangle + \langle Z^* - Z_{s+1}, \tilde{R}_{s+1}^{(3)} \rangle \\
= & \langle X^* - X_{s+1}, B_1(R_{s+1}^{(1)})^T A_1 + A_2^T R_{s+1}^{(2)} B_2^T + A_3^T R_{s+1}^{(3)} B_3^T \rangle \\
& + \langle Y^* - Y_{s+1}, C_1^T R_{s+1}^{(1)} D_1^T + D_2(R_{s+1}^{(2)})^T C_2 + C_3^T R_{s+1}^{(3)} D_3^T \rangle \\
& + \langle Z^* - Z_{s+1}, E_1^T R_{s+1}^{(1)} F_1^T + E_2^T R_{s+1}^{(2)} F_2^T + F_3(R_{s+1}^{(3)})^T E_3 \rangle \\
= & \langle X^* - X_{s+1}, B_1(R_{s+1}^{(1)})^T A_1 \rangle + \langle X^* - X_{s+1}, A_2^T R_{s+1}^{(2)} B_2^T \rangle + \langle X^* - X_{s+1}, A_3^T R_{s+1}^{(3)} B_3^T \rangle \\
& + \langle Y^* - Y_{s+1}, C_1^T R_{s+1}^{(1)} D_1^T \rangle + \langle Y^* - Y_{s+1}, D_2(R_{s+1}^{(2)})^T C_2 \rangle + \langle Y^* - Y_{s+1}, C_3^T R_{s+1}^{(3)} D_3^T \rangle \\
& + \langle Z^* - Z_{s+1}, E_1^T R_{s+1}^{(1)} F_1^T \rangle + \langle Z^* - Z_{s+1}, E_2^T R_{s+1}^{(2)} F_2^T \rangle + \langle Z^* - Z_{s+1}, F_3(R_{s+1}^{(3)})^T E_3 \rangle \\
= & \langle A_1(X^* - X_{s+1})^T B_1, R_{s+1}^{(1)} \rangle + \langle A_2(X^* - X_{s+1}) B_2, R_{s+1}^{(2)} \rangle + \langle A_3(X^* - X_{s+1}) B_3, R_{s+1}^{(3)} \rangle \\
& + \langle C_1(Y^* - Y_{s+1}) D_1, R_{s+1}^{(1)} \rangle + \langle C_2(Y^* - Y_{s+1})^T D_2, R_{s+1}^{(2)} \rangle + \langle C_3(Y^* - Y_{s+1}) D_3, R_{s+1}^{(3)} \rangle \\
& + \langle E_1(Z^* - Z_{s+1}) F_1, R_{s+1}^{(1)} \rangle + \langle E_2(Z^* - Z_{s+1}) F_2, R_{s+1}^{(2)} \rangle + \langle E_3(Z^* - Z_{s+1})^T F_3, R_{s+1}^{(3)} \rangle \\
= & \langle A_1(X^* - X_{s+1})^T B_1 + C_1(Y^* - Y_{s+1}) D_1 + E_1(Z^* - Z_{s+1}) F_1, R_{s+1}^{(1)} \rangle \\
& + \langle A_2(X^* - X_{s+1}) B_2 + C_2(Y^* - Y_{s+1})^T D_2 + E_2(Z^* - Z_{s+1}) F_2, R_{s+1}^{(2)} \rangle \\
& + \langle A_3(X^* - X_{s+1}) B_3 + C_3(Y^* - Y_{s+1}) D_3 + E_3(Z^* - Z_{s+1})^T F_3, R_{s+1}^{(3)} \rangle \\
= & \langle R_{s+1}^{(1)}, R_{s+1}^{(1)} \rangle + \langle R_{s+1}^{(2)}, R_{s+1}^{(2)} \rangle + \langle R_{s+1}^{(3)}, R_{s+1}^{(3)} \rangle \\
= & \|R_{s+1}\|^2.
\end{aligned}$$

Thus, according to the principle of induction, Eq.(2.29) holds. \square

Remark 2.3. Lemma 2.7 implies that if the generalized coupled Sylvester-transpose matrix equations (1.6) are consistent over bisymmetric matrices, then $\|P_k^{(1)}\|^2 + \|P_k^{(2)}\|^2 + \|P_k^{(3)}\|^2 \neq 0$ while $R_k \neq 0$. Conversely, we can easily see if there exists a positive number k such that $\|P_k^{(1)}\|^2 + \|P_k^{(2)}\|^2 + \|P_k^{(3)}\|^2 = 0$ but $R_k \neq 0$, then the generalized coupled Sylvester-transpose matrix equations (1.6) are not consistent over bisymmetric matrices.

Remark 2.4. The Lemmas 2.5–2.7 are obtained under the assumption that the initial matrices are the bisymmetric matrices. Analogously, if the initial matrices are skew-anti-symmetric matrices, the same results can be obtained. Therefore, we no longer present these results.

Theorem 2.1. *Suppose that the generalized coupled Sylvester-transpose matrix equations (1.6) are consistent over bisymmetric matrices. Then, for any initial bisymmetric matrix pair (X_1, Y_1, Z_1) , an exact solution of the system (1.6) can be derived within at most $3m^2 + 1$ iteration steps by Algorithm 2.1.*

Proof. If $R_k = 0$ ($k = 1, 2, \dots, 3m^2$), then (X_k, Y_k, Z_k) is the solution of Eqs.(1.6).

If $R_k \neq 0$ ($k = 1, 2, \dots, 3m^2$), it follows from Lemma 2.7 that $\|P_k^{(1)}\|^2 + \|P_k^{(2)}\|^2 + \|P_k^{(3)}\|^2 \neq 0$ ($k = 1, 2, \dots, 3m^2$). We can obtain R_{3m^2+1} by Algorithm

2.1. According to Lemma 2.6, we have

$$\langle R_i, R_{3m^2+1} \rangle = 0, (i = 1, 2, \dots, 3m^2), \langle R_i, R_j \rangle = 0, \quad (i \neq j; \quad i, j = 1, 2, \dots, 3m^2). \tag{2.30}$$

Therefore $\{R_1, R_2, \dots, R_{3m^2}\}$ is an orthogonal basis in the subspace

$$\mathbb{M} = \left\{ M | M = \begin{pmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{pmatrix}, \text{ where } M_i \in \mathbb{R}^{m \times m}, \text{ for } i = 1, 2, 3. \right\}.$$

Hence, $R_{3m^2+1} = 0$, that is $(X_{3m^2+1}, Y_{3m^2+1}, Z_{3m^2+1})$ is the solution of Eqs.(1.6). The proof is completed. \square

Note that if the generalized coupled Sylvester-transpose matrix equations (1.6) are consistent over bisymmetric matrices and the solution may not be unique, we need to find the least norm solution of Eqs.(1.6). In the following section, we will discuss the least norm solution of Eqs.(1.6).

3. The least norm solution

Firstly, we give the following lemmas in order to show that Eqs.(1.6) can obtain the least norm solution within finite iterative steps.

Lemma 3.1. *The generalized coupled Sylvester-transpose matrix equations (1.6) have a bisymmetric solution if and only if the matrix equations*

$$\left\{ \begin{array}{l} A_1 X^T B_1 + C_1 Y D_1 + E_1 Z F_1 = G_1 \\ B_1^T X^T A_1^T + D_1^T Y C_1^T + F_1^T Z E_1^T = G_1^T \\ A_1 S X^T S B_1 + C_1 S Y S D_1 + E_1 S Z S F_1 = G_1 \\ B_1^T S X^T S A_1^T + D_1^T S Y S C_1^T + F_1^T S Z S E_1^T = G_1^T \\ A_2 X B_2 + C_2 Y^T D_2 + E_2 Z F_2 = G_2 \\ B_2^T X A_2^T + D_2^T Y^T C_2^T + F_2^T Z E_2^T = G_2^T \\ A_2 S X S B_2 + C_2 S Y^T S D_2 + E_2 S Z S F_2 = G_2 \\ B_2^T S X S A_2^T + D_2^T S Y^T S C_2^T + F_2^T S Z S E_2^T = G_2^T \\ A_3 X B_3 + C_3 Y D_3 + E_3 Z^T F_3 = G_3 \\ B_3^T X A_3^T + D_3^T Y C_3^T + F_3^T Z^T E_3^T = G_3^T \\ A_3 S X S B_3 + C_3 S Y S D_3 + E_3 S Z^T S F_3 = G_3 \\ B_3^T S X S A_3^T + D_3^T S Y S C_3^T + F_3^T S Z^T S E_3^T = G_3^T \end{array} \right. \tag{3.1}$$

are consistent.

Proof. Firstly, suppose that the generalized coupled Sylvester-transpose matrix equations (1.6) have a bisymmetric solution pair (X^*, Y^*, Z^*) , then (X^*, Y^*, Z^*) satisfy $X^* = (X^*)^T = SX^*S$, $Y^* = (Y^*)^T = SY^*S$, $Z^* = (Z^*)^T = SZ^*S$ and

$$\begin{cases} A_1(X^*)^T B_1 + C_1 Y^* D_1 + E_1 Z^* F_1 = G_1, \\ A_2 X^* B_2 + C_2 (Y^*)^T D_2 + E_2 Z^* F_2 = G_2, \\ A_3 X^* B_3 + C_3 Y^* D_3 + E_3 (Z^*)^T F_3 = G_3. \end{cases} \quad (3.2)$$

Hence, we have

$$\begin{aligned} & B_1^T (X^*)^T A_1^T + D_1^T Y^* C_1^T + F_1^T Z^* E_1^T \\ &= B_1^T X^* A_1^T + D_1^T (Y^*)^T C_1^T + F_1^T (Z^*)^T E_1^T \\ &= G_1^T. \end{aligned} \quad (3.3)$$

$$\begin{aligned} & A_1 S (X^*)^T S B_1 + C_1 S Y^* S D_1 + E_1 S Z^* S F_1 \\ &= A_1 S X^* S B_1 + C_1 S Y^* S D_1 + E_1 S Z^* S F_1 \\ &= A_1 X^* B_1 + C_1 Y^* D_1 + E_1 Z^* F_1 \\ &= A_1 (X^*)^T B_1 + C_1 Y^* D_1 + E_1 Z^* F_1 \\ &= G_1. \end{aligned} \quad (3.4)$$

$$\begin{aligned} & B_1^T S (X^*)^T S A_1^T + D_1^T S Y^* S C_1^T + F_1^T S Z^* S E_1^T \\ &= B_1^T S X^* S A_1^T + D_1^T S Y^* S C_1^T + F_1^T S Z^* S E_1^T \\ &= B_1^T X^* A_1^T + D_1^T (Y^*)^T C_1^T + F_1^T (Z^*)^T E_1^T \\ &= G_1^T. \end{aligned} \quad (3.5)$$

$$\begin{aligned} & B_2^T X^* A_2^T + D_2^T (Y^*)^T C_2^T + F_2^T Z^* E_2^T \\ &= B_2^T (X^*)^T A_2^T + D_2^T Y^* C_2^T + F_2^T (Z^*)^T E_2^T \\ &= G_2^T. \end{aligned} \quad (3.6)$$

$$\begin{aligned} & A_2 S X^* S B_2 + C_2 S (Y^*)^T S D_2 + E_2 S Z^* S F_2 \\ &= A_2 X^* B_2 + C_2 S Y^* S D_2 + E_2 S Z^* S F_2 \\ &= A_2 X^* B_2 + C_2 Y^* D_2 + E_2 Z^* F_2 \\ &= A_2 X^* B_2 + C_2 (Y^*)^T D_2 + E_2 Z^* F_2 \\ &= G_2. \end{aligned} \quad (3.7)$$

$$\begin{aligned} & B_2^T S X^* S A_2^T + D_2^T S (Y^*)^T S C_2^T + F_2^T S Z^* S E_2^T \\ &= B_2^T X^* A_2^T + D_2^T S Y^* S C_2^T + F_2^T S Z^* S E_2^T \\ &= B_2^T (X^*)^T A_2^T + D_2^T Y^* C_2^T + F_2^T (Z^*)^T E_2^T \\ &= G_2^T. \end{aligned} \quad (3.8)$$

$$B_3^T X^* A_3^T + D_3^T Y^* C_3^T + F_3^T (Z^*)^T E_3^T$$

$$\begin{aligned}
&= B_3^T (X^*)^T A_3^T + D_3^T (Y^*)^T C_3^T + F_3^T Z^* E_3^T \\
&= G_3^T.
\end{aligned} \tag{3.9}$$

$$\begin{aligned}
&A_3 S X^* S B_3 + C_3 S Y^* S D_3 + E_3 S (Z^*)^T S F_3 \\
&= A_3 X^* B_3 + C_3 Y^* D_3 + E_3 S Z^* S F_3 \\
&= A_3 X^* B_3 + C_3 Y^* D_3 + E_3 Z^* F_3 \\
&= A_3 X^* B_3 + C_3 Y^* D_3 + E_3 (Z^*)^T F_3 \\
&= G_3.
\end{aligned} \tag{3.10}$$

$$\begin{aligned}
&B_3^T S X^* S A_3^T + D_3^T S Y^* S C_3^T + F_3^T S (Z^*)^T S E_3^T \\
&= B_3^T X^* A_3^T + D_3^T Y^* C_3^T + F_3^T S Z^* S E_3^T \\
&= B_3^T (X^*)^T A_3^T + D_3^T (Y^*)^T C_3^T + F_3^T Z^* E_3^T \\
&= G_3^T.
\end{aligned} \tag{3.11}$$

It can be seen from (3.2)–(3.11) that the bisymmetric solution pair (X^*, Y^*, Z^*) is a solution of Eqs.(3.1). On the other hand, suppose that the matrix Eqs.(3.1) are consistent. Let (X, Y, Z) be a solution pair of the matrix Eqs.(3.1). Set

$$\begin{cases} X_0 = \frac{1}{4}(X + X^T + SXS + SX^T S), \\ Y_0 = \frac{1}{4}(Y + Y^T + SY S + SY^T S), \\ Z_0 = \frac{1}{4}(Z + Z^T + SZS + SZ^T S). \end{cases} \tag{3.12}$$

So, $X_0, Y_0, Z_0 \in \mathbb{BSR}^{n \times n}$,

$$\begin{aligned}
&A_1 (X_0)^T B_1 + C_1 Y_0 D_1 + E_1 Z_0 F_1 \\
&= A_1 \left(\frac{1}{4}(X + X^T + SXS + SX^T S) \right)^T B_1 + C_1 \left(\frac{1}{4}(Y + Y^T + SY S + SY^T S) \right) D_1 \\
&\quad + E_1 \left(\frac{1}{4}(Z + Z^T + SZS + SZ^T S) \right) F_1 \\
&= \frac{1}{4} [(A_1 X^T B_1 + C_1 Y D_1 + E_1 Z F_1) + (A_1 X B_1 + C_1 Y^T D_1 \\
&\quad + E_1 Z^T F_1) \\
&\quad + (A_1 S X^T S B_1 + C_1 S Y S D_1 + E_1 S Z S F_1) + (A_1 S X S B_1 \\
&\quad + C_1 S Y^T S D_1 + E_1 S Z^T S F_1)]
\end{aligned} \tag{3.13}$$

$$\begin{aligned}
&= \frac{1}{4} [(A_1 X^T B_1 + C_1 Y D_1 + E_1 Z F_1) + (B_1^T X^T A_1^T + D_1^T Y C_1^T \\
&\quad + F_1^T Z E_1^T)^T \\
&\quad + (A_1 S X^T S B_1 + C_1 S Y S D_1 + E_1 S Z S F_1) + (B_1^T S X^T S A_1^T \\
&\quad + D_1^T S Y S C_1^T + F_1^T S Z S E_1^T)^T]
\end{aligned} \tag{3.14}$$

$$\begin{aligned}
&= \frac{1}{4} [(A_1 X^T B_1 + C_1 Y D_1 + E_1 Z F_1) + (B_1^T X^T A_1^T + D_1^T Y C_1^T \\
&\quad + F_1^T Z E_1^T)^T \\
&\quad + (A_1 S X^T S B_1 + C_1 S Y S D_1 + E_1 S Z S F_1) + (B_1^T S X^T S A_1^T \\
&\quad + D_1^T S Y S C_1^T + F_1^T S Z S E_1^T)^T]
\end{aligned} \tag{3.15}$$

$$\begin{aligned}
&= \frac{1}{4} [G_1 + (G_1^T)^T + G_1 + (G_1^T)^T] \\
&= G_1.
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
&= \frac{1}{4} [G_1 + (G_1^T)^T + G_1 + (G_1^T)^T] \\
&= G_1.
\end{aligned} \tag{3.17}$$

and similarly, we can obtain

$$A_2 X_0 B_2 + C_2 (Y_0)^T D_2 + E_2 Z_0 F_2 = G_2. \tag{3.18}$$

$$A_3X_0B_3 + C_3Y_0D_3 + E_3(Z_0)^T F_3 = G_3. \tag{3.19}$$

Thus, (X_0, Y_0, Z_0) is a bisymmetric solution pair of the generalized coupled Sylvester-transpose matrix equations (1.6). Which completes the proof. \square

Lemma 3.2 ([50]). *Assume that the linear matrix equation $Ax = b$ has a solution $x^* \in \mathcal{R}(A^T)$, then x^* is an unique least Frobenius norm solution of the system of linear equations.*

Theorem 3.1. *Suppose that the generalized coupled Sylvester-transpose matrix equations (1.6) are consistent over bisymmetric matrices. If we take the initial bisymmetric matrices*

$$\begin{aligned} X_1 = & B_1H_1^T A_1 + A_1^T H_1 B_1^T + SB_1H_1^T A_1 S + SA_1^T H_1 B_1^T S \\ & + A_2^T H_2 B_2^T + B_2 H_2^T A_2 + SA_2^T H_2 B_2^T S + SB_2 H_2^T A_2 S \\ & + A_3^T H_3 B_3^T + B_3 H_3^T A_3 + SA_3^T H_3 B_3^T S + SB_3 H_3^T A_3 S, \end{aligned} \tag{3.20}$$

$$\begin{aligned} Y_1 = & C_1^T H_1 D_1^T + D_1 H_1^T C_1 + SC_1^T H_1 D_1^T S + SD_1 H_1^T C_1 S \\ & + D_2 H_2^T C_2 + C_2^T H_2 D_2^T + SD_2 H_2^T C_2 S + SC_2^T H_2 D_2^T S \\ & + C_3^T H_3 D_3^T + D_3 H_3^T C_3 + SC_3^T H_3 D_3^T S + SD_3 H_3^T C_3 S, \end{aligned} \tag{3.21}$$

and

$$\begin{aligned} Z_1 = & E_1^T H_1 F_1^T + F_1 H_1^T E_1 + SE_1^T H_1 F_1^T S + SF_1 H_1^T E_1 S \\ & + E_2^T H_2 F_2^T + F_2 H_2^T E_2 + SE_2^T H_2 F_2^T S + SF_2 H_2^T E_2 S \\ & + F_3 H_3^T E_3 + E_3^T H_3 F_3^T + SF_3 H_3^T E_3 S + SE_3^T H_3 F_3^T S, \end{aligned} \tag{3.22}$$

where $H_1, H_2, H_3 \in \mathbb{R}^{n \times n}$ are arbitrary matrices (especially $X_1 = 0, Y_1 = 0$ and $Z_1 = 0$), then the bisymmetric solution (X^*, Y^*, Z^*) obtained by the Algorithm 2.1 is the unique least Frobenius norm bisymmetric solution.

Proof. The matrix Eqs.(3.1) are equivalent to

$$\underbrace{\begin{pmatrix} (B_1^T \otimes A_1)P_{nn} & D_1^T \otimes C_1 & F_1^T \otimes E_1 \\ A_1 \otimes B_1^T & C_1 \otimes D_1^T & E_1 \otimes F_1^T \\ (B_1^T S^T \otimes A_1 S)P_{nn} & D_1^T S^T \otimes C_1 S & F_1^T S^T \otimes E_1 S \\ A_1 S^T \otimes B_1^T S & C_1 S^T \otimes D_1^T S & E_1 S^T \otimes F_1^T S \\ B_2^T \otimes A_2 & (D_2^T \otimes C_2)P_{nn} & F_2^T \otimes E_2 \\ A_2 \otimes B_2^T & C_2 \otimes D_2^T & E_2 \otimes F_2^T \\ B_2^T S^T \otimes A_2 S & D_2^T S^T \otimes C_2 S & F_2^T S^T \otimes E_2 S \\ A_2 S^T \otimes B_2^T S & C_2 S^T \otimes D_2^T S & E_2 S^T \otimes F_2^T S \\ B_3^T \otimes A_3 & D_3^T \otimes C_3 & (F_3^T \otimes E_3)P_{nn} \\ A_3 \otimes B_3^T & C_3 \otimes D_3^T & E_3 \otimes F_3^T \\ B_3^T S^T \otimes A_3 S & D_3^T S^T \otimes C_3 S & F_3^T S^T \otimes E_3 S \\ A_3 S^T \otimes B_3^T S & C_3 S^T \otimes D_3^T S & E_3 S^T \otimes F_3^T S \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} \text{vec}(X) \\ \text{vec}(Y) \\ \text{vec}(Z) \end{pmatrix}}_{\mathbf{x}} = \underbrace{\begin{pmatrix} \text{vec}(G_1) \\ \text{vec}(G_1^T) \\ \text{vec}(G_1) \\ \text{vec}(G_1^T) \\ \text{vec}(G_2) \\ \text{vec}(G_2^T) \\ \text{vec}(G_2) \\ \text{vec}(G_2^T) \\ \text{vec}(G_3) \\ \text{vec}(G_3^T) \\ \text{vec}(G_3) \\ \text{vec}(G_3^T) \end{pmatrix}}_{\mathbf{b}}.$$

From relations (3.16) – (3.18) and Lemma 2.1, we can obtain

$$\begin{pmatrix} \text{vec}(X_1) \\ \text{vec}(Y_1) \\ \text{vec}(Z_1) \end{pmatrix} = \mathbf{A}^T \mathbf{c} \in \mathcal{R}(\mathbf{A}^T),$$

where

$$\mathbf{c} = [\text{vec}(H_1)^T, \text{vec}(H_1^T)^T, \text{vec}(H_2)^T, \text{vec}(H_2^T)^T, \text{vec}(H_3)^T, \text{vec}(H_3^T)^T, \text{vec}(H_3^T)^T, \text{vec}(H_3)^T, \text{vec}(H_3)^T, \text{vec}(H_3^T)^T]^T.$$

So, it is obvious that if we give (X_1, Y_1, Z_1) by the relations (3.16) – (3.18) respectively, then (X_k, Y_k, Z_k) generated by Algorithm 2.1 will satisfy

$$\begin{pmatrix} \text{vec}(X_k) \\ \text{vec}(Y_k) \\ \text{vec}(Z_k) \end{pmatrix} \in \mathcal{R}(\mathbf{A}^T).$$

Therefore from Lemma 3.2, with the initial bisymmetric matrices (3.16) – (3.18) (especially $X_1 = 0, Y_1 = 0$ and $Z_1 = 0$), then the bisymmetric solution (X^*, Y^*, Z^*) obtained by the Algorithm 2.1 is the unique least Frobenius norm bisymmetric solution. \square

4. Numerical experiments

In this section, we give some numerical examples to support our Algorithms 2.1 and 2.2. The iterations have been carried out on MATLAB R2016a (9.0.). Due to the influence of round-off errors, we use a stopping criterion $\langle R_k, R_k \rangle < 10^{-11}$ or the number of iteration proposed exceeds 2000. In addition T denotes a zero matrix if $\|T\|_F \leq 10^{-11}$.

Example 4.1. Consider the Eqs.(1.6) with the following parameters:

$$A_1 = \begin{pmatrix} 1 & 1 & 2 & -3 & 4 \\ 3 & 4 & 2 & 2 & 1 \\ 0 & 4 & 7 & 2 & 4 \\ -1 & -1 & -1 & 2 & 4 \\ 4 & 4 & 3 & 2 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 2 & 3 & 1 & 2 \\ 0 & 1 & 2 & 3 & 1 \\ 4 & 4 & 2 & 1 & 3 \\ 1 & 0 & 0 & -1 & 0 \\ 2 & 4 & 5 & 3 & 2 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & 5 & 3 & 2 & 1 \\ 2 & 1 & 3 & 4 & 2 \\ 2 & 5 & 3 & 4 & 3 \\ 1 & 2 & 0 & 3 & 0 \\ -3 & -2 & 4 & 1 & 2 \end{pmatrix}, B_1 = \begin{pmatrix} 2 & 2 & 3 & 1 & 1 \\ 0 & 5 & 4 & -2 & -2 \\ 2 & 3 & 4 & 1 & 1 \\ 2 & 0 & 2 & 0 & 1 \\ -3 & -3 & 1 & 2 & 2 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 1 & 2 & 3 & -1 & -1 \\ 3 & 0 & 0 & 3 & 3 \\ 1 & 2 & 3 & 0 & -2 \\ -1 & -1 & 2 & 2 & 3 \\ 5 & 4 & 5 & 4 & 4 \end{pmatrix}, B_3 = \begin{pmatrix} 5 & 1 & 2 & 4 & 3 \\ 0 & 0 & 1 & 3 & 5 \\ -2 & -4 & 2 & 3 & 0 \\ 0 & 0 & 2 & 5 & 3 \\ 1 & 1 & 2 & 0 & 3 \end{pmatrix},$$

$$C_1 = \begin{pmatrix} 4 & 3 & 4 & 4 & 1 \\ -2 & -2 & 3 & 4 & 4 \\ 5 & 6 & 5 & 0 & 1 \\ 5 & 4 & 5 & 3 & 3 \\ 1 & 2 & 0 & 0 & 1 \end{pmatrix}, C_2 = \begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ 3 & 3 & 3 & 1 & 2 \\ 1 & 2 & 3 & -4 & -4 \\ 5 & 5 & 5 & 4 & 4 \\ 2 & -2 & -2 & -2 & 1 \end{pmatrix},$$

$$C_3 = \begin{pmatrix} 0 & 0 & 0 & 2 & 3 \\ 1 & 4 & 2 & 3 & -2 \\ 3 & 3 & 2 & 1 & 4 \\ 0 & 2 & 4 & 3 & 2 \\ 1 & 1 & 2 & 2 & 3 \end{pmatrix}, D_1 = \begin{pmatrix} -2 & -1 & -2 & 3 & 2 \\ 6 & 5 & 4 & 4 & 3 \\ 2 & 3 & 2 & 1 & 1 \\ 1 & 1 & 2 & 4 & 1 \\ 0 & 0 & 2 & 3 & 2 \end{pmatrix},$$

$$D_2 = \begin{pmatrix} 2 & 4 & 3 & 2 & 1 \\ 0 & -3 & -3 & -3 & 2 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 & 2 \end{pmatrix}, D_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ -2 & -2 & 3 & 1 & 0 \\ 5 & 2 & 3 & 1 & 4 \\ 3 & 3 & 5 & 1 & 0 \\ 2 & 5 & 1 & 1 & 3 \end{pmatrix},$$

$$E_1 = \begin{pmatrix} 2 & 0 & 1 & 0 & 3 \\ 3 & 3 & 2 & 0 & 1 \\ 1 & 4 & 2 & 0 & 3 \\ -1 & -1 & -1 & 2 & 4 \\ 2 & 3 & 1 & 0 & 4 \end{pmatrix}, E_2 = \begin{pmatrix} 2 & 2 & 3 & -1 & 0 \\ 3 & 4 & 2 & 1 & 2 \\ 1 & 2 & 3 & 0 & 4 \\ 0 & 0 & 2 & 1 & 3 \\ -3 & -3 & -2 & 0 & 1 \end{pmatrix},$$

$$E_3 = \begin{pmatrix} 4 & 3 & 4 & 4 & 1 \\ -2 & -2 & 3 & 4 & 4 \\ 5 & 6 & 5 & 0 & 1 \\ 5 & 4 & 5 & 3 & 3 \\ 1 & 2 & 0 & 0 & 1 \end{pmatrix}, F_1 = \begin{pmatrix} 9 & 1 & -5 & 9 & -6 \\ -6 & 1 & 9 & 4 & -3 \\ 0 & -4 & 4 & -1 & -6 \\ -3 & -3 & 3 & 2 & -4 \\ 8 & -5 & -5 & 0 & -3 \end{pmatrix},$$

$$\begin{aligned}
 F_2 &= \begin{pmatrix} 1 & 2 & 3 & 2 & 1 \\ 3 & 2 & 3 & 1 & 2 \\ 1 & 2 & 0 & -4 & -1 \\ 5 & 5 & 2 & 4 & 4 \\ 2 & -2 & -2 & -2 & 1 \end{pmatrix}, F_3 = \begin{pmatrix} 0 & 2 & 8 & -5 & -3 \\ -7 & -11 & 0 & 11 & -1 \\ 1 & 10 & -3 & 1 & 8 \\ 11 & -1 & 0 & 4 & -8 \\ 2 & -6 & -6 & -7 & -7 \end{pmatrix}, \\
 G_1 &= \begin{pmatrix} 310 & 211 & 531.25 & 462.5 & -105.5 \\ 252.25 & -33.25 & 636.75 & 436 & -393.25 \\ 591.75 & 294.75 & 983 & 634.25 & -212.75 \\ 269.5 & 260.5 & 381.75 & 440.5 & 132 \\ 247 & -13.75 & 654.25 & 349.5 & -402.5 \end{pmatrix}, \\
 G_2 &= \begin{pmatrix} 425.5 & 374.25 & 390 & 152 & 238.25 \\ 560.75 & 461.25 & 413 & 122.75 & 361.25 \\ 587.5 & 431 & 438.5 & 59.5 & 294.75 \\ 341.25 & 268 & 238.25 & 76 & 270.75 \\ 17.5 & 36.5 & 336 & 168.25 & -59.75 \end{pmatrix}, \\
 G_3 &= \begin{pmatrix} 489 & -34.25 & 211 & 602.75 & 41.5 \\ -38.5 & -236.5 & 412 & 721.5 & 468 \\ 893.25 & 24 & 329 & 768.75 & 177.75 \\ 694.75 & 93 & 220.25 & 544.5 & -40.25 \\ 295.75 & 32.5 & 194.75 & 242 & 73 \end{pmatrix}.
 \end{aligned}$$

It can be verified that the matrix equations are consistent over bisymmetric matrices. We give the initial matrix pair $X_1 = \text{zeros}(5)$, $Y_1 = \text{zeros}(5)$ and $Z_1 = \text{zeros}(5)$, by using Algorithm 2.1 and after 47 iterations, we obtain an approximation solution:

$$X_{47} = \begin{pmatrix} 1.0000 & 2.5000 & 2.2500 & 0.2500 & 4.0000 \\ 2.5000 & 3.0000 & 1.7500 & 0.5000 & 0.2500 \\ 2.2500 & 1.7500 & 7.0000 & 1.7500 & 2.2500 \\ 0.2500 & 0.5000 & 1.7500 & 3.0000 & 2.5000 \\ 4.0000 & 0.2500 & 2.2500 & 2.5000 & 1.0000 \end{pmatrix},$$

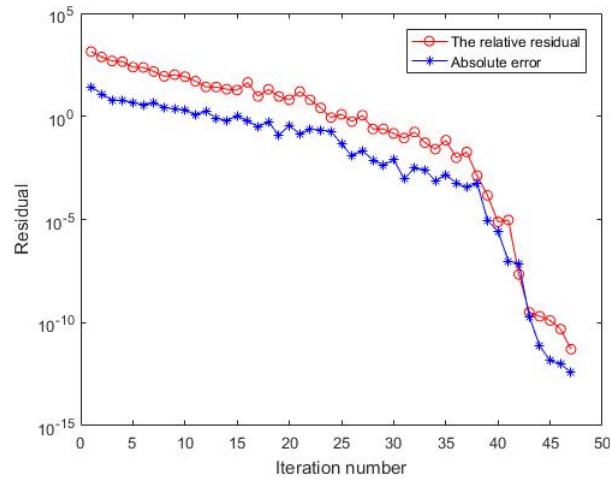


Figure 1. The relative residual of solution and absolute error for Example 4.1.

$$Y_{47} = \begin{pmatrix} 2.0000 & 1.2500 & 1.7500 & -0.5000 & -1.0000 \\ 1.2500 & 2.5000 & 2.5000 & -1.0000 & -0.5000 \\ 1.7500 & 2.5000 & 4.0000 & 2.5000 & 1.7500 \\ -0.5000 & -1.0000 & 2.5000 & 2.5000 & 1.2500 \\ -1.0000 & -0.5000 & 1.7500 & 1.2500 & 2.0000 \end{pmatrix},$$

$$Z_{47} = \begin{pmatrix} 2.5000 & 1.0000 & 2.5000 & 3.7500 & 1.0000 \\ 1.0000 & 0.5000 & 3.5000 & 4.0000 & 3.7500 \\ 2.5000 & 3.5000 & 5.0000 & 3.5000 & 2.5000 \\ 3.7500 & 4.0000 & 3.5000 & 0.5000 & 1.0000 \\ 1.0000 & 3.7500 & 2.5000 & 1.0000 & 2.5000 \end{pmatrix}.$$

It can be found that the solutions are bisymmetric matrices. The corresponding residual and norm are $\langle R_{47}, R_{47} \rangle = 5.1130 \times 10^{-12} < 10^{-11}$, $\|X_{47}\|_F = 12.6194$, $\|Y_{47}\|_F = 9.2195$, $\|Z_{47}\|_F = 14.3614$. Obviously, the residual is decreasing, and converges to zero as k increases.

Example 4.2. Consider the Eqs.(1.6) with the following parameters:

$$\begin{aligned} A_1 &= rand(6, 5), A_2 = rand(6, 5), A_3 = rand(6, 5), \\ C_1 &= rand(6, 5), C_2 = rand(6, 5), C_3 = rand(6, 5), \\ E_1 &= rand(6, 5), E_2 = rand(6, 5), E_3 = rand(6, 5), \\ B_1 &= rand(5, 6), B_2 = rand(5, 6), B_3 = rand(5, 6), \\ D_1 &= rand(5, 6), D_2 = rand(5, 6), D_3 = rand(5, 6), \\ F_1 &= rand(5, 6), F_2 = rand(5, 6), F_3 = rand(5, 6). \end{aligned}$$

$$\begin{aligned}
 G_1 &= \begin{pmatrix} -0.1098 & -0.1430 & 0.0171 & 0.3653 & 0.5944 & 0.1093 \\ -0.1287 & 0.1074 & 0.1152 & 0.6360 & 0.7853 & 0.0062 \\ 0.2176 & 0.2162 & 0.4111 & 0.8265 & 1.0628 & 0.2049 \\ -0.5194 & -0.3471 & -0.2358 & 0.5251 & 0.7770 & -0.3318 \\ -0.0405 & -0.3553 & 0.0746 & 0.3147 & 0.6653 & 0.0267 \\ -0.1072 & -0.2015 & 0.1530 & 0.3512 & 0.8265 & -0.1847 \end{pmatrix}, \\
 G_2 &= \begin{pmatrix} 0.3491 & -0.1726 & 0.7970 & 0.2929 & 0.1943 & 0.5616 \\ -0.3121 & -0.6128 & 0.0282 & -0.2286 & -0.2219 & -0.2461 \\ -0.3330 & -0.8524 & 0.4417 & 0.0495 & -0.2532 & -0.1729 \\ -0.0718 & -0.3082 & 0.5566 & 0.2659 & 0.0397 & -0.1756 \\ -0.1929 & -0.7605 & 0.5109 & 0.0474 & -0.2386 & -0.1568 \\ -0.1362 & -0.6230 & 0.5392 & 0.2721 & -0.0512 & 0.0933 \end{pmatrix}, \\
 G_3 &= \begin{pmatrix} -0.0215 & -0.3264 & -0.1229 & -0.2918 & 0.0055 & 0.0098 \\ -0.2465 & -0.2979 & -0.1955 & -0.4256 & -0.3940 & -0.0131 \\ 0.6081 & -0.0480 & 0.3560 & 0.0798 & 0.1328 & 0.5455 \\ 0.4044 & 0.0270 & 0.4137 & -0.3010 & 0.0788 & 0.3229 \\ 0.2152 & -0.3293 & -0.1993 & -0.1434 & -0.2349 & 0.2529 \\ 0.2297 & -0.2451 & -0.1258 & -0.0539 & -0.2423 & 0.3041 \end{pmatrix}.
 \end{aligned}$$

It can be verified that the matrix equations are consistent over skew-anti-bisymmetric matrices. We also give the initial matrix pair $X_1 = \text{zeros}(5)$, $Y_1 = \text{zeros}(5)$ and $Z_1 = \text{zeros}(5)$, by using Algorithm 2.2 and after 26 iterations, we obtain an approximation solution:

$$X_{26} = \begin{pmatrix} 0.1871 & 0.3671 & 0.0903 & -0.0184 & 0 \\ 0.3671 & 0.1333 & 0.2814 & 0 & 0.0184 \\ 0.0903 & 0.2814 & 0 & -0.2814 & -0.0903 \\ -0.0184 & 0 & -0.2814 & -0.1333 & -0.3671 \\ 0 & 0.0184 & -0.0903 & -0.3671 & -0.1871 \end{pmatrix},$$

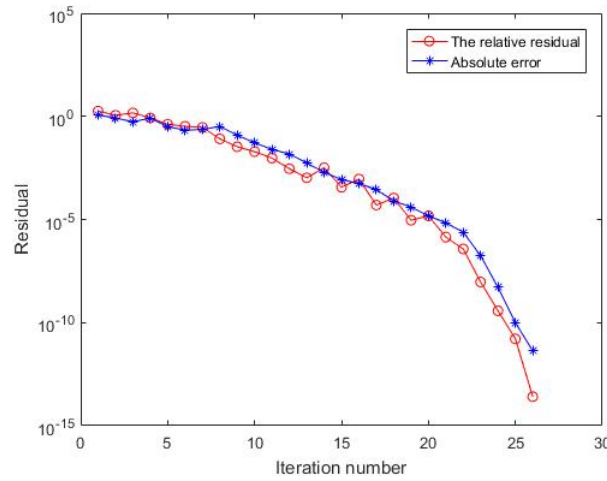


Figure 2. The relative residual of solution and absolute error for Example 4.2.

$$Y_{26} = \begin{pmatrix} 0.3562 & -0.0104 & -0.1042 & -0.3494 & 0 \\ -0.0104 & 0.0368 & 0.1493 & 0 & 0.3494 \\ -0.1042 & 0.1493 & 0 & -0.1493 & 0.1042 \\ -0.3494 & 0 & -0.1493 & -0.0368 & 0.0104 \\ 0 & 0.3494 & 0.1042 & 0.0104 & -0.3562 \end{pmatrix},$$

$$Z_{26} = \begin{pmatrix} 0.2944 & 0.1036 & -0.3227 & -0.0681 & 0 \\ 0.1036 & -0.0707 & 0.0706 & 0 & 0.0681 \\ -0.3227 & 0.0706 & 0 & -0.0706 & 0.3227 \\ -0.0681 & 0 & -0.0706 & 0.0707 & -0.1036 \\ 0 & 0.0681 & 0.3227 & -0.1036 & -0.2944 \end{pmatrix}.$$

It can be found that the solutions are skew-anti-symmetric matrices. The corresponding residual and norm are $\langle R_{26}, R_{26} \rangle = 2.5495 \times 10^{-14} < 10^{-11}$, $\|X_{26}\|_F = 0.9977$, $\|Y_{26}\|_F = 0.9369$, $\|Z_{26}\|_F = 0.8254$. The results show that Algorithm 2.2 is very effective.

Example 4.3. Consider the Eqs.(1.6) with the following parameters:

$$\begin{aligned} A_1 &= 5 * rand(8, 6), A_2 = 8 * rand(8, 6), A_3 = 10 * rand(8, 6), \\ C_1 &= 5 * rand(8, 6), C_2 = 8 * rand(8, 6), C_3 = 10 * rand(8, 6), \\ E_1 &= 5 * rand(8, 6), E_2 = 8 * rand(8, 6), E_3 = 10 * rand(8, 6), \\ B_1 &= 3 * rand(6, 8), B_2 = 5 * rand(6, 8), B_3 = 7 * rand(6, 8), \\ D_1 &= 3 * rand(6, 8), D_2 = 5 * rand(6, 8), D_3 = 7 * rand(6, 8), \\ F_1 &= 3 * rand(6, 8), F_2 = 5 * rand(6, 8), F_3 = 7 * rand(6, 8). \end{aligned}$$

$$G_1 = \begin{pmatrix} 748.2600 & 863.3073 & 918.3507 & 839.0616 & 748.3320 & 752.8479 & 689.9533 & 822.9110 \\ 493.1317 & 587.7734 & 621.4529 & 549.6430 & 509.0016 & 504.7149 & 482.6137 & 556.3684 \\ 720.0591 & 841.2069 & 889.2125 & 787.9809 & 720.2724 & 725.1263 & 682.5750 & 787.3118 \\ 669.9631 & 768.4453 & 818.7982 & 718.7071 & 645.2052 & 636.4028 & 631.3675 & 716.7971 \\ 617.2761 & 714.6759 & 761.2506 & 653.3713 & 601.7524 & 598.0074 & 605.6415 & 674.2656 \\ 679.9893 & 790.4894 & 842.8873 & 751.1477 & 677.9819 & 686.6785 & 635.8603 & 746.0114 \\ 579.0211 & 702.0206 & 714.1677 & 688.9691 & 622.2845 & 659.9964 & 512.9552 & 647.0491 \\ 514.0650 & 616.6019 & 644.4001 & 575.7067 & 533.4392 & 544.1444 & 497.7058 & 579.7017 \end{pmatrix}$$

$$G_2 = 1.0e+03 * \begin{pmatrix} 1.7810 & 1.3005 & 1.7502 & 1.1795 & 2.0822 & 1.6568 & 1.3947 & 1.4106 \\ 2.4573 & 1.7930 & 2.3957 & 1.7245 & 2.8116 & 2.2767 & 1.9491 & 1.9706 \\ 2.3510 & 1.7129 & 2.2985 & 1.5961 & 2.7428 & 2.1974 & 1.8540 & 1.8945 \\ 2.4917 & 1.7884 & 2.3987 & 1.7945 & 2.8196 & 2.2914 & 1.9655 & 2.0450 \\ 2.4116 & 1.7816 & 2.3182 & 1.7087 & 2.8115 & 2.1896 & 1.9281 & 1.9924 \\ 1.6493 & 1.1771 & 1.5579 & 1.2154 & 1.8183 & 1.4677 & 1.2889 & 1.4140 \\ 2.1586 & 1.5655 & 2.0887 & 1.5295 & 2.4072 & 1.9810 & 1.6709 & 1.7458 \\ 1.9742 & 1.4146 & 1.9282 & 1.3901 & 2.2053 & 1.8480 & 1.5389 & 1.5711 \end{pmatrix}$$

$$G_3 = 1.0e+03 * \begin{pmatrix} 2.6757 & 3.1779 & 3.6413 & 3.7805 & 3.8546 & 3.6195 & 4.1103 & 3.8252 \\ 2.2667 & 2.7039 & 3.0800 & 3.2902 & 3.3277 & 3.0012 & 3.4147 & 3.1766 \\ 1.7489 & 2.0667 & 2.4074 & 2.4462 & 2.5321 & 2.3942 & 2.7319 & 2.5216 \\ 2.2732 & 2.6145 & 3.1487 & 3.2032 & 3.3153 & 3.0691 & 3.5142 & 3.2655 \\ 2.5985 & 2.9795 & 3.4590 & 3.3938 & 3.7368 & 3.5901 & 4.0812 & 3.5882 \\ 2.4032 & 2.8133 & 3.3150 & 3.3022 & 3.5112 & 3.3378 & 3.8406 & 3.4637 \\ 2.0856 & 2.5210 & 2.8650 & 2.9030 & 3.0484 & 2.9101 & 3.2886 & 2.9677 \\ 1.7460 & 2.0679 & 2.4001 & 2.4785 & 2.5951 & 2.4019 & 2.7504 & 2.4305 \end{pmatrix}$$

It can be verified that the matrix equations are consistent over bisymmetric matrices. We give the initial matrix pair $X_1 = \text{zeros}(6)$, $Y_1 = \text{zeros}(6)$ and $Z_1 = \text{zeros}(6)$, by using Algorithm 2.1 and after 86 iterations, we get an approximation

solution:

$$\begin{aligned}
 X_{86} &= \begin{pmatrix} 0.6441 & 1.0011 & 0.9958 & 0.9093 & 0.6229 & 0.8441 \\ 1.0011 & 1.5290 & 0.6341 & 1.0364 & 1.2445 & 0.6229 \\ 0.9958 & 0.6341 & 0.2405 & 0.8783 & 1.0364 & 0.9093 \\ 0.9093 & 1.0364 & 0.8783 & 0.2405 & 0.6341 & 0.9958 \\ 0.6229 & 1.2445 & 1.0364 & 0.6341 & 1.5290 & 1.0011 \\ 0.8441 & 0.6229 & 0.9093 & 0.9958 & 1.0011 & 0.6441 \end{pmatrix}, \\
 Y_{86} &= \begin{pmatrix} 1.8429 & 2.0126 & 2.4309 & 1.4086 & 2.4384 & 2.8272 \\ 2.0126 & 2.3984 & 2.4191 & 0.6986 & 1.0152 & 2.4384 \\ 2.4309 & 2.4191 & 1.5214 & 1.3339 & 0.6986 & 1.4086 \\ 1.4086 & 0.6986 & 1.3339 & 1.5214 & 2.4191 & 2.4309 \\ 2.4384 & 1.0152 & 0.6986 & 2.4191 & 2.3984 & 2.0126 \\ 2.8272 & 2.4384 & 1.4086 & 2.4309 & 2.0126 & 1.8429 \end{pmatrix}, \\
 Z_{86} &= \begin{pmatrix} 1.5436 & 1.4309 & 1.8725 & 2.0589 & 1.8345 & 1.2618 \\ 1.4309 & 1.5217 & 1.4434 & 2.5959 & 2.0213 & 1.8345 \\ 1.8725 & 1.4434 & 3.6372 & 2.8915 & 2.5959 & 2.0589 \\ 2.0589 & 2.5959 & 2.8915 & 3.6372 & 1.4434 & 1.8725 \\ 1.8345 & 2.0213 & 2.5959 & 1.4434 & 1.5217 & 1.4309 \\ 1.2618 & 1.8345 & 2.0589 & 1.8725 & 1.4309 & 1.5436 \end{pmatrix}.
 \end{aligned}$$

It can be found that the solutions are bisymmetric matrices. The corresponding residual and norm are $\langle R_{86}, R_{86} \rangle = 9.5088 \times 10^{-12} < 10^{-11}$, $\|X_{86}\|_F = 5.5160$, $\|Y_{86}\|_F = 11.8934$, $\|Z_{86}\|_F = 12.3229$. We can observe that the residual is decreasing, and converges to zero as k increases.

5. Conclusions

In this paper, we studied MCG algorithm for solving a class of generalized coupled Sylvester-transpose matrix equations (1.6) over bisymmetric or skew-anti-symmetric matrices. We showed that if the generalized coupled Sylvester-transpose matrix equations (1.6) are consistent over bisymmetric matrices, an exact solution of the system (1.6) can be derived within finite iteration steps by Algorithm 2.1. Simultaneously, we also proved that if the matrix equations (1.6) are consistent over bisymmetric matrices, we can obtain the unique least Frobenius norm bisymmetric solution for the arbitrary initial bisymmetric matrices. Finally, some numerical examples were presented to demonstrate our theoretical analysis.

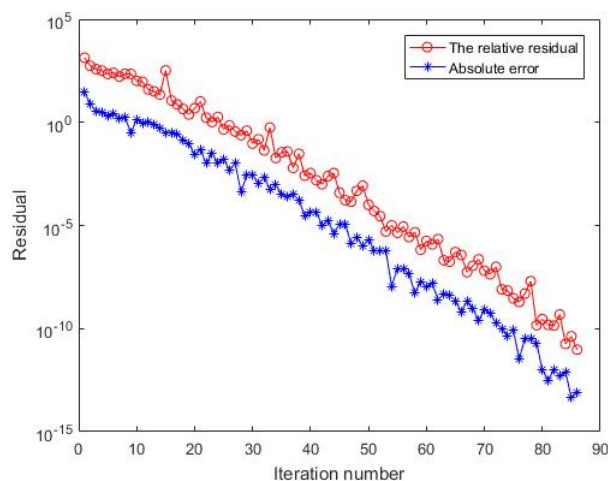


Figure 3. The relative residual of solution and absolute error for Example 4.3.

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